# Quantum Serre Relations

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**Abstract**

Let $\mathbb{Z}I$ be the free abelian group with basis $I$, let $\chi$ be a pair of integral bilinear forms on $\mathbb{Z}I$. We will endow the free $K$-algebra $K\langle I\rangle$ generated by $I$ with a comultiplication which depends on $\chi$. This yields an associated bilinear form on $K\langle I\rangle$ which may be called the Drinfeld form. We are going to show that certain elements of $K\langle I\rangle$ which are similar to the well-known quantum Serre relations belong to the left radical of the Drinfeld form, provided certain integrality conditions are satisfied.

**Résumé**

Soit $\mathbb{Z}I$ le group abélien libre de base $I$ et soit $\chi$ un couple de formes bilinéaires entières sur $\mathbb{Z}I$. Nous allons munir la $K$-algèbre libre $K\langle I\rangle$ engendrée par $I$ d'un coproduit qui dépend de $\chi$. On obtient alors une forme bilinéaire associée sur $K\langle I\rangle$ que l'on peut appeler forme de Drinfeld. Sous l'hypothèse de certaines conditions d'intégralité, nous mettons en évidence certains éléments de $K\langle I\rangle$ semblables aux relations de Serre quantiques bien connues et nous montrons qu'ils appartiennent au radical à gauche de la forme de Drinfeld.

Serre exhibited a presentation of the finite dimensional semisimple complex Lie algebras by generators and relations in 1966. This presentation uses only the data given by the Cartan matrix of the Lie algebra. Of great importance was the introduction of correspondingly defined Lie algebras by Kac and Moody in 1968, starting with a generalized Cartan matrix or, what is equivalent, with a suitable bilinear form on a free abelian group $\mathbb{Z}I$, where $I$ is some finite set. Presentations of a Lie algebra $\mathbf{g}$ by generators and relations are also presentations of the universal enveloping algebra $U(\mathbf{g})$; of course, one has to rewrite the Lie bracket operation in terms of commutators.

The quantum groups as investigated by Drinfeld and Jimbo in 1985 are associative algebras which are presented by relations similar to the Serre relations used in the definition of a Kac-Moody algebra. Here, an additional parameter, say $v$, is involved: the binomial coefficients are replaced by the corresponding Gauß polynomials in the variable $v$. Even the most enthusiastic mathematician could not have foreseen the

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large variety of connections which have been encountered in the meantime relating these algebras and quite different areas of mathematics and physics.

Now, let us consider in more detail the positive part $U^+$ of such a quantum group $U$. Using the representation theory of finite dimensional hereditary algebras one can realize $U^+$ in different ways: as a sort of Grothendieck ring of perverse sheaves [L] or as a twisted generic composition algebra [G]. Closely related algebras such as Hall algebras [R1,G] should be treated at the same time. Lusztig [L] has stressed the importance of algebraic structures $A$ which are similar to bialgebras, but where multiplication and comultiplication are compatible only up to a twist on $A \otimes A$. Extending this approach we may start with a pair $\chi$ of bilinear forms on the free abelian group $\mathbb{Z}I$ and consider what we call $(K, v, \chi)$-bialgebras. Algebras such as $U^+$, the Hall algebras, and also the free $K$-algebra $K\langle I\rangle$ generated by $I$ can then be regarded as $(K, v, \chi)$-bialgebras. Recall that $U^+$ can be defined as the factor algebra of $K\langle I\rangle$ modulo the radical of the Drinfeld form, which is a bilinear form uniquely determined by the $(K, v, \chi)$-bialgebra structure of $K\langle I\rangle$.

The present note will consider the Drinfeld form in the general case where $\chi$ is an arbitrary pair of bilinear forms. Elements of $K\langle I\rangle$ which are similar to quantum Serre relations will be shown to belong to the left radical of the Drinfeld form, provided certain integrality conditions are satisfied. So we recover the result that $U^+$, defined as the factor algebra of $K\langle I\rangle$ modulo the radical of the Drinfeld form, satisfies the quantum Serre relations ([L], Proposition 1.4.3). The general form of the quantum Serre relations as exhibited below occurs quite naturally as the fundamental relations of the Hall algebras [R2]. Similar considerations produce elements in the right radical of the Drinfeld form.

