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FOURIER INTEGRAL OPERATORS IN SG CLASSES I COMPOSITION THEOREMS AND ACTION ON SG SOBOLEV SPACES

Abstract. A new class of Fourier Integral Operators (FIOs, for short) is defined. Phase and amplitude functions are chosen in **SG** symbol classes, the former with the additional requirements of being of order $(1, 1)$, real-valued and suitably growing at infinity. These FIOs turn out to be continuous on the spaces $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions. Results about the composition of **SG** FIOs with **SG** pseudodifferential operators and about the composition of a **SG** FIO with its L^2-adjoint are proved. These allow to obtain results about the existence of parametrices for elliptic FIOs, the continuity on the **SG** Sobolev Spaces and the wave front sets. As an example, the action of a **SG**-compatible change of variable on a **SG** pseudodifferential operator is reconsidered in terms of **SG** FIOs.

1. Introduction

Fourier Integral Operators† were systematically treated by Hörmander for the first time in [21], after having been initially used by Lax, Maslov, Egorov and others. The results in [21] were expanded in the paper [15] by Duistermaat and Hörmander, where they studied parametrices of ψdos of principal type and propagation of singularities. In the meantime, FIOs had also been applied to the study of hyperbolic equations and spectral theory.

The standard theory of FIOs is based on symbol classes which satisfy uniform estimates in compact sets over $\mathbb{R}^n$. It is very well suited for studying operators on compact manifolds, and also the Cauchy problem can be solved in a satisfactory way. A collection of techniques and results in this environment can be found, e.g., in Kumano-go [26]. However, problems arise when one tries to solve the same problems on noncompact manifolds. Also in the simple case of $\mathbb{R}^n$, we need decay of the symbol both in the variables and in the covariables if we want to achieve compactness of the remainder operators. A possible approach to the solution of the problem is based on amplitude classes satisfying, for all $\alpha, \beta, \gamma \in \mathbb{N}^n$ and $x, y, \xi \in \mathbb{R}^n$, the estimates

$$|\partial_\xi^\alpha \partial_\beta^x \partial_\gamma^y a(x, y, \xi)| \leq C_{\alpha\beta\gamma} \langle\xi\rangle^{m_1-|\alpha|} \langle x\rangle^{m_2-|\beta|} \langle y\rangle^{m_3-|\gamma|} . \tag{1}$$

$m = (m_1, m_2, m_3) \in \mathbb{R}^3$ is the "order" of the amplitude. Analogously, we can consider left-symbols $a(x, \xi)$ of double-order $m = (m_1, m_2) \in \mathbb{R}^2$. The associated Sobolev spaces H^s,

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†From now on, FIO will stand for Fourier Integral Operator and ψdo for pseudodifferential operator.

$s \in \mathbb{R}^2$ are defined in the canonical way. These concepts date back to the works of Shubin [43], Parenti [35] and Cordes [10] and the ψdos theory so obtained is very precise: in fact, the residual elements of the calculus associated to the amplitudes in (1) are the integral operators with Schwartz kernels, i.e., kernels in the Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. In Schrohe [38] the whole theory has been named "**SG**" calculus and transferred to a class of noncompact manifolds with a compatible structure, the so-called **SG** manifolds. Applications concerned the analysis of complex powers of elliptic operators on noncompact manifolds (Schrohe [39]) and the solution of boundary value problems for manifolds with noncompact boundary (Cordes and Erkip [12], Schrohe and Erkip [40], Schrohe [37]). A detailed discussion of the **SG** theory is given in Cordes [11]: the definition of the **SG** symbol classes and some of their properties are recalled in section 2.

We will be concerned here with a class of FIOs with phase and amplitude functions chosen in **SG** classes. As standard, these operators present two forms with analogous properties. Type I operators, for functions $u \in \mathcal{S}(\mathbb{R}^n)$, have the form

$$Au(x) = A_{\varphi,a}u(x) = \int đ\xi \ \mathrm{e}^{i\varphi(x,\xi)}\, a(x,\xi)\, \hat{u}(\xi), \tag{2}$$

while Type II operators are defined by

$$\widehat{Bu}(\xi) = \widehat{B_{\varphi,b}u}(\xi) = \int dx \ \mathrm{e}^{-i\varphi(x,\xi)}\, \overline{b(x,\xi)}\, u(x). \tag{3}$$

Here, as usual, $đ\xi = (2\pi)^{-n} d\xi$, while the phase function φ and the amplitudes a and b are chosen in **SG** symbol classes. More precisely, the phase function φ is chosen real valued in the $\mathbf{SG}_l^{(1,1)}$ class, with the additional property that its first derivatives satisfy a growth condition: there exist constants $c, C > 0$ such that

$$\begin{array}{rrcccl} & c\,\langle x\rangle & \le & \left\langle \nabla_\xi \varphi\right\rangle & \le & C\,\langle x\rangle \\ \text{and} & c\,\langle \xi\rangle & \le & \langle d_x\varphi\rangle & \le & C\,\langle \xi\rangle\,. \end{array} \tag{4}$$

The amplitudes a and b can be chosen in any $\mathbf{SG}_l^m$ class. We have set

$$\begin{aligned} \langle x\rangle^2 &= 1 + |x|^2 \text{ for } x \text{ vector or covector in } \mathbb{R}^n, \\ d_x\varphi &= \left(\frac{\partial\varphi}{\partial x^1}, \dots, \frac{\partial\varphi}{\partial x^n}\right) = (\partial_1\varphi, \dots, \partial_n\varphi) = (\partial_1^x\varphi, \dots, \partial_n^x\varphi), \\ \nabla_\xi\varphi &= \begin{pmatrix} \dfrac{\partial\varphi}{\partial\xi_1} \\ \dots \\ \dfrac{\partial\varphi}{\partial\xi_n} \end{pmatrix} = \begin{pmatrix} \partial^1\varphi \\ \dots \\ \partial^n\varphi \end{pmatrix} = \begin{pmatrix} \partial_\xi^1\varphi \\ \dots \\ \partial_\xi^n\varphi \end{pmatrix} \end{aligned}$$

and $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the integer, real and complex numbers sets, while $\mathbb{N}^\star$, $\mathbb{R}^\star$ and $\mathbb{C}^\star$ are the same sets without 0. It can be shown that ψdos of the form

$$Pu(x) = \int đ\xi \ \mathrm{e}^{i<x|\xi>}\, p(x,\xi)\, \hat{u}(\xi)$$

(for[‡] $< x|\xi >= x_i\xi^i$ and $p \in \mathbf{SG}_l^m$) map $\mathcal{S}(\mathbb{R}^n)$ continuously into itself and are extendable to linear continuous operators from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ (see [11] and references therein). The same is true for FIOs defined in (2) and (3), as we show in section 3.

[‡]Whenever it will be convenient, we will use the convention that expressions with repeated upper and lower indices denote summation over such indices, e.g., $x_i\xi^i = \sum_{i=1}^n x_i\xi^i$.

One of the main results proved in section 4 is the following theorem.

THEOREM 1 (COMPOSITION THEOREM). *Given a FIO $A = A_{\varphi,a}$ of Type I such that the real valued phase $\varphi \in \mathbf{SG}_l^{(1,1)}(\mathbb{R}^n)$ satisfies (4) with $a \in \mathbf{SG}_l^m(\mathbb{R}^n)$ and a ψdo $P = \mathrm{Op}\,(p)$ with $p \in \mathbf{SG}_l^t(\mathbb{R}^n)$, then the composed operator $H = PA$ is, modulo smoothing operators, a FIO of Type I. In fact, $H = H_{\varphi,h}$ where φ is the same phase function and the amplitude $h \in \mathbf{SG}_l^{m+t}(\mathbb{R}^n)$ admits the following asymptotic expansion:*

$$h(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}(\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi)) D_\alpha^y \left[\mathrm{e}^{i\psi(x,y,\xi)} a(y,\xi) \right]_{y=x} .$$

Here

$$\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \langle y - x | d_x\varphi(x,\xi) \rangle ,$$

and, as usual, $D_\alpha^y = (-i)^{|\alpha|}\partial_\alpha^y$.

As a first application, we reconsider in subsection 4.4 the **SG**-compatible change of variables for ψdos with symbol in $\mathbf{SG}_l^m(\mathbb{R}^n)$, cf. Schrohe [38]. In subsection 4.5 we analyze the action of these FIOs on the **SG** Sobolev spaces H^s, $s \in \mathbb{R}^2$, recovering the expected continuity results. In particular, FIOs with amplitude $a \in \mathbf{SG}_l^m$ map H^s continuously into H^{s-m} for all $m, s \in \mathbb{R}^2$. Finally, in subsection 4.3 elliptic FIOs are defined and in subsection 4.6 we consider the action of FIOs on wave fronts. Section A.1 of the appendix contains the proof by induction of formula (27) while section A.2 contains an alternative proof of the continuity of FIOs in the **SG** Sobolev spaces. For the detailed proofs of most of the cumbersome formulae used throughout the text, see, e.g., the appendix of Coriasco [14].

In a subsequent paper (Coriasco [13]), the solution of hyperbolic Cauchy problems in this environment will be given in terms of FIOs.

From now on, we will use the notations $f \prec g$ and $f(x) \prec g(x)$ to mean $\exists C > 0 \;:\; \forall x \; |f(x)| \le C|g(x)|$, while the notations $f \sim g$ and $f(x) \sim g(x)$ will mean $f \prec g \,\wedge\, g \prec f$. Notations concerning **SG** classes are recalled in section 2, where, for the sake of completeness, we also recall the notion of **SG** manifold. For convenience, when dealing with orders of symbols, we will often use the obvious notations $e = (1,1)$, $e_1 = (1,0)$ and $e_2 = (0,1)$. Analogously, $e = (1,1,1)$, $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ when the reference is to orders of amplitudes. If not explicitly otherwise stated, A will always stand for a Type I FIO of the form (2) with phase φ and amplitude a while B will stand for a Type II FIO of the form (3) with the same phase and amplitude b. In general, ψdos will be denoted by capital letters and their symbols or amplitudes by the corresponding small letter (i.e., $P = \mathrm{Op}\,(p)$, $q = \mathrm{Sym}\,(Q)$, etc.). The other notations are standard.

Comparing our results with the existing literature, we finally observe that general FIOs calculi exist already, see for example Liess and Rodino [27] and Bony [5], but they seem not applicable to the present situation. A natural question is whether our results may keep valid if **SG** symbol classes are replaced by more general Beals or Weyl-Hörmander classes in $\mathbb{R}^n$, see Beals [2], Hörmander [22] Vol. 3. As it concerns Theorem 1, it is certainly possible to extend it in some way to such situations, however the developments of the FIOs calculus in subsection 4.2 take advantage of the peculiarities of the **SG** structure, and of course **SG** changes of variables, treated in subsection 4.4, have not a counterpart in the Beals-Hörmander frame.

2. Definition and basic properties of SG symbol classes, operators and manifolds

DEFINITION 1. *For $m = (m_1, \ldots, m_N), r = (r_1, \ldots, r_N) \in \mathbb{R}^N$ write:*

$$\begin{aligned} m \geq r, \quad &\textit{if} \quad m_j \geq r_j,\ j = 1, \ldots, N; \\ m > r, \quad &\textit{if} \quad m_j > r_j,\ j = 1, \ldots, N. \end{aligned}$$

DEFINITION 2. *For $m = (m_1, m_2, m_3) \in \mathbb{R}^3$ we denote by $\mathbf{SG}^m(\mathbb{R}^n)$ the space of all amplitudes functions $a \in C^\infty\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\right)$ which satisfy the condition*

$$\forall \alpha, \beta, \gamma \in \mathbb{N}^n \ : \ \partial_\xi^\alpha \partial_\beta^x \partial_\gamma^y a(x, y, \xi) \prec \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|} \langle y \rangle^{m_3 - |\gamma|}. \tag{5}$$

$\mathbf{SG}^m(\mathbb{R}^n)$ is given the usual Fréchet topology based upon the seminorms implicit in (5). *Moreover, let us set*

$$\mathbf{SG}^\infty = \bigcup_{m \in \mathbb{R}^3} \mathbf{SG}^m, \quad \mathbf{SG}^{-\infty} = \bigcap_{m \in \mathbb{R}^3} \mathbf{SG}^m.$$

The functions $a \in \mathbf{SG}^m(\mathbb{R}^n)$ can be $(\nu \times \nu)$-matrix-valued (this will be useful to deal with systems) and the estimates must be valid for each entry of the matrix. We will write simply $\mathbf{SG}^m$ instead of $\mathbf{SG}^m(\mathbb{R}^n)$: the dimension of the base space is from now on fixed to n, and the base space is specified only if it is a manifold, a submanifold (according to the definitions in the following) or an open subset of $\mathbb{R}^n$.

DEFINITION 3. *For $m = (m_1, m_2) \in \mathbb{R}^2$ denote by $\mathbf{SG}_l^m(\mathbb{R}^n) = \mathbf{SG}_l^m$ the double-order symbol space of functions $a \in \mathbf{SG}^{(m_1, m_2, 0)}$ which are independent of y. As in Definition 2, let us set*

$$\mathbf{SG}_l^\infty = \bigcup_{m \in \mathbb{R}^2} \mathbf{SG}_l^m, \quad \mathbf{SG}_l^{-\infty} = \bigcap_{m \in \mathbb{R}^2} \mathbf{SG}_l^m.$$

DEFINITION 4. *A formal infinite sum $\sum_{j=1}^\infty a_j$ is an asymptotic expansion if*

1. $\forall j \in \mathbb{N}\ a_j \in \mathbf{SG}_l^{m_j}$;
2. $\forall j \in \mathbb{N}\ m_{j+1} \leq m_j$;
3. $\lim_{j \to \infty} m_j = (-\infty, -\infty)$.

We further write $a \sim \sum_{j=1}^\infty a_j$ when

$$\forall N \in \mathbb{N} \quad a - \sum_{j=1}^N a_j \in \mathbf{SG}_l^{m_{N+1}}.$$

DEFINITION 5. *We denote by $\Xi^\Delta(k)$ with $k > 0$ the set of all $\mathbf{SG}$-compatible cut-off functions which are equal to one in a suitable neighbourhood of the diagonal Δ, more precisely the set of all $\chi = \chi(x, y) \in \mathbf{SG}^{(0,0,0)}$ such that*

$$\begin{aligned} |y - x| \leq \frac{k}{2} \langle x \rangle \quad &\Rightarrow \quad \chi(x, y) = 1, \\ |y - x| > k \langle x \rangle \quad &\Rightarrow \quad \chi(x, y) = 0. \end{aligned}$$

If not otherwise stated, we will always assume $k \in (0; 1)$, which is what we will generally need when we will make use of these cut-off functions. $\Xi(R)$ with $R > 0$ will instead denote *the set of all* **SG***-compatible cut-off functions which vanish near the origin, i.e., the set of all* $\phi = \phi(x, \xi) \in \mathbf{SG}_l^{(0,0)}$ *such that*

$$|x| + |\xi| \geq R \quad \Rightarrow \quad \phi(x, \xi) = 1,$$
$$|x| + |\xi| \leq \frac{R}{2} \quad \Rightarrow \quad \phi(x, \xi) = 0.$$

PROPOSITION 1. *The sets* $\Xi^{\Delta}(k)$ *and* $\Xi(R)$ *are non-empty for any* $k, R > 0$.

DEFINITION 6. *To each amplitude* $p \in \mathbf{SG}^m$ *associate a linear operator* $P = \mathrm{Op}\,(p) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *defined as*

$$Pu(x) = \mathrm{Op}\,(p)\, u(x) = \int \int dy\, đ\xi \; \mathrm{e}^{i<x-y|\xi>}\, p(x, y, \xi)\, u(y). \tag{6}$$

Let us denote by $\mathbf{LG}^m$ *the space of all these operators, i.e.,*

$$\mathbf{LG}^m = \mathrm{Op}\left(\mathbf{SG}^m\right) = \left\{P \in \mathrm{Hom}\left(\mathcal{S}(\mathbb{R}^n)\right) \mid \exists p \in \mathbf{SG}^m \;:\; P = \mathrm{Op}\,(p)\right\}.$$

An element $P \in \mathbf{LG}^m$ *is called a* ψ*do , of order less or equal to* m.

DEFINITION 7. *For* $P \in \mathbf{LG}^m$ *we denote by* $p = \mathrm{Sym}\,(P) \in \mathbf{SG}_l^m$ *the symbol of* P*, that is* $P = \mathrm{Op}\,(p)$*. Moreover, we denote by* $\mathrm{Sym_p}\,(P)$ *the principal symbol of* P*, that is a* $p' \in \mathbf{SG}_l^m$ *such that* $p - p' \in \mathbf{SG}_l^{m-e}$.

DEFINITION 8. *Let* $\mathcal{K}$ *denote the space of linear integral operators having kernels in* $\mathcal{S}(\mathbb{R}^{2n})$*, i.e.,*

$$\mathcal{K} = \mathcal{K}(\mathbb{R}^n) = \left\{K \in \mathrm{Hom}(\mathcal{S}(\mathbb{R}^n)) \mid \exists k \in \mathcal{S}(\mathbb{R}^{2n}) \;:\; Kf(x) = \int dy\, k(x, y)\, f(y)\right\}$$

PROPOSITION 2. *Every* $K \in \mathcal{K}$ *extends to a linear continuous map* $K \;:\; \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

DEFINITION 9. *A symbol* $p \in \mathbf{SG}_l^m$ *and the corresponding operator* $P = \mathrm{Op}\,(p)$ *are called md-elliptic if there exists* $R > 0$ *such that*

$$|x| + |\xi| > R \Rightarrow p(x, \xi) \neq 0$$

and

$$|x| + |\xi| > R \Rightarrow [p(x, \xi)]^{-1} \prec \langle\xi\rangle^{-m_1} \langle x\rangle^{-m_2}.$$

Let us denote by $\mathbf{ESG}_l^m(\mathbb{R}^n) = \mathbf{ESG}_l^m$ *the subset of* $\mathbf{SG}_l^m$ *of all md-elliptic symbols of order m and by* $\mathbf{ELG}^m = \mathrm{Op}\left(\mathbf{ESG}_l^m\right)$ *the corresponding subset of md-elliptic operators. Analogously, an amplitude* $p \in \mathbf{SG}^m$ *and the corresponding operator* $P = \mathrm{Op}\,(p)$ *are called md-elliptic and we write* $p \in \mathbf{ESG}^m(\mathbb{R}^n) = \mathbf{ESG}^m$ *if*

$$\exists \widetilde{p} \in \mathbf{ESG}_l^{m'}, m' = (m_1, m_2 + m_3) \mid \mathrm{Op}\,(p) - \mathrm{Op}\,(\widetilde{p}) \in \mathcal{K}.$$

DEFINITION 10. *A* $\mathcal{K}$*-parametrix (or simply parametrix) of a* ψ*do* $P \in \mathbf{LG}^m$ *is a* ψ*do* Q *such that*

$$PQ - I,\, QP - I \in \mathcal{K},$$

where I *denotes the identity operator.*

In the following propositions we state some basic properties of the symbols and operators defined above. We also define and state some properties of the corresponding Sobolev spaces. Proofs can be found in [11], [38] and the references quoted therein.

PROPOSITION 3.

$$\forall m, m' \in \mathbb{R}^3 \quad : \quad m \le m' \Rightarrow \mathbf{SG}^m \subseteq \mathbf{SG}^{m'},$$
$$\forall m, m' \in \mathbb{R}^2 \quad : \quad m \le m' \Rightarrow \mathbf{SG}_l^m \subseteq \mathbf{SG}_l^{m'}.$$

Moreover, as direct consequence of the Leibniz rule and the Definition 2,

$$\forall p \in \mathbf{SG}^m, q \in \mathbf{SG}^r \ : \ pq \in \mathbf{SG}^{m+r}. \tag{7}$$

$$\forall p \in \mathbf{SG}^m \ \forall \alpha, \beta, \gamma \in \mathbb{N}^n \ : \ \partial_\xi^\alpha \partial_\beta^x \partial_\gamma^y p \in \mathbf{SG}^{m-|\alpha|e_1-|\beta|e_2-|\gamma|e_3} \tag{8}$$

PROPOSITION 4.

1. *The integral in* (6) *makes sense and, as already observed, defines an element of $\mathcal{L}(\mathcal{S})$ (i.e.,* $\mathrm{Op}\,(p)$ *is a linear continuous operator from $\mathcal{S}$ in itself), which is extendable to a linear continuous operator from $\mathcal{S}'$ in itself.*
2. *If in* (6) $p \in \mathbf{SG}_l^m$*, Definition 6 coincides with the usual one, i.e.,*

$$Op(p)u(x) = \int đ\xi \ \mathrm{e}^{i<x|\xi>} \ p(x,\xi)\,\hat{u}(\xi)$$

where $\hat{u}(\xi) = \mathcal{F}_{x\to\xi}(u)(\xi)$ is the Fourier transform of u.

3. $\mathbf{SG}^{-\infty} = \mathcal{S}(\mathbb{R}^{3n})$*,* $\mathbf{SG}_l^{-\infty} = \mathcal{S}(\mathbb{R}^{2n})$ *and* $\mathbf{LG}^{-\infty} = \mathcal{K}$*.*

DEFINITION 11. *For $s = (s_1, s_2) \in \mathbb{R}^2$ the symbol π_s denotes the product*

$$\pi_s(x,\xi) = \langle\xi\rangle^{s_1}\,\langle x\rangle^{s_2} \tag{9}$$

and $\Pi_s = \mathrm{Op}\,(\pi_s)$ the corresponding operator.

DEFINITION 12. *The associated family of weighted Sobolev spaces $H^s(\mathbb{R}^n) = H^s$, $s = (s_1, s_2) \in \mathbb{R}^2$, is defined in the canonical way:*

$$H^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \Pi_s u \in L^2(\mathbb{R}^n) \right\} \tag{10}$$

with the norm $\|u\|_s = \|\Pi_s u\|_{L^2} = \|\Pi_s u\|_0$.

PROPOSITION 5. *The following results govern the asymptotic expansions of symbols.*

1. *(Identification of symbols).*
For every asymptotic expansion $\sum_{j=1}^\infty p_j$ we have:

$$\begin{array}{ll} 1) & \exists p \in \mathbf{SG}_l^{m_1} \mid p \sim \sum_{j=1}^\infty p_j; \\ 2) & p' \sim \sum_{j=1}^\infty p_j \ \Rightarrow \ p - p' \in \mathcal{S}(\mathbb{R}^{2n}). \end{array} \tag{11}$$

2. *(Simplified criterion).*
If $p \in C^\infty(\mathbb{R}^{2n})$ satisfies

$$\forall \alpha, \beta \in \mathbb{N}^n \quad \exists k_1(\alpha), k_2(\beta) \in R \mid \partial_\xi^\alpha \partial_\beta^x p(x,\xi) \prec \langle \xi \rangle^{k_1(\alpha)} \langle x \rangle^{k_2(\beta)}$$

and

$$\begin{array}{rl} \exists \{l_r\}_{r \in \mathbb{N}, l_r \in \mathbb{R}} \mid & l_r \to -\infty \\ \textit{with} & p(x,\xi) - \sum_{j=1}^r p_j(x,\xi) \prec \langle \xi \rangle^{l_r} \langle x \rangle^{l_r}, \end{array}$$

then we have $p \sim \sum_{j=1}^\infty p_j$.

3. *(Existence of the symbol*[§] *)*

$$\forall p \in \mathbf{SG}^m \; \exists \widetilde{p} \in \mathbf{SG}_l^{m'}, m' = (m_1, m_2 + m_3) \mid \mathrm{Op}\,(p) = \mathrm{Op}\,(\widetilde{p})$$

and

$$\widetilde{p}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_\alpha^y \, p(x,y,\xi)|_{y=x}. \tag{12}$$

4. *(Symbol of the composition).*

$$\forall p \in \mathbf{SG}_l^m, q \in \mathbf{SG}_l^r \; \exists s \in \mathbf{SG}_l^{m+r} \mid \mathrm{Op}\,(p)\,\mathrm{Op}\,(q) = \mathrm{Op}\,(s)$$

and

$$s(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi) D_\alpha^x q(x,\xi). \tag{13}$$

In particular, the composition of two ψdos is a ψdo the order of which is the sum of the orders of the two operators.

