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COMPATIBILITY CONDITIONS FOR DISCRETE ELASTIC STRUCTURES

Abstract. The theory of plane, elastic trusses is reconsidered from the viewpoint of the continuum theory of elastic media. A main difference between continuum and discrete theories is the following: In the continuous case all quantities are declared throughout the whole body, whereas in the discrete case each quantity has its own "carrier". In a truss, for instance, displacements and applied forces are declared in the nodes while strain and stress live in the members of the truss.

The paper deals with the proper formulation of compatibility conditions for a truss. They are derived along the same lines as St.-Venant's compatibility conditions of plane elasticity, i. e. by stipulating that Cesàro's integrals are path independent. Compatibility conditions can be formulated at each *inner* node of a truss, and they relate the strains of all members which constitute the rosette surrounding the inner node.

1. Continuous and discrete elastic systems

Continuum theories are usually developed from physical models that are discrete in nature. A continuous distribution of dislocations, for instance, would hardly be conceivable, if we had not a clear idea of an *isolated* dislocation. Even the notion of stress as a distributed force follows the example of a single force. Within the framework of a continuum theory, however, discrete quantities appear as singularities and are formally less convenient to handle than their continuous counterparts.

By the process of *homogenization* the underlying discrete ideas are transformed into a continuum theory. The resulting partial differential equations do not admit closed-form solutions, in general. To solve them numerically a *discretization* process is invoked, which approximates the continuum by a discrete system. In this sense a continuum theory is squeezed between the underlying discrete *physical* model and the discrete *numerical* approximation.

The general structure of a physical theory should be perceptible independently of the discrete or continuum formulation. A balance equation, for instance, has a genuine physical meaning whether the model is continuous or discrete. The theory of a discrete elastic structure, be it a crystal lattice, a finite-element system or an elastic truss, should exhibit the same fundamental laws as continuum elasticity theory. The general form of the fundamental equations can be represented most suggestively by a so-called TONTI diagram [6, 7]. Figure 1 shows the TONTI diagram of plane, linear elasticity theory. If we consider a plane, *discrete* elastic system, we should encounter the same physical laws, although in a rather different formal garment.

This paper deals with the governing equations of plane, elastic trusses with special emphasize of the compatibility conditions, which are derived along the same lines as ST.-VENANT's compatibility conditions of plane elasticity, i. e. by stipulating that CESÀRO's integrals are path independent.

The theory of plane, elastic trusses is reconsidered from the viewpoint of the continuum theory of elastic media. Mathematically a truss is considered as an oriented 2-complex, on which displacement, strain, etc. are defined. In contrast to the continuous body the mechanical quantities in a truss are not available everywhere in the body, each quantity resides on its own "carrier": Displacements and applied forces are declared in the nodes while strain and stress live in the members of the truss. It will be shown that the compatibility conditions are attached to "rosettes", i. e. inner nodes that are completely surrounded by triangles of truss members.

To consider trusses from the point of view of elasticity theory is not at all new. KLEIN and WIEGHARDT [4] have presented such an exposition even in 1905, and they rely on earlier works of MAXWELL and CREMONA. Meanwhile, however, trusses have become more a subject of structural mechanics and the more theoretical aspects have been banned from textbooks. As an exception a manuscript by RIEDER [5] should be mentioned, in which the cross-relations between electrical and mechanical frameworks are studied in great detail.

2. Trusses

Mechanically a truss is a system of elastic *members* joint to each other in hinges or *nodes* without friction. The truss is loaded by forces acting on the nodes only.

The appropriate mathematical model of a truss is a 1-complex consisting of 0-simplexes (nodes) and 1-simplexes (members), which are "properly joined" [3]. The subsequent analysis gives rise to two extensions of this model, namely (i) each member is given an orientation, which