# 1 $(K, v, \chi)$-bialgebras

The data which are given are as follows: Let $K$ be a commutative ring and $v \in K$ an invertible element. Let $I$ be a set. We denote by $\mathbb{Z}I$ the free abelian group with basis $I$, and $\chi', \chi''$ are two bilinear forms on $\mathbb{Z}I$ with values in $\mathbb{Z}$.

Let $A = \bigoplus_{x \in \mathbb{Z}I} A_x$ be a $\mathbb{Z}I$-graded $K$-algebra. Given a bilinear form $\phi$ on $\mathbb{Z}I$ and an invertible element $v \in K$, we denote by $A_{[v,\phi]}$ the algebra obtained from $A$ by twisting the muliplication using $v^\phi$; to be precise: the new multiplication $*$ is defined on the same underlying $K$-module $\bigoplus_{x \in \mathbb{Z}I} A_x$ by

$$a * b = v^{\phi(x,y)}\, ab,$$

where $a, b$ are homogeneous elements of degree $x, y \in \mathbb{Z}I$ respectively. Note that $A_{[v,\phi]}$ is again a $K$-algebra [R3], and of course also $\mathbb{Z}I$-graded. Let $A$ be a $\mathbb{Z}I$-

graded algebra. Given the pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$ and $v \in K$, we consider a corresponding map $(\mathbb{Z}I)^4 \to \mathbb{Z}$, which we also denote by $\chi$ and which is defined as follows:

$$\chi(x_1, x_2, x_3, x_4) = \chi'(x_1, x_4) + \chi''(x_2, x_3).$$

Note that this map $\chi$ is a bilinear form on $(\mathbb{Z}I)^2$ (the maps $(\mathbb{Z}I)^4 \to \mathbb{Z}$ obtained in this way may be characterized by certain bilinearity properties, see [R4]). We may consider $A \otimes A$ as a $(\mathbb{Z}I)^2$-graded algebra, where for $x, y \in \mathbb{Z}I$, we have $(A \otimes A)_{(x,y)} = A_x \otimes A_y$. Thus, given a pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$, we may use the bilinear form $\chi$ on $(\mathbb{Z}I)^2$ in order to twist the multiplication of $A \otimes A$ and we obtain in this way the algebra $(A \otimes A)_{[v,\chi]}$ with multiplication

$$(a_1 \otimes a_2) * (b_1 \otimes b_2) = v^{\chi'(a_1,b_2)+\chi''(a_2,b_1)}\, a_1 b_1 \otimes a_2 b_2.$$

Here we see in which way the two bilinear forms are used: the form $\chi''$ draws attention to the interchange of the inner elements $a_2, b_1$, whereas the form $\chi'$ is called in for the outer elements $a_1$ and $b_2$.

By definition, a $(K, v, \chi)$-*bialgebra* is of the form $A = (A, \mu, \delta)$, where $A$ is a $K$-module with a direct decomposition $A = \bigoplus_{x \in \mathbb{Z}I} A_x$, such that $(A, \mu)$ is a $\mathbb{Z}I$-graded algebra, $(A, \delta)$ is a $\mathbb{Z}I$-graded coalgebra and such that on the one hand, the counit $\epsilon$ satisfies $\epsilon(1) = 1$, and, on the other hand,

$$\delta\colon A \to (A \otimes A)_{[v,\chi]}$$

is an algebra homomorphism.

## 2 The free $K$-algebra $K\langle I \rangle$ as $(K, v, \chi)$-bialgebra.

Let $F = K\langle I \rangle$ be the free $K$-algebra generated by $I$; we may consider it as the semigroup algebra of the free semigroup $\langle I \rangle$ generated by $I$. The generator corresponding to $i \in I$ will be denoted by $\theta_i$. Thus, the elements of $\langle I \rangle$ are words in the letters $\theta_i$ ($i \in I$): there is the empty word which is denoted by 1, and there are the words $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_n}$ of length $n \geq 1$, with $i_1, i_2, \ldots, i_n \in I$. The multiplication in $\langle I \rangle$ is just the concatenation of words. We consider $F$ as a $\mathbb{Z}I$-graded algebra, with the generator $\theta_i$ being of degree $i$ and we denote by $\epsilon\colon F \to F_0 = K$ the canonical projection. In order to define a comultiplication $\delta$, we consider the algebra $(F \otimes F)_{[v,\chi]}$. Let $\delta$ be the algebra homomorphism $F \to (F \otimes F)_{[v,\chi]}$ defined by

$$\delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

With this comultiplication $\delta$ the $K$-algebra $F$ becomes a $(K, v, \chi)$-bialgebra, see [R4]; its counit is $\epsilon$.

