Division Algebras on $\mathbb{P}^2$ of Odd Index, Ramified Along a Smooth Elliptic Curve Are Cyclic

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Abstract

The simplest non-trivial division algebras that can be constructed over a rational function field in two variables are those that ramify along a divisor of degree three. In this note we give a precise structure theorem for such division algebras. It follows in particular that they are cyclic if the ramification locus is singular or if the index is odd.

Résumé

Les corps gauches non-triviaux les plus simples que l'on peut construire sur un corps de fonctions rationnelles à deux variables sont ceux qui se ramifient lelong d'un diviseur de degré trois. Dans cette note, nous donnons un théorème de structure précis pour de tels corps gauches. En particulier, il en résulte qu'ils sont cycliques si le lieu de ramification est singulier ou si l'indice est impair.

1 Introduction

Let R be a discrete valuation ring with quotient field K and residue field l. We assume that both l and K are of characteristic zero. Then it is classical [5] that there is an exact sequence

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \xrightarrow{\mathrm{ram}} H^1(l, \mathbb{Q}/\mathbb{Z}) \to 0$$

Here $H^1(l, \mathbb{Q}/\mathbb{Z})$ is the set of couples (l', σ) where l' is a cyclic extension of l and σ is a generator of $\mathrm{Gal}(l'/l)$. The *ramification map*, denoted by ram, is as described in [5]. Assume $[D] \in \mathrm{Br}(K)$. Then there is an unramified finite Galois extension L/K splitting D. Let S be the integral closure of R in L. S is a semi-local Dedekind domain. Let $\mathrm{Div}(S)$ be the group of divisors of S. Associating to $f \in L^*$ its divisor S yields a homomorphism

$$L^* \to \mathrm{Div}(S) \tag{1.1}$$

AMS 1980 *Mathematics Subject Classification* (1985 *Revision*): 16K20, 13A20

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Clearly $\mathrm{Div}(S) = \mathbb{Z}G/G_{\mathfrak{p}}$ where $G = \mathrm{Gal}(L/K)$ and $G_{\mathfrak{p}}$ is the stabilizer of a prime divisor $\mathfrak{p}$ of S. Alternatively $G_{\mathfrak{p}} = \mathrm{Gal}\big((S/\mathfrak{p})/l\big)$. Taking Galois cohomology of (1.1) yields a map

$$H^2(G, L^*) \to H^2(G, \mathrm{Div}(S)) \cong H^2(G_{\mathfrak{p}}, \mathbb{Z}) \cong H^1(G_{\mathfrak{p}}, \mathbb{Q}/\mathbb{Z}) \tag{1.2}$$

where the first isomorphism is Shapiro's lemma. The composition of the maps in (1.2) is the ramification map. Now let k be an algebraically closed field of characteristic zero and let Y be a simply connected surface over k. According to [2] there is a long exact sequence

(1.3)

$$0 \to \mathrm{Br}(Y) \to \mathrm{Br}(K(Y)) \xrightarrow{\oplus\,\mathrm{ram}_C} \bigoplus_{\substack{C\subset Y\\ \text{irr. curve}}} H^1(K(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\bigoplus_{x\in C} r_{C,x}} \bigoplus_{x\in Y} \mu^{-1} \xrightarrow{\Sigma} \mu^{-1} \to 0$$

Here $\mu^{-1} = \bigcup_n \mathrm{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$ where μ_n is the group of n'th roots of unity. Hence, non-canonically, $\mu^{-1} \cong \mathbb{Q}/\mathbb{Z}$. As above $H^1(K(C), \mathbb{Q}/\mathbb{Z}) \to \mu^{-1}$ is given by the cyclic extensions of $K(C)$. Given such a cyclic extension one may measure its ramification at a point y of the normalization $\bar{C}$ of C in terms of an element of μ^{-1}. $r_{C,x}$ is defined as the sum of the ramifications of the points $y \in \bar{C}$ lying above x. For $D \in \mathrm{Br}(K(Y))$ we write

$$R = \bigcup_C \{C \subset Y \mid \mathrm{ram}_C(D) \neq 0\}$$

and we call R the *ramification locus* of D. By construction R is a reduced divisor in Y. In the rest of this note we specialize to $Y = \mathbb{P}^2_k$. In that case $\mathrm{Br}(Y) = 0$ and so (1.3) allows us to compute $\mathrm{Br}(K(Y)) = \mathrm{Br}(k(u, v))$. The following result easily follows

