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MULTIPLICATION OF SECTIONS OF STABLE VECTOR BUNDLES: THE INJECTIVITY RANGE

Abstract. Let X be a smooth curve of genus g. For any vector bundles E, F on X, let $\mu_{E,F} : H^0(X, E) \otimes H^0(X, F) \to H^0(X, E \otimes F)$ be the multiplication map. Here we study the injectivity of $\mu_{E,F}$ when E, F are general stable bundles with $h^1(X, E) = h^1(X, F) = 0$.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ and E, F vector bundle on X. Let $\mu_{E,F} : H^0(X, E) \otimes H^0(X, F) \to H^0(X, E \otimes F)$ be the multiplication map. When E and F are spanned by their global sections several geometric properties of the pair (E, F) may be translated in terms of the rank of $\mu_{E,F}$. For instance if $E, F \in Pic(X)$, $u := h^0(X, E) - 1$, $v := h^0(X, F) - 1$ and $f : X \to \mathbf{P}^u \times \mathbf{P}^v$ is the map associated to the pair (E, F), the linear map $\mu_{E,F}$ is injective if and only if $f(X)$ spans $\mathbf{P}^t$, $t := uv + u + v$, where $\mathbf{P}^u \times \mathbf{P}^v$ is embedded in $\mathbf{P}^t$ using the Plücker embedding. For other uses of the multiplication map, see [2], [3] and [8]. If $h^1(X, E) = h^1(X, F) = 0$ we have $h^1(X, E \otimes F) = 0$ and hence $h^0(X, E) = deg(E) + rank(E)(1 - g)$ and similarly for F and $E \otimes F$ (Riemann - Roch). Thus if $h^1(X, E) = h^1(X, F) = 0$, the possible pairs $(deg(E), deg(F))$ for which $\mu_{E,F}$ may be injective is quite small (even if E and F are line bundles). For all integers e, f with $e > 0$ the moduli scheme $M(X; e, f)$ of all rank e stable vector bundles on X with degree f is an irreducible smooth variety with $dim(M(X; e, f)) = e2(g-1)+1$. In this paper we work over an arbitrary algebraically closed base field $\mathbf{K}$ and prove the following result.

THEOREM 1. *Let X be a general smooth curve of genus $g \geq 4$. Fix positive integers r, s, x_i, $1 \geq i \geq r$, and y_j, $1 \geq j \geq s$. Assume $x_i y_j \geq g$ for all pairs (i, j). Let E (resp. F) be the general rank r (resp. rank s) stable vector bundle on X with $deg(E) = r(g-1) + x_1 + ... + x_r$(resp. $deg(F) = s(g-1) + y_1 + ... + y_s$). Then $h^1(X, E) = h^1(X, F) = 0$, $h^0(X, E) = x_1 + ... + x_r$, $h^0(X, F) = y_1 + ... + y_s$ and the multiplication map $\mu_{E,F}$ is injective.*

We think that Theorem 1 is quite strong even for non-special line bundles (see Remark 1). Theorem 1 will be proved in section 2 by reduction to the case of line bundles. This case will be proved using Gieseker - Petri theorem for special divisors ([4]). It is the use of this theorem which force us to assume that X has general moduli.

We do not know if the corresponding result is true for arbitrary smooth curves; guess: yes. For the case of nodal curves, see Remark 4.

2. Proof of Theorem 1

First, we will prove the following result, i.e. the case $rank(E) = rank(F) = 1$ of Theorem 1.

PROPOSITION 1. *Let X be a general smooth curve of genus $g \geq 4$. Fix integers x, y with $x \geq 2$, $y \geq 2$ and $xy \leq g$. Set $a := x + g - 1$ and $b := y + g - 1$. Let (L, M) be a general element of $Pic^a(X) \times Pic^b(X)$. Then the multiplication map $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ is injective.*