5. *(Order of the commutator).*

$$\forall p \in \mathbf{SG}_l^m, q \in \mathbf{SG}_l^r \; R = [P, Q] \in \mathbf{LG}^{m+r-e}$$

and

$$\mathrm{Sym_p}\,(R)\,(x,\xi) = \sum_{|\alpha|=1} i(D_\xi^\alpha p(x,\xi) D_\alpha^x q(x,\xi) - D_\alpha^x p(x,\xi) D_\xi^\alpha q(x,\xi)).$$

6. *(Symbol of the L^2-adjoint*[¶] *).*

$$\forall p \in \mathbf{SG}_l^m \; \exists q \in \mathbf{SG}_l^m \mid \mathrm{Op}\,(p)^\star = \mathrm{Op}\,(q)$$

and

$$q(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_\alpha^x \overline{p(x,\xi)}. \tag{14}$$

[§] The equalities of operators like in points 3, 4 and 6 of Proposition 5 are to be understood "modulo $\mathcal{K}$".

[¶] The symbol * will also denote, in some parts of the sequel, the pull-back of an operator or of a function and the "adjoint" function $a^\star(x,\xi) = \overline{a(\xi,x)}$. The meaning of the symbol in the various situations is generally clear by the context, since we will never use pull-backs of adjoint operators and functions or adjoints of pull-backs of operators and functions.

PROPOSITION 6. *(Action on Sobolev spaces).*

1. $\forall P \in \mathbf{LG}^m \; : \; P \in \mathcal{L}\left(H^s, H^{s-m}\right)$.
2. *In particular, if* $s \geq r$ *then* H^s *is continuously embedded in* H^r*. If* $s > r$ *the embedding* $H^s \hookrightarrow H^r$ *is compact.*

PROPOSITION 7. *(Parametrix of md-elliptic operators).*
Every $P \in \mathbf{ELG}^m$ *admits a parametrix* $Q \in \mathbf{ELG}^{-m}$ *and is a Fredholm operator from* H^s *to* H^{s-m} *for every* $s \in \mathbb{R}^2$.

An important property of **SG** classes of symbols and operators is their invariance under coordinate changes in a suitable class of diffeomorphisms of open sets of $\mathbb{R}^n$, the **SG**-compatible diffeomorphisms (or **SG** diffeomorphisms). This is the content of Proposition 8 and Theorem 2 below. They allow us to transport the whole structure to a class of manifolds, the **SG** manifolds, which are all those manifolds having an **SG**-compatible atlas. The precise meaning of this is given in Definition 17. We introduce now all the necessary notions for these statements.

DEFINITION 13. *(Push-forward and pull-back of functions on open subsets of* $\mathbb{R}^n$*).*
Let us denote by $\mathcal{O}(\mathbb{R}^n)$ *the set of all open sets of* $\mathbb{R}^n$*. Let* $U, V \in \mathcal{O}(\mathbb{R}^n)$ *and let* $\phi \in C^\infty(U, V)$ *be a diffeomorphism of* U *onto* V *with inverse* $\overline{\phi} = \phi^{-1}$*. Let us denote this by* $\phi \in \mathrm{Diffeo}(U, V)$*. For any* $f \in C^\infty(U)$ *and* $g \in C^\infty(V)$ *define:*

$$f_\star = \phi_\star f \in C^\infty(V) \quad by \quad f_\star(y) = f \circ \overline{\phi}(y) = f(\overline{\phi}(y)); \tag{15}$$

$$g^\star = \phi^\star g \in C^\infty(U) \quad by \quad g^\star(x) = g \circ \phi(x) = g(\phi(x)). \tag{16}$$

DEFINITION 14. *(Push-forward and pull-back of functions on open subsets of* $T^\star\mathbb{R}^n$*).*
Let us take $U, V \in \mathcal{O}(\mathbb{R}^n)$*,* $\phi \in \mathrm{Diffeo}(U, V)$ *and* $f \in C^\infty(T^\star U)$*,* $g \in C^\infty(T^\star V)$*. Denote by* $\frac{\partial \phi}{\partial x}$ *the Jacobian matrix of the function* ϕ *and define* $f_\star \in C^\infty(T^\star V)$ *and* $g^\star \in C^\infty(T^\star U)$ *by*

$$\begin{aligned} f_\star(y, \eta) = \phi_\star f(y, \eta) &= f\left(\overline{\phi}(y), \eta \left.\frac{\partial \phi}{\partial x}\right|_{x=\overline{\phi}(y)}\right); \\ g^\star(x, \xi) = \phi^\star g(x, \xi) &= g\left(\phi(x), \xi \left.\frac{\partial \overline{\phi}}{\partial y}\right|_{y=\phi(x)}\right). \end{aligned}$$

The operators * and $_\star$ induce actions on $\mathrm{Hom}(C^\infty(U))$ and $\mathrm{Hom}(C^\infty(V))$ as described in the following definition.

DEFINITION 15. *For all* $P \in \mathrm{Hom}(C^\infty(U))$*,* $Q \in \mathrm{Hom}(C^\infty(V))$*,* $f \in C^\infty(U)$ *and* $g \in C^\infty(V)$ *define:*

$$\begin{aligned} P_\star = \phi_\star P \in \mathrm{Hom}(C^\infty(V)) \;\; by \;\; (P_\star g)(y) = (Pg^\star)(\overline{\phi}(y)) = (Pg^\star)_\star(y); \\ Q^\star = \phi^\star Q \in \mathrm{Hom}(C^\infty(U)) \;\; by \;\; (Q^\star f)(x) = (Qf_\star)(\phi(x)) = (Qf_\star)^\star(x). \end{aligned}$$

DEFINITION 16. *(**SG**-compatible diffeomorphisms).*
Let $\phi \in \mathrm{Diffeo}(U^\#, V^\#)$ *with* $U^\#, V^\# \in \mathcal{O}(\mathbb{R}^n)$ *satisfy*

$$\forall x \in U^\# \; \partial^x_\alpha \phi(x) \prec \langle x \rangle^{1-|\alpha|} \quad and \quad \forall y \in V^\# \; \partial^y_\alpha \overline{\phi}(y) \prec \langle y \rangle^{1-|\alpha|}. \tag{17}$$

Assume also

$$\exists U \subseteq U^{\#}, V \subseteq V^{\#}, \delta > 0 \mid \begin{array}{l} \phi|_U \in \mathrm{Diffeo}(U, V) \\ \forall x \in U \;\; B(x, \delta \langle x \rangle) \subset U^{\#} \\ \forall y \in V \;\; B(y, \delta \langle y \rangle) \subset V^{\#} \end{array} \tag{18}$$

where $B(x, r), x \in \mathbb{R}^n, r > 0$ *is the euclidean ball of* $\mathbb{R}^n$ *of center* x *and radius* r. *We will then say that* ϕ *is an* **SG***-compatible diffeomorphism (or* **SG** *diffeomorphism) and we will write* $\phi \in \mathbf{SG}\mathrm{Diffeo}(U^{\#}, V^{\#}; U, V; \delta)$.

PROPOSITION 8. *(Invariance of* $\mathbf{SG}^m_l$ *by the action of* **SG** *diffeomorphisms).*
For all $\phi \in \mathbf{SG}\mathrm{Diffeo}(U^{\#}, V^{\#}; U, V; \delta)$ *and all* $p \in \mathbf{SG}^m_l \mid \mathrm{supp}\,(p) \subseteq U \times \mathbb{R}^n$, $q \in \mathbf{SG}^m_l \mid \mathrm{supp}\,(q) \subseteq V \times \mathbb{R}^n$ *we have* $p_\star, q^\star \in \mathbf{SG}^m_l$.

THEOREM 2. *Let* $\phi \in \mathbf{SG}\mathrm{Diffeo}(U^{\#}, V^{\#}; U, V; \delta)$. *Then:*

$$\begin{aligned} Q = \mathrm{Op}\,(q) \mid q \in \mathbf{SG}^m_l, \mathrm{supp}\,(q) \subseteq V \times \mathbb{R}^n \Rightarrow & \\ \Rightarrow \exists p \in \mathbf{SG}^m_l \; \exists K \in \mathcal{K} \;\; \mid \;\; & \mathrm{supp}\,(p) \subseteq U \times \mathbb{R}^n, \\ & Q^\star = \mathrm{Op}\,(p) + K. \end{aligned}$$

Moreover, $p - q^\star \in \mathbf{SG}^{m'}_l$ *with* $m' < m$.

To complete this short survey on the **SG** calculus, we now describe the concept of **SG** manifold. In the following sections we will always work on $\mathbb{R}^n$.

DEFINITION 17. *(***SG** *manifolds).*
Let X *be an* n*-dimensional manifold. We will say that* X *is an* **SG***-compatible manifold (or an* **SG** *manifold) if*

1. X *has a finite atlas* $\mathcal{A}^{\#} = \left\{(X^{\#}_j, \phi^{\#}_j)\right\}_{j \in \{1,\dots,N\}}$, $\phi^{\#}_j : X^{\#}_j \to U^{\#}_j \in \mathcal{O}(\mathbb{R}^n)$*;*
2. $\mathcal{A}^{\#}$ *is shrinkable, i.e.,* $\exists \mathcal{A} = \left\{(X_j, \phi_j)\right\}_{j \in \{1,\dots,N\}}$*, atlas of* X *such that*

$$\begin{aligned} \forall j \in \{1, \dots, N\} \quad : \quad & X_j \subseteq X^{\#}_j \\ & \phi_j = \phi^{\#}_j\Big|_{X_j} \in \mathrm{Diffeo}(X_j, U_j), \;\; U_j \in \mathcal{O}(\mathbb{R}^n), \end{aligned}$$

$$\exists \delta_X > 0 \mid \forall j \in \{1, \dots N\} \; \forall x \in U_j \; : \; B(x, \delta_X \langle x \rangle) \subset U^{\#}_j.$$

   The atlas $\mathcal{A}$ *is called a "good" shrinking of* $\mathcal{A}^{\#}$.
3. *The changes of coordinates* $\phi^{\#}_{ij} = \phi^{\#}_j \circ \overline{\phi}^{\#}_i$, $i, j \in \{1, \dots, N\}$, $i \neq j$ *satisfy (17) on the corresponding open sets* $\phi_i(X^{\#}_i \cap X^{\#}_j)$ *where they are defined.*

These notations and those introduced in the next Lemma 1 will be used repeatedly in the sequel.

EXAMPLE 1. (Manifold with finitely many cilindrical ends).
Suppose X is an n-dimensional manifold of the following form:

$$X = X_0 \cup X_1 \cup \dots \cup X_N \cup \partial X_1 \cup \dots \cup \partial X_N$$

with disjoint union, where $X_0, \dots, X_N$ are n-dimensional submanifold and $\partial X_0, \dots, \partial X_N$ are connected $(n-1)$-dimensional submanifolds. Assume that X_0 is relatively compact and its boundary ∂X_0 satisfies $\partial X_0 = \partial X_1 \cup \dots \cup \partial X_N$. Moreover, for all $j = 1, \dots, N$, let X_j be diffeomorphic to $\partial X_j \times (1, +\infty)$. Then X is an **SG** manifold. In particular, all compact manifolds are **SG** manifolds.

LEMMA 1. *Let X be an* **SG** *manifold. Let us set $U^{\#}_{ij} = \phi_i(X^{\#}_i \cap X^{\#}_j)$, $V^{\#}_{ij} = \phi_j(X^{\#}_i \cap X^{\#}_j)$, $U_{ij} = \phi_i(X_i \cap X_j)$ and $V_{ij} = \phi_j(X_i \cap X_j)$. Then all the changes of coordinates satisfy $\phi^{\#}_{ij} \in$* **SG**Diffeo$(U^{\#}_{ij}, V^{\#}_{ij}; U_{ij}, V_{ij}; \delta_X)$.

DEFINITION 18. *(Transfer operators).*
Let X be an n-dimensional manifold with atlas $\left\{(X^{\#}_j, \phi_j)\right\}_{j\in J}$ and corresponding open sets $U_j = \phi_j(X^{\#}_j) \subseteq \mathbb{R}^n$. Denote by $_\star : C^\infty(X^{\#}_j) \to C^\infty(U_j)$ the transfer operator from smooth functions on the manifold to functions on $\mathbb{R}^n$, defined as in (15):

$$\forall f \in C^\infty(X^{\#}_j)\, \forall x \in U_j \;:\; f_\star(x) = f \circ \overline{\phi}(x).$$

Similarly, we denote by $^\star : C^\infty(U_j) \to C^\infty(X^{\#}_j)$ the transfer operator from smooth functions on $\mathbb{R}^n$ to functions on the manifold, defined as in (16):

$$\forall f \in C^\infty(U_j)\, \forall x \in X^{\#}_j \;:\; f^\star(x) = f \circ \phi(x).$$

DEFINITION 19. *(Extension operator).*
Let f be a function defined on $U \subset \mathbb{R}^n$. Denote by e *the extension operator defined by*

$$\mathrm{e}\, f(x) = \begin{cases} f(x) & x \in U \\ 0 & x \notin U. \end{cases}$$

THEOREM 3. *Any* **SG** *manifold X admits an* **SG***-compatible partition of unity subordinate to the atlas $\mathcal{A}^{\#}$, i.e., there are functions Φ_j, $j = 1, \dots, N$, such that*

1. $\Phi_j \in C^\infty(X; [0,1])$, $\operatorname{supp}\left(\Phi_j\right) \subset X^{\#}_j$, $\sum_{j=1}^N \Phi_j = 1$;
2. $\forall j = 1, \dots, N \;:\; \mathrm{e}(\Phi_j)_\star \in \mathbf{SG}^0_l$, *where the transfer $_\star$ is performed via the corresponding chart map ϕ_j.*

LEMMA 2. *Let $\left\{\Phi_j\right\}_{j\in\{1,\dots,N\}}$ be the partition of unity of Theorem 3. Then there are functions $\Theta_j = \Theta^0_j$, $j = 1, \dots, N$, defined on X such that*

1. $\Theta_j \in C^\infty(X; [0,1])$, $\operatorname{supp}\left(\Theta_j\right) \subset X^{\#}_j$, $\Theta_j|_{\operatorname{supp}(\Phi_j)} \equiv 1$;
2. $\forall j = 1, \dots, N \;:\; \mathrm{e}(\Theta_j)_\star \in \mathbf{SG}^0_l$.

Moreover, it is possible to build a sequence $\left\{\Theta^k_j\right\}_{k\in\mathbb{N}}$, $j = 1, \dots, N$, such that for all $k \in \mathbb{N}^\star$

1. $\Theta^k_j \in C^\infty(X; [0,1])$, $\operatorname{supp}\left(\Theta^k_j\right) \subset X^{\#}_j$, $\Theta^k_j|_{\operatorname{supp}\left(\Theta^{k-1}_j\right)} \equiv 1$;
2. $\forall j = 1, \dots, N \;:\; \mathrm{e}(\Theta^k_j)_\star \in \mathbf{SG}^0_l$.

DEFINITION 20. *(Space $\mathcal{S}(X)$).*
Let X be an **SG** *manifold. Define $\mathcal{S}(X)$ by*

$$\mathcal{S}(X) = \left\{ u \in C^\infty(X) \mid \forall j = 1, \dots, N \; u|_{X_j} \in \mathcal{S}(X_j) \right\}$$

where

$$\begin{aligned} \mathcal{S}(X_j) \; = \; &\{u \in C^\infty(X_j) \mid \forall \alpha, \beta \in \mathbb{N}^n \; \exists C_{\alpha\beta} > 0 \; : \\ &\forall x \in U_j \; x^\alpha \partial_\beta (\phi_j)_\star u(x) \prec C_{\alpha\beta}\} \end{aligned}$$

i.e., $u \in \mathcal{S}(X)$ if all its local coordinate expressions satisfy $\mathcal{S}(\mathbb{R}^n)$ estimates on their domains.

REMARK 1. We may introduce as standard the space of the distributions on the manifold X, cf. [22]. In view of Definitions 17 and 20, we may also refer in the same way to the space $\mathcal{S}'(X)$. Once a C^∞ density $d\mu$ is fixed on X, we may identify $\mathcal{D}'(X)$ with the space of the continuous linear forms on $C_0^\infty(X)$. Basing on Definition 4 and Theorem 2 we may also easily define $H^s(X)$ for $s \in \mathbb{R}^2$.

DEFINITION 21. *(Symbols on* **SG** *manifolds).*
Let X be an **SG** *manifold and let $p_j \in C^\infty(U_j^\# \times \mathbb{R}^n)$, $j = 1, \dots, N$.*

1. $p \in \mathbf{SG}_l^m(U_j^\#) \Leftrightarrow \forall \Theta \in \mathbf{SG}_l^0 \mid \operatorname{supp}(\Theta) \subset U_j^\# \times \mathbb{R}^n \; : \; \mathrm{e}(p\,\Theta) \in \mathbf{SG}_l^m$*;*
2. $\mathbf{SG}_l^m(X) = \left\{ p = (p_1, \dots, p_N) \mid \forall j = 1, \dots, N \; : \; p_j \in \mathbf{SG}_l^m(U_j^\#) \right\}$.

DEFINITION 22. *(Smoothing operators and ψdos on* **SG** *manifolds).*
Let X be an **SG** *manifold. We denote by $\mathcal{K}(X)$ the set of smoothing operators on X:*

$$\mathcal{K}(X) = \left\{ K \in \operatorname{Hom}(\mathcal{S}(X)) \mid \exists k \in \mathcal{S}(X \times X) \; : \; Ku(x) = \int_X k(x, y) u(y) d\mu(y) \right\},$$

where $d\mu(y)$ is a C^∞ density of class $\mathbf{SG}_l^0$ in local coordinates. We say that $P : \mathcal{S}(X) \to \mathcal{S}(X)$ is in $\mathbf{LG}^m(X)$ with symbol $p = \operatorname{Sym}(P) \in \mathbf{SG}_l^m(X)$ if $\forall \Theta \in \mathbf{SG}_l^0(X)$ such that Θ is ξ-independent and $\operatorname{supp}(\Theta) \subset X_j^\#$ there exists $K_\Theta \in \mathcal{K}(X)$ such that:

$$\forall u \in \mathcal{S}(X) \; : \; P(\Theta u) = K_\Theta u + (\operatorname{Op}(\mathrm{e} p_j)(\mathrm{e}(\Theta u)_*))^*.$$

Analogously we may define $P \in \mathbf{ELG}^m(X)$.

LEMMA 3. *By construction, the ψdos defined above satisfy the usual properties, i.e.:*

1. *$P \in \mathbf{LG}^r(X)$ and $Q \in \mathbf{LG}^s(X)$ imply $P + Q \in \mathbf{LG}^{\max(r,s)}(X)$;*
2. *$P \in \mathbf{LG}^r(X)$ and $Q \in \mathbf{LG}^s(X)$ imply $R = PQ \in \mathbf{LG}^{r+s}(X)$ and, in local coordinates, $\operatorname{Sym}(R)$ has the asymptotic expansion given by (13);*
3. *$P \in \mathbf{ELG}^r(X)$ admits a parametrix and extends to a bounded Fredholm operator $P : H^s(X) \to H^{s-m}(X)$ for all $s \in \mathbb{R}^2$.*

3. Continuity in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$

In subsection 3.1 we give the precise definition of the class of phase functions to be considered in our context. We further show the relation between Type I and Type II FIOs and explain why it suffices to make explicit calculation only with Type I operators throughout this section. Explicit expressions for the transpose operators of our FIOs will also be given in subsection 3.1; they will be used in subsection 3.2 to show the continuity in $\mathcal{S}(\mathbb{R}^n)$ of our FIOs and their extension to $\mathcal{S}'(\mathbb{R}^n)$, as well as in the next subsection.

3.1. Phase functions. Transpose and adjoint operators

DEFINITION 23. *(Phase functions. Regular phase functions). We will call phase function or simply phase any real valued* $\varphi \in \mathbf{SG}_l^e$ *satisfying*

$$\begin{array}{rrcccl} & c\,\langle x\rangle & \leq & \left\langle \nabla_\xi \varphi \right\rangle & \leq & C\,\langle x\rangle \\ \textit{and} & c\,\langle \xi\rangle & \leq & \langle d_x \varphi\rangle & \leq & C\,\langle \xi\rangle \end{array} \tag{19}$$

for suitable constants $C, c > 0$ *and denote by* $\mathcal{P}$ *the set of all such phases. Moreover, we define the set* $\mathcal{P}^\varepsilon$, $\varepsilon > 0$ *of all regular phases as follows:*

$$\mathcal{P}^\varepsilon = \left\{ \varphi \in \mathcal{P} \mid \forall x, \xi \ : \ \left| \det\left(\partial_i^x \partial_\xi^j \varphi \right) \right| \geq \varepsilon \right\}. \tag{20}$$

DEFINITION 24. *(Transpose and adjoint functions.) Let us set, from now on:*

$$\forall x, \xi \qquad {}^t\varphi(x, \xi) = \varphi(\xi, x)$$

and

$$\forall x, \xi \qquad a^\star(x, \xi) = \overline{a(\xi, x)}.$$

Using the standard properties of the oscillatory integrals, see for example Boggiatto, Buzano, Rodino [4], we see easily that Type I and Type II FIOs define continuous maps from $\mathcal{S}$ to $\mathcal{S}'$. This allows us to state the following proposition.

PROPOSITION 9. *If A is a Type I FIO as defined in* (2)*, then its transpose* ${}^tA$ *is given by*

$$ {}^tA_{\varphi,a} = \mathcal{F} \circ A_{{}^t\varphi, {}^ta} \circ \mathcal{F}^{-1}.$$

Here transposition is formed with respect to the customary pairing of elements of $\mathcal{S}$, namely $< u, v >= \int uv$*, so that* $< {}^tAu, v > = < u, Av >$ *holds for all* $u, v \in \mathcal{S}$*. Moreover,* ${}^t\mathcal{F} = \mathcal{F}$*,* ${}^t(\mathcal{F}^{-1}) = \mathcal{F}^{-1}$ *and, by definition of the transpose of a linear operator on $\mathcal{S}$, for any couple of such linear operators* ${}^t(PQ) = {}^tQ\,{}^tP$.

Proof. By Definition 2 we have for $u, v \in \mathcal{S}$

$$\begin{aligned} < {}^tA_{\varphi,a}u, v > &= < u, A_{\varphi,a}v >= \int dx\ u(x) \int d\!\!{}^-\!\xi\ \mathrm{e}^{i\varphi(x,\xi)} a(x,\xi)\hat{v}(\xi) \\ &= \int dx d\!\!{}^-\!\xi dy\ \mathrm{e}^{i\varphi(x,\xi)} a(x,\xi) \mathrm{e}^{-i<y|\xi>} u(x) v(y) \\ &= \int dy\ \left(\int d\xi\ \mathrm{e}^{-i<\xi|y>} \int d\!\!{}^-\!x \mathrm{e}^{i\varphi(x,\xi)} a(x,\xi)\hat{w}(x) \right) v(y) \\ &= < (\mathcal{F} \circ A_{{}^t\varphi,{}^ta} \circ \mathcal{F}^{-1})u, v > \end{aligned}$$

where we used standard properties of oscillatory integrals, $u(x) = \hat{w}(x) \Leftrightarrow w = \mathcal{F}^{-1}(u)$ and Definition 24. For the transpose of the Fourier transform we have obviously:

$$< {}^t\mathcal{F}u, v >=< \mathcal{F}u, v >$$

and analogously

$$< {}^t(\mathcal{F}^{-1})u, v >=< \mathcal{F}^{-1}u, v > .$$

The last result follows immediately from

$$< {}^t(PQ)u, v >=< u, PQv >=< {}^tPu, Qv >=< {}^tQ^tPu, v > .$$

$\square$

REMARK 2. By Definition 24 and Proposition 9 above, we have immediately

$${}^t({}^tA_{\varphi,a}) = {}^tA_{{}^t\varphi,{}^ta} = A_{{}^t({}^t\varphi),{}^t({}^ta)} = A_{\varphi,a},$$

as expected.

REMARK 3. Since, in particular, $P = \mathrm{Op}\,(p) = A_{<.|..>,p}$ for any $P \in \mathbf{LG}^m$, we also have ${}^tP = \mathcal{F} \circ \mathrm{Op}\left({}^tp\right) \circ \mathcal{F}^{-1}$.

REMARK 4. By operating in a completely similar way to that used in Proposition 9, for all Type II FIOs we also have ${}^tB_{\varphi,b} = \mathcal{F} \circ B_{{}^t\varphi,{}^tb} \circ \mathcal{F}^{-1}$.

PROPOSITION 10. *The Type I FIO $A_{\varphi,a}$ defined in* (2) *is the L^2-adjoint of the Type II FIO $B_{\varphi,a}$ defined as in* (3) *and viceversa.*

Proof. Immediate by the definitions, operating as in the proof of Proposition 9.