Lemma 1.1 — *Let D, R, Y be as above and assume that D is non-trivial. Then*

1. $\deg R \geq 3$.
2. *If* $\deg R = 3$ *then there are the following possibilities*
   (a) *R is a union of three lines, not passing through one point.*
   (b) *R is a union of a line and a conic, not tangent to one another.*
   (c) *R is a nodal elliptic curve.*
   (d) *R is a smooth elliptic curve.*

A long standing question, due to Albert, is whether every division algebra of prime index is cyclic. Given the seemingly rather tractable nature of division algebras ramified along a cubic divisor, some people have suggested that these might be used to answer Albert's question negatively. See for example [11]. In this note we show that this is not so. That is, we show

Proposition 1.2 — *Let D be a non-trivial central division algebra over $K(\mathbb{P}^2_k)$ and let R be its ramification divisor. Assume that* $\deg R = 3$ *and that one of the following hypotheses holds.*

1. *R is singular.*
2. *R is smooth and the period of D in the Brauer group is odd.*

Then D is cyclic and has period equal to index.

Part (1) of this proposition has already been proved by T. Ford using somewhat different methods [9]. Furthermore in [15] it is shown that if R is smooth then D is similar to a tensor product of three cyclic algebras. Finally, with R still smooth, it has been shown in [11] (under considerably weaker hypotheses on k) that D is cyclic if its period is 5 or 7. Proposition 1.2 is a corollary of the following theorem

Theorem 1.3 — *Let D be a central division algebra over $K(\mathbb{P}^2_k)$ and let R be its ramification locus. Assume that* $\deg R = 3$. *Then the following holds*

1. *If R is singular then as k-algebras*

$$D \cong k(x, y; yx = \omega xy) \tag{1.4}$$

   where ω is a root of unity.

2. *If R is smooth then as k-algebras*

$$D \cong K(S)(x, \tau)^H \tag{1.5}$$

   where

   – *S is an unramified cyclic covering of R (hence in particular S is an elliptic curve).*
   – *τ is a generator for* $\mathrm{Gal}(S/R)$.
   – *$H = \{1, \sigma\}$ with $\sigma(u) = -u$ for $u \in S$ (for a choice of group law on S) and $\sigma(x) = x^{-1}$.*

That Proposition 1.2 follows from Theorem 1.3 is clear in the singular case, and in the smooth case it follows from [14]. I wish to thank Burt Fein, Zinovy Reichstein for some valuable comments and for pointing out an error in an earlier version of this note. I also wish to thank Colliot-Thélène for some private communication concerning the case where k is not algebraically closed. This is reproduced in the appendix.

2 Proof of Theorem 1.3

Let us first recall the following result

Proposition 2.1 — *Let l be a field of characteristic zero. Then there is an exact sequence*

(2.1)

$$0 \to \mathrm{Br}(l) \to \mathrm{Br}(K(\mathbb{P}^1_l)) \xrightarrow{\oplus \mathrm{ram}_x} \bigoplus_{x \in \mathbb{P}^1_l} H^1(l(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\oplus \mathrm{cor}_{l(x)/l}} H^1(l, \mathbb{Q}/\mathbb{Z}) \to 0$$

Here $x \in \mathbb{P}^1_l$ runs through the closed points of $\mathbb{P}^1_l$.