Given a $\mathbb{Z}I$-graded $K$-module $A = \bigoplus_{x\in\mathbb{Z}I} A_x$, a bilinear form $\langle -,-\rangle : A\otimes A \to K$ is said to respect the grading provided we have $\langle a,b\rangle = 0$ for $a \in A_x,\ b \in A_y$ and $x \neq y$ in $\mathbb{Z}I$.

***Proposition 2.1*** — *There exists a unique bilinear form $\langle -,-\rangle$ on the $(K,v,\chi)$-bialgebra $F = K\langle I\rangle$ with the following properties:*

(0) *The bilinear form $\langle -,-\rangle$ respects the grading.*

(1) *We have $\langle \theta_i,\theta_i\rangle = 1$ for all $i \in I$,*

(2) *If $a, b_1, b_2 \in F$ and $\delta(a) = \sum a_{i1} \otimes a_{i2}$, with $a_{i1}, a_{i2} \in F$, then*

$$\langle a, b_1 b_2\rangle = \sum \langle a_{i1}, b_1\rangle\langle a_{i2}, b_2\rangle.$$

The last condition may be rewritten as follows: We may introduce a bilinear form on $F \otimes F$ which works componentwise and we denote it again by $\langle -,-\rangle$; to be precise: for $a_1, a_2, b_1, b_2 \in F$, let $\langle a_1 \otimes a_2, b_1 \otimes b_2\rangle = \langle a_1, b_1\rangle\langle a_2, b_2\rangle$. Then we can write:

$$\langle a, b_1 b_2\rangle = \langle \delta(a), b_1 \otimes b_2\rangle,$$

The bilinear form $\langle -,-\rangle$ on $K\langle I\rangle$ has been introduced by Drinfeld [D] in the case where $K\langle I\rangle$ is a quantum group; thus we will call $\langle -,-\rangle$ the *Drinfeld form* of the $(K,v,\chi)$-bialgebra $K\langle I\rangle$. A proof of the proposition can be found in the book of Lusztig [L], at least in the special case of a quantum group. For the general case, we refer to [R4]. We are interested in elements of $K\langle I\rangle$ which are in the left radical of the Drinfeld form; by definition, the *left radical* is the set of all elements $a \in K\langle I\rangle$ which satisfy $\langle a,b\rangle = 0$ for all $b \in K\langle I\rangle$.

# 3 The main result

For $0 \le t \le n$, let

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad [n]! = \prod_{t=1}^{n} [t]\,, \quad \text{and} \quad \begin{bmatrix} n \\ t \end{bmatrix} = \frac{[n]!}{[t]![n-t]!}.$$

These are Laurent polynomials in $v$ with integer coefficients. Given such a Laurent polynomial $\varphi = \varphi(v)$, and an integer $d$, one denotes by $\varphi_d$ the Laurent polynomial which is obtained from $\varphi$ by inserting $v^d$, thus $\varphi_d(v) = \varphi(v^d)$. For example, $[n]_d = \frac{v^{dn}-v^{-dn}}{v^d-v^{-d}}$.

***Theorem 3.1*** — *Let $i \neq j$ be elements of $I$. Let us define $\eta(i,j) = \chi'(i,j) - \chi''(j,i)$. Assume that $m$ is a natural number with*

$$m \cdot \eta(i,i) = -\eta(i,j) - \eta(j,i).$$

*Define*

$$e = \frac{1}{2}\big(-\eta(i,j) + \eta(j,i)\big),$$
$$d = \frac{1}{2}\eta(i,i).$$

*and assume that* $v^e, v^d$ *are defined. Then the element*

$$\sum_{p=0}^{m+1} (-1)^p \begin{bmatrix} m+1 \\ p \end{bmatrix}_d v^{pe}\theta_i^p\theta_j\theta_i^{m+1-p}$$

*belongs to the left radical of the Drinfeld form.*

We may reformulate the assumptions as follows: First of all, if $\eta(i,i) = 0$, then we have to assume that $\eta(i,j) + \eta(j,i) = 0$ and then we can choose $m$ arbitrarily. If $\eta(i,i) \neq 0$, then we have to assume that

$$m = -\frac{\eta(i,j) + \eta(j,i)}{\eta(i,i)}$$

is a natural number. The condition that $v^e, v^d$ are defined, is always satisfied in case $e, d$ both are integers; otherwise, we have to assume the existence of a square root of $v$ in $K$.