$\square$

REMARK 5. By comparing the two definitions (2) and (3), we also have $B_{\varphi,b} = (2\pi)^n\mathcal{F}^{-1}\circ A_{-{}^t\varphi,b^\star} \circ \mathcal{F}^{-1} \Leftrightarrow A_{\varphi,a} = (2\pi)^{-n}\mathcal{F} \circ B_{-{}^t\varphi,a^\star} \circ \mathcal{F}$.

3.2. Continuity in $\mathcal{S}(\mathbb{R}^n)$. Extension to $\mathcal{S}'(\mathbb{R}^n)$

THEOREM 4. *The FIO defined in (2) with $\varphi \in \mathcal{P}$ and $a \in \mathbf{SG}^m_l$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ in itself.*

For the proof we need the following two lemmas.

LEMMA 4. $\varphi \in \mathbf{SG}^e_l \Rightarrow \partial^\alpha_\xi\partial^x_\beta e^{i\varphi} = b^\alpha_\beta e^{i\varphi}$ *with* $b^\alpha_\beta \in \mathbf{SG}^{(|\beta|,|\alpha|)}_l$.

Proof. By induction on $|\alpha|$ and $|\beta|$.

$\square$

LEMMA 5. *Let us consider the operator L defined by:*

$$L = \frac{1-\Delta_\xi}{\left\langle\nabla_\xi\varphi\right\rangle^2 - i\Delta_\xi\varphi} \tag{21}$$

such that $L e^{i\varphi} = e^{i\varphi}$. *Assume also* $\varphi \in \mathcal{P}$ *and denote by* $\mathcal{D}$ *the division by* $d = \langle \nabla_\xi \varphi \rangle^2 - i \Delta_\xi \varphi$ *operator (i.e.,* $\mathcal{D} q = \frac{q}{d}$*), so that* $L = \mathcal{D}(1 - \Delta_\xi)$*. Then, for any* $s \in \mathbb{N}^\star$*:*

$$({}^tL)^s = \underbrace{(1-\Delta_\xi)\mathcal{D} \ldots (1-\Delta_\xi)\mathcal{D}}_{s \text{ times}} = \mathcal{D}^s + Q(\mathcal{D}, \Delta_\xi) \tag{22}$$

where Q *is a suitable polynomial of total degree* $2s$ *in the variables* $\mathcal{D}, \Delta_\xi$*, whose terms contains exactly* s $\mathcal{D}$ *factors and at least one* Δ_ξ*. Then we have, for all orders* $m \in \mathbb{R}^2$*,* $({}^tL)^s : \mathbf{SG}^m_l \to \mathbf{SG}^{m-2se_2}_l$ *and* $Q(\mathcal{D}, \Delta_\xi) : \mathbf{SG}^m_l \to \mathbf{SG}^{m-2e_1-2se_2}_l$.

Proof. We obviously have

$${}^tL = {}^t(1-\Delta_\xi)\, {}^t\mathcal{D} = (1-\Delta_\xi)\mathcal{D}$$

which implies the first part of (22). The second part of (22) is obtained immediately by induction. To prove the last part of the lemma it is enough to observe that

$$\begin{aligned} \varphi \in \mathcal{P} \quad &\Rightarrow \quad |d| = |\langle \nabla_\xi \varphi \rangle^2 - i \Delta_\xi \varphi| \geq \langle \nabla_\xi \varphi \rangle^2 \succ \langle x \rangle^2 \Rightarrow d \in \mathbf{ESG}^{(0,2)}_l \Leftrightarrow \\ &\Leftrightarrow \quad \frac{1}{d} \in \mathbf{ESG}^{(0,-2)}_l \\ &\Rightarrow \quad \mathcal{D} : \mathbf{SG}^m_l \to \mathbf{SG}^{m-2e_2}_l \end{aligned} \tag{23}$$

by (7). (23) gives the desired conclusions, since obviously, by (8), $1 - \Delta_\xi : \mathbf{SG}^m_l \to \mathbf{SG}^m_l$ and $\Delta_\xi : \mathbf{SG}^m_l \to \mathbf{SG}^{m-2e_1}_l$.

□

Proof of Theorem 4. Since it is possible to differentiate under the integral sign in (2), we find, with the notations of Lemma 4,

$$\begin{aligned} \forall \alpha, \beta \in \mathbb{N}^n \quad : \quad & x^\alpha \partial^x_\beta \int e^{i\varphi(x,\xi)} a(x,\xi) \hat{u}(\xi) d\!\!{}^-\xi = \\ & \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \int e^{i\varphi(x,\xi)} x^\alpha (b^0_\gamma \partial^x_{\beta-\gamma} a)(x,\xi) \hat{u}(\xi) d\!\!{}^-\xi, \end{aligned}$$

so we only need to show that for any $\widetilde{a} \in \mathbf{SG}^{\widetilde{m}}_l$

$$\int e^{i\varphi(x,\xi)} \widetilde{a}(x,\xi) \hat{u}(\xi) d\!\!{}^-\xi \prec |u|_{\mathcal{S},k} = \sup_{|\gamma+\delta| \leq k, x \in \mathbb{R}^n} |x^\gamma \partial_\delta u(x)|$$

for suitable k. Let us use the operator L defined in (22). Since integration by parts is admissible, by Lemma 5 we find:

$$\begin{aligned} \forall r \in \mathbb{N} \qquad & \int e^{i\varphi(x,\xi)} \widetilde{a}(x,\xi) \hat{u}(\xi) d\!\!{}^-\xi = \int (L)^r e^{i\varphi(x,\xi)} \widetilde{a}(x,\xi) \hat{u}(\xi) d\!\!{}^-\xi = \\ & = \int e^{i\varphi(x,\xi)} \left\{ \frac{\widetilde{a}(x,\xi)}{(d(x,\xi))^r} \hat{u}(\xi) + Q(\mathcal{D}, \Delta_\xi) \left[\widetilde{a}(x,\xi) \hat{u}(\xi) \right] \right\} d\!\!{}^-\xi \\ & = \int e^{i\varphi(x,\xi)} \left[\frac{\widetilde{a}(x,\xi)}{(d(x,\xi))^r} \hat{u}(\xi) + \sum_{|\gamma| \leq 2r} c_\gamma(\widetilde{a}, d) \partial^\gamma_\xi \hat{u}(\xi) \right] d\!\!{}^-\xi \end{aligned} \tag{24}$$

with coefficients $c_\gamma \in \mathbf{SG}_l^{\widetilde{m}-2re_2}$ depending only on $\widetilde{a}$ and d. In fact, the maximum order of differentiation of $\hat{u}$ in (24) is $2r$, and, by Lemma 5, every monomial of Q contains exactly r $\mathcal{D}$-factors. So, $c_\gamma \in \mathbf{SG}_l^{\widetilde{m}-2re_2}$ follows by (23), Leibniz rule and Proposition 3. Then, recalling $u \in \mathcal{S}$, it is easily seen by means of (24) that:

$$\forall r \in \mathbb{N} \ \int e^{i\varphi(x,\xi)}\widetilde{a}(x,\xi)\hat{u}(\xi)đ\xi \ \prec \ \langle x\rangle^{\widetilde{m}_2-2r}\,|u|_{\mathcal{S},k}\int\langle\xi\rangle^{-n-1}đ\xi \ \prec \ |u|_{\mathcal{S},k},$$

choosing $r \geq \widetilde{m}_2/2$ and[‖] $k \geq 2r + [n+1+\widetilde{m}_1]_+$. In fact, as already said, the maximum order of derivatives of $\hat{u}$ in (24) is $2r$ and for the convergence of the integral we can use $\forall\gamma\ |\partial_\xi^\gamma\hat{u}(\xi)| \prec \langle\xi\rangle^{-n-1-\widetilde{m}_1}$. Summing up, we have proved that

$$\forall p \in \mathbb{N}\ \exists k \in \mathbb{N} \mid |A_{\varphi,a}u|_{\mathcal{S},p} \prec |u|_{\mathcal{S},k}$$

and we can conclude invoking the Closed Graph Theorem.

□

THEOREM 5. *$A_{\varphi,a}$ with $\varphi \in \mathcal{P}$ and $a \in \mathbf{SG}_l^m$ extends countinously from $\mathcal{S}'(\mathbb{R}^n)$ in itself.*

Proof. Since $\mathcal{S} \hookrightarrow \mathcal{S}'$ and is dense in $\mathcal{S}'$, it is enough to prove the continuity of ${}^tA_{\varphi,a}$ restricted to $\mathcal{S}$. Using Proposition 9:

$$\begin{aligned} < {}^tA_{\varphi,a}u, v > &= < u, A_{\varphi,a}v > \\ &= < (\mathcal{F}\circ A_{{}^t\varphi,{}^ta}\circ\mathcal{F}^{-1})u, v > . \end{aligned}$$

Since ${}^t\varphi$ and ${}^ta$ behave like φ and a (symmetry in the role of variable and covariable, with simple exchange of the order components for the amplitude), $A_{{}^t\varphi,{}^ta}$ is continuous from $\mathcal{S}$ in itself, as we have proved in Theorem 4. So, the same is true for ${}^tA_{\varphi,a}$, since it turns out to be a composition of operators which are all continuous from $\mathcal{S}$ in itself.

□

THEOREM 6. *$B_{\varphi,b}$ is an element of $\mathcal{L}(\mathcal{S})$ extendable to an element of $\mathcal{L}(\mathcal{S}')$.*

Proof. Immediate, by Remark 5 and Theorems 4 and 5.

□

4. Composition theorems

In subsection 4.1 we prove the Composition Theorem already quoted in the introduction. In subsection 4.2 we deal with other composition theorems. Those which involve FIOs and ψdos are consequences of the Composition Theorem 7 while the results about the composition of FIOs of Type I and Type II will be needed in particular in subsection 4.3, where elliptic FIOs and their parametrices are introduced. In subsection 4.4 an example of application of all the composition theorems is given, analyzing the action of **SG**-compatible change of variables on operators in **LG** classes. In subsection 4.5 we analyze the action of our FIOs on the Sobolev spaces of Definition 11, which also will require the use of the composition theorems. In section 4.6 an adapted version of the Egorov Theorem is obtained and used to recover the expected result about the action of FIOs with regular phase on wave front sets.

[‖] $[a]_+ = \max\{a, 0\}$ and $[a]_- = \max\{-a, 0\}$ denote respectively the positive and negative part of $a \in \mathbb{R}$.

4.1. The main composition theorem

THEOREM 7. *Given a FIO $A = A_{\varphi,a}$ of Type I such that $\varphi \in \mathcal{P}$ and $a \in \mathbf{SG}^m_l(\mathbb{R}^n)$ and a ψdo $P = \mathrm{Op}(p)$ with $p \in \mathbf{SG}^t_l(\mathbb{R}^n)$, then the composed operator $H = PA$ is, modulo smoothing operators, a FIO of Type I. In fact, $H = H_{\varphi,h}$ where φ is the same phase function and the amplitude $h \in \mathbf{SG}^{m+t}_l(\mathbb{R}^n)$ admits the following asymptotic expansion:*

$$h(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}(\partial^\alpha_\xi p)(x, d_x\varphi(x,\xi)) D^y_\alpha \left[e^{i\psi(x,y,\xi)} a(y,\xi)\right]_{y=x}. \tag{25}$$

Here

$$\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \langle y - x | d_x\varphi(x,\xi)\rangle, \tag{26}$$

and, as usual, $D^y_\alpha = (-i)^{|\alpha|}\partial^y_\alpha$.

To prove Theorem 7 we will need many lemmas. In particular, Lemma 6 below, dealing with the y-derivatives of the exponential function involved in the asymptotic expansion (25), will be important also in future developments concerning the hyperbolic Cauchy problems (see [13]).

LEMMA 6. *Let us set $\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - < y - x | d_x\varphi(x,\xi) >$ as in (26). Then we have, for $|\alpha| \geq 1$:*

$$\begin{aligned} D^y_\alpha e^{i\psi} &= \sigma_\alpha e^{i\psi} \\ &= \Big[\left(d_y\varphi - d_x\varphi\right)^\alpha + \\ &+ \sum_{j_1} c_{j_1} \left(d_y\varphi - d_x\varphi\right)^{\theta_{j_1}} \prod_{j_2=1}^{n_{1j_1}} \partial^y_{\beta_{j_1 j_2}}\varphi + \\ &+ \sum_{j_1} c'_{j_1} \prod_{j_2=1}^{n_{2j_1}} \partial^y_{\gamma_{j_1 j_2}}\varphi \Big] e^{i\psi} \end{aligned} \tag{27}$$

with suitable $c_{j_1}, c'_{j_1}, \beta_{j_1 j_2}$ and $\gamma_{j_1 j_2}$ such that:

$$|\beta_{j_1 j_2}|, |\gamma_{j_1 j_2}| \geq 2 \tag{28}$$

$$\theta_{j_1} + \sum_{j_2=1}^{n_{1j_1}} \beta_{j_1 j_2} = \sum_{j_2=1}^{n_{2j_1}} \gamma_{j_1 j_2} = \alpha \tag{29}$$

where $d_x\varphi = d_x\varphi(x,\xi)$, $d_y\varphi = d_y\varphi(y,\xi)$, $\partial^x_\alpha\varphi = \partial^x_\alpha\varphi(x,\xi)$ and $\partial^y_\alpha\varphi = \partial^y_\alpha\varphi(y,\xi)$ is to be understood.

Proof. By induction on $|\alpha|$ (see section A.1.)

□

REMARK 6. Note that, by (28) and (29), we have, in any term of (27) where $n_{1j_1}, n_{2j_1} \geq 1$:

$$|\alpha| \geq \sum_{j_2=1}^{n_{1j_1}} |\beta_{j_1 j_2}| \geq 2n_{1j_1}, |\alpha| \geq \sum_{j_2=1}^{n_{2j_1}} |\gamma_{j_1 j_2}| \geq 2n_{2j_1} \Rightarrow n_{1j_1}, n_{2j_1} \leq \frac{|\alpha|}{2}.$$

LEMMA 7. *With the same ψ of Lemma 6, for $\varphi \in \mathbf{SG}^e_l$ we have:*

$$\partial^y_\alpha e^{i\psi(x,y,\xi)}\Big|_{y=x} \in \mathbf{SG}_l^{([|\alpha|/2],[-|\alpha|/2])} \Rightarrow \partial^y_\alpha e^{i\psi(x,y,\xi)}\Big|_{y=x} \prec \langle\xi\rangle^{\frac{|\alpha|}{2}} \langle x\rangle^{-\frac{|\alpha|}{2}} \tag{30}$$

($[a]$ denotes the integer part of a) and

$$\begin{gathered} |y-x| \le k\langle x\rangle\,, k \in (0;1) \\ \Rightarrow \partial^y_\alpha e^{i\psi(x,y,\xi)} \prec (1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}} \end{gathered} \tag{31}$$

Proof. (30) is immediate by Lemma 6, Remark 6 and $\varphi \in \mathbf{SG}^e_l$, observing that the first term and the first sum of (27) vanish for $y = x$, as well as $\psi(x, x, \xi) = 0$, and, of course, $n_{1j_1}, n_{2j_1} \le |\alpha|/2 \Rightarrow n_{1j_1}, n_{2j_1} \le [|\alpha|/2]$. For what concerns (31), we obviously have:

$$\begin{aligned} \partial_i\varphi(y,\xi) - \partial_i\varphi(x,\xi) &= \int_0^1 dt\ \ \partial_{ij}\varphi(x+t(y-x),\xi)(y^j-x^j) \Rightarrow \\ &\Rightarrow \partial^y_i\varphi - \partial^x_i\varphi \prec \\ &\prec |y-x| \sup_{t\in[0;1];i,j} |\partial_{ij}\varphi(x+t(y-x),\xi)| \prec \\ &\prec |y-x| \sup_{t\in[0;1]} \langle\xi\rangle\langle x+t(y-x)\rangle^{-1} \\ &\prec |y-x|\langle y-x\rangle\langle\xi\rangle\langle x\rangle^{-1}\,. \end{aligned}$$

Moreover, note that

$$|y-x| \le k\langle x\rangle\,, k \in (0;1) \Rightarrow \langle x\rangle \sim \langle y\rangle$$

We then have the following estimates:

$$\begin{aligned} \left(d_y\varphi - d_x\varphi\right)^\alpha &\prec (|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle x\rangle^{-|\alpha|} \\ &\prec (1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}} \\ \left(d_y\varphi - d_x\varphi\right)^{\theta_{j_1}} \prod_{j_2=1}^{n_{1j_1}} \partial^y_{\beta_{j_1j_2}}\varphi &\prec (|y-x|\langle y-x\rangle\langle\xi\rangle\langle x\rangle^{-1})^{|\theta_{j_1}|}\langle\xi\rangle^{n_{1j_1}} \\ &\quad\ \langle y\rangle^{n_{1j_1}-\sum_{j_2=1}^{n_{1j_1}}|\beta_{j_1j_2}|} \\ &\prec (|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}} \\ &\quad\ \langle x\rangle^{n_{1j_1}-(|\theta_{j_1}|+\sum_{j_2=1}^{n_{1j_1}}|\beta_{j_1j_2}|)} \\ &\prec (1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}} \\ \prod_{j_2=1}^{n_{2j_1}} \partial^y_{\gamma_{j_1j_2}}\varphi &\prec \langle\xi\rangle^{n_{2j_1}}\langle y\rangle^{n_{2j_1}-\sum_{j_2=1}^{n_{2j_1}}|\gamma_{j_1j_2}|} \\ &\prec \langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}} \\ &\prec (1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{\frac{|\alpha|}{2}}\langle x\rangle^{-\frac{|\alpha|}{2}} \end{aligned}$$

which prove (31).

□

LEMMA 8. *If $\varphi \in \mathcal{P}$, ψ is defined as in Lemma 6, $p \in \mathbf{SG}_l^t$ and $a \in \mathbf{SG}_l^m$, the expression*

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} c_\alpha(x,\xi) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}(\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi)) D_\alpha^y \left[\mathrm{e}^{i\psi(x,y,\xi)} a(y,\xi) \right]_{y=x} \tag{32}$$

is an asymptotic expansion which defines an amplitude $h \in \mathbf{SG}_l^{m+t}$.

Proof. Using Lemmas 6 and 7 with $a \in \mathbf{SG}_l^m$, we see that

$$\begin{aligned} D_\alpha^y \left[\mathrm{e}^{i\psi(x,y,\xi)} a(y,\xi) \right]_{y=x} &= \left. \sum_{0\le\beta\le\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D_\beta^y \mathrm{e}^{i\psi(x,y,\xi)} D_{\alpha-\beta}^y a(y,\xi) \right|_{y=x} \\ &\prec \sum_{0\le\beta\le\alpha} \langle\xi\rangle^{\frac{|\beta|}{2}} \langle x\rangle^{-\frac{|\beta|}{2}} \langle\xi\rangle^{m_1} \langle x\rangle^{m_2-|\alpha|+|\beta|} \\ &\prec \langle\xi\rangle^{m_1+\frac{|\alpha|}{2}} \langle x\rangle^{m_2-\frac{|\alpha|}{2}} . \end{aligned}$$

Using (19), we also easily have:

$$(\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi)) \prec \langle d_x\varphi(x,\xi)\rangle^{t_1-|\alpha|} \langle x\rangle^{t_2} \prec \langle\xi\rangle^{t_1-|\alpha|} \langle x\rangle^{t_2} .$$

So, we obtain:

$$\forall\alpha\in\mathbb{N}^n \quad c_\alpha(x,\xi) \prec \langle\xi\rangle^{m_1+t_1-\frac{|\alpha|}{2}} \langle x\rangle^{m_2+t_2-\frac{|\alpha|}{2}}$$

which proves the lemma, invoking the first point of Proposition 5.

□

LEMMA 9. *Let us consider the operator $M = -i\sum_{j=1}^n \frac{x^j - y^j}{|x-y|^2}\partial_\eta^j$. M is well defined on* $\operatorname{supp}(1-\chi)$ *for $\chi \in \Xi^\Delta(k)$, $k \in (0;1)$, it has the property $M\mathrm{e}^{i<x-y|\xi>} = \mathrm{e}^{i<x-y|\xi>}$ and, $\forall r \in \mathbb{N}^\star$:*

$$({}^tM)^r = (-i)^r \sum_{|\theta|=r} c_\theta \frac{(x-y)^\theta}{|x-y|^{2r}} \partial_\eta^\theta \tag{33}$$

for suitable $c_\theta \in \mathbb{N}^\star$. Moreover, it is possible to show that

$$|y-x| \ge k\langle x\rangle \Rightarrow \exists k'>0 \mid |y-x| \ge k'\langle y\rangle \Rightarrow |y-x| \succ \langle x\rangle + \langle y\rangle \ge (\langle x\rangle\langle y\rangle)^{\frac{1}{2}}. \tag{34}$$

Proof. (33) can be proved by induction on r. For its proof and some hints about (34) see the appendix of [14].

□

LEMMA 10. *Let $\omega = \omega(y)$ be a smooth function such that $|d_y\omega| \neq 0$ and let us set*

$$U = \frac{i}{|d_y\omega|^2} \sum_{k=1}^n \partial_k^y \omega \partial_k^y \tag{35}$$

so that $U\mathrm{e}^{-i\omega} = \mathrm{e}^{-i\omega}$. Then

$$\forall r\in\mathbb{N} \;\; ({}^tU)^r = \frac{1}{|d_y\omega|^{4r}} \sum_{|\alpha|\le r} P_{\alpha,r}\,\partial_\alpha^y \tag{36}$$

with

$$P_{\alpha,r} = \sum c^{\alpha,r}_{\gamma\delta_1\ldots\delta_r} (d_y\omega)^\gamma \partial^y_{\delta_1}\omega \ldots \partial^y_{\delta_r}\omega \tag{37}$$

where in the sum

$$\begin{aligned} &|\gamma| = 2r, \\ &|\delta_j| \geq 1, \quad \sum_{j=1}^r |\delta_j| + |\alpha| = 2r \end{aligned} \tag{38}$$

and $c^{\alpha,r}_{\gamma\delta_1\ldots\delta_r}$ *are suitable constants.*

Proof. By induction on r.

□

LEMMA 11. *If* $\varphi \in \mathcal{P}$, $\chi \in \Xi^\Delta(k)$, $a \in \mathbf{SG}^m_l$ *and* $p \in \mathbf{SG}^t_l$ *then the function* $h_2 = h_2(x,\xi)$ *defined by*

$$h_2(x,\xi) = \int dy đ\eta \; \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)}(1-\chi(x,y))a(y,\xi)p(x,\eta)$$

is in $\mathcal{S}$.