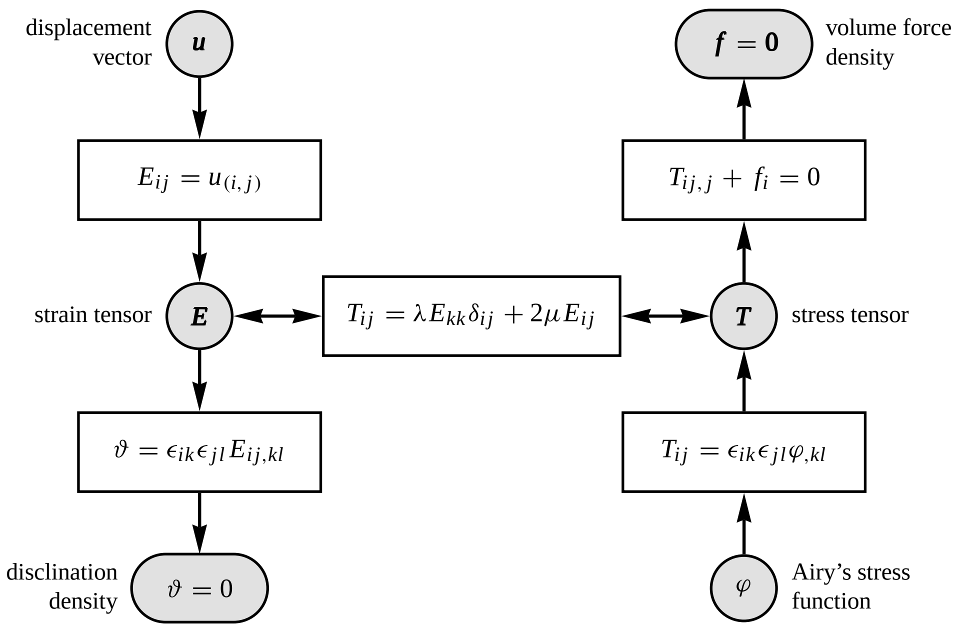

Figure 1: Tonti diagram of plane linear elasticity

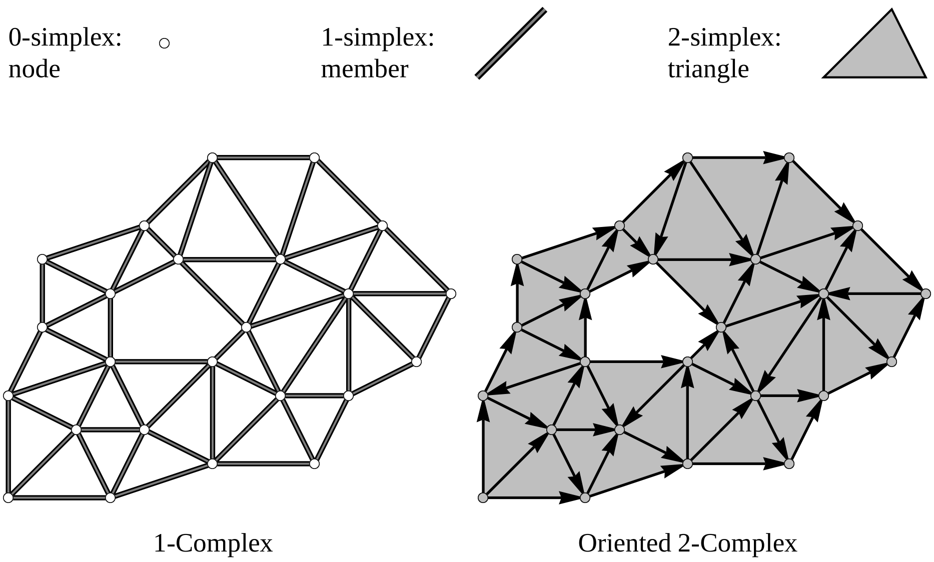

Figure 2: Plane truss as a geometric complex

may be prescribed arbitrarily, and (ii) the triangles or 2-simplexes formed by the members are taken into account. Thus the mathematical model of a truss is extended to an *oriented 2-complex* (Figure 2). Only the non-oriented 1-complex is reproduced in hardware while the imposed orientation and the appended triangular patches are mere mathematical constructs which facilitate the formulation of the theory.

Subsequently nodes will be designated by Latin letters $i, j, \ldots$ while Greek letters $\alpha, \beta, \ldots$ denote the members. The connectivity of the truss is described by *incidence numbers* $[\alpha, k]$, which are defined as

$$[\alpha, k] = \begin{cases} -1 & \text{if member } \alpha \text{ starts at node } k, \\ +1 & \text{if member } \alpha \text{ ends at node } k, \\ 0 & \text{else.} \end{cases}$$

The distinction between start and end point of a member provides its orientation. The matrix of all incidence numbers describes the topological structure of the truss.