Proof. This is a version of the Faddeev-Auslander-Brumer sequence where one keeps track of the point at infinity. It is also very closely related to various exact sequences occurring in [7]. Let us quickly recall the proof. Let $\bar{l}$ be the algebraic closure of l and let $\mathrm{Prin}(\mathbb{P}^1_{\bar{l}})$, $\mathrm{Div}(\mathbb{P}^1_{\bar{l}})$ respectively stand for the principal divisors and the Weil divisors on $\mathbb{P}^1_{\bar{l}}$. We have exact sequences of $G = \mathrm{Gal}(\bar{l}/l)$ modules

$$0 \to \bar{l}^* \to K(\mathbb{P}^1_{\bar{l}})^* \to \mathrm{Prin}(\mathbb{P}^1_{\bar{l}}) \to 0$$
$$0 \to \mathrm{Prin}(\mathbb{P}^1_{\bar{l}}) \to \mathrm{Div}(\mathbb{P}^1_{\bar{l}}) \xrightarrow{\deg} \mathbb{Z} \to 0$$

Both these sequences are (non-canonically) split. This is clear for the second one. For the first one we send $f \in K(\mathbb{P}^1_{\bar{l}})^*$ to the first non-zero coefficient of the Taylor series expansion of f around 0 (for a G invariant uniformizing element). Hence applying $H^2(G, -)$ to these exact sequences, and afterwards combining them, yields a long exact sequence

$$0 \to \mathrm{Br}(l) \to \mathrm{Br}(K(\mathbb{P}^1_l)) \to H^2(G, \mathrm{Div}(\mathbb{P}^1_l)) \xrightarrow{\deg} H^2(G, \mathbb{Z}) \to 0$$

taking into account that $H^2(G, \bar{l}^*) = \mathrm{Br}(l)$ and by Tsen's theorem $H^2(G, K(\mathbb{P}^1_{\bar{l}})^*) = \mathrm{Br}(K(\mathbb{P}^1_l))$. Now $\mathrm{Div}(\mathbb{P}^1_{\bar{l}}) = \bigoplus_{x \in \mathbb{P}^1_l} \mathbb{Z}G/G_x$ where $G_x = \mathrm{Gal}(\bar{l}/l(x))$. So by Shapiro's lemma $H^2(G, \mathrm{Div}(\mathbb{P}^1_{\bar{l}})) = \bigoplus_x H^2(G_x, \mathbb{Z})$. It is now clear that the resulting map

$$\bigoplus_x H^2(G_x, \mathbb{Z}) \xrightarrow{\deg} H^2(G, \mathbb{Z})$$

is obtained by applying $H^2(G, -)$ to the "sum map" $\mathbb{Z}G/G_x \to \mathbb{Z}$ and then invoking Shapiro's lemma. It follows from [6, Prop. III.6.2] that this is precisely the corestriction. To obtain the exact form of (2.1) we use $H^2(G, \mathbb{Z}) = H^1(l, \mathbb{Q}/\mathbb{Z})$, $H^2(G_x, \mathbb{Z}) = H^1(l(x), \mathbb{Q}/\mathbb{Z})$. That the map $\mathrm{Br}(K(\mathbb{P}^1_l)) \to \bigoplus H^1(l(x), \mathbb{Q}/\mathbb{Z})$ is $\bigoplus \mathrm{ram}_x$ follows by looking at the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l^* & \longrightarrow & K(\mathbb{P}^1_{\bar{l}})^* & \longrightarrow & \mathrm{Prin}(\mathbb{P}^1_{\bar{l}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}^*_{\mathbb{P}^1_{\bar{l}},x} & \longrightarrow & K(\mathbb{P}^1_{\bar{l}})^* & \longrightarrow & \mathrm{Div}(\mathcal{O}_{\mathbb{P}^1_{\bar{l}},x}) & \longrightarrow & 0 \end{array}$$

□

Theorem 2.2 — *Assume that l is a field of characteristic zero with trivial Brauer group, containing a primitive n^{th} root of unity. Let D be a central division algebra of period n over $K(\mathbb{P}^1_l)$.*

1. *If D is ramified in at most two points of degree one then as l-algebras*

$$D \cong L(x, \tau) \tag{2.2}$$

where L/l is cyclic of dimension n and τ is a generator of $\mathrm{Gal}(L/l)$.