Proof. Using the operators $L = \frac{1-\Delta_y}{\langle\nabla_y\varphi\rangle^2 - i\Delta_y\varphi}$, analogous to that defined in (21), and M, defined in Lemma 9, we have, for any $r, s \in \mathbb{N}^\star$:

$$\begin{aligned} h_2(x,\xi) &= \int dy đ\eta \; \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)}(1-\chi(x,y))a(y,\xi)\left[({}^tM)^r p\right](x,\eta) \\ &= \int dy đ\eta \; \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)+<x|\eta>)}({}^tL)^s\left[\mathrm{e}^{-i<y|\eta>)}q(x,y,\xi,\eta)\right] \end{aligned} \tag{39}$$

having set

$$q(x,y,\xi,\eta) = (1-\chi(x,y))a(y,\xi)\left[({}^tM)^r p\right](x,\eta).$$

Let us analyze the y derivatives of q. By Lemma 9, we find **

$$\begin{aligned} &\partial^\alpha_y q(x,y,\xi,\eta) = \\ &= \partial^\alpha_y\left[(1-\chi(x,y))a(y,\xi)(-i)^r \sum_{|\theta|=r} c_\theta \frac{(x-y)^\theta}{|x-y|^{2r}}(\partial^\theta_\eta p)(x,\eta)\right] \\ &= \sum_{|\theta|=r}(\partial^\theta_\eta p)(x,\eta) \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}(\delta_{|\alpha_1|,0} - (\partial^y_{\alpha_1}\chi)(x,y)) \\ &(\partial^y_{\alpha_2}a)(y,\xi) \sum_{\beta_1+\beta_2=\alpha_3} \frac{\alpha_3!}{\beta_1!\beta_2!} c_{\theta\beta_1}(x-y)^{\theta-\beta_1} \frac{P_{\beta_2}(x-y)}{|x-y|^{2(r+|\beta_2|)}} \end{aligned}$$

**We denote by $\delta_{j,k}$ the Kronecker symbol such that

$$\delta_{j,k} = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases}$$

with P_{β_2} homogeneous polynomial of degree $|\beta_2|$. So, by obvious calculations:

$$\begin{aligned}
&\partial_y^\alpha q(x,y,\xi,\eta) \prec \\
&\prec \sum_{|\theta|=r} \langle x\rangle^{t_2} \langle\eta\rangle^{t_1-|\theta|} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \langle y\rangle^{-|\alpha_3|} \langle\xi\rangle^{m_1} \langle y\rangle^{m_2-|\alpha_2|} \\
&\sum_{\beta_1+\beta_2=\alpha_3} |x-y|^{|\theta|-|\beta_1|+|\beta_2|-2r-2|\beta_2|} \\
&\prec \langle x\rangle^{t_2} \langle\eta\rangle^{t_1-r} \langle\xi\rangle^{m_1} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \langle y\rangle^{m_2-|\alpha_1|-|\alpha_2|} |x-y|^{-r-|\alpha_3|}.
\end{aligned}$$

Since only the domain in which $|y-x| \geq \frac{k}{2}\langle x\rangle$ is relevant here (q identically vanishes elsewhere) and since from (34)

$$|y-x| \geq \frac{k}{2}\langle x\rangle \Rightarrow |y-x| \succ \langle y\rangle \Rightarrow |y-x| \succ \langle x\rangle + \langle y\rangle \geq (\langle x\rangle \langle y\rangle)^{\frac{1}{2}}$$

we can conclude

$$\partial_y^\alpha q(x,y,\xi,\eta) \prec \langle\xi\rangle^{m_1} \langle\eta\rangle^{t_1-r} \langle x\rangle^{t_2-\frac{r}{2}} \langle y\rangle^{m_2-\frac{r}{2}-|\alpha|} \tag{40}$$

so that q has a **SG** behaviour also with respect to y. Let us now analyze the integrand of (39). As shown in Lemma 5, once set $d = \langle\nabla_y\varphi(y,\xi)\rangle^2 - i\Delta_y\varphi(y,\xi) \succ \langle\xi\rangle^2$, we have:

$$\begin{aligned}
&({}^tL)^s \left[\mathrm{e}^{-i<y|\eta>} q(x,y,\xi,\eta)\right] = \\
&= \frac{\mathrm{e}^{-i<y|\eta>} q(x,y,\xi,\eta)}{d^s} + Q(\mathcal{D},\Delta_y)\left[\mathrm{e}^{-i<y|\eta>} q(x,y,\xi,\eta)\right]
\end{aligned}$$

as in (24). Due to the presence of the exponential in the argument of $Q(\mathcal{D},\Delta_y)$, in the second term there are powers of η of heigth not greater than $2s$. Owing to (40) we have at last:

$$h_2(x,\xi) \prec \langle\xi\rangle^{m_1-2s} \langle x\rangle^{t_2-\frac{r}{2}} \int dy\, \langle y\rangle^{m_2-\frac{r}{2}} \int d\eta\, \langle\eta\rangle^{t_1-r+2s}.$$

so that

$$\begin{aligned}
\forall\alpha,\beta\in\mathbb{N}^n \quad \xi^\alpha x^\beta h_2(x,\xi) \quad \prec \quad & \langle\xi\rangle^{m_1-2s+|\alpha|} \langle x\rangle^{t_2-\frac{r}{2}+|\beta|} \\
& \int dy\, \langle y\rangle^{m_2-\frac{r}{2}} \int d\eta\, \langle\eta\rangle^{t_1-r+2s} \prec 1
\end{aligned}$$

provided

$$\begin{aligned}
s &> \frac{m_1+|\alpha|}{2}, \\
r &> \max\{2(t_2+|\beta|),\, t_1+2s+n,\, 2(n+m_2)\}.
\end{aligned}$$

Since then, differentiating under the integral sign,

$$\begin{aligned}
&\forall\alpha,\beta\in\mathbb{N}^n \quad \partial_\xi^\alpha \partial_\beta^x h_2(x,\xi) = \\
&\sum_j \int dy\, d\!\!{}^-\!\eta\, \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)} \chi_j(x,y) a_j(y,\xi) p_j(x,\eta)
\end{aligned} \tag{41}$$

with suitable χ_j, a_j and p_j in some **SG** classes and χ_j having support in the domain $|y-x| \geq \frac{k}{2}\langle x\rangle$, we can also conclude

$$\forall\alpha,\beta,\gamma,\delta \in \mathbb{N}^n \quad \xi^\alpha x^\beta \partial_\xi^\gamma \partial_\delta^x h_2(x,\xi) \prec 1,$$

by applying the same procedure illustrated above to every integral in the sum (41).

□

Proof of Theorem 7. We can now prove the Composition Theorem. Writing explicitly $PA_{\varphi,a}u(x)$ with $P = \mathrm{Op}\,(p) \in \mathbf{LG}^t$, we find:

$$\begin{aligned}
& PA_{\varphi,a}u(x) = \\
= \; & \int d\!\!\bar{}\,\xi \mathrm{e}^{i<x|\xi>} p(x,\xi) \int dy \mathrm{e}^{-i<y|\xi>} \int d\!\!\bar{}\,\eta \mathrm{e}^{i\varphi(y,\eta)} a(y,\eta)\hat{u}(\eta) \\
= \; & \int d\!\!\bar{}\,\eta \mathrm{e}^{i\varphi(x,\eta)} \left[\int dy d\!\!\bar{}\,\xi \mathrm{e}^{i(\varphi(y,\eta)-\varphi(x,\eta)-<y-x|\xi>)} a(y,\eta) p(x,\xi)\right] \hat{u}(\eta) \\
= \; & \int d\!\!\bar{}\,\xi \mathrm{e}^{i\varphi(x,\xi)} \left[\int dy d\!\!\bar{}\,\eta \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)} a(y,\xi) p(x,\eta)\right] \hat{u}(\xi) \\
= \; & \int d\!\!\bar{}\,\xi \mathrm{e}^{i\varphi(x,\xi)} h(x,\xi)\hat{u}(\xi).
\end{aligned}$$

We have to show

$$h(x,\xi) = \int dy d\!\!\bar{}\,\eta \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)} a(y,\xi) p(x,\eta) \quad \in \mathbf{SG}_l^{m+t}.$$

Choosing $\chi \in \Xi^\Delta(k)$ we can write

$$\begin{aligned}
h(x,\xi) &= h_1(x,\xi) + h_2(x,\xi) \\
&= \int dy d\!\!\bar{}\,\eta \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)} \chi(x,y) a(y,\xi) p(x,\eta) \\
&+ \int dy d\!\!\bar{}\,\eta \mathrm{e}^{i(\varphi(y,\xi)-\varphi(x,\xi)-<y-x|\eta>)} (1-\chi(x,y)) a(y,\xi) p(x,\eta),
\end{aligned}$$

with $h_2 \in \mathcal{S}$, by Lemma 11. We will prove $h_1 \in \mathbf{SG}_l^{m+t}$ by showing that it admits the asymptotic expansion already studied in Lemma 8. In fact, setting $\eta = d_x\varphi(x,\xi) + \theta$ in the expression of h_1 and using the Taylor expansion

$$\begin{aligned}
p(x,\eta) &= \sum_{|\alpha|<M} \frac{\theta^\alpha}{\alpha!} (\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi)) + \sum_{|\alpha|=M} \frac{M}{\alpha!} \theta^\alpha r_\alpha(x,\xi,\theta) \\
r_\alpha(x,\xi,\theta) &= \int_0^1 dt (1-t)^{M-1} (\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi) + t\theta),
\end{aligned}$$

we have:

$$
\begin{aligned}
& h_1(x,\xi) = \\
= & \sum_{|\alpha|<M} \frac{(\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi))}{\alpha!} \mathcal{F}^{-1}_{\theta\to x}\left[\theta^\alpha \mathcal{F}_{y\to\theta}\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right] \\
+ & \sum_{|\alpha|=M} \frac{M}{\alpha!}\mathcal{F}^{-1}_{\theta\to x}\left[\theta^\alpha r_\alpha(x,\xi,\theta)\mathcal{F}_{y\to\theta}\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right] \\
= & \sum_{|\alpha|<M} \frac{(\partial_\xi^\alpha p)(x, d_x\varphi(x,\xi))}{\alpha!} D^y_\alpha\left[e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right]_{y=x} \\
+ & \sum_{|\alpha|=M} \frac{M}{\alpha!}\int đ\theta e^{i<x|\theta>} r_\alpha(x,\xi,\theta)\mathcal{F}_{y\to\theta}\left[D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right].
\end{aligned}
$$

Now, since every derivative of χ vanishes in a neighbourhood of the diagonal $\Delta = \{(x,y) \in \mathbb{R}^{2n} \mid x = y\}$ and $\chi(x,x) = 1$, by an obvious use of the Leibniz rule in the last formula, we can write:

$$
h_1(x,\xi) = \sum_{|\alpha|<M} \frac{1}{\alpha!} c_\alpha(x,\xi) + \sum_{|\alpha|=M} \frac{M}{\alpha!} R_\alpha(x,\xi)
$$

where the c_α are the terms of the asymptotic expansion (32) and

$$
R_\alpha = \int đ\theta e^{i<x|\theta>} r_\alpha(x,\xi,\theta)\mathcal{F}_{y\to\theta}\left[D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right].
$$

Let us now estimate R_α: these estimates will prove the convergence of the integral defining h_1 and will allow the use of the point 2 of Proposition 5 to complete our proof. To our aim, let us choose $\chi^\star \in C_0^\infty$ such that

$$
\chi^\star(x) = \begin{cases} 1 & |x| \le \dfrac{\varepsilon}{2} \\ 0 & |x| \ge \varepsilon \end{cases}
$$

with $\varepsilon > 0$ to be fixed later. Let us denote by $E_{\varepsilon,\xi}$ the set $\left\{\theta \in \mathbb{R}^n \mid |\theta| \le \varepsilon\langle\xi\rangle\right\}$, so that $\operatorname{supp}\left(\chi^\star\left(\frac{\cdot}{\langle\xi\rangle}\right)\right) \subseteq E_{\varepsilon,\xi}$. R_α can obviously be expressed by the sum of the two following integrals:

$$
\begin{aligned}
I &= \int đ\theta e^{i<x|\theta>} r_\alpha(x,\xi,\theta)\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\mathcal{F}_{y\to\theta}\left[D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right]; \\
K &= \int đ\theta e^{i<x|\theta>} r_\alpha(x,\xi,\theta)\left[1-\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right]\cdot \\
&\quad \cdot\mathcal{F}_{y\to\theta}\left[D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right].
\end{aligned}
$$

1. **Estimate of I.**
   Let us set

$$
f_\alpha(x,\xi,.) = \mathcal{F}^{-1}_{\theta\to .}\left[r_\alpha(x,\xi,\theta)\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right].
$$

Then, we have:

$$\begin{aligned} I &= \int đ\theta dy e^{i<x|\theta>} r_\alpha(x,\xi,\theta)\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\cdot \\ &\quad \cdot e^{-i<y|\theta>} D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right) \\ &= \int dy \left[\int đ\theta e^{i<x-y|\theta>} r_\alpha(x,\xi,\theta)\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right]\cdot \\ &\quad \cdot D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right) \\ &= \int dy f_\alpha(x,\xi,x-y) D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right). \end{aligned}$$

Remembering our choice of $\chi^\star$ and $\varphi \in \mathcal{P}$, we have:

$$\begin{aligned} \forall \alpha,\beta \in \mathbb{N}^n \qquad & \partial^\beta_\theta r_\alpha(x,\xi,\theta) \\ \prec \quad & \langle x\rangle^{t_2} \int_0^1 dt\, \langle d_x\varphi(x,\xi)+t\theta\rangle^{t_1-|\alpha|-|\beta|}(1-t)^{M-1}\, t^{|\beta|} \\ \prec \quad & \langle\xi\rangle^{t_1-|\alpha|-|\beta|}\langle x\rangle^{t_2}. \end{aligned} \tag{42}$$

In fact, the presence of $\chi^\star$ in the integrand of I and $t \in [0;1]$ imply $|\theta| \le \varepsilon\langle\xi\rangle \Rightarrow |t\theta| \le \varepsilon\langle\xi\rangle$. Moreover

$$\begin{aligned} \varphi\in\mathcal{P} \;&\Rightarrow\; \langle d_x\varphi(x,\xi)\rangle \sim \langle\xi\rangle \Rightarrow \\ &\Rightarrow \begin{cases} \langle d_x\varphi(x,\xi)+t\theta\rangle^2 = \langle d_x\varphi(x,\xi)\rangle^2 + t^2|\theta|^2 \le (C_2^2+\varepsilon^2)\langle\xi\rangle^2 \\ \langle d_x\varphi(x,\xi)+t\theta\rangle^2 = \langle d_x\varphi(x,\xi)\rangle^2 + t^2|\theta|^2 \ge C_1^2\langle\xi\rangle^2. \end{cases} \end{aligned}$$

We have also:

$$\begin{aligned} \forall\alpha,\beta\in\mathbb{N}^n \quad & \left|u^\beta f_\alpha(x,\xi,u)\right| = \left|\mathcal{F}^{-1}_{\theta\to u}\left[D^\beta_\theta\left(r_\alpha(x,\xi,\theta)\chi^\star\left(\tfrac{\theta}{\langle\xi\rangle}\right)\right)\right]\right| \\ & \prec \mu_\theta(E_{\varepsilon,\xi}) \sup_{\theta\in E_{\varepsilon,\xi}}\left|D^\beta_\theta\left(r_\alpha(x,\xi,\theta)\chi^\star\left(\tfrac{\theta}{\langle\xi\rangle}\right)\right)\right|. \end{aligned} \tag{43}$$

In view of (42), a good estimate for the last expression in (43) can be easily found. In fact:

$$\begin{aligned} & \partial^\beta_\theta\left(r_\alpha(x,\xi,\theta)\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right) \\ \prec \quad & \sum_{\gamma\le\beta}\left|\partial^\gamma_\theta r_\alpha(x,\xi,\theta)\right|\left|\partial^{\beta-\gamma}_\theta\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right| \\ \prec \quad & \sum_{\gamma\le\beta}\langle\xi\rangle^{t_1-|\alpha|-|\gamma|}\langle x\rangle^{t_2}\langle\xi\rangle^{|\gamma|-|\beta|} \\ \prec \quad & \langle\xi\rangle^{t_1-|\alpha|-|\beta|}\langle x\rangle^{t_2} \end{aligned} \tag{44}$$

and also

$$\mu_\theta(E_{\varepsilon,\xi}) = \int_{|\theta|\le\varepsilon\langle\xi\rangle} d\theta = \langle\xi\rangle^n \int_{|\eta|\le\varepsilon} d\eta \prec \langle\xi\rangle^n \tag{45}$$

by the linear change of variable $\theta = \langle\xi\rangle\,\eta$.

So, (42), (44) and (45) imply

$$\begin{aligned}
&\forall\alpha,\beta\in\mathbb{N}^n \left|u^\beta f_\alpha(x,\xi,u)\right| \prec \langle\xi\rangle^{t_1+n-|\alpha|-|\beta|}\langle x\rangle^{t_2} \Leftrightarrow\\
&\forall\alpha,\beta\in\mathbb{N}^n \left|(u\langle\xi\rangle)^\beta\right| |f_\alpha(x,\xi,u)| \prec \langle\xi\rangle^{t_1+n-|\alpha|}\langle x\rangle^{t_2} \Rightarrow\\
&\forall\alpha\in\mathbb{N}^n,\ \forall j\in\mathbb{N}\,(|u|\langle\xi\rangle)^j\,|f_\alpha(x,\xi,u)| \prec \langle\xi\rangle^{t_1+n-|\alpha|}\langle x\rangle^{t_2}
\end{aligned}$$

which finally implies

$$\begin{aligned}
\forall\alpha\in\mathbb{N}^n,\ \forall L>0 \quad & (1+|u|\langle\xi\rangle)^L\,|f_\alpha(x,\xi,u)|\\
&\le (1+|u|\langle\xi\rangle)^{[L]+1}\,|f_\alpha(x,\xi,u)| \prec \langle\xi\rangle^{t_1+n-|\alpha|}\langle x\rangle^{t_2} \Rightarrow\\
\forall\alpha\in\mathbb{N}^n,\ \forall L>0 \quad & |f_\alpha(x,\xi,u)| \prec (1+|u|\langle\xi\rangle)^{-L}\langle\xi\rangle^{t_1+n-|\alpha|}\langle x\rangle^{t_2}.
\end{aligned}$$

So, setting $L=L_1+L_2$ with $L_1,L_2>0$, we can say that

$$\begin{aligned}
&\forall L_1,L_2>0\ \forall\alpha\in\mathbb{N}^n\\
&I \prec \langle\xi\rangle^{t_1+n-|\alpha|}\langle x\rangle^{t_2}\cdot\\
&\quad\cdot\sup_y\left[\left|D^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right|(1+|y-x|\langle\xi\rangle)^{-L_1}\right]\cdot\\
&\quad\cdot\int dy(1+|y-x|\langle\xi\rangle)^{-L_2}.
\end{aligned}\tag{46}$$

For what concerns the integral in (46), by the translation $y-x\to y$ and the transform $\theta=\langle\xi\rangle\,\eta$, it turns out to be estimated by $\langle\xi\rangle^n$, by choosing L_2 large enough to assure its convergence. The $\sup_y$ is easily estimated by observing that

$$\begin{aligned}
&\partial^y_\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)=\\
&=\sum_{\beta+\gamma+\delta=\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}\partial^y_\beta e^{i\psi(x,y,\xi)}\,(\partial^y_\gamma\chi)(x,y)\,(\partial^y_\delta a)(y,\xi)\\
&\prec\sum_{\beta+\gamma+\delta=\alpha}(1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\beta|}\langle\xi\rangle^{\frac{|\beta|}{2}}\langle x\rangle^{-\frac{|\beta|}{2}}\cdot\\
&\quad\cdot\langle y\rangle^{-|\gamma|}\langle\xi\rangle^{m_1}\langle y\rangle^{m_2-|\delta|}\\
&\prec (1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}\langle\xi\rangle^{m_1+\frac{|\alpha|}{2}}\cdot\\
&\quad\cdot\sum_{\beta+\gamma+\delta=\alpha}\langle x\rangle^{m_2-\frac{|\beta|}{2}-\frac{|\gamma|}{2}-\frac{|\delta|}{2}}\\
&\prec \langle\xi\rangle^{m_1+\frac{|\alpha|}{2}}\langle x\rangle^{m_2-\frac{|\alpha|}{2}}(1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|},
\end{aligned}$$

where we used (31) and the fact that $\langle x\rangle\sim\langle y\rangle$ (owing to the presence of χ). We conclude that

$$\begin{aligned}
I &\prec \langle\xi\rangle^{m_1+t_1+2n-\frac{|\alpha|}{2}}\langle x\rangle^{m_2+t_2-\frac{|\alpha|}{2}}\sup_y\frac{(1+|y-x|\langle y-x\rangle\langle\xi\rangle)^{|\alpha|}}{(1+|y-x|\langle\xi\rangle)^{L_1}}\\
&\prec \langle\xi\rangle^{m_1+t_1+2n-\frac{|\alpha|}{2}}\langle x\rangle^{m_2+t_2-\frac{|\alpha|}{2}}
\end{aligned}$$

for $L_1>2|\alpha|$.

2. **Estimate of K.**
Let us set

$$\begin{aligned}\omega(x,y,\xi,\theta) &= < y|\theta > -\psi(x,y,\xi)\\ &= < y|\theta > -(\varphi(y,\xi)-\varphi(x,\xi) - < y-x|d_x\varphi(x,\xi) >)\end{aligned} \tag{47}$$

which implies

$$\begin{aligned} d_y\omega(x,y,\xi,\theta) &= \theta-(d_y\varphi(y,\xi)-d_x\varphi(x,\xi))\\ &= \theta-(d_y\varphi-d_x\varphi)\\ &\prec \langle\theta\rangle+\langle\xi\rangle\,. \end{aligned}$$

We begin by using the operator $W = \frac{1-\Delta_\theta}{\langle x\rangle^2} = {}^tW$, such that $\forall s_1 \in \mathbb{N}$ $W^{s_1}\mathrm{e}^{i<x|\theta>} = \mathrm{e}^{i<x|\theta>}$, in the integral defining K, to obtain, for all $s_1\in\mathbb{N}$,

$$\begin{aligned} K &= \int d\!\!\bar{}\,\theta \mathrm{e}^{i<x|\theta>} W^{s_1}\left\{r_\alpha(x,\xi,\theta)\left[1-\chi^\star\left(\frac{\theta}{\langle\xi\rangle}\right)\right]\cdot\right.\\ &\quad\left.\cdot\,\mathcal{F}_{y\to\theta}\left[D^y_\alpha\left(\mathrm{e}^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right]\right\}\\ &= \sum_j\int d\!\!\bar{}\,\theta \mathrm{e}^{i<x|\theta>} r^j_\alpha(x,\xi,\theta)\chi^\star_j\left(\frac{\theta}{\langle\xi\rangle}\right)\cdot\\ &\quad\cdot\mathcal{F}_{y\to\theta}\left[y^{\beta_j}D^y_\alpha\left(\mathrm{e}^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\right] \end{aligned} \tag{48}$$

with, for any j,

$$\chi^\star_j \prec 1,\ \mathrm{supp}\left(\chi^\star_j\left(\frac{\cdot}{\langle\xi\rangle}\right)\right)\subseteq\left\{\theta \mid |\theta|\geq\frac{\varepsilon}{2}\langle\xi\rangle\right\}; \tag{49}$$

$$r^j_\alpha \prec \langle x\rangle^{t_2-2s_1} \text{ if } \alpha \text{ satisfies } t_1-|\alpha|\leq 0\,; \tag{50}$$

$$|\beta_j|\leq 2s_1. \tag{51}$$

This can be proved by induction on s_1. From now on, we will consider only one of the integrals in the sum (48), since all the estimates we will find will not depend on j. Writing explicitly the Fourier transform and the derivative by y in one of such integrals and using Definition (47) and the notation in Lemma 6, we have to estimate

$$\begin{aligned} K &= \sum_{\beta+\gamma+\delta=\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}\int d\!\!\bar{}\,\theta dy \mathrm{e}^{-i\omega(x,y,\xi,\theta)} r^j_\alpha(x,\xi,\theta)\chi^\star_j\left(\frac{\theta}{\langle\xi\rangle}\right)\cdot\\ &\quad\cdot\sigma_\beta(x,y,\xi)\,\partial^y_\gamma\chi(x,y)\,y^{\beta_j}\partial^y_\delta a(y,\xi) \end{aligned} \tag{52}$$

under the conditions (49), (50) and (51). We will write

$$f^j_{\beta\gamma\delta}(x,y,\xi) = \sigma_\beta(x,y,\xi)\,\partial^y_\gamma\chi(x,y)\,y^{\beta_j}\partial^y_\delta a(y,\xi)$$

for the sake of brevity. Note that

$$f^j_{\beta\gamma\delta}\in\mathbf{SG}^{(|\alpha|+m_1,0,m_2+2s_1)}, \tag{53}$$

owing to

- (27), which implies $\sigma_\beta \in \mathbf{SG}^{(|\beta|,0,0)} \subseteq \mathbf{SG}^{(|\alpha|,0,0)}$;
- $\chi \in \mathbf{SG}^{(0,0,0)} \Rightarrow \partial^y_\gamma \chi \in \mathbf{SG}^{(0,0,-|\gamma|)} \subseteq \mathbf{SG}^{(0,0,0)}$;
- (51) and $a \in \mathbf{SG}^m_l \Rightarrow y^{\beta_j} a(y,\xi) \in \mathbf{SG}^{(m_1,0,m_2+|\beta_j|)} \subseteq \mathbf{SG}^{(m_1,0,m_2+2s_1)}$.

