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# PERIODIC FACTORIZATION OF A FINITE ABELIAN 2-GROUP

**Abstract.**

Let $G$ be a finite abelian 2-group that is a direct product of a cyclic group and an elementary group. Suppose that $G$ is a direct product of its subsets $A_1, \ldots, A_n$ of cardinality two or four. Then one of the subsets $A_1, \ldots, A_n$ is periodic. The subset $A_i$ is periodic if $A_i g = A_i$ holds with a nonidentity element $g$ of $G$. This is a generalization of an earlier result of A. D. Sands and S. Szabó.

## 1. Introduction

Throughout the paper $G$ will be a finite abelian group. We use multiplicative notation. The identity element is denoted by $e$. The symbol "$\subset$" denotes a not necessarily strict inclusion, $|a|$ denotes the order of the element $a \in A$, $|A|$ denotes the cardinality of the subset $A$ of $G$. If $G$ is a direct product of its subsets $A_1, \ldots, A_n$, then we express this fact saying that the equality $G = A_1 \cdots A_n$ is a factorization of $G$. If $e \in A_i$, then we say that the subset $A_i$ is normed. We call the factorization $G = A_1 \cdots A_n$ normed if each $A_i$ is normed. A subset $A$ of $G$ is called periodic if there is a $g \in G \setminus \{e\}$ such that $Ag = A$. The element $g$ is a period of $A$. If $G$ is a direct product of cyclic groups of orders $t_1, \ldots, t_s$ respectively, then we say $G$ is of type $(t_1, \ldots, t_s)$. A. D. Sands and S. Szabó [2] proved that if $G$ is of type $(2, \ldots, 2)$ and $G = A_1 \cdots A_n$ is a factorization, where $|A_1| = \cdots = |A_n| = 4$, then one of the factors $A_1, \ldots, A_n$ is periodic. We will prove the following generalization of this theorem. Let $G$ be a finite abelian 2-group and let $G = A_1 \cdots A_n$ be a factorization of $G$, where each $|A_i|$ is either 2 or 4. If $G$ is of type $(2^\lambda, 2, \ldots, 2)$, then one of the factors $A_1, \ldots, A_n$ is periodic. We accomplish this using characters of $G$.

If $\chi$ is a character and $A$ is a subset of $G$, then we denote the sum

$$\sum_{a \in A} \chi(a)$$

by $\chi(A)$. If $\chi(A) = 0$, then $\chi$ annihilates $A$. We denote by $\mathrm{Ann}(A)$ the set of characters of $G$ that annihilates $A$.

If $A$ and $A'$ are subsets of $G$ such that given any subset $B$ of $G$, if $G = AB$ is a factorization of $G$, then $G = A'B$ is also a factorization of $G$, then we say that $A$ is replaceable by $A'$. There is a character test for replaceability due to L. Rédei [1] which reads as follows. If $|A| = |A'|$ and $\mathrm{Ann}(A) \subset \mathrm{Ann}(A')$, then $A$ can be replaced by $A'$

## 2. The result

Let $G$ be a finite abelian group and let $A = \{e, a, b, c\}$ be a subset of $G$. We define a subset $A'$ by $A' = \{e, a\}\{e, b\}$. Since the equation $c = abd$ is solvable for $d$, $A$ can be written in the form $A = \{e, a, b, abd\}$. We need the next lemma.

LEMMA 1. *If* $|a| = 2$*, then*
*(a)* $\mathrm{Ann}(A) \subset \mathrm{Ann}(A')$,
*(b)* $A$ *is periodic if and only if* $d = e$,
*(c)* $\chi(A) = 0$ *implies* $\chi(d) = 1$.

