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ON THE ANTIMAXIMUM PRINCIPLE FOR PARABOLIC PERIODIC PROBLEMS WITH WEIGHT

Abstract. We prove that an antimaximum principle holds for the Neumann and Dirichlet periodic parabolic linear problems of second order with a time periodic and essentially bounded weight function. We also prove that an uniform antimaximum principle holds for the one dimensional Neumann problem which extends the corresponding elliptic case.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{2+\gamma}$ boundary, $0 < \gamma < 1$ and let $\tau > 0$. Let $\{a_{i,j}(x,t)\}_{1\le i,j\le N}$ $\{b_j(x,t)\}_{1\le j\le N}$ and $a_0(x,t)$ be τ-periodic functions in t such that $a_{i,j}, b_j$ and a_0 belong to $C^{\gamma,\gamma/2}\left(\overline{\Omega}\times\mathbb{R}\right)$, $a_{i,j} = a_{j,i}$ for $1 \le i, j \le N$ and $\sum_{i,j} a_{i,j}(x,t)\xi_i\xi_j \ge c\sum_i \xi_i^2$ for some $c > 0$ and all $(x,t) \in \overline{\Omega}\times\mathbb{R}$, $(\xi_1, ..., \xi_N) \in \mathbb{R}^N$. Let L be the periodic parabolic operator given by

$$Lu = \frac{\partial u}{\partial t} - \sum_{i,j} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_j b_j \frac{\partial u}{\partial x_j} + a_0 u \tag{1}$$

Let $B(u) = 0$ denote either the Dirichlet boundary condition $u_{|\partial\Omega\times\mathbb{R}} = 0$ or the Neumann condition $\partial u/\partial\nu = 0$ along $\partial\Omega\times\mathbb{R}$.

Let us consider the problem

$$\left(P_{\lambda,h}\right) \qquad \begin{cases} Lu = \lambda m u + h \text{ in } \Omega\times\mathbb{R}, \\ u \quad \tau - \text{periodic in } t \\ B(u) = 0 \text{ on } \partial\Omega\times\mathbb{R} \end{cases}$$

where the weight function $m = m(x,t)$ is a τ - periodic and essentially bounded function, $h = h(x,t)$ is τ - periodic in t and $h \in L^p(\Omega\times(0,\tau))$ for some $p > N+2$.

We say that $\lambda^* \in \mathbb{R}$ is a *principal eigenvalue* for the weight m if $\left(P_{\lambda^*,h}\right)$ has a positive solution when $h \equiv 0$. The antimaximum principle can be stated as follows:

DEFINITION 1. *We will say that the* antimaximum principle *(AMP) holds to the right (respectively to the left) of a principal eigenvalue λ^* if for each $h \ge 0$, $h \ne 0$ (with*

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$h \in L^p(\Omega \times (0,\tau))$ for some $p > N+2$) there exists a $\delta(h) > 0$ such that $\left(P_{\lambda,h}\right)$ has a negative solution for each $\lambda \in (\lambda^*, \lambda^* + \delta(h))$ (respectively $\lambda \in (\lambda^* - \delta(h), \lambda^*)$).

We prove that, depending on m, these two possibilities happen and that in some cases the AMP holds left and right of λ^*, similarly to the purely stationary case where all data are independent of t (but in that case the period becomes artificial)

Our results are described by means of the real function $\mu_m(\lambda)$, $\lambda \in \mathbb{R}$, defined as the unique $\mu \in \mathbb{R}$ such that the homogeneous problem

$$\left(P_\mu\right) \qquad \begin{cases} Lu - \lambda m u = \mu u \text{ in } \Omega \times \mathbb{R}, \\ u \quad \tau - \text{periodic in } t \\ B(u) = 0 \text{ on } \partial\Omega \times \mathbb{R} \end{cases}$$

has a positive solution.

This function was first studied by Beltramo - Hess in [2] for Hölder continuous weight and Dirichlet boundary condition. They proved that μ_m is a concave and real analytic function, for Neumann the same holds ([8], Lemmas 15.1 and 15.2). A given $\lambda \in \mathbb{R}$ is a principal eigenvalue for the weight m iff μ_m has a zero at λ

We will prove that if μ_m is non constant and if λ^* is a principal eigenvalue for the weight m then the AMP holds to the left of λ^* if $\mu_m'(\lambda^*) > 0$, holds to the right of λ^* if $\mu_m'(\lambda^*) < 0$ and holds right and left of λ^* if $\mu_m'(\lambda^*) = 0$. As a consequence of these results we will give (see section 3, Theorem 1), for the case $a_0 \geq 0$, conditions on m that describe completely what happens respect to the AMP, near each principal eigenvalue.

The notion of AMP is due to Ph. Clement and L. A. Peletier [3]. They proved an AMP to the right of the first eigenvalue for $m = 1$, with all data independent of t and $a_0(x) \geq 0$, i.e. the elliptic case. Hess [7] proves the same, in the Dirichlet case, for $m \in C\left(\overline{\Omega}\right)$. Our aim is to extend these results to periodic parabolic problems covering both cases, Neumann and Dirichlet. In section 2 we give a version of the AMP for a compact family of positive operators adapted to our problem and in section 3 we state the main results.