## 4 Preliminary calculations.

We fix an element $i \in I$. Let $\theta = \theta_i$ and define $a' = \chi'(\theta,\theta)$ and $a'' = \chi''(\theta,\theta)$. Then $d = \frac{1}{2}(a' - a'')$, and we consider also $c = \frac{1}{2}(a' + a'')$.

***Lemma 4.1*** —

$$\delta(\theta^p) = \sum_{t+t'=p} \begin{bmatrix} p \\ t \end{bmatrix}_d v^{tt'c}\ \theta^t \otimes \theta^{t'}.$$

*Proof.* For $p = 0$, we have $\delta(1) = 1 \otimes 1$. For $p = 1$, we have $\begin{bmatrix} p \\ t \end{bmatrix} = 1$ and $tt' = 0$ for all $t$, and thus the right hand side is just $\theta \otimes 1 + 1 \otimes \theta$. Let $p \geq 1$. We use the multiplication rule in $F \otimes F$

$$(\theta^s \otimes \theta^{s'})(\theta^t \otimes \theta^{t'}) = v^{st'a'+s'ta''}\ \theta^{s+t} \otimes \theta^{s'+t'}.$$

and induction:

$$\begin{aligned}
\delta(\theta^{p+1}) &= \delta(\theta)\delta(\theta^p) \\
&= (\theta\otimes 1 + 1\otimes\theta)\sum_{t=0}^{p} \begin{bmatrix} p \\ t \end{bmatrix}_d v^{t(p-t)c}\theta^t\otimes\theta^{p-t} \\
&= \sum_{t=0}^{p} v^{(p-t)a'} \begin{bmatrix} p \\ t \end{bmatrix}_d v^{t(p-t)c}\theta^{t+1}\otimes\theta^{p-t} \\
&\qquad + \sum_{t=0}^{p} v^{ta''} \begin{bmatrix} p \\ t \end{bmatrix}_d v^{t(p-t)c}\theta^{t}\otimes\theta^{p-t+1} \\
&= \sum_{t=1}^{p+1} v^{(p-t+1)a'} \begin{bmatrix} p \\ t-1 \end{bmatrix}_d v^{(t-1)(p-t+1)c}\theta^{t}\otimes\theta^{p-t+1} \\
&\qquad + \sum_{t=0}^{p} v^{ta''} \begin{bmatrix} p \\ t \end{bmatrix}_d v^{t(p-t)c}\theta^{t}\otimes\theta^{p-t+1} \\
&= \sum_{t=0}^{p} \gamma(t)\ \theta^t\otimes\theta^{p+1-t}.
\end{aligned}$$

where the coefficients $\gamma(t)$ are given by $\gamma(0)=1$, and, for $t\geq 1$, by

$$\gamma(t) = v^{(p-t+1)a'+(t-1)(p-t+1)c} \begin{bmatrix} p \\ t-1 \end{bmatrix}_d + v^{ta''+t(p-t)c} \begin{bmatrix} p \\ t \end{bmatrix}_d .$$

Note that we can rewrite the two exponents as follows:

$$(p-t+1)a' + (t-1)(p-t+1)c = (p-t+1)d + t(p-t+1)c$$

and

$$ta'' + t(p-t)c = -td + t(p-t+1)c.$$

As a consequence, we can continue the calculation

$$\begin{aligned}
\gamma(t) &= v^{t(p+1-t)c}\left( v^{(p-t+1)d} \begin{bmatrix} p \\ t-1 \end{bmatrix}_d + v^{-td} \begin{bmatrix} p \\ t \end{bmatrix}_d \right) \\
&= v^{t(p+1-t)c}\left( v^{p-t+1} \begin{bmatrix} p \\ t-1 \end{bmatrix} + v^{-t} \begin{bmatrix} p \\ t \end{bmatrix} \right)_d \\
&= v^{t(p+1-t)c} \begin{bmatrix} p+1 \\ t \end{bmatrix}_d .
\end{aligned}$$