*Proof.* (a) Let $\chi$ be a character of $G$ for which $0 = \chi(A) = 1 + \chi(a) + \chi(b) + \chi(c)$. As $|a| = 2$, it follows that $\chi(a) = -1$ or $\chi(a) = 1$. If $\chi(a) = 1$, then $\chi(A) = 0$ gives that $\chi(b) = \chi(c) = -1$. Using this we have

$$\chi(A') = 1 + \chi(a) + \chi(b) + \chi(a)\chi(b) = 1 + 1 - 1 - 1 = 0.$$

If $\chi(a) = -1$, then $\chi(A) = 0$ gives that $\chi(b) = \rho$ and $\chi(c) = -\rho$, where $\rho$ is a root of unity. Using this we have

$$\chi(A') = 1 + \chi(a) + \chi(b) + \chi(a)\chi(b) = 1 - 1 + \rho - \rho = 0.$$

(b) If $d = e$, then $A = A'$ and so $A$ is periodic with period $a$. Conversely, assume that $A$ is periodic with period $g$. Note that $g^2$ is also a period of $A$ if $g^2 \neq e$. Using this observation we may assume that $|g| = 2$. From $e \in A$ it follows that $g \in A$.

If $g = a$, then

$$Aa = \{a, e, ab, bd\} = \{e, a, b, abd\} = A$$

gives that $\{ab, bd\} = \{b, abd\}$. Here either $ab = b$ or $bd = b$. The first one leads to the contradiction $a = e$. The second one gives $d = e$.

If $g = b$, then

$$Ab = \{b, ab, e, ad\} = \{e, a, b, abd\} = A$$

gives that $\{ab, ad\} = \{a, abd\}$. Hence either $ab = a$ or $ad = a$. The first one leads to the contradiction $b = e$. the second one gives $d = e$.

If $g = abd$, then

$$Aabd = \{abd, bd, ab^2d, e\} = \{e, a, b, abd\} = A$$

gives that $\{ab^2d, bd\} = \{a, b\}$. Now either $ab^2d = b$ or $bd = b$. The first equality gives the contradiction $abd = e$, the second one provides $d = e$.

(c) If $\chi(A) = 0$, then by part (a), $\chi(A') = 0$ and so

$$0 = \chi(A) - \chi(A') = \chi(ab)\chi(d) - \chi(ab) = \chi(ab)\big[\chi(d) - 1\big].$$

This completes the proof.

□

After this preparation we are ready to prove the main result of the paper.

THEOREM 1. *Let* $G$ *be a finite group of type* $(2^\lambda, 2, \ldots, 2)$*. If* $G = A_1 \cdots A_n$ *is a normed factorization of* $G$*, where* $|A_i|$ *is either* 2 *or* 4 *for each* $i$*,* $1 \leq i \leq n$*, then at least one of the factors* $A_1, \ldots, A_n$ *is periodic.*

*Proof.* The $|G| = 2$ case is trivial. So we assume that $|G| \geq 4$ and proceed by induction on $|G|$. Clearly $G$ is a direct product of its subgroups $H$ and $K$ of types $(2^\lambda)$ and $(2, \ldots, 2)$ respectively. If $\lambda = 1$, then $G$ is of type $(2, \ldots, 2)$. This special case is covered by [2] Theorem 9. So for the remaining part of the proof we may assume that $\lambda \geq 2$. Let $H = \langle x \rangle$ and $K = \langle y_1, \ldots, y_s \rangle$, where $|x| = 2^\lambda$ and $|y_1| = \cdots = |y_s| = 2$. Consider a character $\chi$ of $G$ that is faithful on $H$ or equivalently for which $\chi(x) = \rho$, where $\rho$ is a primitive $(2^\lambda)$th root of unity.

Let $A_i = \{e, a_i\}$ be a factor of order 2. If $0 = \chi(A_i) = 1 + \chi(a_i)$, then $\chi(a_i) = -1$ or $\chi(a_i^2) = 1$ and so $a_i^2 = e$. Therefore $A_i$ is periodic with period $a_i$. So in the remaining part of the proof we may assume that $\chi(A_i) \neq 0$ when $\chi$ is faithful on $H$ and $|A_i| = 2$. As $\chi$ is not the principal character of $G$, it follows that $0 = \chi(G) = \chi(A_1) \cdots \chi(A_n)$ and so $\chi(A_i) = 0$ for some $i$, $1 \leq i \leq n$. Thus we may assume that $|A_i| = 4$ for some $i$.