Proof. Set $d := g - 1 + x - y$. By the existence theorem for special divisors and Gieseker - Petri theorem ([4] or [1], Ch. IV and Ch. VII) there is $R \in Pic^d(X)$ such that $h^0(X, R) = x$ and the multiplication map $\mu_{R,\omega_X \otimes R^*} : H^0(X, R) \otimes H^0(X, \omega_X \otimes R^*) \to H^0(X, \omega_X)$ is injective. By Riemann - Roch and the choice of d we have $h^0(X, \omega_X \otimes R^*) = y$. Take $y + x$ general points of X, say $P_1, \dots, P_y, Q_1, \dots, Q_x$ and set $L' := R(P_1 + ... + P_y)$ and $M' := \omega_X \otimes R^*(Q_1 + \dots + Q_x)$. By the generality of the points P_i and Q_j we have $h^1(X, L') = h^1(X, M') = 0$, $h^0(X, L') = x$ and $h^0(X, M') = y$. The injectivity of $\mu_{R,\omega_X \otimes R^*}$ implies the injectivity of $\mu_{L',M'}$ because $H^0(X, L')$ may be identified (after deleting the base locus) with $H^0(X, R)$, while $H^0(X, M')$ may be identified with a linear subspace of $H^0(X, \omega_X \otimes R^*)$. Hence we conclude by semicontinuity.

□

REMARK 1. Proposition bal:prop2.1 is almost the best a priori possible result. Indeed, by Riemann - Roch the best range in which $\mu L, M$ may be injective is the range $xy \leq g - 1 + x + y$.

REMARK 2. Fix positive integers r, s, a projective curve X and vector bundles $A(i)$, $1 \leq i \leq r$ and $B(j)$, $1 \leq j \leq s$, on X. Set $A := \bigotimes_{1 \leq i \leq r} A(i)$ and $B := \bigotimes_{1 \leq j \leq s} B(j)$. Assume that for every pair (i, j) with $1 \leq i \leq r$ and $1 \leq j \leq s$ the multiplication map $\mu_{A(i),B(j)} : H^0(X, A(i)) \otimes H^0(X, B(j)) \to H^0(X, A(i) \otimes B(j))$ is injective. Then $\mu_{A,B}$ is injective.

REMARK 3. Let X be a smooth projective curve and E a vector bundle on X. Let F be the general vector bundle obtained from E making a positive elementary transformation, i.e. the general vector bundle fitting in an exact sequence

$$0 \to E \to F \to \mathbf{K}^P \to 0$$

with $P \in X$ and $\mathbf{K}^P$ skyscraper sheaf supported by P and with $h^0(X, \mathbf{K}^P) = 1$. Alternatively, $F*$ may be obtained from $E*$ in the following way. Fix any $P \in X$ and consider a general surjection $a : E * \mathbf{K}^P$, i.e. a general linear map $E * |P \to \mathbf{K}$, where $E * |P$ is the fiber of $E*$ over P; then set $F* := Ker(a)$. We have $rank(E) =$

$rank(F) = r$, $deg(F) = deg(E) + 1$ and E is isomorphic to a subsheaf of F. It is easy to check that $h^1(X, F) = max 0, h^1(X, E) - 1$ (see [6], proof of 1.6 at p. 101, for a characteristic free proof). Thus by Riemann - Roch we have $h^0(X, F) = h^0(X, E)$ if $h^1(X, E) > 0$ and $h^0(X, F) = h^0(X, E) + 1$ if $h^1(X, E) = 0$.

Proof. The values for $h^i(X, E)$ and $h^i(X, F)$ are well-known ([6], Cor. 1.7, or [9] or just apply several times Remark 3). Let $A(i)$ (resp. $B(j)$) be the general line bundle of degree $g - 1 + x_i$ (resp. $g - 1 + y_j$). Set $A := \bigotimes_{1 \le i \le r} A(i)$ and $B := \bigotimes_{1 \le j \le s} B(j)$. Notice that $h^i(X, A) = h^i(X, E)$ and $h^i(X, B) = h^i(X, F)$, $i = 0, 1$. By Proposition 1 and Remark 3 the multiplication map $\mu_{A,B}$ is injective. Since $h^1(X, A) = h^1(X, B) = 0$ we may apply semicontinuity and obtain the injectivity of $\mu_{E,F}$ when E (resp F) is a sufficiently general deformation of A (resp. B). Since any vector bundle on X is the flat limit of a family of stable vector bundles ([7], Prop. 2.6, or, in arbitrary characteristic, [5], Cor. 2.2), we conclude.

□

REMARK 4. Fix an integer q with $0 \le q < g$. Let Y be the general curve with $pa(Y) = g$ and exactly q nodes as only singularities. By [4], Prop. 1.2, we may apply the proof of Proposition 1 and hence of Theorem 1 to Y.

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