This completes the proof. □

***Lemma 4.2*** —

$$\langle \theta^p, \theta^p \rangle = [p]!_d \cdot v^{\binom{p}{2}c}.$$

*Proof.* Let $p \geq 1$. We use Lemma 4.1 and induction:

$$\begin{aligned}
\langle \theta^{p+1}, \theta^{p+1} \rangle &= \langle \delta(\theta^{p+1}), \theta^p \otimes \theta \rangle \\
&= \langle \sum_{t+t'=p+1} \begin{bmatrix} p+1 \\ t \end{bmatrix}_d v^{tt'c} \cdot \theta^t \otimes \theta^{t'}, \theta^p \otimes \theta \rangle \\
&= \begin{bmatrix} p+1 \\ p \end{bmatrix}_d v^{pc} \cdot \langle \theta^p \otimes \theta, \theta^p \otimes \theta \rangle \\
&= [p+1]_d \cdot v^{pc} \cdot [p]!_d \cdot v^{\binom{p}{2}c} \\
&= [p+1]!_d \cdot v^{\binom{p+1}{2}c}.
\end{aligned}$$

Here, we have used that $p + \binom{p}{2} = \binom{p+1}{2}$. This completes the proof. □

## 5 The calculation of $\langle \theta_i^p \theta_j \theta_i^{p'}, \theta_i^q \theta_j \theta_i^{q'} \rangle$.

Consider now a pair $\theta_i, \theta_j$ where $i \neq j$. We want to calculate $\langle \theta_i^p \theta_j \theta_i^{p'}, \theta_i^q \theta_j \theta_i^{q'} \rangle$ in case $p + p' = q + q'$. Again, we write $\theta = \theta_i$ and we use the previous notation.

***Proposition 5.1*** — *Let $i \neq j$ and $p + p' = q + q'$. Then*

$$\langle \theta^p \theta_j \theta^{p'}, \theta^q \theta_j \theta^{q'} \rangle = h \cdot [p]!_d [p']!_d \sum_J f(s) g(t)$$

*where the summation is over the set $J$ of all quadrupels $(s, t, s', t')$ such that $s+t = p$, $s' + t' = p'$, $s + s' = q$, $t + t' = q'$ and where*

$$\begin{aligned}
h &= [q]!_d [q']!_d \cdot v^{\left(\binom{q}{2} + \binom{q'}{2}\right)c + q\left(\frac{1}{2}q'a'' + \chi''(j,i)\right) + q'\left(\frac{1}{2}qa' + \chi'(j,i)\right)}, \\
f(s) &= \frac{1}{[s]!_d [s']!_d} v^{s(q'd + \chi'(i,j) - \chi''(j,i))}, \\
g(t) &= \frac{1}{[t]!_d [t']!_d} v^{t(-qd + \chi''(i,j) - \chi'(j,i))}.
\end{aligned}$$

We consider first the special case $q' = 0$. Thus, let $p + p' = q$. We have $\delta(\theta^p) = \theta^p \otimes 1 + x$, and $\delta(\theta^{p'}) = \theta^{p'} \otimes 1 + x'$, where $x, x'$ are linear combinations of elements of the form $\theta^s \otimes \theta^t$ with $t \geq 1$. Therefore

$$\begin{aligned}\langle \theta^p\theta_j\theta^{p'}, \theta^q\theta_j\rangle &= \langle \delta(\theta^p)\delta(\theta_j)\delta(\theta^{p'}), \theta^q \otimes \theta_j\rangle \\ &= \langle (\theta^p \otimes 1 + x)(\theta_j \otimes 1 + 1 \otimes \theta_j)(\theta^{p'} \otimes 1 + x'), \theta^q \otimes \theta_j\rangle \\ &= \langle (\theta^p \otimes 1)(1 \otimes \theta_j)(\theta^{p'} \otimes 1), \theta^q \otimes \theta_j\rangle \\ &= v^{p\chi'(i,j)+p'\chi''(j,i)}\langle \theta^q, \theta^q\rangle\langle \theta_j, \theta_j\rangle \\ &= v^{p\chi'(i,j)+p'\chi''(j,i)}\langle \theta^q, \theta^q\rangle.\end{aligned}$$

This will be used in the next calculation.