Let $A_i = \{e, a_i, b_i, c_i\}$ be a factor of order 4. If

$$0 = \chi(A_i) = 1 + \chi(a_i) + \chi(b_i) + \chi(c_i),$$

then one of $\chi(a_i)$, $\chi(b_i)$, $\chi(c_i)$ must be $-1$ and so one of $a_i^2$, $b_i^2$, $c_i^2$ must be $e$. Thus there is at least one factor of order 4 that contains at least one second order element. We choose the notation such that $A_1, \ldots, A_m$ are all the factors of order 4 containing at least one second order element. If $m = 1$, then $\chi(A_1) = 0$ for each $\chi$ that is faithful on $H$. Now, by [2] Theorem 1, $A_1$ is periodic. So we may assume that $m \geq 2$.

Let us consider an $A_i = \{e, a_i, b_i, c_i\}$ with $1 \leq i \leq m$. We choose the notation such that $|a_i| = 2$. Further $c_i$ can be written in the form $c_i = a_i b_i d_i$ with a suitable $d_i \in G$. By Lemma 1 $A_i$ is periodic if and only if $d_i = e$. Thus we may assume that $d_i \neq e$. Also by Lemma 1 $\chi(A_i) = 0$ implies $\chi(d_i) = 1$. From this it follows that $A_i$ can be replaced by

$$\{e, a_i, b_i, a_i b_i d_i^k\}$$

for each integer $k$. In particular, we may assume that $|d_i| = 2$ for each $i$, $1 \leq i \leq m$. Also $A_i$ can be replaced by

$$\{e, a_i, b_i, a_i b_i\} = \{e, a_i\}\{e, b_i\}.$$

If $b_i^2 = e$, then each element of $A_i \setminus \{e\}$ is of order 2. We will say that $A_i$ is a type 1 factor. Now $A_i$ can be replaced by $H_i B_i$, where $H_i = \langle a_i, b_i \rangle$ and $B_i = \{e\}$. If $b_i^2 \neq e$, then $a_i$ is the only second order element in $A_i$. We will say that $A_i$ is a type 2 factor. In this case $A_i$ can be replaced by $H_i B_i$, where $H_i = \{e, a_i\} = \langle a_i \rangle$ and $B_i = \{e, b_i\}$.

The subgroup $H$ has a unique subgroup $L = \langle x^{2^{\lambda-1}} \rangle$ of order 2. From the factorization

$$G = H_1 B_1 \cdots H_m B_m A_{m+1} \cdots A_n$$

it follows that the product $H_1 \cdots H_m$ is direct. So there can be only one subgroup $H_i$ for which $L \subset H_i$. Such an $H_i$ does not necessarily exists. But if it does, then we choose the notation such that $L \subset H_1$. We claim that $L \not\subset H_1$ may be assumed.

In order to prove this claim let us consider $A_1 = \{e, a_1, b_1, c_1\}$ and distinguish two cases depending on whether $A_1$ is of type 1 or type 2.

If $A_1$ is of type 1, then it can be written in the forms

$$A_1 = \{e, a_1, b_1, a_1 b_1 d_1\}, \quad A_1 = \{e, b_1, c_1, b_1 c_1 d_1'\}, \quad A_1 = \{e, a_1, c_1, a_1 c_1 d_1''\}$$

and can be replaced by the subgroups

$$H_1 = \langle a_1, b_1 \rangle, \quad H_1' = \langle b_1, c_1 \rangle, \quad H_1'' = \langle a_1, c_1 \rangle$$

respectively. If $L \subset H_1$, then one of $a_1$, $b_1$, $a_1b_1$ is equal to $x^{2^{\lambda-1}}$. In the $a_1 = x^{2^{\lambda-1}}$ case $a_1 \notin H_1'$ since obviously $a_1 \neq e$, $a_1 \neq b_1$, $a_1 \neq c_1$ and $a_1 = b_1c_1$ combined with $c_1 = a_1b_1d_1$ leads to the $d_1 = e$ contradiction. In the $b_1 = x^{2^{\lambda-1}}$ case $b_1 \notin H_1''$ since clearly $b_1 \neq e$, $b_1 \neq a_1$, $b_1 \neq c_1$ and $b_1 = a_1c_1$ leads to the $d_1 = e$ contradiction. In the $a_1b_1 = x^{2^{\lambda-1}}$ case $a_1b_1 \notin H_1'$ since $a_1b_1 = e$, $a_1b_1 = b_1$, $a_1b_1 = c_1$, $a_1b_1 = b_1c_1$ leads in order to the $a_1 = b_1$, $a_1 = e$, $d_1 = e$, $a_1 = c_1$ contradictions.