The geometry may be specified by prescribing the position vectors $\boldsymbol{x}_k$ of all nodes in the unloaded, stress-free state of the truss. The edge vector of a member α can then be represented by

$$\boldsymbol{a}_\alpha = \sum_k [\alpha, k]\boldsymbol{x}_k, \tag{1}$$

where the summation index may run over all nodes, since the incidence numbers single out the proper starting and terminating points, thus reducing the sum to a simple difference. The decomposition

$$\boldsymbol{a}_\alpha = \ell_\alpha \boldsymbol{e}_\alpha$$

yields the length ℓ_α and the direction vector $\boldsymbol{e}_\alpha$ of a member.

It has been tacitly assumed that there exists an unloaded, stress-free state of the truss. In continuum elasticity theory this corresponds to the assumption that the unloaded elastic body is free of initial stresses. In a more general setting one has to start from the lengths ℓ_α of the undeformed members rather than from a given initial placement $k \mapsto \boldsymbol{x}_k$ of the nodes. This approach within a nonlinear theory is indicated in [2].

3. Displacement and strain

When loads are applied to the truss, each node k is displaced by a certain vector $\boldsymbol{u}_k$ from its original position. The *strain* or relative elongation ε of a member due to displacements $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of its endpoints is

$$\varepsilon = \frac{1}{\ell}\boldsymbol{e} \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1),$$

if only linear terms are retained. Using again the incidence numbers $[\alpha, k]$ the strain ε_α of an arbitrary member α can be represented by

$$\varepsilon_\alpha = \frac{1}{\ell_\alpha}\boldsymbol{e}_\alpha \cdot \sum_k [\alpha, k]\boldsymbol{u}_k \,. \tag{2}$$

As in (1) above, the incidence numbers single out the end nodes of the member and the sum reduces to a simple difference.

Due to the nodal displacements each member α undergoes also a rotation ω_α. Restriction to linear approximation yields

$$\omega_\alpha = \frac{1}{\ell_\alpha}\boldsymbol{e}_\alpha \wedge \sum_k [\alpha, k]\boldsymbol{u}_k \,,$$

where $\wedge$ denotes the outer product of two plane vectors. An approach starting from displacements does not need these rotations explicitly, since they do not enter the stress-strain relation. However, if the displacements have to be reconstructed from given strains, the rotations are needed as well.

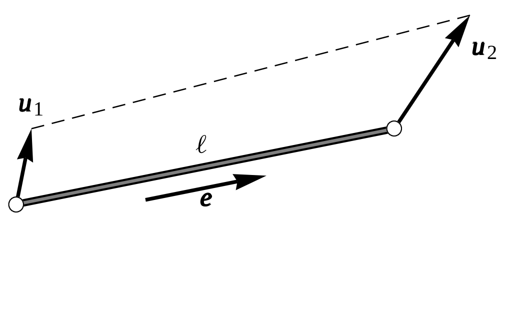

Figure 3: Elongation of a member

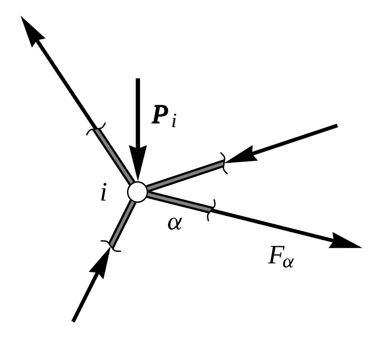

Figure 4: Equilibrium

4. Equilibrium condition

Once the strain of a member α is known, one obtains the transmitted force by HOOKE's law

$$F_\alpha = (EA)_\alpha \varepsilon_\alpha \,, \tag{3}$$

where EA denotes the axial rigidity of the member, i. e. the product of YOUNG's modulus and cross-sectional area.

At any node i of the truss the applied external force $\boldsymbol{P}_i$ and the member forces acting on that node must be in equilibrium (Figure 4). The force acted upon the node by the member α is $-[\alpha, i] F_\alpha \boldsymbol{e}_\alpha$. Thus the equilibrium condition can be formulated as

$$\sum_\alpha [\alpha, i] F_\alpha \boldsymbol{e}_\alpha = \boldsymbol{P}_i \,. \tag{4}$$

The sum may be taken over all members of the truss, since the incidence numbers single out only those which start or end at node i.