2. Preliminaries

Let Y be an ordered real Banach space with a total positive cone P_Y with norm preserving order, i.e. $u, v \in Y$, $0 < u \leq v$ implies $\|u\| \leq \|v\|$. Let P_Y° denote the interior of P_Y in Y. We will assume, from now on, that $P_Y^\circ \neq \emptyset$. Its dual Y' is an ordered Banach space with positive cone

$$P' = \left\{y' \in Y' : \left\langle y', y\right\rangle \geq 0 \text{ for all } y \in P\right\}$$

For $y' \in Y'$ we set $y'^{\perp} = \left\{y \in Y : \left\langle y', y\right\rangle = 0\right\}$ and for $r > 0$, $B_r^Y(y)$ will denote the open ball in Y centered at y with radius r. For $v, w \in Y$ with $v < w$ we put (v, w) and $[u, v]$ for the order intervals $\{y \in Y : v < y < w\}$ and $\{y \in Y : v \leq y \leq w\}$

respectively. $B(Y)$ will denote the space of the bounded linear operators on Y and for $T \in B(Y)$, T^* will be denote its adjoint $T^* : Y' \to Y'$.

Let us recall that if T is a compact and strongly positive operator on Y and if ρ is its spectral radius, then, from Krein - Rutman Theorem, (as stated, e.g., in [1], Theorem 3.1), ρ is a positive algebraically simple eigenvalue with positive eigenvectors associated for T and for its adjoint T^*.

We will also need the following result due to Crandall - Rabinowitz ([4], Lemma 1.3) about perturbation of simple eigenvalues:

LEMMA 1. *If T_0 is a bounded operator on Y and if r_0 is an algebraically simple eigenvalue for T_0, then there exists $\delta > 0$ such that $\|T - T_0\| < \delta$ implies that there exists a unique $r(T) \in R$ satisfying $|r(T) - r_0| < \delta$ for which $r(T)I - T$ is singular. Moreover, the map $T \to r(T)$ is analytic and $r(T)$ is an algebraically simple eigenvalue for T. Finally, an associated eigenvector $v(T)$ can be chosen such that the map $T \to v(T)$ is also analytic.*

We start with an abstract formulation of the AMP for a compact family of operators. The proof is an adaptation, to our setting, of those in [3] and [7].

LEMMA 2. *Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a compact family of compact and strongly positive operators on Y. Denote by $\rho(\lambda)$ the spectral radius of T_λ and $\sigma(\lambda)$ its spectrum. Then for all $0 < u \le v$ in Y there exists $\delta_{u,v} > 0$ in $\mathbb{R}$ such that*

$$\left(\rho(\lambda) - \delta_{u,v}, \rho(\lambda)\right) \cap \sigma(\lambda) = \emptyset \text{ and } (\theta I - T_\lambda)^{-1} h < 0$$

uniformly in $h \in [u, v] \subset Y$ and $\theta \in \left(\rho(\lambda) - \delta_{u,v}, \rho(\lambda)\right) \subset \mathbb{R}$.

Proof. We have that

(1) $\rho(\lambda)$ is an algebraically simple eigenvalue for T_λ with a positive eigenvector Φ_λ

(2) T_λ^* has an eigenvector Ψ_λ associated to the eigenvalue $\rho(\lambda)$ such that $\langle \Psi_\lambda, x \rangle > 0$ for all $x \in P - \{0\}$.

(3) Φ_λ normalized by $\|\Phi_\lambda\| = 1$ and Ψ_λ normalized by $\langle \Psi_\lambda, \Phi_\lambda \rangle = 1$ imply that $\left\{(\Phi_\lambda, \Psi_\lambda, \rho(\lambda)) \in Y \times Y' \times \mathbb{R}\right\}$ is compact.

(4) There exists $r > 0$ such that $B_r^Y(0) \subset (-\Phi_\lambda, \Phi_\lambda)$ for all $\lambda \in \Lambda$.

(1) and (2) follow from Krein - Rutman Theorem.

For (3) $\{\rho(\lambda), \lambda \in \Lambda\}$ is compact in $(0, \infty)$ because given $\rho(\lambda_n)$, the sequence T_{λ_n} has a subsequence (still denoted) $T_{\lambda_n} \to T_{\lambda_\infty} \in \{T_\lambda\}_{\lambda \in \Lambda}$ in $B(Y)$. Taking into account (1), Lemma 1 provides an $r > 0$ such that $T \in B(Y)$, $\|T - T_\infty\| < r$ imply that $0 \notin (\rho(\lambda_\infty) - r, \rho(\lambda_\infty) + r) \cap \sigma(T) = \{\rho(T)\}$, so $\rho(\lambda_n) \to \rho(\lambda_\infty) > 0$. This lemma also gives $\{\Phi_\lambda : \lambda \in \Lambda\}$ and $\{\Psi_\lambda : \lambda \in \Lambda\}$ compact in Y and Y' respectively.

(4) follows remarking that $\{\Phi_\lambda, \lambda \in \Lambda\}$ has a lower bound $v \le \Phi_\lambda$, $v \in P_Y^\circ$. Indeed, $\frac{1}{2}\Phi_\lambda \in P_Y^\circ$ so $w - \frac{1}{2}\Phi_\lambda = \frac{1}{2}\Phi_\lambda + w - \Phi_\lambda \in P_Y^\circ$ for $w \in B_{r(\lambda)}(\Phi_\lambda)$ with

$r(\lambda) > 0$. The open covering $\left\{B_{r(\lambda)}(\Phi_\lambda)\right\}$ of $\{\Phi_\lambda : \lambda \in \Lambda\}$ admits a finite subcovering $\left\{B_{r(\lambda_j)}\left(\Phi_{\lambda_j}\right), j = 1, 2, ..., l\right\}$ and it is simple to obtain $r_j \in (0,1)$ such that $\Phi_{\lambda_j} > r_j \frac{1}{2}\Phi_{\lambda_1}$ $j = 1, 2, ..., l$, so $v = r\Phi_{\lambda_1} \leq \frac{1}{2}\Phi_{\lambda_j} < \Phi_\lambda$ for all $\lambda \in \Lambda$ and some j (j depending on λ).