If $A_1$ is of type 2, then $a_1^2 = e$, $b_1^2 \neq e$ and $A_1$ can be written in the form $A_1 = \{e, a_1, b_1, a_1b_1d_1\}$ and can be replaced by $H_1B_1$, where $H_1 = \{e, a_1\}$, $B_1 = \{e, b_1\}$. If $L \subset H_1$, then $a_1 = x^{2^{\lambda-1}}$. Now replace $A_1$ by

$$A_1' = b_1^{-1}A_1 = \{b_1^{-1}, b_1^{-1}a_1, e, a_1d_1\} = \{e, a_1', b_1', a_1'b_1'd_1'\},$$

where $a_1' = a_1d_1$, $b_1' = b_1^{-1}a_1$, $d_1' = d_1$. The only second order element in $A_1'$ is $a_1' = a_1d_1$ which is not equal to $x^{2^{\lambda-1}}$. Here $A_1'$ is replaceable by $H_1'B_1'$, where

$$H_1' = \langle a_1' \rangle = \langle a_1d_1 \rangle, \quad B_1' = \{e, b_1'\}.$$

Thus in each case we may assume that $L \not\subset H_1$.

Replace $A_i$ by $H_iB_i$ in the factorization $G = A_1 \cdots A_n$ to get the factorization

$$G = A_1 \cdots A_{i-1}(H_iB_i)A_{i+1} \cdots A_n,$$

where $1 \leq i \leq m$. This leads to the factorization

$$\overline{G} = \overline{A}_1 \cdots \overline{A}_{i-1}\overline{B}_i\overline{A}_{i+1} \cdots \overline{A}_n$$

of the factor group $\overline{G} = G/H_i$. Here

$$\begin{aligned} \overline{A}_j &= \{H_i, a_jH_i, b_jH_i, c_jH_i\} \quad &\text{or} \quad \overline{A}_j &= \{H_i, a_jH_i\}, \\ \overline{B}_i &= \{H_i, b_iH_i\} \quad &\text{or} \quad \overline{B}_i &= \{H_i\}. \end{aligned}$$

As $|\overline{G}| < |G|$, by the inductive assumption it follows that either $\overline{B}_i$ or $\overline{A}_j$ is periodic for some $j$, $1 \leq j \leq n$, $j \neq i$.

If $\overline{B}_i$ is periodic, then $|\overline{B}_i| = |B_i|$ must be 2 and consequently $A_i$ must be of type 2. Since $\overline{B}_i$ is periodic, it follows that $(b_iH_i)^2 = b_i^2H_i = H_i$ and so $b_i^2 \in H_i = \{e, a_i\}$. We know that $b_i^2 \neq e$ and hence $b_i^2 = a_i$. Let

$$b_i = x^\beta y_1^{\beta_1} \cdots y_s^{\beta_s} \qquad \text{and} \qquad a_i = x^\alpha y_1^{\alpha_1} \cdots y_s^{\alpha_s},$$

where $\alpha = 2^{\lambda-1}$, $0 \leq \beta \leq 2^\lambda - 1$, $0 \leq \alpha_1, \beta_1, \ldots, \alpha_s, \beta_s \leq 1$. Now

$$b_i^2 = (x^\beta y_1^{\beta_1} \cdots y_s^{\beta_s})^2 = x^{2\beta} = x^\alpha y_1^{\alpha_1} \cdots y_s^{\alpha_s} = a_i$$

gives that $\alpha_1 = \cdots = \alpha_s = 0$ and so $L \subset H_i$ and this is a contradiction.

If $A_j$ is periodic and $|\overline{A}_j| = |A_j| = 2$, then in a similar way $a_j^2 \in H_i$ and $a_j^2 \neq e$ lead to the contradiction $L \subset H_i$. Therefore if $A_j$ is periodic, then $|\overline{A}_j| = |A_j| = 4$. The periodicity of $\overline{A}_j$ implies that $\overline{A}_j$ contains a second order element, say $(a_jH_i)^2 = a_j^2H_i = H_i$. Hence $a_j^2 \in H_i$. As $a_j^2 \neq e$ in the known way leads to the contradiction $L \subset H_i$, it follows that $a_j^2 = e$.