Let us now use in each integral in the sum (52) the operator U defined in (35). This is admissible since

$$\begin{aligned} \exists C>0 \mid |d_y\omega| &= |\theta-(d_y\varphi-d_x\varphi)| \\ &\geq |\theta|-|d_y\varphi-d_x\varphi)| \\ &\geq C(\langle\theta\rangle+\langle\xi\rangle) \succ (\langle\theta\rangle\langle\xi\rangle)^{\frac{1}{2}}, \end{aligned} \tag{54}$$

provided $k \in (0;1)$ in the definition of χ is suitably small. In fact, owing to the presence of the $\chi^\star_j$, we have here $|\theta| \geq \frac{\varepsilon}{2}\langle\xi\rangle$ and also

$$\forall C_1 \in (0;1) \;\; |\theta| \geq \frac{C_1}{\sqrt{1-C_1^2}} \Rightarrow |\theta| \geq C_1 \langle\theta\rangle\,;$$
$$|\theta| \geq \frac{\varepsilon}{2}\langle\xi\rangle\,,\;\; \varepsilon \in (0;1) \Rightarrow \langle\theta\rangle \geq \frac{\varepsilon}{2}\langle\xi\rangle\,.$$

Choosing, as is possible,

$$C_1 \mid \frac{\varepsilon}{2} \geq \frac{C_1}{\sqrt{1-C_1^2}}$$

we have $|\theta| \geq \frac{\varepsilon}{2}\langle\xi\rangle \Rightarrow |\theta| \geq \frac{\varepsilon}{2} \Rightarrow |\theta| \geq C_1\langle\theta\rangle$, which gives

$$\begin{aligned} |d_y\omega| &\geq |\theta|-|d_y\varphi-d_x\varphi)| \\ &\geq C_1\langle\theta\rangle - C_2 k\langle\xi\rangle \\ &= \frac{C_1}{2}\langle\theta\rangle + \frac{C_1}{2}\langle\theta\rangle - C_2k\langle\xi\rangle \\ &\geq \frac{C_1}{2}\langle\theta\rangle + \left(\frac{C_1\varepsilon}{4}-C_2k\right)\langle\xi\rangle \end{aligned}$$

which implies (54) for $0<k<\frac{C_1\varepsilon}{4C_2}$. Note also that ${}^tU$ acts only on $f^j_{\beta\gamma\delta}$, leaving r^j_α and $\chi^\star_j$ unchanged, so that we can use the estimates (49) and (50) for them. By applying formulae (36), (37), (52) and (53) we find:

$$\begin{aligned} &\int dy \mathrm{e}^{-i\omega} f^j_{\beta\gamma\delta} = \\ &= \int dy \mathrm{e}^{-i\omega} ({}^tU)^{s_2} f^j_{\beta\gamma\delta} \\ &= \int dy \mathrm{e}^{-i\omega} \frac{1}{|d_y\omega|^{4s_2}} \sum_{|\tau|\leq s_2} P_{\tau,s_2}\partial^y_\tau f^j_{\beta\gamma\delta} \end{aligned}$$

which implies

$$
\begin{aligned}
K &\prec \langle x\rangle^{t_2-2s_1} \sum_{j,\beta+\gamma+\delta=\alpha} \int d\theta dy (\langle\theta\rangle + \langle\xi\rangle)^{-4s_2} \cdot \\
&\quad \cdot \sum_{|\tau|\le s_2} \langle\xi\rangle^{m_1+|\alpha|} \langle y\rangle^{m_2+2s_1-|\tau|} \cdot \\
&\quad \cdot \sum (\langle\theta\rangle + \langle\xi\rangle)^{2s_2} \langle y\rangle^{|\tau|-s_2} (\langle\theta\rangle + \langle\xi\rangle)^{s_2} \\
&\prec \langle x\rangle^{t_2-2s_1} \langle\xi\rangle^{m_1+|\alpha|} \int dy \, \langle y\rangle^{m_2+2s_1-s_2} \int d\theta (\langle\theta\rangle + \langle\xi\rangle)^{-s_2} \\
&\prec \langle x\rangle^{t_2-2s_1} \langle\xi\rangle^{m_1+|\alpha|-\frac{s_2}{2}} \int d\theta \, \langle\theta\rangle^{-\frac{s_2}{2}} \\
&\prec \langle x\rangle^{t_2-2s_1} \langle\xi\rangle^{m_1+|\alpha|-\frac{s_2}{2}}
\end{aligned}
$$

provided

$$
s_2 > \max\{2n, m_2 + 2s_1 + n\}.
$$

By all the estimates we showed above, it is now possible to conclude as follows. For an arbitrary $\rho \in \mathbb{N}$, fix α such that

$$
\begin{aligned}
&t_1 - |\alpha| \le 0; \\
&\rho + m_1 + t_1 + 2n - \frac{|\alpha|}{2} \le 0; \\
&\rho + m_2 + t_2 - \frac{|\alpha|}{2} \le 0.
\end{aligned}
$$

Then, with k and ε fixed by the above discussion about the estimate of K, fix s_1 such that

$$
\rho + t_2 - 2s_1 \le 0
$$

and s_2 such that

$$
\begin{aligned}
&s_2 > m_2 + n + 2s_1; \\
&s_2 > 2n; \\
&\rho + m_1 + |\alpha| - \frac{s_2}{2} \le 0.
\end{aligned}
$$

This shows that $\forall \rho \in \mathbb{N}\ \exists M \in \mathbb{N}$ such that:

$$
\begin{aligned}
(\langle\xi\rangle \langle x\rangle)^{\rho} \left(h_1(x,\xi) - \sum_{|\alpha|<M} \frac{1}{\alpha!} c_\alpha(x,\xi) \right) = \\
= \quad (\langle\xi\rangle \langle x\rangle)^{\rho} \sum_{|\alpha|=M} \frac{M}{\alpha!} R_\alpha(x,\xi) \prec 1
\end{aligned}
$$

which gives the desired result, invoking point 2 of Proposition 5.

□

4.2. Further composition theorems

The next three theorems are immediate consequences of the Composition Theorem 7.

THEOREM 8. *Under the hypotheses of Theorem 7, the composed operator $V = A_{\varphi,a} P$ is, modulo smoothing operators, a FIO of Type I. In fact, $V = V_{\varphi,v}$ where φ is the same phase function and the transpose ${}^t v$ of the amplitude $v \in \mathbf{SG}_l^{m+t}$ admits the asymptotic expansion (25) with p changed in ${}^t p$, a changed in ${}^t a$ and φ changed in ${}^t \varphi$.*

Proof. Using Proposition 9, Remarks 2 and 3 and Theorem 7, we have

$$\begin{aligned}
A_{\varphi,a} P &= {}^t({}^t P\, {}^t A_{\varphi,a}) \\
&= {}^t \left[(\mathcal{F} \circ \mathrm{Op}\left({}^t p\right) \circ \mathcal{F}^{-1}) \circ (\mathcal{F} \circ A_{{}^t\varphi,{}^t a} \circ \mathcal{F}^{-1}) \right] \\
&= {}^t \left[\mathcal{F} \circ (\mathrm{Op}\left({}^t p\right) \circ A_{{}^t\varphi,{}^t a}) \circ \mathcal{F}^{-1} \right] \\
&= {}^t \left[\mathcal{F} \circ A_{{}^t\varphi,h} \circ \mathcal{F}^{-1} \right] \\
&= {}^t(\mathcal{F}^{-1}) \circ {}^t A_{{}^t\varphi,h} \circ {}^t \mathcal{F} \\
&= \mathcal{F}^{-1} \circ \mathcal{F} \circ A_{\varphi,{}^t h} \circ \mathcal{F}^{-1} \circ \mathcal{F} \\
&= A_{\varphi,{}^t h}.
\end{aligned}$$

□

THEOREM 9. *Given a FIO $B = B_{\varphi,b}$ of Type II such that $\varphi \in \mathcal{P}$ and $b \in \mathbf{SG}_l^m(\mathbb{R}^n)$ and a ψdo $P = \mathrm{Op}\,(p)$ with $p \in \mathbf{SG}_l^t(\mathbb{R}^n)$, the composed operator $G = BP$ is, modulo smoothing operators, a FIO of Type II. In fact $G = G_{\varphi,h}$, where φ is the same phase function and the amplitude $h \in \mathbf{SG}_l^{m+t}(\mathbb{R}^n)$ admits the following asymptotic expansion:*

$$h(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (\partial_\xi^\alpha q)(x, d_x\varphi(x,\xi)) D_\alpha^y \left[\mathrm{e}^{i\psi(x,y,\xi)} b(y,\xi) \right]_{y=x} \tag{55}$$

where

$$\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \langle y - x | d_x \varphi(x,\xi) \rangle$$

and q is given by equation (14).

Proof. By Proposition 10 and Theorem 7 we immediately have

$$\begin{aligned}
((B_{\varphi,b} P)^\star u, v)_{L^2} &= (P^\star B^\star_{\varphi,b} u, v)_{L^2} \\
&= (P^\star A_{\varphi,b} u, v)_{L^2} \\
&= (G^\star_{\varphi,h} u, v)_{L^2}
\end{aligned}$$

which gives the desired result, recalling also point 6 of Proposition 5.

□

THEOREM 10. *Under the hypotheses of Theorem 9, the operator $W = P B_{\varphi,b}$ is, modulo smoothing operators, a FIO of Type II. In fact, $W = W_{\varphi,w}$ where φ is the same phase function and the transpose ${}^t w$ of the amplitude $w \in \mathbf{SG}_l^{m+t}$ admits the asymptotic expansion* (55) *with q changed in ${}^t q$, b changed in ${}^t b$ and φ changed in ${}^t \varphi$.*

Proof. Immediate, using the same technique of Theorem 8 and recalling Remark 4 and Theorem 9.

□

The subsequent Theorems 13 and 14 deal with the composition of a Type I operator with a Type II operator. They will be useful in subsections 4.5, where we will show continuity of the FIOs in the Sobolev spaces of Definition 12, and 4.3, where elliptic FIOs will be introduced and their parametrices computed. First of all, we give a simple sufficient condition for maps to be **SG** diffeomorphisms in open subsets of $\mathbb{R}^n$. Then, we put it in relation with regular phases. In the following lemmas ϕ and its inverse $\phi^{-1} = \overline{\phi}$ are smooth functions defined on open subsets of $\mathbb{R}^n$, vector valued in $\mathbb{R}^n$.

LEMMA 12. *Let $f \in \mathbf{SG}^m$ and g vector valued in $\mathbb{R}^n$ such that $g \in \mathbf{SG}^{e_1}$ and $\langle g\rangle \sim \langle \xi\rangle$. Then $f(x, y, g(x, y, \xi)) \in \mathbf{SG}^m$.*

Proof. The desired estimates can be obtained by induction.

□

REMARK 7. Of course, the requirements for g in Lemma 12 need to be satisfied on supp (f) only. By applying repeatedly Lemma 12, we obtain also:

$$
\begin{aligned}
& f \in \mathbf{SG}^m,\ g_j \in \mathbf{SG}^{e_j} \mid \langle g_1\rangle \sim \langle \xi\rangle,\ \langle g_2\rangle \sim \langle x\rangle,\ \langle g_3\rangle \sim \langle y\rangle \\
& \Rightarrow f(g_2(x, y, \xi), g_3(x, y, \xi), g_1(x, y, \xi)) \in \mathbf{SG}^m.
\end{aligned}
$$

LEMMA 13. *Let $\overline{\phi} = \overline{\phi}(y) \in C^\infty$ be such that $\forall \alpha \in \mathbb{N}^n \mid |\alpha| = 1$: $\partial^y_\alpha \overline{\phi}(y) = a_\alpha(\overline{\phi}(y))$ with $a_\alpha(x) \in \mathbf{SG}^0_l$ and $\langle\overline{\phi}(y)\rangle \sim \langle y\rangle$. Then $\overline{\phi}(y) \in \mathbf{SG}^{e_2}_l$.*

Proof. It is obvious that $\partial^y_\alpha \overline{\phi}(y) \prec \langle y\rangle^{1-|\alpha|}$ for $|\alpha| \le 1$. The other estimates can be obtained by induction on $|\alpha|$.

□

LEMMA 14. *Let $\phi = \phi(x) \in \mathbf{SG}^{e_2}_l$ and $\left|\det \frac{\partial \phi}{\partial x}\right| \ge \varepsilon > 0$. Then the inverse function $\overline{\phi} = \overline{\phi}(y)$ is such that*

$$
\forall \alpha \in \mathbb{N}^n \mid |\alpha| = 1 \ :\ \partial^y_\alpha \overline{\phi}(y) = a_\alpha(\overline{\phi}(y))
$$

with $a_\alpha(x) \in \mathbf{SG}^0_l$.

Proof. Obviously, the hypotheses imply $\det \frac{\partial \phi}{\partial x} \in \mathbf{ESG}^0_l$. Since

$$
\left(\frac{\partial \phi}{\partial x}\right)^{-1} = \left(\det \frac{\partial \phi}{\partial x}\right)^{-1} M
$$

where the adjoint matrix M is made of determinants of submatrices of $\frac{\partial \phi}{\partial x}$, we also obtain $\left(\frac{\partial \phi}{\partial x}\right)^{-1} \in \mathbf{SG}^0_l$. The result is then a consequence of the inverse function theorem and the composition Lemma 12.

□

LEMMA 15. *If* $\phi \in \mathrm{Diffeo}(U^{\#}, V^{\#})$*, satisfies* (18) *and also*

$$\begin{aligned}&\phi(x) \in \mathbf{SG}_l^{e_2};\\ &\langle\phi(x)\rangle \sim \langle x\rangle\,;\\ &\left|\det\frac{\partial\phi}{\partial x}\right| \geq \varepsilon > 0\end{aligned}$$

for a suitable constant $\varepsilon > 0$ *then* $\phi \in \mathbf{SG}\mathrm{Diffeo}(U^{\#}, V^{\#}; U, V; \delta)$.

Proof. We only have to prove $\overline{\phi}(y) \in \mathbf{SG}_l^{e_2}$. This is an immediate consequence of Lemmas 13 and 14.

□

PROPOSITION 11. *Let* $\phi = \phi(x, y; \xi) \in \mathbf{SG}^{e_1}$ *be such that* $\langle\phi\rangle \sim \langle\xi\rangle$ *and* $|\frac{\partial\phi}{\partial\xi}| \geq \varepsilon > 0$*. Then, setting* $\eta = \phi(x, y; \xi) \Leftrightarrow \xi = \overline{\phi}(x, y; \eta)$*,* ϕ *and its inverse both satisfy* $\mathbf{SG}^0$ *estimates with respect to* x *and* y*. We will briefly speak in such case of* **SG** *diffeomorphisms with* $\mathbf{SG}^0$ *parameter dependence.*

Proof. $\overline{\phi}$ satisfies the required estimates with respect to η in view of the obvious variant of Lemma 15 to ξ, η variables. For what concerns the estimates with respect to x and y, it is enough to use the Riemann-Dini theorem about derivatives of implicit functions and an inductive process completely analogous to that used in Lemma 13.

□

PROPOSITION 12. *If* $\varphi \in \mathcal{P}^{\varepsilon}$*, according to Definition 23, then* $\xi \to d_x\varphi(x, \xi)$ *and* $x \to \nabla_\xi\varphi(x, \xi)$ *are two global* **SG** *diffeomorphisms with* $\mathbf{SG}^0$ *parameter dependence.*

Proof. The property of being **SG** diffeomorphisms is immediate from Proposition 11, since all the hypotheses made about ϕ there are expressed by the properties $d_x\varphi \in \mathbf{SG}_l^{e_1}$, $\nabla_\xi\varphi \in \mathbf{SG}_l^{e_2}$, (19) and (20). The globality is a consequence of the following theorem (see Berger [3], page 221):

THEOREM 11. *Let us assume that* $\widetilde{\phi} \in C^1(X, Y)$ *with* X *and* Y *Banach spaces. Then* $\widetilde{\phi}$ *is a diffeomorphism of* X *onto* Y *if and only if* $\widetilde{\phi}$ *is proper and* $\frac{\partial\widetilde{\phi}}{\partial x}(x)$ *is a linear homeomorphism for each* $x \in X$.

The condition on $\frac{\partial\widetilde{\phi}}{\partial x}$ with $\widetilde{\phi} = d_x\varphi(x, .)$ or $\widetilde{\phi} = \nabla_\xi\varphi(., \xi)$ is satisfied, owing to the hypothesis $\varphi \in \mathcal{P}^{\varepsilon}$. The fact that $\widetilde{\phi}$ is proper in the two cases again descends from $\varphi \in \mathcal{P}^{\varepsilon}$. In fact, we have the following characterization of proper mappings in finite dimensional Banach spaces (see [3], page 102):

THEOREM 12. *If* X *and* Y *are finite dimensional Banach spaces and* $\widetilde{\phi} \in C^0(X, Y)$*, then* $\widetilde{\phi}$ *is proper if and only if it is coercive, i.e.,* $\lim_{\|x\|\to+\infty} \|f(x)\| = +\infty$.

For the first case we have, at least for large ξ,

$$|d_x\varphi(x, \xi)| = \sqrt{\langle d_x\varphi(x, \xi)\rangle^2 - 1} \geq \sqrt{C\,\langle\xi\rangle^2 - 1},$$

which implies the required coercivity of the mapping in $\mathbb{R}^n$, so that it is proper and therefore global owing to Theorem 11. The same is obviously true also for $\nabla_\xi\varphi(.,\xi)$.

$\square$

THEOREM 13. *Let $A = A_{\varphi,a}$ be a Type I and $B = B_{\varphi,b}$ a Type II FIO with $\varphi \in \mathcal{P}^\varepsilon$, $a \in \mathbf{SG}^r_l$ and $b \in \mathbf{SG}^s_l$. Then the operator $P = AB$ is, modulo smoothing operators, a ψdo with amplitude $p \in \mathbf{SG}^m$ (given in equation (62) below); m is related to r and s by $m = (r_1 + s_1, r_2, s_2)$.*

Proof. Let us write explicitly the composition for $u \in \mathcal{S}$. We find

$$\begin{aligned} A_{\varphi,a}B_{\varphi,b}u(x) &= \int đ\xi\ \mathrm{e}^{i\varphi(x,\xi)}\, a(x,\xi) \int dy\ \mathrm{e}^{-i\varphi(y,\xi)}\, \overline{b(y,\xi)}\, u(y) \\ &= \int đ\xi dy\ \mathrm{e}^{i\psi(x,y,\xi)}\, c(x,y,\xi)\, u(y) \end{aligned}$$

where we have set $\psi(x,y,\xi) = \varphi(x,\xi) - \varphi(y,\xi)$ and $c(x,y,\xi) = a(x,\xi)\overline{b(y,\xi)} \in \mathbf{SG}^m$. Let us choose $\chi \in \Xi^\Delta(k)$ and write

$$\begin{aligned} A_{\varphi,a}B_{\varphi,b}u(x) &= \int đ\xi dy\ \mathrm{e}^{i\psi(x,y,\xi)}\, q_1(x,y,\xi)\, u(y) \\ &+ \int đ\xi dy\ \mathrm{e}^{i\psi(x,y,\xi)}\, q_2(x,y,\xi)\, u(y) \\ &= (I_1 + I_2)u(x) \end{aligned}$$

with $q_1(x,y,\xi) = \chi(x,y)c(x,y,\xi)$ and $q_2(x,y,\xi) = (1-\chi(x,y))c(x,y,\xi)$. We begin by showing that I_2 is a smoothing operator, then we will show how to rewrite I_1 as an operator in $\mathbf{LG}^m$ with suitable amplitude p.

1. **I_2 is smoothing.**
   First of all, note that supp $(q_2) \subseteq \left\{(x,y,\xi) \mid |x-y| \geq \frac{k}{2}\langle x\rangle\right\} = R^e$. So, the use of the operator $U = \frac{-i}{|\nabla_\xi\psi|^2}\sum_{k=1}^n \partial_\xi^j\psi\,\partial_\xi^j$, analogous to that defined in (35), is allowed in I_2. In fact, let us set $v = \nabla_\xi\varphi(x,\xi)$ and $w = \nabla_\xi\varphi(y,\xi)$. By making use of Proposition 12 and by $\varphi \in \mathbf{SG}^e_l$, we can write

$$\begin{aligned} |x-y| &= |(\nabla_\xi\varphi)^{-1}(v,\xi) - (\nabla_\xi\varphi)^{-1}(w,\xi)| \\ &= \left|\int_0^1 dt\ < v-w|d_x(\nabla_\xi\varphi)^{-1}(tv+(1-t)w,\xi) >\right| \\ &\leq |v-w| \sup_{\mathbb{R}^n\times\mathbb{R}^n} \|d_x(\nabla_\xi\varphi)^{-1}(z,\xi)\| \\ &\leq M|\nabla_\xi\varphi(x,\xi) - \nabla_\xi\varphi(y,\xi)| \\ &= M|\nabla_\xi\psi(x,y,\xi)|. \end{aligned}$$

   So we have

$$|\nabla_\xi\psi(x,y,\xi)| \succ |x-y| \succ \langle x\rangle + \langle y\rangle \tag{56}$$

   in the region R^e. Then, acting as above and using (36), (37), (38) and again $\varphi \in \mathbf{SG}^e_l$, for all $h \in \mathbb{N}$

$$I_2u(x) = \int đ\xi dy\ \mathrm{e}^{i\psi}\ (({}^tU)^h q_2)\, u$$

and

$$\begin{aligned}({}^tU)^h q_2(x,y,\xi) &= \frac{1}{|\nabla_\xi\psi|^{4h}}\sum_{|\alpha|\le h} P_{\alpha,h}\partial_\xi^\alpha q_2(x,y,\xi)\\ &\prec \frac{\sum_{|\alpha|\le h}\langle x\rangle^{m_2}\langle y\rangle^{m_3}\langle\xi\rangle^{m_1-|\alpha|}(\langle x\rangle+\langle y\rangle)^{3h}\langle\xi\rangle^{|\alpha|-h}}{(\langle x\rangle+\langle y\rangle)^{4h}}\\ &= \frac{\sum_{|\alpha|\le h}\langle x\rangle^{m_2}\langle y\rangle^{m_3}\langle\xi\rangle^{m_1-h}}{(\langle x\rangle+\langle y\rangle)^{h}}\\ &\prec \frac{\langle\xi\rangle^{m_1-h}\langle x\rangle^{m_2}\langle y\rangle^{m_3}}{(\langle x\rangle+\langle y\rangle)^{h}}. \end{aligned} \tag{57}$$

Let us write

$$\begin{aligned} I_2u(x) &= \int dy\; u(y)\int đ\xi\; \mathrm{e}^{i\psi(x,y,\xi)}\,({}^tU)^h q_2(x,y,\xi)\\ &= \int dy\; f(x,y)\,u(y). \end{aligned}$$

Since $\langle x\rangle+\langle y\rangle \ge (\langle x\rangle\,\langle y\rangle)^{\frac{1}{2}}$, we easily see that

$$\forall\alpha,\beta,\gamma,\delta\in\mathbb{N}^n\;\; x^\alpha y^\beta\partial_\gamma^x\partial_\delta^y f(x,y)\prec 1 \Leftrightarrow f\in\mathcal{S}(\mathbb{R}^{2n}).$$

This is trivial for $\gamma=\delta=0$ by the estimate (57), choosing $h>\max\{m_1+n, 2(m_2+|\alpha|), 2(m_3+|\beta|)\}$. For what concerns the case $|\gamma+\delta|>0$, since it is possible to differentiate under the integral sign, any derivative produces a sum of terms of analogous form with different **SG** orders, so that the result is also true for any γ and δ: this concludes the proof of $I_2\in\mathcal{K}$.