We prove now the Lemma for each λ and $h \in [u, v]$, i.e. we find $\delta_{u,v}(\lambda)$ and we finish by a compactness argument thanks to (1)-(4).

$\Psi_\lambda^\perp$ is a closed subspace of Y and then, endowed with the norm induced from Y, it is a Banach space. It is clear that $\Psi_\lambda^\perp$ is T_λ invariant. and that $T_{\lambda|\Psi_\lambda^\perp} : \Psi_\lambda^\perp \to \Psi_\lambda^\perp$ is a compact operator. Now, $\rho(\lambda)$ is a simple eigenvalue for T_λ with eigenvector Φ_λ and $\Phi_\lambda \notin \Psi_\lambda^\perp$, but $\rho(\lambda) > 0$ and T_λ is a compact operator, thus $\rho(\lambda) \notin \sigma\left(T_{\lambda|\Psi_\lambda^\perp}\right)$.

We have also $Y = \mathbb{R}\Phi_\lambda \bigoplus \Psi_\lambda^\perp$, a direct sum decomposition with bounded projections P_{Φ_λ}, $P_{\Psi_\lambda^\perp}$ given by $P_{\Phi_\lambda} y = \langle \Psi_\lambda, y\rangle \Phi_\lambda$ and $P_{\Psi_\lambda^\perp} y = y - \langle \Psi_\lambda, y\rangle \Phi_\lambda$ respectively. Let $\widetilde{T}_\lambda : Y \to Y$ be defined by $\widetilde{T}_\lambda = T_\lambda P_{\Psi_\lambda^\perp}$ Thus $\widetilde{T}_\lambda$ is a compact operator. Moreover, $\rho(\lambda)$ does not belongs to its spectrum. (Indeed, suppose that $\rho(\lambda)$ is an eigenvalue for $\widetilde{T}_\lambda$, let v be an associated eigenvector. We write $v = P_{\Phi_\lambda} v + P_{\Psi_\lambda^\perp} v$. Then $\rho(\lambda) v = \widetilde{T}_\lambda(v) = T_\lambda P_{\Psi_\lambda^\perp} v$ and so $v \in \Psi_\lambda^\perp$, but $\rho(\lambda) \notin \sigma\left(T_{\lambda|\Psi_\lambda^\perp}\right)$. Contradiction). Thus, for each λ, $\rho(\lambda) I - \widetilde{T}_\lambda$ has a bounded inverse.

Hence, from the compactness of the set $\{\rho(\lambda) : \lambda \in \Lambda\}$ it follows that there exists $\varepsilon > 0$ such that $\theta I - \widetilde{T}_\lambda$ has a bounded inverse for $(\theta, \lambda) \in D$ where

$$D = \{(\theta, \lambda) : \lambda \in \Lambda,\ \rho(\lambda) - \varepsilon \leq \theta \leq \rho(\lambda) + \varepsilon\}$$

and that $\left\|\left(\theta I - \widetilde{T}_\lambda\right)^{-1}\right\|_{B(Y)}$ remains bounded as (λ, θ) runs on D.

But $\theta I - T_{\lambda|\Psi_\lambda^\perp} : \Psi_\lambda^\perp \to \Psi_\lambda^\perp$ has a bounded inverse given by $\left(\theta I - T_{\lambda|\Psi_\lambda^\perp}\right)^{-1} = \left(\left(\theta I - \widetilde{T}_\lambda\right)^{-1}\right)_{|\Psi_\lambda^\perp}$ and so $\left\|\left(\theta I - T_{\lambda|\Psi_\lambda^\perp}\right)^{-1}\right\|_{B\left(\Psi_\lambda^\perp\right)}$ remains bounded as (λ, θ) runs on D.

For $h \in [u, v]$ we set $w_{\lambda,h} = h - \langle \Psi_\lambda, h\rangle \Phi_\lambda$. As $\theta \notin \sigma(T_\lambda)$ we have

$$(2) \qquad (\theta I - T_\lambda)^{-1} h = \frac{\langle \Psi_\lambda, h\rangle}{\theta - \rho(\lambda)}\left[\Phi_\lambda + \frac{\theta - \rho(\lambda)}{\langle \Psi_\lambda, h\rangle}\left((\theta I - T_\lambda)_{|\Psi_\lambda^\perp}\right)^{-1} w_{h,\lambda}\right]$$

and $u - \langle \Psi_\lambda, v\rangle \Phi_\lambda \leq w_{h,\lambda} \leq v - \langle \Psi_\lambda, u\rangle \Phi_\lambda$ that is $\|w_{h,\lambda}\| \leq c_{u,v}$ for some constant $c_{u,v}$ independent of h. Hence $\left((\theta I - T_\lambda)_{|\Psi_\lambda^\perp}\right)^{-1} w_{h,\lambda}$ remains bounded in Y, uniformly on $(\theta, \lambda) \in D$ and $h \in [u, v]$. Also, $\langle \Psi_\lambda, h\rangle \geq \langle \Psi_\lambda, u\rangle$ and since $\{\langle \Psi_\lambda, u\rangle\}$ is compact in $(0, \infty)$ it follows that $\langle \Psi_\lambda, h\rangle \geq c$ for some positive constant c and all $\lambda \in \Lambda$ and all $h \in [u, v]$. Thus the lemma follows from (4).

□

REMARK 1. The conclusion of Lemma 2 holds if (1)-(4) are fulfilled.