Combining the equilibrium condition (4), the constitutive equation (3), and the definition of strain (2) yields the linear system of equations

$$\sum_k \sum_\alpha [\alpha, i]\,[\alpha, k] \left(\frac{EA}{\ell} \boldsymbol{e} \otimes \boldsymbol{e} \right)_\alpha \boldsymbol{u}_k = \boldsymbol{P}_i \,,$$

which is the discrete analogue of NAVIER's equations. In structural analysis the matrix of this system of equations would be called the global stiffness matrix of the truss. The three constituents of NAVIER's equations can be arranged in a TONTI diagram (Figure 5), which is still incomplete, since the lower part with the compatibility condition and AIRY's stress function is missing.

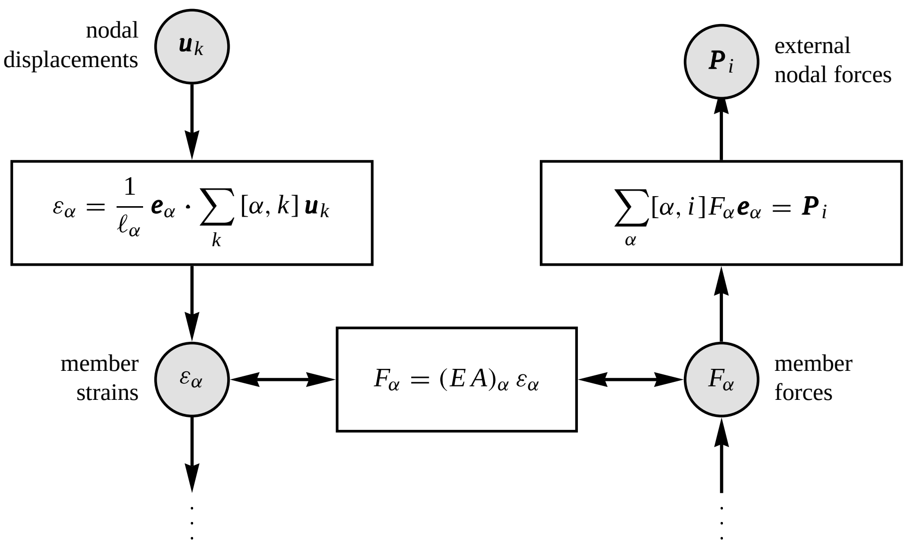

Figure 5: Tonti diagram (upper part) for an elastic truss

5. Compatibility in plane elasticity

Before developing the compatibility conditions for a truss we shall first review the ST.-VENANT compatibility condition of plane elasticity, which may be used as paradigm. Starting from a position $\boldsymbol{x}_0$ with given displacement vector the whole displacement field has to be reconstructed from the strain field $E_{ij} = u_{(i,j)}$. Integration along a curve $\mathcal{C}$ connecting $\boldsymbol{x}_0$ with an arbitrary position $\boldsymbol{x}$ yields the displacement components in terms of CESÀRO's integral,

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{x}_0) + \int_{\mathcal{C}} (E_{ij} + \omega\epsilon_{ij})\, \mathrm{d}x_j \ , \tag{5}$$

where ϵ_{ij} denotes the two-dimensional permutation symbol. The rotation $\omega = \frac{1}{2}(u_{2,1} - u_{1,2})$, however, is still unknown and has to be reconstructed from the strain field too.

For the integral in (5) to be path independent the integrand has to satisfy the integrability condition

$$a_i \equiv E_{ij,k}\epsilon_{kj} - k_i = 0, \tag{6}$$

where $k_i = \omega_{,i}$ denotes the rotation gradient or *structural curvature* [1]. Geometrically this means that the body is free of dislocations. On the other hand the rotation field itself can be reconstructed by another integral,

$$\omega(\boldsymbol{x}) = \omega(\boldsymbol{x}_0) + \int_{\mathcal{C}} k_i \, \mathrm{d}x_i \ . \tag{7}$$

For this integral to be path-independent the integrability condition

$$\vartheta \equiv k_{i,j}\epsilon_{ji} = 0 \tag{8}$$

has to be satisfied, which means that the body is free of disclinations. Combining the two conditions (6) and (8) yields the ST.-VENANT compatibility condition

$$\epsilon_{ik}\epsilon_{jl}E_{ij,kl} = 0, \tag{9}$$

which stipulates that both the dislocation and the disclination densities vanish.