2. **I_1 is in $\mathbf{LG}^m$.**
Here we have supp $(q_1)\subseteq\{(x,y,\xi)\mid |x-y|\le k\langle x\rangle\}=R^i\Rightarrow\langle x\rangle\sim\langle y\rangle$. Let us define

$$\widetilde{d}_x\varphi(x,y;\xi)=\int_0^1 d\theta d_x\varphi(y+\theta(x-y),\xi). \tag{58}$$

We can write

$$\begin{aligned}\widetilde{d}_x\varphi(x,y;\xi) &= d_x\varphi(y,\xi)\\ &+ \int_0^1\int_0^1 d\theta_1 d\theta_2 < \theta_1(x-y)|d_x^2\varphi(y+\theta_1\theta_2(x-y),\xi)>\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial\xi}\widetilde{d}_x\varphi(x,y;\xi)=\frac{\partial}{\partial\xi}d_x\varphi(y,\xi)$$
$$+\int_0^1\int_0^1 d\theta_1 d\theta_2\;\theta_1 < x-y|\frac{\partial}{\partial\xi}d_x^2\varphi(y+\theta_1\theta_2(x-y),\xi)>. \tag{59}$$

The integrand in (59) can be estimated as follows

$$\begin{aligned}&(x^k-y^k)\,\partial_{kj}^x\partial_\xi^i\varphi(y+\theta_1\theta_2(x-y),\xi)\\ &\prec |x-y|\sup_{\theta_1,\theta_2\in[0;1]}\langle y+\theta_1\theta_2(x-y)\rangle^{-1}\\ &\prec k\,\langle x\rangle\,\langle y\rangle^{-1}\prec k,\end{aligned}$$

so that the jacobian of $\widetilde{d}_x\varphi(x, y; \xi)$ is a small perturbation of that of $d_x\varphi(y, \xi)$. Then by choosing k suitably small and recalling $\varphi \in \mathcal{P}^\varepsilon$, we can assume (see, e.g., the appendix of [14])

$$\left|\det \frac{\partial}{\partial\xi}\widetilde{d}_x\varphi(x, y; \xi)\right| \geq \frac{\varepsilon}{2} > 0.$$

Moreover

$$\begin{aligned}
&\partial_\xi^\alpha \partial_\beta^x \partial_\gamma^y \widetilde{d}_x\varphi(x, y; \xi) = \\
&\qquad = \int d\theta\ \theta^{|\beta|}(1-\theta)^{|\gamma|}(\partial_\xi^\alpha \partial_{\beta+\gamma}^x \varphi)(y + \theta(x-y), \xi) \\
&\qquad \prec \langle\xi\rangle^{1-|\alpha|}\ \langle y\rangle^{-|\beta+\gamma|} = \langle\xi\rangle^{1-|\alpha|}\ \langle x\rangle^{-|\beta|}\ \langle y\rangle^{-|\gamma|}\ ,
\end{aligned} \tag{60}$$

since $\langle x\rangle \sim \langle y\rangle$, so that $\widetilde{d}_x\varphi(x, y; \xi)$ satisfies $\mathbf{SG}^{e_1}$ estimates in the region R^i. We want now to prove

$$\left\langle\widetilde{d}_x\varphi(x, y; \xi)\right\rangle \sim \langle\xi\rangle\ . \tag{61}$$

The first inequality $\left\langle\widetilde{d}_x\varphi(x, y; \xi)\right\rangle \prec \langle\xi\rangle$ is easily proved by

$$\begin{aligned}
\left\langle\widetilde{d}_x\varphi(x, y; \xi)\right\rangle^2 &= 1 + \sum_{j=1}^n \left|\int_0^1 d\theta\ \partial_j^x\varphi(y + \theta(x-y), \xi)\right|^2 \leq \\
&\leq 1 + \sum_{j=1}^n \left(\int_0^1 d\theta\ |\partial_j^x\varphi(y + \theta(x-y), \xi)|\right)^2 \\
&\prec 1 + \sum_{j=1}^n \langle\xi\rangle^2 \prec \langle\xi\rangle^2\ .
\end{aligned}$$

We also have

$$\begin{aligned}
&\left\langle\widetilde{d}_x\varphi(x, y; \xi)\right\rangle^2 = 1 + \\
&\sum_{j=1}^n \left(\partial_j^x\varphi(y, \xi) + \int_0^1\int_0^1 d\theta_1 d\theta_2\ \theta_1 < x - y|d_x\partial_j\varphi(y + \theta_1\theta_2(x-y), \xi) >\right)^2 \\
&= \langle d_x\varphi(y, \xi)\rangle^2 \\
&+2\sum_{j=1}^n \overbrace{\partial_j^x\varphi(y, \xi)\int_0^1\int_0^1 d\theta_1 d\theta_2\ \theta_1 < x - y|d_x\partial_j\varphi(y + \theta_1\theta_2(x-y), \xi) >}^{F_j} \\
&+ \sum_{j=1}^n \underbrace{\left(\int_0^1\int_0^1 d\theta_1 d\theta_2\ \theta_1 < x - y|d_x\partial_j\varphi(y + \theta_1\theta_2(x-y), \xi) >\right)^2}_{G_j}.
\end{aligned}$$

Let us now estimate F_j and G_j:

$$\begin{aligned} F_j &\prec \langle\xi\rangle\,|x-y|\,\langle\xi\rangle \int_0^1\int_0^1 d\theta_1 d\theta_2\ \langle y+\theta_1\theta_2(x-y)\rangle^{-1} \\ &\prec k\,\langle\xi\rangle^2\,\langle x\rangle\,\langle y\rangle^{-1} \\ &\prec k\,\langle\xi\rangle^2 \end{aligned}$$

and similarly

$$G_j \prec k^2\,\langle\xi\rangle^2 .$$

We have now all we need to show (61). Since $\langle d_x\varphi(y,\xi)\rangle^2 \geq C_1\,\langle\xi\rangle^2$ we have

$$\begin{aligned} &\left\langle \widetilde{d}_x\varphi(x,y;\xi)\right\rangle^2 \geq \\ &\quad\geq C_1\,\langle\xi\rangle^2 - 2\sum_{j=1}^n |F_j| - \sum_{j=1}^n G_j \\ &\quad\geq \langle\xi\rangle^2\left[C_1 - kn(2C_2+kC_3)\right] \end{aligned}$$

which implies $\left\langle \widetilde{d}_x\varphi(x,y;\xi)\right\rangle \succ \langle\xi\rangle$ for k suitably small. This completes the proof of (61). Then, with a suitable choice of k, $\widetilde{d}_x\varphi(x,y;\xi)$ satisfies all the requirements of Proposition 11, and, for $(x,y,.)\in R^i$, $\widetilde{d}_x\varphi(x,y;\xi)$ is an **SG** diffeomorphism with $\mathbf{SG}^0$ parameter dependence. With this in mind, the operator I_1 is an integral extended to R^i which we can rewrite as

$$\begin{aligned} I_1u(x) &= \int đ\xi dy\ \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}\,q_1(x,y,\xi)\,u(y) \\ &= \int đ\eta dy\ \mathrm{e}^{i<x-y|\widetilde{d}_x\varphi(x,y;\eta)>}\,q_1(x,y,\eta)\,u(y). \end{aligned}$$

In the region R^i we can perform the substitution

$$\xi = \widetilde{d}_x\varphi(x,y;\eta) \Leftrightarrow \eta = (\widetilde{d}_x\varphi)^{-1}(x,y;\xi)$$

so that we can conclude

$$\begin{aligned} I_1u(x) &= \int dyđ\xi\ e^{i<x-y|\xi>}\,q_1(x,y,(\widetilde{d}_x\varphi)^{-1}(x,y;\xi)) \\ &\qquad \left|\det\frac{\partial}{\partial\xi}(\widetilde{d}_x\varphi)^{-1}(x,y;\xi)\right|\ u(y) \\ &= \int dyđ\xi\ e^{i<x-y|\xi>}\ p(x,y,\xi)\,u(y) \end{aligned}$$

setting

$$p(x,y,\xi) = q_1(x,y,(\widetilde{d}_x\varphi)^{-1}(x,y;\xi))\left|\det\frac{\partial}{\partial\xi}(\widetilde{d}_x\varphi)^{-1}(x,y;\xi)\right|. \tag{62}$$

By (60), Lemma 12 and Proposition 11 we find $p\in\mathbf{SG}^m$, which concludes the proof.

□

THEOREM 14. *Let $A = A_{\varphi,a}$ be a Type I and $B = B_{\varphi,b}$ a Type II FIO with $\varphi \in \mathcal{P}^{\varepsilon}$, $a \in \mathbf{SG}^r_l$ and $b \in \mathbf{SG}^s_l$. Then the operator $P = BA$ is, modulo smoothing operators, a ψdo with symbol $p \in \mathbf{SG}^m_l$, $m = r + s$, which admits the asymptotic expansion given in equation* (66) *below.*

Proof. Again, let us begin by writing explicitly the composition for $u \in \mathcal{S}$. We find

$$\begin{aligned} B_{\varphi,b}A_{\varphi,a}u(x) &= \\ &= \int d\!\!\!^{-}\xi \; \mathrm{e}^{i<x|\xi>} \int dy \; \mathrm{e}^{-i\varphi(y,\xi)}\, \overline{b(y,\xi)} \int d\!\!\!^{-}\eta \; \mathrm{e}^{i\varphi(y,\eta)}\, a(y,\eta)\, \hat{u}(\eta) \\ &= \int d\!\!\!^{-}\eta \; \mathrm{e}^{i<x|\eta>} \left(\int dy d\!\!\!^{-}\xi \; \mathrm{e}^{i(\varphi(y,\eta)-\varphi(y,\xi)-<x|\eta-\xi>)}\, a(y,\eta)\overline{b(y,\xi)} \right) \hat{u}(\eta) \\ &= \int d\!\!\!^{-}\xi \; \mathrm{e}^{i<x|\xi>} \left(\int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, c(\xi,\eta,y) \right) \hat{u}(\xi) \end{aligned}$$

where we set $\psi(\xi,\eta,y) = \varphi(y,\xi) - \varphi(y,\eta)$, $\omega(x,y,\xi,\eta) = \psi(\xi,\eta,y) - < x|\xi - \eta >$ and $c(\xi,\eta,y) = a(y,\xi)\overline{b(y,\eta)} \in \mathbf{SG}^t$, with $t = (r_2 + s_2, r_1, s_1)$. The theorem is proved if we can show that

$$p(x,\xi) \quad = \quad \int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, c(\xi,\eta,y) \in \mathbf{SG}^m_l.$$

Let us choose $\chi \in \Xi^{\Delta}(k)$ as above and set

$$\begin{aligned} p(x,\xi) &= \int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, c(\xi,\eta,y)\, \chi(\xi,\eta) \\ &+ \int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, c(\xi,\eta,y)\, (1-\chi(\xi,\eta)) \\ &= \int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, q_1(\xi,\eta,y) \\ &+ \int dy d\!\!\!^{-}\eta \; \mathrm{e}^{i\omega(x,y,\xi,\eta)}\, q_2(\xi,\eta,y) \\ &= (I_1 + I_2). \end{aligned}$$

Again, we analyze separately I_1 and I_2.

1. $I_2 \in \mathcal{S}(\mathbb{R}^{2n})$.
   The proof is very similar to the one in Theorem 13 above, showing that the operator associated with I_2 is smoothing. In fact, we are in the region R^e, so that we have, analogously to (56),
   $$|d_y\psi(\xi,\eta,y)| \succ |\xi - \eta| \succ \langle\xi\rangle + \langle\eta\rangle\,.$$
   This implies that the operator $U = \frac{-i}{|d_y\psi|^2}\sum_{j=1}^n \partial^y_j\psi\,\partial^y_j$, identical to that used above (apart a change of names of variables) can be used in I_2. Then for all $\alpha, \beta \in \mathbb{N}^n$ and

arbitrary $s \in \mathbb{N}$,

$$\begin{aligned}
I_2 &= \xi^\alpha x^\beta \int đ\eta dy\, \mathrm{e}^{i<x|\eta-\xi>}\, \mathrm{e}^{i\psi(\xi,\eta,y)}\, q_2(\xi,\eta,y) \\
&= \int đ\eta dy\, x^\beta \mathrm{e}^{i<x|\eta>}\, \mathrm{e}^{i(\psi(\xi,\eta,y)-<x|\xi>)}\, \xi^\alpha q_2(\xi,\eta,y) \\
&= \int đ\eta dy\, D^\beta_\eta \mathrm{e}^{i<x|\eta>}\, \mathrm{e}^{i(\psi(\xi,\eta,y)-<x|\xi>)}\, \xi^\alpha q_2(\xi,\eta,y) \\
&= (-1)^{|\beta|} \int đ\eta dy\, \mathrm{e}^{i<x|\eta>}\, D^\beta_\eta \left(\mathrm{e}^{i(\psi(\xi,\eta,y)-<x|\xi>)}\, \xi^\alpha q_2(\xi,\eta,y)\right) \\
&= \sum_j \int đ\eta dy\, \mathrm{e}^{i<x|\eta-\xi>}\, \mathrm{e}^{i\psi(\xi,\eta,y)}\, \widetilde{q}_{2j}(\xi,\eta,y) \qquad (63)
\end{aligned}$$

for suitable $\widetilde{q}_{2j} \in \mathbf{SG}^{m_j}_l$, where m_j depends on t, α and β. We find, just as above,

$$\xi^\alpha x^\beta I_2 \prec 1$$

if we use $U^s \mathrm{e}^{i\psi} = \mathrm{e}^{i\psi}$ in each of the integrals of the sum (63) with s large enough. Again, for $\xi^\alpha x^\beta \partial^\gamma_\xi \partial^x_\delta I_2$ we have to act in the same way on a sum of integrals similar to (63), since differentiation under the sign simply produces a sum of terms in the integrands which are still amplitudes with orders depending on the multi-indices of the derivatives.

2. I_1 **defines a symbol in** $\mathbf{SG}^m_l$**.**
In the region $R^i = \{(\xi,\eta,y) \mid |\xi-\eta| \le k\,\langle\xi\rangle\}$ defined as in the proof of Theorem 13 it is possible to consider the **SG** diffeomorphism with $\mathbf{SG}^0$ parameter dependence

$$\widetilde{\nabla}_\xi \varphi(\xi,\eta;y) = \int_0^1 d\theta\, \nabla_\xi \varphi(y, \eta+\theta(\xi-\eta))$$

analogous to that defined in (58) (symmetry in the role of variable and covariable for $\varphi \in \mathcal{P}^\varepsilon$). Let us perform the change of variables

$$z = \widetilde{\nabla}_\xi \varphi(\xi,\eta;y) \Leftrightarrow y = (\widetilde{\nabla}_\xi \varphi)^{-1}(\xi,\eta;z)$$

recalling that obviously

$$\psi(\xi,\eta,y) =< \widetilde{\nabla}_\xi \varphi(\xi,\eta;y)|\xi-\eta> =< z|\xi-\eta>,$$

and set

$$I_1 = \int dz đ\eta\, \mathrm{e}^{i<z-x|\xi-\eta>}\, \widetilde{q}_1(\xi,\eta,z)$$

with

$$\begin{aligned}
\widetilde{q}_1(\xi,\eta,z) &= q_1(\xi,\eta,(\widetilde{\nabla}_\xi\varphi)^{-1}(\xi,\eta;z))\, \chi(\xi,\eta)\, \cdot \\
&\quad \cdot \left|\det \frac{\partial}{\partial z} (\widetilde{\nabla}_\xi \varphi)^{-1}(\xi,\eta;z)\right| \in \mathbf{SG}^t
\end{aligned}$$

(again, we are following the proof of Theorem 13). We can now show that $I_1 \in \mathbf{SG}^m_l$ by writing for it an asymptotic expansion. Let us set $\zeta = \eta - \xi \Leftrightarrow \eta = \zeta + \xi$, so that

$$
\begin{aligned}
I_1 &= \int đ\zeta\, e^{i<x|\zeta>} \int dz e^{-i<z|\zeta>} \widetilde{q}_1(\xi, \xi+\zeta, z) \\
&= \sum_{|\alpha|<M} \frac{1}{\alpha!} \mathcal{F}^{-1}_{\zeta\to x}\left[\zeta^\alpha \mathcal{F}_{z\to\zeta}\left((\partial^\alpha_\eta \widetilde{q}_1)(\xi,\xi,z)\right)\right] + \sum_{|\alpha|=M} \frac{M}{\alpha!} R_\alpha \\
&= \sum_{|\alpha|<M} \frac{i^{|\alpha|}}{\alpha!} (D^z_\alpha D^\alpha_\eta \widetilde{q}_1)(\xi,\xi,x) + \sum_{|\alpha|=M} \frac{M}{\alpha!} R_\alpha
\end{aligned}
\tag{64}
$$

having used the Taylor expansion with respect to the second variable η of $\widetilde{q}_1$ with

$$
R_\alpha = \int đ\zeta dz\, e^{i<x-z|\zeta>}\, \zeta^\alpha \int_0^1 d\theta\, (1-\theta)^{M-1}\, (\partial^\alpha_\eta \widetilde{q}_1)(\xi, \xi+\theta\zeta, y).
$$

Now, the sum for $|\alpha| < M$ in (64) has the same behaviour (apart a change in role of variables and covariables) of the asymptotic expansion associated to a generic amplitude defined in (12). The first term obviously has order $m = (r_1 + s_1, r_2 + s_2)$. So, we simply have to estimate R_α, to make possible the use of the simplified criterion for asymptotic expansions in point 6 of Proposition 5. Let us note, first of all, that the presence of χ in $\widetilde{q}_1$ implies

$$
\begin{aligned}
&|\zeta| = |\xi - \eta| \le k\,\langle\xi\rangle \Rightarrow \\
&\qquad |\,\langle\xi+\theta\zeta\rangle - \langle\xi\rangle\,| \le \theta|\zeta| \le |\zeta| \le k\,\langle\xi\rangle \\
&\qquad \Rightarrow \langle\xi+\theta\zeta\rangle \sim \langle\xi\rangle\,.
\end{aligned}
\tag{65}
$$

Then set $z - x = w$ in R_α, to find

$$
R_\alpha(x,\xi) = \int đ\zeta dw\, e^{i<w|\zeta>} \int_0^1 d\theta\, (1-\theta)^{M-1} (D^z_\alpha \partial^\alpha_\eta \widetilde{q}_1)(\xi, \xi+\theta\zeta, x+w).
$$

Let us now take into account the operators $W = \frac{1-\Delta_\zeta}{\langle w\rangle^2} = {}^tW$ and $\widetilde{W} = \frac{1-\Delta_w}{\langle\zeta\rangle^2} = {}^t\widetilde{W}$ (having obviously the same properties of the W used in the proof of Theorem 7). We have, for all $s_1, s_2 \in \mathbb{N}$,

$$
\begin{aligned}
R_\alpha(x,\xi) &= \int đ\zeta\, dw\ \mathrm{e}^{i<w|\zeta>}\,\cdot \\
&\quad \int_0^1 d\theta\ (1-\theta)^{M-1} ({}^tW)^{s_1} ({}^t\widetilde{W})^{s_2} \Big[(D^z_\alpha \partial^\alpha_\eta \widetilde{q}_1)(\xi, \xi+\theta\zeta, x+w) \Big] \\
&= \int đ\zeta\, dw\ \mathrm{e}^{i<w|\zeta>} \int_0^1 d\theta\ (1-\theta)^{M-1}\,\cdot \\
&\quad \Bigg\{ \frac{(D^z_\alpha \partial^\alpha_\eta \widetilde{q}_1)(\xi,\xi+\theta\zeta,x+w)}{\langle w\rangle^{2s_1}\,\langle\zeta\rangle^{2s_2}} \\
&+ \frac{1}{\langle w\rangle^{2s_1}} \sum_{j_1=1}^{s_1} \binom{s_1}{j_1} (-\Delta_\zeta)^{j_1}\,\cdot \\
&\quad \Bigg[\frac{1}{\langle\zeta\rangle^{2s_2}} \sum_{j_2=1}^{s_2} \binom{s_2}{j_2} (-\Delta_w)^{j_2} (D^z_\alpha \partial^\alpha_\eta \widetilde{q}_1)(\xi,\xi+\theta\zeta,x+w) \Bigg] \Bigg\} \\
&\prec \int d\zeta\, dw \int_0^1 d\theta \frac{\langle x+w\rangle^{t_1-|\alpha|}\,\langle\xi\rangle^{t_2}\,\langle\xi+\theta\zeta\rangle^{t_3-|\alpha|}}{\langle w\rangle^{2s_1}\,\langle\zeta\rangle^{2s_2}} \\
&\prec \langle x\rangle^{t_1-|\alpha|}\,\langle\xi\rangle^{t_2+t_3-|\alpha|} \int dw\ \langle w\rangle^{|t_1-|\alpha||-2s_1} \int dw\ \langle\zeta\rangle^{-2s_2},
\end{aligned}
$$

since all the terms in the sum have lower order than the first term, due to the action of the Laplacians. We have used Peetre inequality and (65) in the last estimate. Then, with $M = |\alpha|$ fixed, choose s_1 and s_2 such that

$$
\begin{aligned}
|t_1 - |\alpha|| - 2s_1 &< -n; \\
-2s_2 &< -n.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\sum_{|\alpha|=M} \frac{M}{\alpha!} R_\alpha \prec \langle\xi\rangle^{m_1-M}\,\langle x\rangle^{m_2-M} \\
&\Rightarrow I_1 \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D^z_\alpha D^\alpha_\eta \widetilde{q}_1)(\xi,\eta,x)|_{\eta=\xi}. \qquad (66)
\end{aligned}
$$

From the calculations described above, it turns out that $p(x,\xi)$ is a symbol with asymptotic expansion given by (66), i.e., a symbol in $\mathbf{SG}^m_l$ as desired.

□

4.3. Elliptic FIOs and parametrices

DEFINITION 25. *A Type I $A_{\varphi,a}$ or a Type II $B_{\varphi,b}$ FIO is said md-elliptic if $\varphi \in \mathcal{P}^\varepsilon$ and the amplitude a or b is md-elliptic.*

LEMMA 16. *If $A_{\varphi,a}$ is md-elliptic, then the two ψdos $A_{\varphi,a}A^\star_{\varphi,a}$ and $A^\star_{\varphi,a}A_{\varphi,a}$ are md-elliptic as well.*

Proof. It is enough to prove that the principal part of the asymptotic expansion of the two symbols is md-elliptic. In fact, let us pick a generic symbol r with md-elliptic principal part $r_0 \in \mathbf{SG}_l^m$ such that $r - r_0 = r_1 \in \mathbf{SG}_l^{m-\delta e}$ with $\delta > 0$. We have

$$r^{-1} = (r_0 + r_1)^{-1} = r_0^{-1}(1 + \frac{r_1}{r_0})^{-1}$$

which implies

$$r^{-1} \prec \langle \xi \rangle^{-m_1} \langle x \rangle^{-m_2} .$$

Now, from Theorem 13, cf. (62), the symbol of $A_{\varphi,a} A^\star_{\varphi,a}$ has principal part

$$p_0(x, \xi) = a(x, \eta)\overline{a(x, \eta)}\Big|_{\eta=(d_x\varphi)^{-1}(x,\xi)} \left|\det \tfrac{\partial}{\partial \xi}(d_x\varphi)^{-1}(x, \xi)\right| = h(x, \xi)E(x, \xi).$$

$h(x, \xi) = a(x, \eta)\overline{a(x, \eta)}\Big|_{\eta=(d_x\varphi)^{-1}(x,\xi)}$ is md-elliptic of order $2m$, owing to the hypothesis on the amplitude a, to the property of $d\varphi(., \xi)$ of being an **SG** diffeomorphism with $\mathbf{SG}^0$ parameter dependence and to the composition properties in **SG** classes expressed by Lemma 12. $E \in \mathbf{ESG}_l^0$ because it is the jacobian determinant of an **SG** diffeomorphism (see the proof of the next Proposition 13). With similar arguments, it turns out that also $A^\star_{\varphi,a} A_{\varphi,a}$ is md-elliptic.

□

THEOREM 15. *Any md-elliptic FIO A admits a parametrix, A^{-1}. If A is of Type I, A^{-1} is of Type II and viceversa.*

Proof. Let us denote by P^{-1} the parametrix of $P = AA^\star$ and by Q^{-1} the parametrix of $Q = A^\star A$, which exist owing to Lemma 16. We have

$$\begin{aligned} PP^{-1} &= I - R_1, \qquad & P^{-1}P &= I - R_2, \\ QQ^{-1} &= I - R_3, \qquad & Q^{-1}Q &= I - R_4, \end{aligned}$$

with $R_1, R_2, R_3, R_4 \in \mathcal{K}$. Let us set $F_l = Q^{-1}A^\star$ and $F_r = A^\star P^{-1}$. We then have

$$\begin{aligned} &F_l A = (A^\star A)^{-1}(A^\star A) = I - R_4, \\ &AF_r = (AA^\star)(AA^\star)^{-1} = I - R_1, \\ &F_l AF_r = (I - R_4)F_r \Rightarrow F_l - F_l R_1 = F_r - R_4 F_r \Leftrightarrow F_l = F_r \bmod \mathcal{K}, \end{aligned}$$

so that F_r or F_l can be chosen as parametrices of A. With similar arguments it is possible to find a parametrix for $A^\star$, namely setting $G_r = AQ^{-1}$ and $G_l = P^{-1}A$. The second part of the theorem follows from the composition Theorems 7, 8, 9 and 10.

□

4.4. Example: the action of SG-compatible change of variables on SG operators

As an example, we reexamine here the pull-back of a ψdo in $\mathbf{LG}^m$ in terms of FIOs. In all this subsection $\phi \in \mathbf{SG}\mathrm{Diffeo}(U^\#, V^\#; U, V; \delta)$ with $U^\#, V^\#, U, V \in \mathcal{O}(\mathbb{R}^n)$, $\delta > 0$ and $p \in \mathbf{SG}_l^m \mid \mathrm{supp}\,(p) \subset U \times \mathbb{R}^n$. Note also that the properties required to the phase φ in the various composition theorems examined in the preceding subsections need to be fulfilled only on the supports of the various amplitudes and symbols involved, as it will be in our calculations here.