We turn to the general case $p + p' = q + q'$. We observe that

$$\langle (\theta^s \otimes \theta^t)\delta(\theta_j)(\theta^{s'} \otimes \theta^{t'}), \theta^q\theta_j \otimes \theta^{q'}\rangle = \langle (\theta^s \otimes \theta^t)(\theta_j \otimes 1)(\theta^{s'} \otimes \theta^{t'}), \theta^q\theta_j \otimes \theta^{q'}\rangle,$$

and this element is non-zero only in case $s + s' = q$ and $t + t' = q'$. As a consequence

$$\begin{aligned}&\langle \theta^p\theta_j\theta^{p'}, \theta^q\theta_j\theta^{q'}\rangle = \langle \delta(\theta^p)\delta(\theta_j)\delta(\theta^{p'}), \theta^q\theta_j \otimes \theta^{q'}\rangle \\ &= \sum_{s+t=p}\sum_{s'+t'=p'} \begin{bmatrix} p \\ s \end{bmatrix}_d \begin{bmatrix} p' \\ s' \end{bmatrix}_d v^{stc+s't'c}\langle (\theta^s \otimes \theta^t)\delta(\theta_j)(\theta^{s'} \otimes \theta^{t'}), \theta^q\theta_j \otimes \theta^{q'}\rangle \\ &= \sum_J \begin{bmatrix} p \\ s \end{bmatrix}_d \begin{bmatrix} p' \\ s' \end{bmatrix}_d v^{stc+s't'c}\langle (\theta^s \otimes \theta^t)(\theta_j \otimes 1)(\theta^{s'} \otimes \theta^{t'}), \theta^q\theta_j \otimes \theta^{q'}\rangle \\ &= \sum_J \begin{bmatrix} p \\ s \end{bmatrix}_d \begin{bmatrix} p' \\ s' \end{bmatrix}_d v^{stc+s't'c}v^{st'a'+s'ta''+t\chi''(i,j)+t'\chi'(j,i)}\langle \theta^s\theta_j\theta^{s'} \otimes \theta^{t+t'}, \theta^q\theta_j \otimes \theta^{q'}\rangle \\ &= \sum_J \begin{bmatrix} p \\ s \end{bmatrix}_d \begin{bmatrix} p' \\ s' \end{bmatrix}_d v^r v^{t\chi''(i,j)+t'\chi'(j,i)}v^{s\chi'(i,j)+s'\chi''(j,i)}\langle \theta^q, \theta^q\rangle\langle \theta^{q'}, \theta^{q'}\rangle \\ &= \sum_J \begin{bmatrix} p \\ s \end{bmatrix}_d \begin{bmatrix} p' \\ s' \end{bmatrix}_d v^{r+r'}[q]!_d \cdot v^{\binom{q}{2}c} \cdot [q']!_d \cdot v^{\binom{q'}{2}c} \\ &= \sum_J \left(\frac{[p]![p']![q]![q']!}{[s]![s']![t]![t']!}\right)_d v^{\left(\binom{q}{2}+\binom{q'}{2}\right)c}v^{r+r'}\end{aligned}$$

where

$$\begin{aligned}r &= stc + s't'c + st'a' + s'ta'' \\ &= \frac{1}{2}sta' + \frac{1}{2}sta'' + \frac{1}{2}s't'a' + \frac{1}{2}s't'a'' + st'a' + s'ta'' \\ &= \frac{1}{2}s(t+t')a' + \frac{1}{2}s'(t+t')a'' + \frac{1}{2}t(s+s')a'' + \frac{1}{2}t'(s+s')a' \\ &= \frac{1}{2}(sq'a' + s'q'a'' + tqa'' + t'qa').\end{aligned}$$