The compatibility condition emerges from a two-stage process and combines two independent conditions. To unwrap this combination the geometric part of the TONTI diagram, Figure 1, has to be extended to show all the details, see Figure 6.

6. Compatibility condition for a plane truss

The displacement difference $\Delta\boldsymbol{u}$ between the terminating nodes of a single member can be reconstructed from the strain ε and the rotation ω of that member. According to Figure 7 one obtains

$$\Delta\boldsymbol{u} = \varepsilon\boldsymbol{a} + \omega\boldsymbol{a}^{\wedge}, \tag{10}$$

where the vector $\boldsymbol{a}$ is aligned with the member and $\boldsymbol{a}^{\wedge}$ denotes the vector obtained by rotation through $+\pi/2$.

The role of the path $\mathcal{C}$ in CESÀRO's integrals is adopted by an oriented 1-chain of truss members (Figure 9). A 1-chain $\mathcal{C}$ can be specified by incidence numbers

$$[\mathcal{C},\alpha] = \begin{cases} +1 & \text{if } \mathcal{C} \text{ contains } \alpha \text{ and has the same orientation,} \\ -1 & \text{if } \mathcal{C} \text{ contains } \alpha \text{ and has opposite orientation,} \\ 0 & \text{if } \mathcal{C} \text{ does not contain } \alpha. \end{cases}$$

Extending (10) to an oriented 1-chain $\mathcal{C}$ yields the rotation difference

(11) $$\Delta \boldsymbol{u}_{\mathcal{C}} = \sum_{\alpha} [\mathcal{C},\alpha](\varepsilon_\alpha \boldsymbol{a}_\alpha + \omega_\alpha \boldsymbol{a}_\alpha^{\wedge}).$$

This is the discrete analogue to CESÀRO's first integral (5).