Finally, it could appear that we cannot use here Theorems 13 and 14 to compose FIOs of Type I and Type II (which is required at some point in this subsection), since we cannot use Theorems 11 and 12 to show that $\widetilde{\nabla}_\xi \varphi(x;\xi,\eta)$ is a $\mathbf{SG}$-diffeomorphism with $\mathbf{SG}^0$ parameter dependence (x lives in an open subset of $\mathbb{R}^n$). However, in this case we can show directly this property (see (68) below), so that results analogous to those expressed by Theorems 13 and 14 hold.

PROPOSITION 13. *Let us set* $\varphi_1(x,\xi) =< \phi(x)|\xi >$, $\varphi_2(y,\eta) =< \overline{\phi}(y)|\eta >$ *for* $x \in U^\#, y \in V^\#, \xi, \eta \in \mathbb{R}^n$. *Then* φ_1 *and* φ_2 *have the same properties of the phases* $\varphi \in \mathcal{P}^\varepsilon$, *for* $x \in U^\#, y \in V^\#$.

Proof. We will prove the result only for φ_1, since calculations are identical for φ_2. $\varphi_1 \in \mathbf{SG}^e_l$ is immediate from $\phi \in \mathbf{SG}$Diffeo, owing to (17). $\left\langle \nabla_\xi \varphi(x,\xi) \right\rangle = \langle \phi(x) \rangle \sim \langle x \rangle$ again follows from the hypotheses on ϕ. In fact, $\phi(x) \prec \langle x \rangle \Rightarrow \langle \phi(x) \rangle \prec \langle x \rangle$ is the case $\alpha = 0$ of the first of (17). From the second of (17):

$$\overline{\phi}(y) \prec \langle y \rangle \Rightarrow \left\langle \overline{\phi}(y) \right\rangle \prec \langle y \rangle \Rightarrow \langle x \rangle = \left\langle \overline{\phi}(\phi(x)) \right\rangle \prec \langle \phi(x) \rangle \Rightarrow \langle \phi(x) \rangle \succ \langle x \rangle \,.$$

Since

$$d_x \varphi_1(x,\xi) = \xi \, \frac{\partial \phi}{\partial x}(x),$$

the inequality $\langle d_x \varphi_1(x,\xi) \rangle \prec \langle \xi \rangle$ is trivial from (17), since the components of $\frac{\partial f}{\partial x}$, are bounded from above. For the other estimate, since $\frac{\partial \phi}{\partial x}$ is invertible, we can write

$$\xi = d_x \varphi_1(x,\xi) \, \frac{\partial \overline{\phi}}{\partial y}(\phi(x))$$

and, recalling the same argument as above, this implies $\langle \xi \rangle \prec \langle d_x \varphi_1(x,\xi) \rangle$ as required. Now, let us note that $E(x) = \det \frac{\partial \phi}{\partial x}(x) \in \mathbf{ESG}^0_l$. In fact, we obviously have $E \in \mathbf{SG}^0_l$. Moreover, owing to the invertibility of $\frac{\partial \phi}{\partial x}(x)$ on all $U^\#$, we also find:

$$I = \frac{\partial \phi}{\partial x}(x) \, \frac{\partial \overline{\phi}}{\partial y}(\phi(x)), \ \det \frac{\partial \overline{\phi}}{\partial y} \prec 1 \Rightarrow \frac{1}{\det \frac{\partial \overline{\phi}}{\partial y}(\phi(x))} = \det \frac{\partial \phi}{\partial x} \succ 1.$$

This gives the regularity of the phase. In fact,

$$\left(\partial^x_j \partial^i_\xi \varphi_1(x,\xi) \right) = \left(\partial_j \phi^i \right) = \frac{\partial \phi}{\partial x},$$

and

$$E \in \mathbf{ESG}^0_l \Rightarrow \left| \det \left(\partial^x_j \partial^i_\xi \varphi_1(x,\xi) \right) \right| \geq \varepsilon > 0$$

with suitable ε.

□

We have thus showed that the operators $A_{\varphi_1,1}$ and $A_{\varphi_2,1}$ are well defined md-elliptic FIOs with regular phases. The following lemma is immediate.

LEMMA 17. *With the notations of Proposition 13,* $A_{\varphi_1,1} = A^{-1}_{\varphi_2,1}$.

Proof. We have, for all $u \in \mathcal{S}$ supported in U,

$$
\begin{aligned}
&(A_{\varphi_1,1}\, A_{\varphi_2,1}u)(x) = \\
&\quad = \int đ\xi\ \mathrm{e}^{i<\phi(x)|\xi>} \int dw\ \mathrm{e}^{-i<w|\xi>} \int đ\zeta\ \mathrm{e}^{i<\overline{\phi}(w)|\zeta>} \int dy\ \mathrm{e}^{-i<y|\zeta>} u(y) \\
&\quad = \mathcal{F}^{-1}_{\cdot\to\phi(x)}\left(\widehat{u\circ\overline{\phi}}\right) \\
&\quad = u(\overline{\phi}\circ\phi(x)) = u(x)
\end{aligned}
$$

and analogous result for the composition $A_{\varphi_2,1}\, A_{\varphi_1,1}$.

□

We obtain now the following more precise version of Theorem 2.

EXAMPLE 2. For all $P \in \mathbf{LG}^m$ we have for $P^\star$ as in Definition 15:

$$
P^\star = A_{\varphi_1,1} P A_{\varphi_2,1} \tag{67}
$$

and Sym $\left(P^\star\right)$ can be described by means of Theorems 8 and 14.

Proof. (67) is immediate from the definition $P^\star u = (Pu_\star)^\star$ if we note that

$$
\begin{aligned}
v^\star(x) &= (\mathcal{F}^{-1}_{\xi\to\phi(x)}\mathcal{F}_{\cdot\to\xi})v = A_{\varphi_1,1}v(x), \\
u_\star(y) &= (\mathcal{F}^{-1}_{\zeta\to\overline{\phi}(y)}\mathcal{F}_{\cdot\to\zeta})u = A_{\varphi_2,1}u(y).
\end{aligned}
$$

From Theorem 15 and Lemma 17 we also find

$$
P^\star = A_{\varphi_1,1} P A^{-1}_{\varphi_1,1} = A_{\varphi_1,1}(P\,C)A^\star_{\varphi_1,1} \bmod \mathcal{K}
$$

where C is the parametrix of the md-elliptic ψdo $Q = A^\star_{\varphi_1,1}A_{\varphi_1,1}$. From (66), with an arbitrary $\chi \in \Xi^\Delta(k)$

$$
q(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\eta D^z_\alpha \widetilde{q}(\xi,\xi,x)
$$

with

$$
\begin{aligned}
\widetilde{q}(\xi,\eta,x) &= \chi(\xi,\eta)\left|\det\frac{\partial}{\partial x}(\widetilde{\nabla}_\xi\varphi_1)^{-1}(x;\xi,\eta)\right| \\
&= \chi(\xi,\eta)\left|\det\frac{\partial}{\partial w}\widetilde{\nabla}_\xi\varphi_1(w;\xi,\eta)\right|^{-1}_{w=(\widetilde{\nabla}_\xi\varphi_1)^{-1}(x;\xi,\eta)}.
\end{aligned}
$$

Since in this case

$$
\begin{aligned}
&\widetilde{\nabla}_\xi\varphi_1(w;\xi,\eta) = \int_0^1 d\theta\ \nabla_\xi\varphi_1(w,\xi+\theta(\eta-\xi)) = \phi(w) \\
&\Leftrightarrow (\widetilde{\nabla}_\xi\varphi_1)^{-1}(x;\xi,\eta) = \overline{\phi}(x),
\end{aligned} \tag{68}
$$

so that $\widetilde{q}$ depends on ξ and η only through χ, we have, modulo $\mathcal{K}$,

$$
q(x,\xi) = \left|\det\frac{\partial\phi}{\partial w}\right|^{-1}_{w=\overline{\phi}(x)}.
$$

Q is then a multiplication operator, whose inverse (and parametrix) has symbol

$$c = c(x) = \left| \det \frac{\partial \phi}{\partial w} \right|_{w=\overline{\phi}(x)}.$$

Let us set $S = PC$. Then, from (13),

$$s(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi) D_\alpha^x c(x).$$

Let us compute the symbol of ${}^tSA_{{}^t\varphi_1,1} = A_{{}^t\varphi_1,h}$. From Theorem 7, choosing an arbitrary $\chi \in \Xi^\Delta(k)$,

$$h(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} (\partial_\xi^\alpha({}^ts))(x,(d_x({}^t\varphi_1))(x,\xi)) D_\alpha^y \left(\mathrm{e}^{i\psi(x,y,\xi)} \chi(x,y) \right)_{y=x},$$

where

$$\begin{aligned} \psi(x,y,\xi) &= ({}^t\varphi_1)(y,\xi) - ({}^t\varphi_1)(x,\xi) - < y-x | (d_x({}^t\varphi_1)(x,\xi)) > \\ &= < \phi(\xi)|y > - < \phi(\xi)|x > - < y-x|(d_x < \phi(\xi)|x >) > \\ &= < y-x|\phi(\xi) > - < y-x|\phi(\xi) >= 0. \end{aligned}$$

Then

$$D_\alpha^y(\mathrm{e}^{i\psi}\chi) = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases}$$

so that

$$h(x,\xi) \sim ({}^ts)(x,\phi(\xi)) = s(\phi(\xi),x)$$

and, from Theorem 8,

$$A_{\varphi_1,1}S = {}^t({}^tSA_{{}^t\varphi_1,1}) = A_{\varphi_1,{}^th}$$

with

$${}^th(x,\xi) = h(\xi,x) \sim s(\phi(x),\xi).$$

At last, the amplitude $\widetilde{p}$ of $P^\star$ can be then expressed using Theorem 14, as the amplitude of $A_{\varphi_1,{}^th}A^\star_{\varphi_1,1}$. Let us set

$$\begin{aligned} &M(x,y) = \int_0^1 d\theta \, \frac{\partial \phi}{\partial x}(y + \theta(x-y)) \\ &\Rightarrow (\widetilde{d}_x\varphi_1)^{-1}(x,y;\xi) = \xi M^{-1}(x,y), \end{aligned}$$

Since M is obviously invertible for $|x-y| \le k\langle x \rangle$. We have

$$\begin{aligned} \widetilde{p}(x,y,\xi) &= \chi(x,y)^t h(x,(\widetilde{d}_x\varphi_1)^{-1}(x,y;\xi)) \left| \det \frac{\partial}{\partial \xi} (\widetilde{d}_x\varphi_1)^{-1}(x,y;\xi) \right| \\ &\sim \chi(x,y) \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\theta p)(\phi(x), \xi M^{-1}(x,y)) \, |\det M(x,y)|^{-1} D_\alpha^x c(x) \end{aligned}$$

From this we may deduce the asymptotic expansion of the symbol. It is easy to verify that the first term is the same of the analogous one described in [38], cf. the previous Theorem 2.

□

4.5. Action on the Sobolev spaces

Theorem 16 here below could be proved as in [4], that is, as an adapted version of the general L^2-boundedness result of Asada-Fujiwara [1]: the proof which follows these lines is in section A.2 of the appendix. However, we can make here full use of Theorem 13 and of the usual L^2-boundedness result for ψdos in $\mathbf{LG}^0$ (special case of Proposition 6) to skip long calculations completely.

THEOREM 16. *Let $A = A_{\varphi,a}$ be a Type I FIO with $\varphi \in \mathcal{P}^{\varepsilon}$ and $a \in \mathbf{SG}^0_l$. Then $A \in \mathcal{L}(L^2)$.*

Proof. We have easily, for $u \in L^2$,

$$\|Au\|^2 = (Au, Au) = (A^\star Au, u) \le \|A^\star Au\| \, \|u\| \le \|A^\star A\|_{\mathcal{L}(L^2)} \, \|u\|^2,$$

and the result follows immediately since, by hypothesis and Theorem 13, $A^\star A \in \mathbf{LG}^0 \subset \mathcal{L}(L^2)$. □

To prove the general continuity Theorem 17 below we have to examine the inverse of the operator Π used in (10). We also give, for sake of completeness, an alternative equivalent definition of Sobolev spaces (10). The proof of Lemma 19 is contained in [11].

LEMMA 18. *For $t \in \mathbb{R}^2$ let us set $\widetilde{\pi}_t(y,\xi) = \langle\xi\rangle^{t_1} \langle y\rangle^{t_2}$, which we consider as x-independent amplitude, and apply (6). Then, for all $s \in \mathbb{R}^2$, $\widetilde{\Pi}_{-s} = \mathrm{Op}\,(\widetilde{\pi}_{-s}) = \Pi_s^{-1}$ where Π_s is defined in (9).*

Proof. The proof is immediate. In fact, for an arbitrary $u \in \mathcal{S}$ we have

$$\begin{aligned}
\widetilde{\Pi}_{-s}\Pi_s u(x) &= \int d\!\!\bar{}\xi dy \; \mathrm{e}^{i<x-y|\xi>} \; \langle\xi\rangle^{-s_1} \langle y\rangle^{-s_2} \int d\!\!\bar{}\eta \; \mathrm{e}^{i<y|\eta>} \; \langle\eta\rangle^{s_1} \langle y\rangle^{s_2} \; \hat{u}(\eta) \\
&= \int d\!\!\bar{}\xi \; \mathrm{e}^{i<x|\xi>} \; \langle\xi\rangle^{-s_1} \int dy \; \mathrm{e}^{-i<y|\xi>} \int d\!\!\bar{}\eta \; \mathrm{e}^{i<y|\eta>} \; \langle\eta\rangle^{s_1} \; \hat{u}(\eta) \\
&= \int d\!\!\bar{}\xi \; \mathrm{e}^{i<x|\xi>} \; \langle\xi\rangle^{-s_1} \langle\xi\rangle^{s_1} \; \hat{u}(\xi) \\
&= u(x)
\end{aligned}$$

and an analogous result for $\Pi_s\widetilde{\Pi}_{-s}$.

□

LEMMA 19. *For all $s \in \mathbb{R}^2$ the space*

$$\widetilde{H}^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \widetilde{\Pi}_s u \in L^2(\mathbb{R}^n) \right\}$$

has the same elements of the Sobolev space of equation (10) and equivalent norm $\|u\|_{\widetilde{H}^s} = \|\widetilde{\Pi}_s u\|_{L^2}$.

Proof. Using Lemma 18 we have

$$u \in H^s \quad \Leftrightarrow \quad \Pi_s u \in L^2 \Rightarrow \widetilde{\Pi}_s u = (\widetilde{\Pi}_s \widetilde{\Pi}_{-s})(\Pi_s u) \in L^2, \tag{69}$$

$$u \in \widetilde{H}^s \quad \Leftrightarrow \quad \widetilde{\Pi}_s u \in L^2 \Rightarrow \Pi_s u = (\Pi_s \Pi_{-s})(\widetilde{\Pi}_s u) \in L^2, \tag{70}$$

since $\widetilde{\Pi}_s \widetilde{\Pi}_{-s}$ as well as $\Pi_s \Pi_{-s}$ are in $\mathcal{L}(L^2)$, because they are order 0 ψdos . The equivalence of the norms is a consequence of (69) and (70) since

$$\begin{aligned}
\|u\|_{\widetilde{H}^s} &= \|\widetilde{\Pi}_s u\|_{L^2} = \|(\widetilde{\Pi}_s \widetilde{\Pi}_{-s})(\Pi_s u)\|_{L^2} \\
&\leq \|\widetilde{\Pi}_s \widetilde{\Pi}_{-s}\|_{\mathcal{L}(L^2)} \|\Pi_s u\|_{L^2} \leq M\|u\|_{H^s} \\
\|u\|_{H^s} &= \|\Pi_s u\|_{L^2} = \|(\Pi_s \Pi_{-s})(\widetilde{\Pi}_s u)\|_{L^2} \\
&\leq \|\Pi_s \Pi_{-s}\|_{\mathcal{L}(L^2)} \|\widetilde{\Pi}_s u\|_{L^2} \leq M\|u\|_{\widetilde{H}^s}
\end{aligned}$$

for a suitable $M > 0$.

□

THEOREM 17. *For all $s \in \mathbb{R}^2$, $a \in \mathbf{SG}^m_l$, $\varphi \in \mathcal{P}^\varepsilon$: $A_{\varphi,a} \in \mathcal{L}(H^s, H^{s-m})$.*

Proof. For any $u \in H^s$ we find

$$\begin{aligned}
\|A_{\varphi,a}u\|_{H^{s-m}} &= \|\Pi_{s-m} A_{\varphi,a} u\|_{L^2} = \|(\Pi_{s-m} A_{\varphi,a} \widetilde{\Pi}_{-s})(\Pi_s u)\|_{L^2} \\
&\leq \|\Pi_{s-m} A_{\varphi,a} \widetilde{\Pi}_{-s}\|_{\mathcal{L}(L^2)} \|\Pi_s u\|_{L^2} \\
&\leq M\|u\|_{H^s},
\end{aligned}$$

where we have used Lemma 19 and Theorems 7 and 8 to get $\Pi_{s-m} A_{\varphi,a} \widetilde{\Pi}_{-s} = A_{\varphi,h}$ with $h \in \mathbf{SG}^0_l$, Theorem 16 to achieve the L^2-continuity of $A_{\varphi,h}$ and $u \in H^s \Leftrightarrow \Pi_s u \in L^2$.

□

REMARK 8. Owing to Remark 5, and since the Fourier transform is an L^2-isometry, Theorems 16 and 17 are also true for a Type II operator $B_{\varphi,b}$ with $\varphi \in \mathcal{P}^\varepsilon$ and $b \in \mathbf{SG}^0_l$ or $b \in \mathbf{SG}^m_l$ respectively.

4.6. Wave front sets

We begin recalling the definition of the so-called "wave front space" (see [11]).

DEFINITION 26. *(Directional compactification of $\mathbb{R}^n$)*
Let $\mathbb{B}^n$ denote the "directional compactification" of $\mathbb{R}^n$, i.e. $\mathbb{B}^n$ consists of $\mathbb{R}^n$ and the "infinite set" $\{\infty x_0 \,:\, x_0 \in \mathbb{R}^n \; |x_0| = 1\}$.

Let $B_1(0)$ denote the unitary open ball centered in the origin. The function

$$s \,:\, \mathbb{R}^n \to B_1(0) \,:\, x \mapsto \frac{x}{\langle x \rangle} \tag{71}$$

is a homeomorphism of $\mathbb{R}^n$ onto $B_1(0)$ which is extendable to a homeomorphism of $\mathbb{B}^n$ onto $\overline{B_1(0)} = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, mapping the boundary $\partial\mathbb{B}^n$ of $\mathbb{B}^n$ onto S^{n-1}. So we can identify the points ∞x of $\partial\mathbb{B}^n$ by means of the corresponding x of S^{n-1}.

DEFINITION 27. *(Wave front space)*
Let us consider the cotangent bundle $T^\star\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and its compatification $\mathbb{B}^n \times \mathbb{B}^n$. We call "wave front space" the subset $\mathcal{W} = \mathbb{R}^n \times \partial\mathbb{B}^n$ of $\partial(\mathbb{B}^n \times \mathbb{B}^n)$.

We will define the wave front set of a temperate distribution u as a subset of $\mathcal{W}$. We now give the definition of **SG**-microregularity.

DEFINITION 28. *(**SG**-microregularity)*
*Let $u \in \mathcal{S}'$ be given. We say that u is **SG**-microregular at a point $(x_0, \infty\xi_0) \in \mathcal{W}$ if there exist*

- *a neighbourhood of $(x_0, \infty\xi_0)$ of the form $U = \Omega \times \Gamma$ where $\Omega \subseteq \mathbb{R}^n$ is an open neighbourhood of x_0 and Γ is of the form $\widetilde{\Gamma} \cap \{\xi \ : \ |\xi| > R \geq 0\}$ with $\widetilde{\Gamma}$ open conical neighbourhood of ξ_0 in $\mathbb{R}^n$;*
- *a ψdo $P = \mathrm{Op}\,(p)$ such that $p \in \mathbf{SG}^0_l$, $p \succ 1$ in U and $Pu \in \mathcal{S}(\mathbb{R}^n)$.*

DEFINITION 29. *(Wave front set of a temperate distribution)*
We define the wave front set $\mathrm{WF}(u)$ of a distribution $u \in \mathcal{S}'$ as

$$\mathrm{WF}(u) = \mathcal{W} \setminus \{(x_0, \infty\xi_0) \in \mathcal{W} \mid u \textit{ is } \mathbf{SG}\textit{-microregular in } (x_0, \infty\xi_0)\}$$

To obtain an adapted version of the Egorov theorem, we first need the expression of $\mathrm{Sym_p}\left(APA^{-1}\right)$, where $A = A_{\varphi,a}$ is an elliptic FIO of Type I while $P = \mathrm{Op}\,(p)$ is a ψdo with $p \in \mathbf{SG}^t_l$. We generalize here the calculations of subsection 4.4.

PROPOSITION 14. *Let $A = A_{\varphi,a}$ be an elliptic FIO of Type I with $a \in \mathbf{ESG}^m_l$ and $P = \mathrm{Op}\,(p)$ a ψdo with $p \in \mathbf{SG}^t_l$. Then, setting $\eta = (d_x\varphi)^{-1}(x, \xi)$ we have*

$$\mathrm{Sym_p}\left(APA^{-1}\right)(x, \xi) = p((\nabla_\xi\varphi)(x, \eta), \eta).$$

Proof. Let us set $C = AA^\star$, so that, in view of Theorem 15, $A^{-1} = A^\star C^{-1}$. By Theorem 13 we immediately have

$$\mathrm{Sym_p}\,(C)\,(x, \xi) = |a(x, (d_x\varphi)^{-1}(x, \xi))|^2 \left|\det \frac{\partial}{\partial\xi}(d_x\varphi)^{-1}(x, \xi)\right|.$$

By Theorem 10 we then have

$$\begin{aligned} {}^t\mathrm{Sym_p}\left(PA^\star\right)(x, \xi) &= ({}^t\mathrm{Sym_p}\left(P^\star\right))(x, (d_x({}^t\varphi))(x, \xi))\ ({}^ta)(x, \xi) \\ &= \overline{p((\nabla_\xi\varphi)(\xi, x), x)}\ a(\xi, x) \Rightarrow \\ \Rightarrow \mathrm{Sym_p}\left(PA^\star\right)(x, \xi) &= \overline{p((\nabla_\xi\varphi)(x, \xi), \xi)}\ a(x, \xi). \end{aligned}$$

Finally, setting $B = PA^\star$ and $\eta = (d_x\varphi)^{-1}(x, \xi)$, owing to Theorem 13 we find

$$\begin{aligned} \mathrm{Sym_p}\left(APA^\star\right)(x, \xi) &= \mathrm{Sym_p}\,(AB)\,(x, \xi) \\ &= |a(x, \eta)|^2 \left|\det \tfrac{\partial}{\partial\xi}(d_x\varphi)^{-1}(x, \xi)\right| p((\nabla_\xi\varphi)(x, \eta), \eta). \end{aligned} \tag{72}$$

Since obviously $\mathrm{Sym_p}\left(C^{-1}\right) = (\mathrm{Sym_p}\,(C))^{-1}$, (72) implies

$$\mathrm{Sym_p}\left(APA^{-1}\right)(x, \xi) = \mathrm{Sym_p}\left(APA^\star\right)(x, \xi)\ \ \mathrm{Sym_p}\left(C^{-1}\right)(x, \xi) = p((\nabla_\xi\varphi)(x, \eta), \eta)$$

as desired.

∎

Let us denote by Φ the canonical transform of $T^\star\mathbb{R}^n$ into itself generated by φ, i.e. $\Phi \; : (x,\xi) \mapsto (y,\eta)$ is defined by

$$\left\{ \begin{array}{lcl} y & = & (\nabla_\xi \varphi)(x,\eta) \\ \xi & = & (d_x\varphi)(x,\eta). \end{array} \right. \tag{73}$$

Let us assume in (73) homogeneity of degree 1 with respect to η for $\varphi(x,\eta)$ (for large η). We denote then by the same symbol Φ the transform that (73) induces on $\mathcal{W}$. We can now state an adapted version of the theorem concerning the action of FIO_S on wave fronts.