Thus $A_j$ contains a second order element, that is $1 \leq j \leq m$. By Lemma 1 the periodicity of $\overline{A}_j$ implies that $d_j \in H_i$.

The summary of the above argument is that for each $i$, $1 \leq i \leq m$ there is a $j$, $1 \leq j \leq m$ such that $d_j \in H_i$ and $i \neq j$. We define a bipartite graph $\Gamma$ whose nodes are $H_1, \ldots, H_m$ and $d_1, \ldots, d_m$ and if $d_j \in H_i$, then $(H_i, d_j)$ is a directed edge of $\Gamma$. If $(H_i, d_j)$, $(H_k, d_j)$ are edges of $\Gamma$ with $i \neq k$, then $d_j \in H_i \cap H_k = \{e\}$ which is a contradiction. Thus for each $d_j$ there is at most one $H_i$ such that $(H_i, d_j)$ is an edge of $\Gamma$. Further, for each $H_i$ there is at least one $d_j$ for which $(H_i, d_j)$ is an edge of $\Gamma$. Therefore there is a ono-to-one map $f$ from $\{H_1, \ldots, H_m\}$ into $\{d_1, \ldots, d_m\}$ such that $\big(H_i,\ f(H_i)\big)$, $1 \leq i \leq m$ are all the edges of $\Gamma$.

Let us consider

$$A_m = \{e, a_m, b_m, c_m\}.$$

(Remember $m \geq 2$.) If $A_m$ is of type 1, then it can be written in the forms

$$A_m = \{e, a_m, b_m, a_m b_m d_m\},\ A_m = \{e, a_m, c_m, a_m c_m d'_m\},\ A_m = \{e, b_m, c_m, b_m c_m d''_m\}$$

and it can be replaced by the subgroups

$$H_m = \langle a_m, b_m \rangle, \qquad H'_m = \langle a_m, c_m \rangle, \qquad H''_m = \langle b_m, c_m \rangle$$

respectively. These replacements give rise to the graphs $\Gamma$, $\Gamma'$, $\Gamma''$ and the maps $f$, $f'$, $f''$ respectively. The nodes $H_1, \ldots, H_{m-1}$ and $d_1, \ldots, d_{m-1}$ are common in these graphs. After removing the edges joining to $H_m, H'_m, H''_m$ and $d_m, d'_m, d''_m$ the remaining parts of the graphs are identical. From this it follows that $f(H_m) = f'(H'_m) = f''(H''_m)$. Let $d_j$ be this common value. This leads to the contradiction $d_j \in H_m \cap H'_m \cap H''_m = \{e\}$.

If $A_m$ is of type 2, then $a_m^2 = e$, $b_m^2 \neq e$ and $A_m$ can be written in the form

$$A_m = \{e, a_m, b_m, a_m b_m d_m\}$$

and can be replaced by $H_m B_m$, where

$$H_m = \{e, a_m\}, \qquad B_m = \{e, b_m\}.$$

The factor $A_m$ can be replaced by

$$A'_m = b_m^{-1} A_m = \{b_m^{-1}, b_m^{-1} a_m, e, a_m d_m\} = \{e, a'_m, b'_m, a'_m b'_m d'_m\},$$

where $a'_m = a_m d_m$, $b'_m = b_m^{-1} a_m$, $d'_m = d_m$. Then $A'_m$ can be replaced by $H'_m B'_m$, where

$$H'_m = \{e, a'_m\}, \qquad B'_m = \{e, b'_m\}.$$

The $A_m \to H_m B_m$ and $A'_m \to H'_m B'_m$ replacements give rise to the graphs $\Gamma$, $\Gamma'$ and the maps $f$, $f'$ respectively. The nodes $H_1, \ldots, H_{m-1}$ and $d_1, \ldots, d_{m-1}, d_m$ are common in these graphs. After removing the edges joining to $H_m, H'_m$ the remaining parts of the graphs are identical. From this it follows that $f(H_m) = f'(H'_m)$. Let $d_j$ be this common value. This gives the contradiction $d_j \in H_m \cap H'_m = \{e\}$.

This completes the proof.

□

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