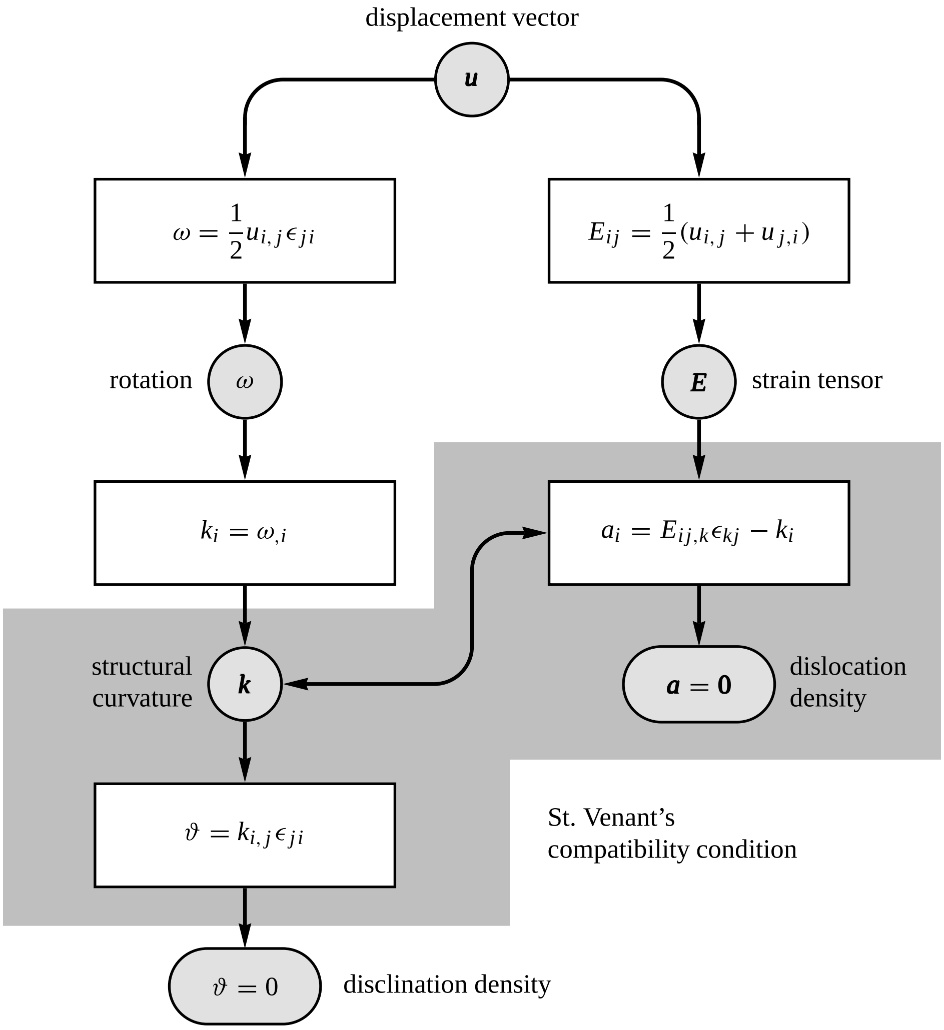

Figure 6: Geometry of continuous deformation

The displacement difference (11) has to vanish for any *closed* 1-chain, or 1-cycle, $\mathcal{C}$. The simplest nontrivial 1-cycle is a triangle formed by three members. With the numbering and orientation as provided in Figure 10 the closing condition for this triangular 1-cycle reads

$$\varepsilon_1 \boldsymbol{a}_1 + \omega_1 \boldsymbol{a}_1^{\wedge} + \varepsilon_2 \boldsymbol{a}_2 + \omega_2 \boldsymbol{a}_2^{\wedge} + \varepsilon_3 \boldsymbol{a}_3 + \omega_3 \boldsymbol{a}_3^{\wedge} = 0.$$

Scalar multiplication by one of the edge vectors, $\boldsymbol{a}_1$ say, yields

$$\omega_2 - \omega_3 = \frac{1}{2A} \left(\varepsilon_1 \boldsymbol{a}_1 + \varepsilon_2 \boldsymbol{a}_2 + \varepsilon_3 \boldsymbol{a}_3 \right) \cdot \boldsymbol{a}_1 \,,$$

where $2A = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 = -\boldsymbol{a}_1 \wedge \boldsymbol{a}_3$ is twice the area of the triangle. From elementary geometric considerations this can also be written as

$$\omega_2 - \omega_3 = (\varepsilon_1 - \varepsilon_2) \cot \alpha_3 + (\varepsilon_1 - \varepsilon_3) \cot \alpha_2 \,, \tag{12}$$

where the angles of the undeformed triangle are denoted as in Figure 10. Within each triangle the rotation difference of two adjacent members can be computed from the strains in the members of that triangle. This corresponds to the local integrability condition (6) of the continuum theory, which expresses the rotation gradient in terms of derivatives of the local strain field.

The simplest 1-chain, for which a rotation difference can be defined, has length 2, it is formed by two adjacent members (Figure 8). An extended 1-chain may be decomposed into a sequence of such elementary 1-chains c. Thus the rotation difference of an arbitrary 1-chain is

$$\Delta\omega_{\mathcal{C}} = \sum_c [\mathcal{C}, c] \Delta\omega_c$$

with appropriately defined incidence numbers $[\mathcal{C}, c]$. Whereas the rotation difference $\Delta\omega_c$ is defined for all pairs c of adjacent members, an explicit formula is available only, if these adjacent members are complemented by a third member to a closed triangle. Therefore, in order to actually compute the rotation difference $\Delta\omega_{\mathcal{C}}$ between the first and the last member of a connected 1-chain $\mathcal{C}$, it has to be accompanied by an appropriate sequence of triangles, i. e. a 2-chain. Also the original 1-chain $\mathcal{C}$ must be extended by certain detours along the edges of the triangles (Figure 11).

For any 1-cycle $\mathcal{C}$ the rotation difference $\Delta\omega_{\mathcal{C}}$ has to vanish,

$$\sum_c [\mathcal{C}, c] \Delta\omega_c = 0 \quad \text{if } \partial\mathcal{C} = \emptyset. \tag{13}$$