THEOREM 18. *For any elliptic FIO A of Type I and any distribution $u \in \mathcal{S}'$ we have*

$$\mathrm{WF}(Au) = \Phi^{-1}(\mathrm{WF}(u)) \tag{74}$$

Proof. Let u be **SG**-microregular in $(y_0, \infty\eta_0)$ and let $P \in \mathbf{LG}^0$ be the operator of Definition 28. Let us set $Q = APA^{-1} \Leftrightarrow QA = AP$. Then, by Definition 28 and Theorem 4, we have $QAu \in \mathcal{S} \Leftrightarrow APu \in \mathcal{S}$. By Proposition 14 and (73), we have $\mathrm{Sym_p}\,(Q) = p \circ \Phi$, so that Au is **SG**-microregular in $\Phi^{-1}(y_0, \infty\eta_0)$. This means that

$$\mathcal{W} \setminus \mathrm{WF}(Au) = \Phi^{-1}(\mathcal{W} \setminus \mathrm{WF}(u)). \tag{75}$$

Complementing (75) with respect to $\mathcal{W}$ and recalling that Φ is a bijection, we obtain (74).

$\square$

Appendix

A.1. Derivatives of the exponential function $\mathrm{e}^{i\psi}$

PROPOSITION 15. *Let us set $\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \, < y - x | d_x\varphi(x,\xi) >$ as in (26). Then we have, for $|\alpha| \geq 1$:*

$$\begin{array}{rcl} D^y_\alpha \mathrm{e}^{i\psi} & = & \sigma_\alpha \mathrm{e}^{i\psi} \\ & = & \mathrm{e}^{i\psi} \Big[\big(d_y\varphi - d_x\varphi\big)^\alpha \; + \\ & + & \displaystyle\sum_{j_1} c_{j_1} \big(d_y\varphi - d_x\varphi\big)^{\theta_{j_1}} \prod_{j_2=1}^{n_{1j_1}} \partial^y_{\beta_{j_1 j_2}} \varphi \; + \\ & + & \displaystyle\sum_{j_1} c'_{j_1} \prod_{j_2=1}^{n_{2j_1}} \partial^y_{\gamma_{j_1 j_2}} \varphi \Big] \end{array} \tag{76}$$

with suitable c_{j_1}, c'_{j_1}, $\beta_{j_1 j_2}$ and $\gamma_{j_1 j_2}$ such that:

$$|\beta_{j_1 j_2}|, |\gamma_{j_1 j_2}| \geq 2 \tag{77}$$

$$\theta_{j_1} + \sum_{j_2=1}^{n_{1j_1}} \beta_{j_1 j_2} = \sum_{j_2=1}^{n_{2j_1}} \gamma_{j_1 j_2} = \alpha \tag{78}$$

where $d_x\varphi = d_x\varphi(x,\xi)$, $d_y\varphi = d_y\varphi(y,\xi)$, $\partial^x_\alpha\varphi = \partial^x_\alpha\varphi(x,\xi)$ and $\partial^y_\alpha\varphi = \partial^y_\alpha\varphi(y,\xi)$ is to be understood.

Proof. For $|\alpha| = 1$ (76) holds with only the first term:

$$-i\partial_j^y e^{i\psi} = e^{i\psi}(\partial_j^y\varphi - \partial_j^x\varphi).$$

For $|\alpha| = 2$ (76) holds without the first sum: in fact

$$-\partial_{jk}^y e^{i\psi} = e^{i\psi}\left[(\partial_j^y\varphi - \partial_j^x\varphi)(\partial_k^y\varphi - \partial_k^x\varphi) - i\partial_{jk}^y\varphi\right].$$

For $|\alpha| = 3$ (76) holds with all the terms:

$$\begin{aligned} i\partial_{jkl}^y e^{i\psi} &= \\ &= e^{i\psi}\Big[(\partial_j^y\varphi - \partial_j^x\varphi)(\partial_k^y\varphi - \partial_k^x\varphi)(\partial_l^y\varphi - \partial_l^x\varphi) - i\partial_{jk}^y\varphi(\partial_l^y\varphi - \partial_l^x\varphi) \\ &\quad -i\partial_{jl}^y\varphi(\partial_k^y\varphi - \partial_k^x\varphi) - i\partial_{kl}^y\varphi(\partial_j^y\varphi - \partial_j^x\varphi) - \partial_{jkl}^y\varphi\Big]. \end{aligned}$$

In all the three cases above (77) and (78) trivially hold. Let us proceed by induction assuming (76), (77) and (78) true for all α such that $1 \le |\alpha| \le p$ with $p \ge 3$. Differentiating (76) we obtain

$$D_{\alpha+e_m}^y e^{i\psi} = -i\partial_m^y\left(\sigma_\alpha e^{i\psi}\right) = \left[\left(\partial_m^y\varphi - \partial_m^x\varphi\right)\sigma_\alpha - i\partial_m^y\sigma_\alpha\right]e^{i\psi}$$

so that

$$
\begin{aligned}
&\sigma_{\alpha+e_m} = \\
&\quad = \left(d_y\varphi - d_x\varphi\right)^{\alpha+e_m} \\
&\quad + \sum_{j_1} c_{j_1}\left(d_y\varphi - d_x\varphi\right)^{\theta_{j_1}+e_m} \prod_{j_2=1}^{n_{1j_1}} \partial^y_{\beta_{j_1 j_2}}\varphi + \sum_{j_1} c'_{j_1}\left(\partial^y_m\varphi - \partial^x_m\varphi\right)\prod_{j_2=1}^{n_{2j_1}} \partial^y_{\gamma_{j_1 j_2}}\varphi \\
&\quad + \sum_{k=1}^{n}(-i)\alpha_k\left\{\left[\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\partial^y_j\varphi - \partial^x_j\varphi\right)^{\alpha_j}\right]\left(\partial^y_k\varphi - \partial^x_k\varphi\right)^{\alpha_k-1}\right\}\partial^y_{km}\varphi \\
&\quad + \sum_{j_1}\sum_{k=1}^{n}(-i)c_{j_1}\theta_{j_1 k}\left\{\left[\prod_{\substack{l=1\\ l\neq k}}^{n}\left(\partial^y_j\varphi - \partial^x_j\varphi\right)^{\theta_{j_1 l}}\right]\left(\partial^y_k\varphi - \partial^x_k\varphi\right)^{\theta_{j_1 k}-1}\right\}\cdot \\
&\quad \cdot\left[\partial^y_{km}\varphi\prod_{j_2=1}^{n_{1j_1}}\partial^y_{\beta_{j_1 j_2}}\varphi\right] \\
&\quad + \sum_{j_1}\sum_{k=1}^{n_{1j_1}}(-i)c_{j_1}\left(d_y\varphi - d_x\varphi\right)^{\theta_{j_1}}\left[\partial^y_{\beta_{j_1 k}+e_m}\varphi\prod_{\substack{j_2=1\\ j_2\neq k}}^{n_{1j_1}}\partial^y_{\beta_{j_1 j_2}}\varphi\right] \\
&\quad + \sum_{j_1}\sum_{k=1}^{n_{2j_1}}(-i)c'_{j_1}\partial^y_{\gamma_{j_1 k}+e_m}\varphi\prod_{\substack{j_2=1\\ j_2\neq k}}^{n_{2j_1}}\partial^y_{\gamma_{j_1 j_2}}\varphi.
\end{aligned}
$$

The terms obtained are all of the correct type, as it is clear by the calculation above. Also the fact that (77) still holds is trivial, so that we only have to check (78) on the above formula. By the inductive hypothesis we have

- for the first sum:

$$\theta'_{j_1} + \sum_{j_2=1}^{n'_{1j_1}} \beta'_{j_1 j_2} = \theta_{j_1} + e_m + \sum_{j_2=1}^{n_{1j_1}} \beta_{j_1 j_2} = \alpha + e_m;$$

- for the second sum:

$$\theta'_{j_1} + \sum_{j_2=1}^{n'_{1j_1}} \beta'_{j_1 j_2} = e_m + \sum_{j_2=1}^{n_{2j_1}} \gamma_{j_1 j_2} = \alpha + e_m;$$

- for the third sum:

$$\theta'_{j_1} + \sum_{j_2=1}^{n'_{1j_1}} \beta'_{j_1 j_2} = \alpha - e_k + e_k + e_m = \alpha + e_m;$$

- for the fourth sum:

$$\theta'_{j_1} + \sum_{j_2=1}^{n'_{1j_1}} \beta'_{j_1 j_2} = \theta_{j_1} - e_k + \sum_{j_2=1}^{n_{1j_1}} \beta_{j_1 j_2} + e_k + e_m = \alpha + e_m;$$

- for the fifth sum:

$$\theta'_{j_1} + \sum_{j_2=1}^{n'_{1j_1}} \beta'_{j_1 j_2} = \theta_{j_1} + \sum_{j_2=1}^{n_{1j_1}} \beta_{j_1 j_2} + e_m = \alpha + e_m.$$

- for the sixth sum:

$$\sum_{j_2=1}^{n'_{2j_1}} \gamma'_{j_1 j_2} = \sum_{j_2=1}^{n_{2j_1}} \gamma_{j_1 j_2} + e_m = \alpha + e_m.$$

Formula (76) with (77) and (78) is then proved for all α.

□

A.2. Direct proof of continuity in Sobolev spaces

We give here an alternative proof of Theorem 16 as an adapted version of a general L^2-boundedness result of Asada-Fujiwara ([1]), very close to the analogous one in [4] (Theorem 12.1). We will need the following classical Schur's lemma.

LEMMA 20. *If $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and*

$$\sup_y \int dx\ |H(x,y)| \leq T, \qquad \sup_x \int dy\ |H(x,y)| \leq T,$$

then the integral operator with kernel H has norm $\leq T$ in $\mathcal{L}(L^2)$.

Proof of Theorem 16. Let us choose a non-increasing $\psi \in C^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t < \frac{1}{2}$ and $\psi(t) = 0$ for $t > 1$. Then set, for $w = (s, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\psi_w(x,\xi) = \frac{\psi(|x-s|)\psi(|\xi-\sigma|)}{\int ds d\sigma\, \psi(|x-s|)\psi(|\xi-\sigma|)},$$

so that

(79) $$\operatorname{supp}(\psi_w) \subseteq U_w = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x-s| \leq 1, |\xi-\sigma| \leq 1\right\},$$

(80) $$\max_{|\alpha+\beta|\leq m} \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} |\partial_\xi^\alpha \partial_\beta^x \psi_w(x,\xi)| \leq C_m,$$

$$\forall x, \xi \int ds d\sigma\ \psi_w(x,\xi) = 1,$$

where the constants C_m do not depend on w. For fixed w, let us set

$$(81)\qquad \begin{aligned} a_w(x,\xi) &= \psi_w(x,\xi)\, a(x,\xi),\\ A_w &= A_{\varphi,a_w}. \end{aligned}$$

(79), (80) and (81) imply $A_w \in \mathrm{Hom}(C_0^\infty, C_0^\infty)$ and $\|A_w u\|_{L^2} \le C\|u\|_{L^2}$ with constant C independent of w. In fact, a_w has compact support and (80) holds. Moreover,

$$\psi_w \in C_0^\infty \Rightarrow \psi_w \in \mathcal{S} \Rightarrow a_w \in \mathbf{SG}_l^0$$

and

$$A_{\varphi,a}u(x) = \lim_{N\to\infty}\int_{|w|\le N} dw\; A_w u(x),$$

where the limit exists pointwise for all $x \in \mathbb{R}^n$ and with respect to the strong topology of L^2. We will prove the theorem if we can show that for all compact sets $K \subset \mathbb{R}^n \times \mathbb{R}^n$

$$(82)\qquad \left\|\int_K dw\; A_w u(.)\right\|_{L^2} \le M\|u\|_{L^2}, \qquad u \in C_0^\infty$$

with constant M independent of u and K. To this aim, we will use Cotlar's lemma (see, e.g., [22]), which, adapted to our operators A_w, can be stated in the following form.

LEMMA 21. *Let $h(w,w')$ and $k(w,w')$ be two positive functions on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that*

$$(83)\qquad \|A_w A^\star_{w'}\| \le h(w,w')^2, \qquad \|A^\star_w A_{w'}\| \le k(w,w')^2.$$

If h and k statisfy

$$(84)\qquad \int dw\; h(w,w') \le M, \qquad \int dw\; k(w,w') \le M,$$

then (82) holds for the same value M.

Here we shall not use Theorem 13, but observe that the kernel $H_{w,w'}(x,y)$ of $A_w A^\star_{w'}$ can be easily written in the form

$$(85)\qquad H_{w,w'}(x,y) = \int đ\xi\; \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, q_{w,w'}(x,y,\xi)$$

with

$$q_{w,w'}(x,y,\xi) = a_w(x,\xi)\overline{a_{w'}(y,\xi)}.$$

We now want to show that $H_{w,w'}$ in (85) satisfies the hypotheses of Lemma 20 for a suitable T. Let us introduce the operator

$$\mathcal{L} = d^{-1}(1-L)$$

where

$$\begin{aligned} L &= i\sum_{j=1}^n \partial_\xi^j\,(\varphi(x,\xi)-\varphi(y,\xi))\,\partial_\xi^j,\\ d &= 1+\left|\nabla_\xi\,(\varphi(x,\xi)-\varphi(y,\xi))\right|^2, \end{aligned}$$

so that

$$\mathcal{L}\mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))} = \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}.$$

Take note that

$$\left|\nabla_\xi\,(\varphi(x,\xi)-\varphi(y,\xi))\right| \succ |x-y| \quad \Rightarrow \quad d \succ \langle x-y\rangle^2$$

(see the first part of the proof of Theorem 13, and also, setting $\mathcal{D}f = f/d$,

$$\begin{aligned} \mathcal{D} &: \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n), \\ L &: \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n), \end{aligned}$$

$$\begin{aligned} &\operatorname{supp}\left(q_{w,w'}\right) \subseteq \{(x,y,\xi) :\ |\,|x-s| \le 1, |y-s'| \le 1, |\xi-\sigma| \le 1, |\xi-\sigma'| \le 1\} \\ &\qquad \Rightarrow q_{w,w'} \in \mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n). \end{aligned}$$

Since for $({}^t\mathcal{L})^m$ a formula analogous to (22) holds, by the hypotheses and the above observations we have, for arbitrary $m\in\mathbb{N}$ and a suitable polynomial Q_m in the variables $\mathcal{D}$, L,

$$\begin{aligned} &H_{w,w'}(x,y) = \\ &= \int d\!\!\bar{}\xi\ \mathcal{L}^m\ \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, q_{w,w'}(x,y,\xi) \\ &= \int d\!\!\bar{}\xi\ \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, ({}^t\mathcal{L})^m\, q_{w,w'}(x,y,\xi) \\ &= \int d\!\!\bar{}\xi\ \mathrm{e}^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, (\mathcal{D}^m + Q_m(\mathcal{D},L))\, q_{w,w'}(x,y,\xi) \end{aligned}$$

$$\Rightarrow H_{w,w'}(x,y) \prec \tau\left(\frac{\sigma-\sigma'}{2}\right)\, \tau(x-s)\, \tau(y-s')\, (1+|x-y|^2)^{-m} \tag{86}$$

where $\tau = \chi_{B(0,1)}$ is the characteristic function of the unit ball in $\mathbb{R}^n$. Then:

$$\begin{aligned} &\sup_y \int dx\ |H_{w,w'}(x,y)| \prec \\ &\quad \prec \tau\left(\frac{\sigma-\sigma'}{2}\right) \sup_{y\in B(s',1)} \int_{u\in B(0,1)} du\ (1+|u+(s-y)|^2)^{-m} \\ &\quad \prec \tau\left(\frac{\sigma-\sigma'}{2}\right) \sup_{y\in B(s',1)} (1+|s-y|^2)^{-m} \\ &\quad \prec \tau\left(\frac{\sigma-\sigma'}{2}\right) (1+|s-s'|^2)^{-m} \end{aligned}$$

and analogously for $\sup_x \int dy\ |H_{w,w'}(x,y)|$, owing to the symmetry in the estimate (86). So, all requirements of Lemma 20 are satisfied and summing up, we have:

$$\begin{aligned} |\sigma-\sigma'| \ge 2 \quad &\Rightarrow \quad A_w A^\star_{w'} = 0 \\ |\sigma-\sigma'| \le 2 \quad &\Rightarrow \quad \|A_w A^\star_{w'}\| \prec (1+|s-s'|^2)^{-m}. \end{aligned}$$

An analogous estimate can be obtained for $A^\star_w A_{w'}$. In fact, using Remark 5,

$$\begin{aligned} A^\star_w A_{w'} &= B_{\varphi,a_w} A_{\varphi,a_{w'}} \\ &= (2\pi)^n \mathcal{F}^{-1} \circ A_{-^t\varphi,a^\star_w} \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ (2\pi)^{-n} B_{-^t\varphi,a^\star_{w'}} \circ \mathcal{F} \\ &= \mathcal{F}^{-1} \circ A_{-^t\varphi,a^\star_w} A^\star_{-^t\varphi,a^\star_{w'}} \circ \mathcal{F} \\ &= \mathcal{F}^{-1} \circ \widetilde{A}_w \widetilde{A}^\star_{w'} \circ \mathcal{F} \end{aligned}$$

which implies

$$\|A^\star_w A_{w'}\|_{\mathcal{L}(L^2)} \le \|\widetilde{A}_w \widetilde{A}^\star_{w'}\|_{\mathcal{L}(L^2)}$$

since the Fourier transform is an L^2-isometry. Of course, also the kernel of $\widetilde{A}_w \widetilde{A}^\star_{w'}$ satisfies estimate (86), due to the usual symmetry in the role of variables and covariables in phases and amplitudes. Then, also the requirements (83) and (84) of Lemma 21 are satisfied, and the theorem is proved.

□

References

[1] ASADA K. AND FUJIWARA D., *On Some Oscillatory Transformation in $L^2(\mathbb{R}^n)$*, Japan J. Math. **4** (1978), 229–361.

[2] BEALS R., *A General Calculus of Pseudodifferential Operators and Applications*, Duke Math. J. **44** (1977), 45–57.

[3] BERGER M. S., *Nonlinearity and Functional Analysis*, Academic Press 1977.

[4] P. Boggiatto, E. Buzano, and L. Rodino. Global Hypoellipticity and Spectral Theory. Akademie Verlag, Berlin, 1996.

[5] BONY J. M., *Fourier Integral Operators and Weyl-Hörmander Calculus*, Technical Report **1147**, Centre de Mathématiques de l'École Poytechnique, U.R.A. 169 du C.N.R.S., 91128 Palaiseau Cedex 1996.

[6] BOVE A., FRANCHI B. AND OBRECHT E., *A Boundary Value Problem for Elliptic Operators with Polynomial Coefficients in a Half Space (I): Pseudodifferential Operators and Function Spaces*, Boll. Univ. Mat. Ital. **18** B (1981), 25–45.

[7] BREZIS H., *Analisi Funzionale*, Liguori Editore 1986.

[8] CHAZARAIN J., *Opérateurs Hyperboliques a Caractéristiques de Multiplicité Constante*, Ann. Inst. Fourier **24** 1 (1974), 173–202.

[9] CHAZARAIN J., *Propagation des Singularités pour une Classe d'Opérateurs Hyperboliques a Caractéristiques Multiples et Résolubilité Locale*, Ann. Inst. Fourier **24** 1 (1974), 203–223.

[10] CORDES H. O., *A Global Parametrix for Pseudodifferential Operators over $\mathbb{R}^n$, with applications*, preprint **90**, SFB 72, Bonn 1976.

[11] CORDES H. O., *The Technique of Pseudodifferential Operators*, Cambridge Univ. Press 1995.

[12] CORDES H. O. AND ERKIP A. K., *The N-th Order Elliptic Boundary Value Problem for Noncompact Boundaries*, Rocky Mountain Math. J. **10** (1980) 7–24.

[13] CORIASCO S., *Fourier Integral Operators in* **SG** *classes II. Application to* **SG** *Hyperbolic Cauchy Problems*, Ann. Univ. Ferrara **44** (1998), 81–122.

[14] CORIASCO S., *Fourier Integral Operators in* **SG** *classes with Applications to Hyperbolic Cauchy Problems*, PhD thesis, Univ. Torino 1998.

[15] DUISTERMAAT J. AND HÖRMANDER L., *Fourier Integral Operators II*, Acta Math. **128** (1972), 183–269.

[16] DUISTERMAAT J., *Fourier Integral Operators*, Birkhäuser, Boston 1996.

[17] ERKIP A. K., *The Elliptic Boundary Problem on the Half Space*, Comm. in PDE **4** (1979), 537–554.

[18] ERKIP A. K., *Normal Solvability of Boundary Value Problems in Half Space*, in Proceedings, Oberwolfach, **1256** (eds. Cordes H. O., Gramsch B., and Widom H.), in Springer LNM, New York 1986, 123–134.

[19] FLASCHKA H. AND STRANG G., *The Correctness of the Cauchy Problem*, Adv. in Math. **6** (1971), 347–379.

[20] GRUBB G., *Functional Calculus of Pseudifferential Boundary Value Problems*, Birkhäuser, Boston 1995 (2^{nd} edition).

[21] HÖRMANDER L., *Fourier Integral Operators I*,Acta Math. 127 (1971), 79–183.

[22] HÖRMANDER L., *The Analysis of Linear Partial Differential Operators*, vol. 1–4, Springer-Verlag, Berlin 1983–85.

[23] HWANG I. L., *The L^2-Boundedness of Pseudodifferential Operators*, Trans. AMS **302** (1987), 55–76.

[24] KITADA H., *Fourier Integral Operators with Weighted Symbols and Micro-Local Resolvent Estimates*, J. Math. Soc. Japan **39** 3 (1987) 455–476.

[25] KUMANO-GO H., *Factorization and Fundamental Solutions for Differential Operators of Elliptic-Hyperbolic Type*, Proc. Japan Acad. **52** (1976), 480–483.

[26] KUMANO-GO H., *Pseudo-Differential Operators*, MIT Press 1981.

[27] LIESS O. AND RODINO L., *Fourier Integral Operators and Inhomogeneous Gevrey Classes*, Ann. Mat. Pura Appl. **150** (1988), 167–252.

[28] MASCARELLO M. AND RODINO L., *Linear Partial Differential Operators with Multiple Characteristics*, Wiley, Akademie Verlag, Berlin 1997.

[29] MIZOHATA S., *The Theory of Partial Differential Equations*, Cambridge Univ. Press 1973.

[30] MIZOHATA S., *Hyperbolic Equations and Related Topics*, Proceedings of the Taniguchi International Symposium, Academic Press, Inc. 1984.

[31] MIZOHATA S., *On the Cauchy Problem*, Academic Press, Inc. 1985.

[32] MIZOHATA S. AND OHYA Y., *Sur la Condition de E. E. Levi Concernant des Équations Hyperboliques*, Publ. RIMS Kyoto Univ. **4** A (1968), 511–526.

[33] MIZOHATA S. AND OHYA Y., *Sur la Condition d'Hyperbolicité pour les Équations à Caractéristiques Multiples. II*, Jap. J. Math. **40** (1971) 63–104.

[34] MORIMOTO Y., *Fundamental Solutions for a Hyperbolic Equation with Involutive Characteristics of Variable Multiplicity*, Comm. in Partial Differential Equations **4** 6 (1979), 609–643.

[35] PARENTI C., *Operatori pseudodifferenziali in* $\mathbb{R}^n$ *e applicazioni*, Ann. Mat. Pura Appl. **93** (1972), 359–389.

[36] SATAKE I., *Linear Algebra*, Marcel Dekker, Inc., New York 1975.

[37] SCHROHE E., *Frèchet Algebra Techniques for Boundary Value Problems on Noncompact Manifolds: Fredholm Criteria and Functional Calculus via Spectral Invariance*, Math. Nachr, to appear.

[38] SCHROHE E., *Spaces of Weighted Symbols and Weighted Sobolev Spaces on Manifolds*, in Proceedings, Oberwolfach, **1256** (eds. Cordes H. O., Gramsch B., and Widom H.), in Springer LNM, New York 1986, 360–377.

[39] SCHROHE E., *Complex Powers on Noncompact Manifolds and Manifolds with Singularities*, Math. Ann. **281** (1988), 393–409.

[40] SCHROHE E. AND ERKIP A. K., *Normal Solvability of Elliptic Boundary Value Problems on Asymptotically Flat Manifolds*, J. Funct. Anal. **109** 1 (1992), 22–51.

[41] SCHULZE B. W., *Pseudodifferential Boundary Value Problems, Conical Singularities and Asymptotics*, Akademie Verlag, Berlin 1994.

[42] SHUBIN M. A., *Pseudodifferential Operators and Spectral Theory*, Springer-Verlag, Berlin 1987.

[43] SHUBIN M. A., *Pseudodifferential Operators in* $\mathbb{R}^n$, Sov. Math. Dokl., **12** (1987)147–151.

[44] TRÈVES F., *Introduction to Pseudodifferential Operators*, vol. 1–2, Plenum Press, New York 1980.

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