We will use the following

COROLLARY 1. *Let $\lambda \to T_\lambda$ be continuous map from $[a,b] \subset \mathbb{R}$ into $B(Y)$. If each T_λ is a compact and strongly positive operator then the conclusion of Lemma 2 holds.*

3. The AMP for periodic parabolic problems

For $1 \leq p \leq \infty$, denote $X = L^p_\tau(\Omega \times \mathbb{R})$ the space of the τ - periodic functions $u : \Omega \times \mathbb{R} \to \mathbb{R}$ (i.e. $u(x,t) = u(x, t+\tau)$ a.e.$(x,t) \in \Omega \times \mathbb{R}$) such whose restrictions to $\Omega \times (0,\tau)$ belong to $L^p(\Omega \times (0,\tau))$. We write also $C^{1+\gamma,\gamma}_{\tau,B}\left(\overline{\Omega} \times \mathbb{R}\right)$ for the space of the τ - periodic Hölder continuous functions u on $\overline{\Omega} \times \mathbb{R}$ satisfying the boundary condition $B(u) = 0$ and $C_\tau\left(\overline{\Omega} \times \mathbb{R}\right)$ for the space of τ - periodic continuous functions on $\overline{\Omega} \times \mathbb{R}$. We set

$$(3) \qquad \begin{array}{ll} Y = C^{1+\gamma,\gamma}_{\tau,B}\left(\overline{\Omega} \times \mathbb{R}\right) & \text{if } B(u) = u_{|\partial\Omega\times\mathbb{R}} \\ \text{and} & \\ Y = C_\tau\left(\overline{\Omega} \times \mathbb{R}\right) & \text{if } B(u) = \partial u/\partial\nu. \end{array}$$

In each case, X and Y, equipped with their natural orders and norms are ordered Banach spaces, and in the first one, Y has compact inclusion into X and the cone P_Y of the positive elements in Y has non empty interior.

Fix $s_0 > \|a_0\|_\infty$. If $s \in (s_0, \infty)$, the solution operator S of the problem

$$Lu + su = f \text{ on } \Omega \times \mathbb{R},\ B(u) = 0,\ u\ \tau - \text{periodic}, \qquad f \in Y$$

defined by $Sf = u$, can be extended to an injective and bounded operator, that we still denote by S, from X into Y (see [9], Lemma 3.1). This provides an extension of the original differential operator L, which is a closed operator from a dense subspace $D \subset Y$ into X (see [9], p. 12). From now on L will denote this extension of the original differential operator.

If $a \in L^\infty_\tau(\Omega \times \mathbb{R})$ and $\delta_1 \leq a + a_0 \leq \delta_2$ for some positive constants δ_1 and δ_2, then $L + aI : X \to Y$ has a bounded inverse $(L + aI)^{-1} : X \to C^{1+\gamma,\gamma}_B\left(\overline{\Omega} \times \mathbb{R}\right) \subset Y$, i.e.

$$(4) \qquad \left\|(L + aI)^{-1} f\right\|_{C^{1+\gamma,\gamma}\left(\overline{\Omega}\times\mathbb{R}\right)} \leq c\,\|f\|_{L^p_\tau(\Omega\times\mathbb{R})}$$

for some positive constant c and all f ([9], Lemma 3.1). So $(L + aI)^{-1} : X \to X$ and its restriction $(L + aI)^{-1}_{|Y} : Y \to Y$ are compact operators. Moreover, $(L + aI)^{-1}_{|Y} : Y \to Y$ is a strongly positive operator ([9], Lemma 3.7).

If $\partial a_{i,j}/\partial x_j \in C\left(\overline{\Omega} \times \mathbb{R}\right)$ for $1 \leq i, j \leq N$, we recall that for $f \in L^p_\tau(\Omega \times \mathbb{R})$,

$(L + aI)^{-1} f$ is a weak solution of the periodic problem

$$\frac{\partial u}{\partial t} - div(A\nabla u) + \sum_j \left(b_j + \sum_i \frac{\partial a_{i,j}}{\partial x_i} \right) \frac{\partial u}{\partial x_j} + (a + a_0) u = f \text{ on } \Omega \times \mathbb{R},$$

$$B(u) = 0 \text{ on } \partial\Omega \times \mathbb{R}$$

$$u(x, t) = u(x, t + \tau)$$

where A is the $N \times N$ matrix whose i, j entry is $a_{i,j}$ (weak solutions defined as, e.g., in [10], taking there, the test functions space adapted to the periodicity and to the respective boundary condition) In fact, this is true for a Hölder continuous f (classical solutions are weak solutions) and then an approximation process, using that L is closed and (4), gives the assertion for a general f.

REMARK 2. Let $m \in L^\infty_\tau(\Omega \times \mathbb{R})$ and let $M : X \to X$ be the operator multiplication by m. Then for each $\lambda \in \mathbb{R}$ there exists a unique $\mu \in \mathbb{R}$ such that the problem (P_μ) from the introduction has a positive solution.

This is shown for $\lambda > 0$, $a_0 \geq 0$ in [9] (see Remark 3.9 and Lemma 3.10). A slight modification of the argument used there shows that this is true for $\lambda \in \mathbb{R}$, $a_0 \in L^\infty_\tau(\Omega \times \mathbb{R})$ (start with $L + r$ instead of L, with $r \in \mathbb{R}$ large enough). Thus $\mu_m(\lambda)$ is well defined for all $\lambda \in \mathbb{R}$, μ_m is a concave function, $\mu_m(\lambda)$ is real analytic in λ, $\mu_m(\lambda)$ is an M simple eigenvalue for L and

$$\mu_m(\lambda) = 0 \text{ if and only if } \lambda \text{ is a principal eigenvalue for the weight } m.$$

Moreover, the positive solution u_λ of $(P_{\mu_m(\lambda)})$ can be chosen real analytic in λ (as a map from $\mathbb{R}$ into Y). As in the case m Hölder continuous we have $\mu_m(-\lambda) = \mu_{-m}(\lambda)$, $\lambda \in \mathbb{R}$. We recall also that for the Dirichlet problem with $a_0 \geq 0$ (and also for the Neumann problem with $a_0 \geq 0$, $a_0 \neq 0$) we have $\mu_m(0) > 0$ (see [8], also [5] and [6]).