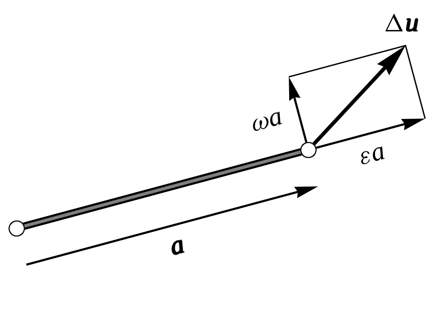

Figure 7: Single member

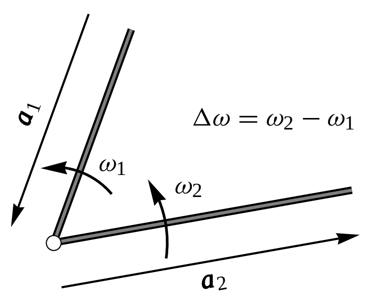

Figure 8: Two members

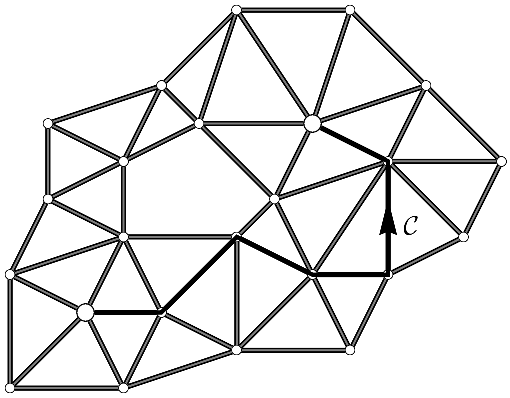

Figure 9: Oriented 1-chain

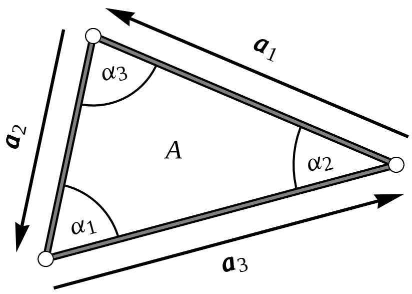

Figure 10: Triangle

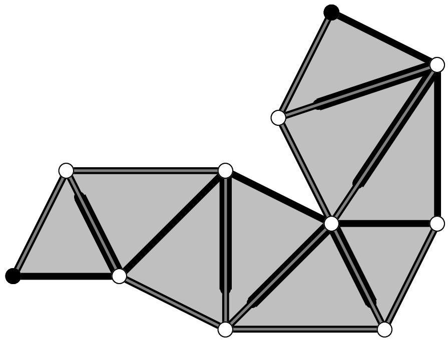

Figure 11: Chain of triangles

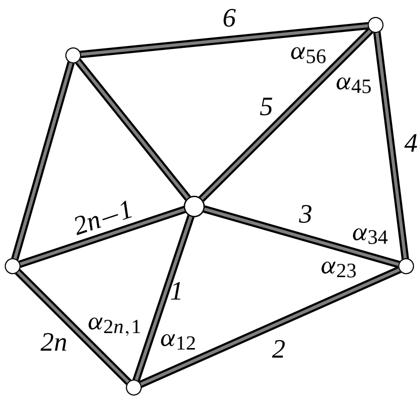

Figure 12: Rosette

The shortest nontrivial 1-cycles are those surrounding an inner node of the truss (Figure 12). For each of these rosette-like substructures we can formulate an appropriate condition, which corresponds to the integrability condition (8) in the continuum case.

The closing condition for a rosette contains the differences $\Delta\omega_C$ of successive members. By use of (12) these can be expressed in the strains of the members of the corresponding triangles. Using the numbering of members and angles indicated in Figure 12 one arrives at a single condition of the form

$$(14) \qquad \sum_{i=1}^{n} \big(\cot \alpha_{\underset{\bmod 2n}{2i-2,2i-1}} + \cot \alpha_{2i-1,2i}\big)\varepsilon_{2i-1} = \sum_{i=1}^{n} \big(\cot \alpha_{2i-1,2i} + \cot \alpha_{\underset{\bmod 2n}{2i,2i+1}}\big)\varepsilon_{2i}\ .$$