and

$$r' = t\chi''(i,j) + t'\chi'(j,i) + s\chi'(i,j) + s'\chi''(j,i).$$

Altogether we have

$$\begin{aligned}
r + r' &= s\left(\frac{1}{2}q'a' + \chi'(i,j)\right) + s'\left(\frac{1}{2}q'a'' + \chi''(j,i)\right) \\
&\qquad + t\left(\frac{1}{2}qa'' + \chi''(i,j)\right) + t'\left(\frac{1}{2}qa' + \chi'(j,i)\right) \\
&= s\left(\frac{1}{2}q'a' + \chi'(i,j)\right) + (q-s)\left(\frac{1}{2}q'a'' + \chi''(j,i)\right) \\
&\qquad + t\left(\frac{1}{2}qa'' + \chi''(i,j)\right) + (q'-t)\left(\frac{1}{2}qa' + \chi'(j,i)\right) \\
&= s\left(q'd + \chi'(i,j) - \chi''(j,i)\right) + t\left(-qd + \chi''(i,j) - \chi'(j,i)\right) \\
&\qquad + q\left(\frac{1}{2}q'a'' + \chi''(j,i)\right) + q'\left(\frac{1}{2}qa' + \chi'(j,i)\right).
\end{aligned}$$

We distribute the various terms in order to form $h$, $f(s)$ and $g(t)$. This completes the proof.

# 6 Proof of Theorem.

Let $p' = m + 1 - p$. Let $q, q'$ be natural numbers such that $q + q' = m + 1$. We claim that

$$\begin{aligned}
v^{se}f(s) &= \frac{1}{[s]!_d[s']!_d}v^{s(-q+1)d} \\
v^{te}g(t) &= \frac{1}{[t]!_d[t']!_d}v^{t(q'-1)d}
\end{aligned}$$

We have to consider the exponent of $v^s$ in $[s]!_d[s']!_d v^{se}f(s)$; it is given by:

$$\begin{aligned}
q'd + \chi'(i,j) - \chi''(j,i) + e &= q'\frac{1}{2}\eta(i,i) + \eta(i,j) + \frac{1}{2}(-\eta(i,j) + \eta(j,i)) \\
&= \frac{1}{2}\left(q'\eta(i,i) + \eta(i,j) + \eta(j,i)\right) \\
&= \frac{1}{2}\left(q'\eta(i,i) - m\cdot\eta(i,i)\right) \\
&= \frac{1}{2}\left(q'\eta(i,i) - (q'+q-1)\eta(i,i)\right) \\
&= \frac{1}{2}(-q+1)\eta(i,i) \\
&= (-q+1)d
\end{aligned}$$

Similarly, we calculate the exponent of $v^t$ in $[t]!_d[t']!_d v^{te} g(t)$, it is of the form:

$$\begin{aligned}
-qd+\chi''(i,j)-\chi'(j,i)+e &= -q\frac{1}{2}\eta(i,i)-\eta(j,i)+\frac{1}{2}(-\eta(i,j)+\eta(j,i)) \\
&= \frac{1}{2}\left(-q\eta(i,i)-\eta(i,j)-\eta(j,i)\right) \\
&= \frac{1}{2}\left(-q\eta(i,i)+m\cdot\eta(i,i)\right) \\
&= \frac{1}{2}\left(-q\eta(i,i)+(q'+q-1)\eta(i,i)\right) \\
&= \frac{1}{2}(q'-1)\eta(i,i) \\
&= (q'-1)d
\end{aligned}$$

We calculate (and we set $p'=m+1-p$ in the definition of $J$):

$$\begin{aligned}
&\langle \sum_{p=0}^{m+1}(-1)^p \begin{bmatrix} m+1 \\ p \end{bmatrix}_d v^{pe}\ \theta_i^p\theta_j\theta_i^{m+1-p}, \theta_i^q\theta_j\theta_i^{q'}\rangle \\
&\qquad = \sum_{p=0}^{m+1}(-1)^p \begin{bmatrix} m+1 \\ p \end{bmatrix}_d v^{pe}\ \langle\theta_i^p\theta_j\theta_i^{m+1-p}, \theta_i^q\theta_j\theta_i^{q'}\rangle \\
&\qquad = \sum_{p=0}^{m+1}(-1)^p \left(\frac{[m+1]!}{[p!][m+1-p]!}\right)_d v^{pe}\ h\ [p]!_d[m+1-p]!_d \sum_J f(s)\cdot g(t) \\
&\qquad = \sum_{p=0}^{m+1}[m+1]!_d\ h\sum_J(-1)^s v^{se} f(s)\cdot(-1)^t v^{te} g(t) \\
&\qquad = [m+1]!_d\ \cdot h\cdot\Big(\sum_{s=0}^{q}(-1)^s v^{se} f(s)\Big)\cdot\Big(\sum_{t=0}^{q'}(-1)^t v^{te} g(t)\Big),
\end{aligned}$$