Given $\lambda \in \mathbb{R}$, we will say that the maximum principle (in brief MP) holds for λ if λ is not an eigenvalue for the weight m and if $h \in X$ with $h \geq 0$, $h \neq 0$ implies that the solution u_λ of the problem $(P_{\lambda,h})$ belongs to P°_Y.

The function μ_m describes what happens, with respect to the MP, at a given $\lambda \in \mathbb{R}$ (for the case m Hölder continuous see [8], Theorem 16.6):

$$\mu_m(\lambda) > 0 \text{ if and only if } \lambda \text{ is not an eigenvalue and } MP \text{ holds for } \lambda$$

Indeed, for $h \in X$ with $h \geq 0$, $h \neq 0$, for $r \in \mathbb{R}$ large enough such that $-\|a_0\|_\infty - \|\lambda m\|_\infty + r > 0$, problem $(P_{\lambda,h})$ is equivalent to $(r^{-1}I - S_\lambda) u = H_\lambda$ with $S_\lambda = (L + r - \lambda M)^{-1}$ and $H_\lambda = r^{-1} S_\lambda h$. Now $H_\lambda > 0$. Also $\mu_m(\lambda) > 0$ if and only if $\widetilde{\rho}(\lambda) < r^{-1}$, where $\widetilde{\rho}(\lambda)$ is the spectral radius of S_λ so Krein - Rutman Theorem ensures, for such a u, that $\mu_m(\lambda) > 0$ is equivalent to $u \in P^\circ_Y$. Moreover, $\mu_m(\lambda) > 0$ implies also that λ is not an eigenvalue for the weight m, since, if λ would be an eigenvalue with an associated eigenfunction Φ and if u_λ is a positive solution of $Lu =$

$\lambda mu + \mu_m(\lambda)\, u,\ B(u) = 0$, then, for a suitable constant $c,\ v = u_\lambda + c\Phi$ would be a solution negative somewhere for the problem $Lv = \lambda mv + \mu_m(\lambda)\, u_\lambda,\ B(v) = 0$.

Next theorem shows that μ_m also describes what happens, with respect to the AMP, near to a principal eigenvalue.

THEOREM 1. *Let L be the periodic parabolic operator given by (1) with coefficients satisfying the conditions stated there, let $B(u) = 0$ be either the Dirichlet condition or the Neumann condition, consider Ygiven by (3), let m be a function in $L^\infty_\tau(\Omega \times \mathbb{R})$ and let λ^* be a principal eigenvalue for the weight m. Finally, let $u, v \in L^p_\tau(\Omega \times \mathbb{R})$ for some $p > N + 2$, with $0 < u \le v$. Then*

- *(a) If $\lambda \to \mu_m(\lambda)$ vanishes identically, then for all $\lambda \in R$ and all $h \ge 0,\ h \ne 0$ in $L^p_\tau(\Omega \times \mathbb{R})$, problem $(P_{\lambda,h})$ has no solution.*
- *(b) If $\mu'_m(\lambda^*) < 0$ (respectively $\mu'_m(\lambda^*) > 0$), then the AMP holds to the right of λ^* (respectivley to the left) and its holds uniformly on $h \in [u, v]$, i.e. there exists $\delta_{u,v} > 0$ such that for each $\lambda \in (\lambda^*, \lambda^* + \delta_{u,v})$ (respectively $\lambda \in (\lambda^* - \delta_{u,v}, \lambda^*)$) and for each $h \in [u, v]$, the solution $u_{\lambda,h}$ of $(P_{\lambda,h})$ satisfies $u_{\lambda,h} \in -P^\circ_Y$.*
- *(c) If $\mu'_m(\lambda^*) = 0$ and if μ_m does not vanishes identically, then the AMP holds uniformly on h for $h \in [u, v]$ right and left of λ^*, i.e. there exists $\delta_{u,v} > 0$ such that for $0 < |\lambda - \lambda^*| < \delta_{u,v},\ h \in [u, v]$, the solution $u_{\lambda,h}$ of $(P_{\lambda,h})$ is in $-P^\circ_Y$.*

Proof. Let $M : X \to X$ be the operator multiplication by m. Given a closed interval I around λ^* we choose $r \in (0, \infty)$ such that $r > \lambda^* \mu'_m(\lambda^*)$ and $r - \|\lambda m\|_\infty - \|a_0\|_\infty > 0$ for all $\lambda \in I$. For a such r and for $\lambda \in I$, let $T_\lambda : Y \to Y$ defined by