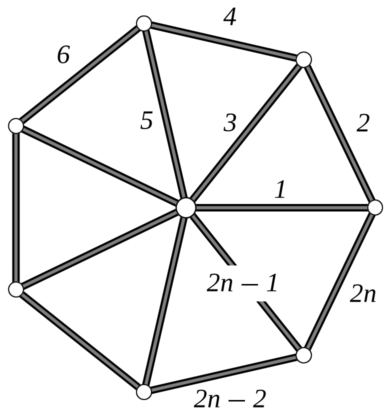

Figure 13: Regular rosette

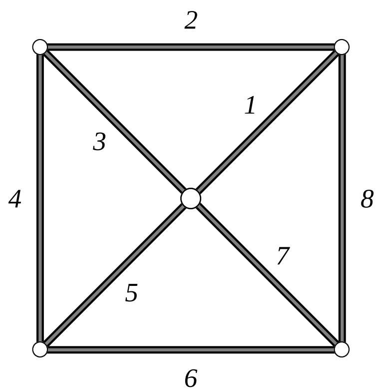

Figure 14: Quadratic rosette

This is the analogue of ST.-VENANT's compatibility condition for a truss. It is obtained by combining the closing condition (13) for the rotations around a rosette with the closing condition (12) for the displacements around a triangle. For the truss to be stress-free in its unloaded state the condition (14) is necessary but not sufficient, in general. If the 2-complex does not contain any holes, the condition is also sufficient. The compatibility conditions are closely connected with the extended model of the truss as an oriented 2-complex, although the 2-simplexes are not material parts of the truss.

In the special case of a regular rosette (Figure 13) all the angles $\alpha_{i,i+1}$ are equal and cancel out. Thus the compatibility condition reduces to

$$\sum_{i=1}^{n} \varepsilon_{2i-1} = \sum_{i=1}^{n} \varepsilon_{2i} \, .$$

The sum of the circumferential strains must be equal to the sum of the radial strains. The general compatibility condition (14) has a similar structure, with the strains being affected by certain geometrical weight factors. For a quadratic rosette the compatibility condition reads

$$\varepsilon_1 + \varepsilon_3 + \varepsilon_5 + \varepsilon_7 = \varepsilon_2 + \varepsilon_4 + \varepsilon_6 + \varepsilon_8 \, .$$

This equation can be interpreted as a discretization of ST.-VENANT's compatibility condition (9).

7. Conclusion

The general structure of elasticity theory is not confined to the continuum version, but holds also for discrete elastic systems such as trusses or finite-element models. A remarkable difference between the theories of plane trusses and of elastic continua is the fact that in the continuous case all quantities are declared throughout the whole body, whereas in the discrete case of the truss each quantity has its own "carrier": Displacements are declared in the nodes, strain and rotation are available in the members, rotation differences need pairs of members, and compatibility conditions can be formulated for "rosettes", i. e. inner nodes that are completely surrounded by triangles of truss members. In this sense the continuum theory could be regarded as "easier", since all quantities are defined in each material point. A closer look shows, however, that the continuum theory can also provide different carriers for different quantities. This becomes manifest, if the mechanical quantities are described in terms of differential forms rather than ordinary field functions.*

The compatibility condition for a truss have been developed using the same ideas as in the continuum. It rests upon the postulation that displacement and rotation can be represented by *path-independent* integrals or, in the discrete case, by path-independet finite sums. To generate *localized* integrability conditions in a continuum the integral around a closed path is transformed via STOKES's theorem into a surface integral, which must vanish identically. In the truss case the local conditions are obtained by choosing the smallest nontrivial closed paths or 1-cycles, namely triangles for the displacements and rosettes for the rotations.

The theory of trusses can be developed further and extended along these lines. The compatibility condition should be complemented by its dual, the representation of member forces by AIRY's stress function. This quantity has the same carrier as the compatibility condition, i. e., it resides in the rosettes surrounding inner nodes of the truss. The generalization to three dimensions is more intricate, especially with respect to the closing condition for the rotation vector.

Quite interesting is the appropriate treatment of frame trusses, with members rigidly clamped to each other. A frame truss allows forces *and couples* to be applied to the nodes, and its members deform under extension, *bending*, and *torsion*. In this case the corresponding continuum theory has to include couple stresses. It might be interesting to compare the common features of continuous and discrete couple-stress theories.

Also a nonlinear theory of trusses can be formulated from the paradigm of nonlinear elasticity theory. The concept of different *placements* is easily transferred to a truss, and also the ESHELBY stress tensor has its counterpart in the discrete case. A first attempt in this direction has been made by the author in [2].

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*This aspect has been pointed out by Professor ANTONIO DI CARLO in the discussion of the paper at the Torino seminar.

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