since any pair $(s,t)$ with $0\le s\le q$ and $0\le t\le q'$ gives a unique pair $(p,(s,t,s',t'))$ with $0\le p\le m+1$ and $(s,t,s',t')\in J$ (namely $p=s+t,\ s'=q-s,\ t'=q'-t$) and any pair is obtained in this way. If $q>0$, then the factor

$$\sum_{s=0}^{q}(-1)^s v^{se} f(s)=\sum_{s=0}^{q}(-1)^s\frac{1}{[s]!_d[s']!_d}v^{ds(-q+1)}$$

is zero, if $q'>0$, then the factor

$$\sum_{t=0}^{q'}(-1)^t v^{te} g(t)=\sum_{t=0}^{q'}(-1)^t\frac{1}{[t]!_d[t']!_d}v^{dt(q'-1)}$$

is zero. Here we use the following well-known formula: for $q \geq 1$,

$$\sum_{s=0}^{q}(-1)^s v^{s(q-1)} \begin{bmatrix} q \\ s \end{bmatrix} = 0 = \sum_{s=0}^{q}(-1)^s v^{-s(q-1)} \begin{bmatrix} q \\ s \end{bmatrix},$$

(see, for example [L], 1.3.1). Since $q+q' = m+1 > 0$, at least one of the two factors is zero.

The algebra $K\langle I\rangle$ is $\mathbb{Z}I$-graded and the element

$$a = \sum_{p=0}^{m+1}(-1)^p \begin{bmatrix} m+1 \\ p \end{bmatrix}_d v^{pe}\ \theta_i^p \theta_j \theta_i^{m+1-p}$$

has degree $x = (m+1)i + j \in \mathbb{Z}I$. Since the Drinfeld form respects the grading, and since the elements $\theta_i^q \theta_j \theta_i^{q'}$ with $q+q' = m+1$ form a basis of $K\langle I\rangle_x$, the calculation above shows that $a$ belongs to the left radical of the Drinfeld form. This completes the proof.

# 7 Applications

## 7.1 The quantum group case

This is the special case when $\chi' = 0$ and $\chi''$ is a Cartan datum as defined in [L]; thus, we assume that $\chi''$ is a symmetric bilinear form on $\mathbb{Z}I$, that $\chi''(i,i)$ is an even positive integer, for any $i \in I$, and finally that $m(i,j) = -2\frac{\chi''(i,j)}{\chi''(i,i)}$ is a non-negative integer, for $i \neq j$ in $I$. In this case, we see that

$$\eta(i,j) = -\chi''(j,i),$$

so that $\eta$ is again a symmetric bilinear form. It follows that $e = 0$. Also, $d = -\frac{1}{2}\chi''(i,i) \neq 0$ is an integer, thus $v^d$ is defined, and the $m$ considered above is equal to $m(i,j)$:

$$m = \frac{-\eta(i,j) - \eta(j,i)}{\eta(i,i)} = \frac{\chi''(j,i) + \chi''(i,j)}{-\chi''(i,i)} = -2\frac{\chi''(i,j)}{\chi''(i,i)} = m(i,j).$$

We note that $\begin{bmatrix} n \\ t \end{bmatrix}_d = \begin{bmatrix} n \\ t \end{bmatrix}_{-d}$. It follows that the elements exhibited above are just the usual quantum Serre relations.

## 7.2 Skew commutation

Let us assume that we have

$$\chi'(i,j) + \chi'(j,i) = \chi''(i,j) + \chi''(j,i)$$

for some pair $i \neq j$ in $I$. Then

$$e = \chi'(j,i) - \chi''(i,j)$$

is an integer and the considerations above show that the element

$$\theta_j \theta_i - v^e\ \theta_i \theta_j$$

belongs to the radical of the Drinfeld form (this is just the case $m = 0$).

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