$$T_\lambda = (L + rI)^{-1}(\lambda M + rI)$$

so each T_λ is a strongly positive and compact operator on Y with a positive spectral radius $\rho(\lambda)$ that is an algebraically simple eigenvalue for T_λ and T^*_λ. Let $\Phi_\lambda, \Psi_\lambda$ be the corresponding positive eigenvectors normalized by $\|\Phi_\lambda\| = 1$ and $\langle \Psi_\lambda, \Phi_\lambda \rangle = 1$. By Lemma 1, $\rho(\lambda)$ is real analytic in λ and $\Phi_\lambda, \Psi_\lambda$ are continuous in λ. As a consequence of Krein - Rutman, we have that $\rho(\lambda) = 1$ iff λ is a principal eigenvalue for the weight m. So $\rho(\lambda^*) = 1$. Since T_λ is strongly positive we have $\Phi_\lambda \in P^\circ_Y$, so there exists $s > 0$ such that $B^Y_s(0) \subset (-\Phi_\lambda, \Phi_\lambda)$ for all $\lambda \in I$. Let $H = (L + r)^{-1} h,\ U = (L + r)^{-1} u$ and $V = (L + r)^{-1} v$. The problem $Lu_\lambda = \lambda mu_\lambda + h$ on $\Omega \times \mathbb{R},\ B(u_\lambda) = 0$ on $\partial\Omega \times \mathbb{R}$ is equivalent to

$$u_\lambda = (I - T_\lambda)^{-1} H \tag{5}$$

and $u \le h \le v$ implies $U \le H \le V$. So, we are in the hypothesis of our Lemma 2 and from its proof we get that $\left\| \left(\left(\rho(\lambda) I - T_{\lambda|\Psi^\perp_\lambda} \right) \right)^{-1} \right\|$ remains bounded for λ near to λ^* and from (2) with $\theta = \rho(\lambda^*) = 1$ we obtain

$$u_\lambda = \frac{\langle \Psi_\lambda, H \rangle}{1 - \rho(\lambda)} \left[\Phi_\lambda + \frac{1 - \rho(\lambda)}{\langle \Psi_\lambda, H \rangle} \left((I - T_\lambda)_{|\Psi^\perp_\lambda} \right)^{-1} w_{H,\lambda} \right] \tag{6}$$

where $w_{H,\lambda} = H - \langle \Psi_\lambda, H \rangle \Phi_\lambda$.

$T_\lambda \Phi_\lambda = \rho(\lambda) \Phi_\lambda$ is equivalent to $L\Phi_\lambda = \frac{\lambda}{\rho(\lambda)} m \Phi_\lambda + r\left(\frac{1}{\rho(\lambda)} - 1\right)\Phi_\lambda$ and, since $\Phi_\lambda > 0$ this implies

$$\mu_m\left(\frac{\lambda}{\rho(\lambda)}\right) = r\left(\frac{1}{\rho(\lambda)} - 1\right). \tag{7}$$

If μ_m vanishes identically then $\rho(\lambda) = 1$ for all λ and (5) has no solution for all $h \geq 0$, $h \neq 0$. This gives assertion (a) of the theorem.

For (b) suppose that $\mu'_m(\lambda^*) \neq 0$. Taking the derivative in (7) at $\lambda = \lambda^*$ and recalling that $\rho(\lambda^*) = 1$ we obtain

$$\mu'_m(\lambda^*)\left(1 - \lambda^* \rho'(\lambda^*)\right) = -r\rho'(\lambda^*)$$

so $\rho'(\lambda^*) = \mu'_m(\lambda^*) / \left(\lambda^* \mu'_m(\lambda^*) - r\right)$. We have chosen $r > \lambda^* \mu'_m(\lambda^*)$, thus $\mu'_m(\lambda^*) > 0$ implies $\rho'(\lambda^*) < 0$ and $\mu'_m(\lambda^*) < 0$ implies $\rho'(\lambda^*) > 0$ and then, proceeding as at the end of the proof of Lemma 2, assertion (b) of the theorem follows from (6).

If $\mu'_m(\lambda^*) = 0$, since μ_m is concave and analytic we have $\mu_m(\lambda) < 0$ for $\lambda \neq \lambda^*$. Then $1/\rho$ has a local maximum at λ^* and (c) follows from (6) as above.

□

To formulate conditions on m to fulfill the assumptions of Theorem 1 we recall the quantities

$$P(m) = \int_0^\tau ess\ sup_{x\in\Omega} m(x,t)\,dt, \qquad N(m) = \int_0^\tau ess\ inf_{x\in\Omega} m(x,t)\,dt.$$

The following two theorems describe completely the possibilities, with respect to the AMP, in Neumann and Dirichlet cases with $a_0 \geq 0$.

THEOREM 2. *Let L be given by* (1). *Assume that either $B(u) = 0$ is the Neumann condition and $a_0 \geq 0$, $a_0 \neq 0$ or that $B(u) = 0$ is the Dirichlet condition and $a_0 \geq 0$. Assume in addition that $\partial a_{i,j}/\partial x_j \in C\left(\overline{\Omega} \times \mathbb{R}\right)$, $1 \leq i, j \leq N$. Then*

(1) *If $P(m) > 0$ $(P(m) \leq 0)$, $N(m) \geq 0$ $(N(m) < 0)$ then there exists a unique principal eigenvalue λ^* that is positive (negative) and the AMP holds to the right (to the left) of λ^**

(2) *If $P(m) > 0$, $N(m) < 0$ then there exist two principal eigenvalues $\lambda_{-1} < 0$ and $\lambda_1 > 0$ and the AMP holds to the right of λ_1 and to the left of λ_{-1}.*

(3) *If $P(m) =$ then $N(m) = 0$ then there are no principal eigenvalues.*

Moreover if $u, v \in L^p_\tau(\Omega \times \mathbb{R})$ satisfy $0 < u < v$, then in (1) *and* (2) *the AMP holds uniformly on h for $h \in [u, v]$.*

Proof. We consider first the Dirichlet problem. If $P(m) > 0$ and $N(m) \geq 0$ then there exists a unique principal eigenvalue λ_1 that is positive ([9]). Since $\mu_m(0) > 0$, $\mu_m(\lambda_1) = 0$ and μ is concave, we have $\mu'_m(\lambda_1) < 0$ and (b) of Theorem 1 applies. If $P(m) > 0$ and $N(m) < 0$ then there exist two eigenvalues $\lambda_{-1} < 0$ and $\lambda_1 > 0$ because $\mu_{-m}(-\lambda) = \mu_m(\lambda)$ and in this case (μ_m is concave) we have $\mu'_m(\lambda_{-1}) > 0$ and $\mu'_m(\lambda_1) < 0$, so Theorem 1 applies. In each case Theorem 1 gives the required uniformity. The other cases are similar. If $P(m) = N(m) = 0$ then $m = m(t)$ and $\mu_m(\lambda) \equiv \mu_m(0) > 0$ ([9]). So (3) holds. Results in [9] for the Dirichlet problem remain valid for the Neumann condition with $a_0 \geq 0$, $a_0 \neq 0$, so the above proof holds.

□

REMARK 3. If $a_0 = 0$ in the Neumann problem then $\lambda_0 = 0$ is a principal eigenvalue and $\mu_m(0) = 0$. To study this case we recall that $(L+1)^{-1*}$ has a positive eigenvector $\Psi \in X' \subset Y'$ provided by the Krein - Rutman Theorem and (4). Then $\mu'_m(0) = -\langle \Psi, m\rangle / \langle \Psi, 1\rangle$ where $\langle \Psi, m\rangle = \int_{\Omega\times(0,\tau)} \psi m$ makes sense because $\Psi \in L^{p'}(\Omega \times (0,\tau))$ ([9], remark 3.8). Indeed, let u_λ be a positive τ - periodic solution of $Lu_\lambda = \lambda m u_\lambda + \mu_m(\lambda) u_\lambda$ on $\Omega \times \mathbb{R}$, $B(u_\lambda) = 0$ with u_λ real analytic in λ and with $u_0 = 1$. Since Ψ vanishes on the range of L we have $0 = \lambda \langle \Psi, m u_\lambda\rangle + \mu_m(\lambda) \langle \Psi, u_\lambda\rangle$. Taking the derivative at $\lambda = 0$ and using that $u_0 = 1$ we get the above expression for $\mu'_m(0)$.

THEOREM 3. *Let L be given by (1). Assume that $B(u) = 0$ is the Neumann condition and that $a_0 = 0$. Assume in addition that $\partial a_{i,j}/\partial x_j \in C(\overline{\Omega} \times \mathbb{R})$, $1 \leq i, j \leq N$. Let Ψ be as in Remark 3.*

Then, if m is not a function of t alone, we have

(1) *If $\langle \Psi, m\rangle < 0$ ($\langle \Psi, m\rangle > 0$), $P(m) \leq 0$ ($N(m) \geq 0$), then 0 is the unique principal eigenvalue and the AMP holds to the left (to the right) of 0.*

(2) *If $\langle \Psi, m\rangle < 0$ ($\langle \Psi, m\rangle > 0$), $P(m) > 0$ ($N(m) < 0$), then there exists two principal eigenvalues, 0 and λ^* wich is positive (negative) and the AMP holds to the left (to the right) of 0 and to the right (to the left) of λ^*.*

(3) *If $\langle \Psi, m\rangle = 0$,hen 0 is the unique principal eigenvalue and the AMP holds left and right of 0.*

If $m = m(t)$ is a function of t alone, then we have

(1') *If $\int_0^\tau m(t)\,dt = 0$ then for all $\lambda \in \mathbb{R}$ the above problem $Lu = \lambda m u + h$ has no solution.*

(2') *If $\int_0^\tau m(t)\,dt \neq 0$ and $\langle \Psi, m\rangle > 0$ ($\langle \Psi, m\rangle < 0$) then 0 is the unique principal eigenvalue and the AMP holds to the right (to the left) of 0.*

Moreover, if $u, v \in X$ satisfy $0 < u < v$, then in each case (except (1')) the AMP holds uniformly on h for $h \in [u, v]$.

Proof. Suppose that m is not a function of t alone. If $\langle \Psi, m\rangle < 0,\ P(m) \le 0$ then 0 is the unique principal eigenvalue and $\mu_m'(0) > 0$. If $\langle \Psi, m\rangle < 0,\ P(m) > 0$ then there exist two principal eigenvalues: 0 and some $\lambda_1 > 0$ and since μ_m is concave we have $\mu_m'(0) > 0$ and $\mu_m'(\lambda_1) < 0$. If $\langle \Psi, m\rangle = 0$ and if m is not funtion of t alone, then μ_m is not a constant and $\mu_m'(0) = 0$ and 0 is the unique principal eigenvalue. In each case, the theorem follows from Theorem 1. The other cases are similar.

If m is a function of t alone then $\mu_m(\lambda) = -\frac{P(m)}{\tau}\lambda$, this implies $\frac{P(m)}{\tau} = \langle \Psi, m\rangle / \langle \Psi, 1\rangle$. If $\int_0^\tau m(t)\,dt = 0$ then $\mu_m = 0$ and (a) of Theorem 1 applies. If $\int_0^\tau m(t)\,dt \neq 0$ and $\langle \Psi, m\rangle > 0$ then $P(m) = N(m) > 0$ so 0 is the unique principal eigenvalue and $\mu_m'(0) < 0$, in this case Theorem 1 applies also. The case $\langle \Psi, m\rangle < 0$ is similar. The remaining case $\int_0^\tau m(t)\,dt \neq 0$ and $\langle \Psi, m\rangle = 0$ is impossible because $\frac{P(m)}{\tau} = \langle \Psi, m\rangle / \langle \Psi, 1\rangle$.

□

For one dimensional Neumann problems, similarly to the elliptic case, a uniform AMP holds.

THEOREM 4. *Suppose $N = 1$, $\Omega = (\alpha, \beta)$ and the Neumann condition. Let L be given by $Lu = u_t - au_{xx} + bu_x + a_0 u$, where $a_0, b \in C_\tau^{\gamma,\gamma/2}\left(\overline{\Omega}\times\mathbb{R}\right)$, $a_0 \ge 0$ and with $a \in C_\tau^1\left(\overline{\Omega}\times\mathbb{R}\right)$, $\min_{x\in\overline{\Omega}\times\mathbb{R}} a(x,t) > 0$. Then the AMP holds uniformly in h (i.e. holds on an interval independent of h) in each situation considered in Theorem 3.*

Proof. Let λ^* be a principal eigenvalue for $Lu = \lambda m u$. Without loss of generality we can assume that $\|h\|_p = 1$ and that the AMP holds to the right of λ^*. Denote M the operator multiplication by m. Let I_{λ^*} be a finite closed interval around λ^* and, for $\lambda \in I_{\lambda^*}$, let T_λ, Φ_λ and Ψ_λ be as in the proof of Theorem 1. Each Φ_λ belongs to the interior of the positive cone in $C\left(\overline{\Omega}\right)$ and $\lambda \to \Phi_\lambda$ is a continuous map from I_{λ^*} into $C\left(\overline{\Omega}\right)$, thus there exist positive constants c_1, c_2 such that

$$c_1 \le \Phi_\lambda(x) \le c_2 \tag{8}$$

for all $\lambda \in I_{\lambda^*}$ and $x \in \overline{\Omega}$. As in the proof of Lemma 2 we obtain (6). Taking into account (8) and that $\left\|\left((I - T_\lambda)_{|\Psi_\lambda^\perp}\right)^{-1}\right\|$ remains bounded for λ near λ^*, in order to prove our theorem, it is enough to see that there exist a positive constant c independent of h, such that $\left\|\left(I - T_{\lambda|\Psi_\lambda^\perp}\right)^{-1} w_{H,\lambda}\right\|_\infty / \langle \Psi_\lambda, H\rangle < c$ for $\lambda \in I_{\lambda^*}$. Since

$$\frac{\left(I - T_{\lambda|\Psi_\lambda^\perp}\right)^{-1} w_{H,\lambda}}{\langle \Psi_\lambda, H\rangle} = \left(I - T_{\lambda|\Psi_\lambda^\perp}\right)^{-1}\left(\frac{H}{\langle \Psi_\lambda, H\rangle} - \Phi_\lambda\right),$$

it suffices to prove that there exists a positive constant c such that

$$\|H\|_\infty \le c\,\langle \Psi_\lambda, H\rangle \tag{9}$$

for all $h \geq 0$ with $\|h\|_p = 1$. To show (9) we proceed by contradiction. If (9) does not holds, we would have for all $j \in \mathbb{N}$, $h_j \in L^p_\tau(\Omega \times \mathbb{R})$ with $h_j \geq 0$, $\|h_j\|_p = 1$ and $\lambda_j \in I_{\lambda^*}$ such that

$$\left\langle \Psi_{\lambda_j}, (L + rI)^{-1} h_j \right\rangle < \frac{1}{j} \left\| (L + rI)^{-1} h_j \right\|_\infty . \tag{10}$$

Thus $\lim_{j\to\infty} \langle \Psi_{\lambda_j}, (L + rI)^{-1} h_j \rangle = 0$. We claim that

$$\left\| (L + rI)^{-1} h \right\|_\infty / \min_{\overline{\Omega}\times[0,\tau]} (L + rI)^{-1} h \leq c \tag{11}$$

for some positive constant c and all nonnegative and non zero $h \in L^p_\tau(\Omega \times \mathbb{R})$. If (11) holds, then

$$\left\langle \Psi_{\lambda_j}, (L + rI)^{-1} h_j \right\rangle \geq \langle \Psi_{\lambda_j}, 1 \rangle \min_{\overline{\Omega}\times[0,\tau]} (L + rI)^{-1} h_j \geq$$

$$\geq c'c \left\| (L + rI)^{-1} h_j \right\|_\infty \langle \Psi_{\lambda_j}, \Phi_{\lambda_j} \rangle = cc' \left\| (L + rI)^{-1} h_j \right\|_\infty$$

for some positive constant c' independent of j, contradicting (10).

It remains to prove (11) which looks like an elliptic Harnack inequality. We may suppose $\alpha = 0$. Extending $u := (L + rI)^{-1} h$ by parity to $[-\beta, \beta]$ we obtain a function $\widetilde{u}$ with $\widetilde{u}(-\beta, t) = \widetilde{u}(\beta, t)$, so we can assume that $\widetilde{u}$ is 2β - periodic in x and τ periodic in t. $\widetilde{u}$ solves weakly the equation $\widetilde{u}_t - \widetilde{a}\widetilde{u}_{xx} + \widetilde{b}\widetilde{u}_x + (a_0 + r)\widetilde{u} = \widetilde{h}$ in $\mathbb{R} \times \mathbb{R}$ where $\widetilde{a}$, $\widetilde{a}_0$, $\widetilde{h}$ are extensions to $\mathbb{R} \times \mathbb{R}$ like $\widetilde{u}$, but b is extended to an odd function $\widetilde{b}$ in $(-\beta, \beta)$ then 2β periodically. In spite of discontinuities of $\widetilde{b}$, a parabolic Harnack inequality holds for $\widetilde{u} = \widetilde{u}(\widetilde{h})$ ([10], Th. 1.1) and using the periodicity of $\widetilde{u}$ in t, we obtain (11).

□

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