2. *If D is ramified in one point u of degree two then as l-algebras*

$$D \cong L(x, \tau)^H \tag{2.3}$$

where L/l is a dihedral extension of dimension $2n$ containing $l(u)$, τ is a generator of $\mathrm{Gal}(L/l(u))$, $H = \mathrm{Gal}(l(u)/l) = \{1, \sigma\}$ *(with action lifted in a arbitrary way to L) and $\sigma(x) = x^{-1}$.*

Proof. The proof consists in showing that the division algebras on the right side of (2.2) and (2.3) have the same ramification as D.

1. This part can be deduced from [8, Prop. 2.1]. For completeness we give a proof. We can choose an affine coordinate y on $\mathbb{P}^1_l$ such that D is ramified on $y = 0, \infty$. Let $(L, \tau) = \mathrm{ram}_0(D)$ and put $E = L(x, \tau)$. Then $Z(E) = k(x^n)$ and if we put $y = x^n$ then E is ramified in $y = 0, \infty$ with $\mathrm{ram}_0(E) = (L, \tau)$. Hence D, E have the same ramification data and thus $D \cong E$.
2. Assume $k(u) = k(\sqrt{t})$. We can now choose an affine coordinate y on $\mathbb{P}^1_l$ such that D is ramified in the prime $(y^2 - t)$. Put $(L, \tau) = \mathrm{ram}_u(D)$. We claim that L/l is dihedral. By Kummer theory $L = l(u)(\sqrt[n]{a})$. Since u is the only place where D ramifies, the corestriction of L must be trivial by (2.1). According to [15, lemma 0.1] this corestriction is given by $l(\sqrt[n]{a\,\sigma a})$ where $\mathrm{Gal}(l(u)/l) = \{1, \sigma\}$. So $a\,\sigma a = q^n$, $q \in l$. This allows us to lift the action of σ to L by putting $\sigma(\sqrt[n]{a}) = q/\sqrt[n]{a}$. Hence L/l is dihedral.

   Put $E_1 = L(x, \tau)$, $E = E_1^H$. Then $Z(E_1) = l(u)(x^n)$ and since H acts non-trivially on $l(u)(x^n)$, $Z(E) = Z(E_1)^H = k\left(\frac{\sqrt{t}(x^n-1)}{x^n+1}\right)$. Put $y = \frac{\sqrt{t}(x^n-1)}{x^n+1}$. Then using the definition of the ramification map, one easily verifies that E is only ramified in $u = (y^2 - t)$ and furthermore $\mathrm{ram}_u(E) = (L, \tau)$. Hence once again D and E have the same ramification data and thus $D \cong E$. □

Proof of Theorem 1.3. As an example we will discuss the cases where R is a nodal or a smooth elliptic curve. The other two cases in lemma 1.1 are similar. Throughout n is the period of D in the Brauer group.

***R* a nodal elliptic curve.** Let $y \in R$ be the singular point and let $B \subset \mathbb{P}^2_k$ be a line not passing through y. Our aim is to project from $\mathbb{P}^2_k$ to B with center y.

To do this properly we first blow up y to obtain a rational surface Y. Let E be the exceptional curve and let $\tilde{R}$ be the strict transform of R in Y. Clearly D defines a Brauer class on $K(Y) = K(\mathbb{P}^2_k)$ ramified on $\tilde{R}$ and possibly on E.

Let l be the function field of B. Then $K(Y) = K(\mathbb{P}^1_l)$. So D gives a Brauer class on $\mathbb{P}^1_l$ ramified in at most two points of degree one (corresponding to the projections $\tilde{R} \to B$, $E \to B$). According to Theorem 2.2 $D = L(x, \tau)$ and $\mathrm{ram}_{\tilde{R}}(D) = (L, \tau)$. Hence to finish the proof in this case we have to determine L.

Now $\tilde{R} \cong E = \mathbb{P}^1_k$ and $|\tilde{R} \cap E| = 2$. Hence L is the function field of a covering of degree n of $\mathbb{P}^1_k$, ramified in two points. From the fact that the fundamental group of $\mathbb{P}^1_k - \{\text{two points}\}$ is $\mathbb{Z}$ we deduce that L is unique. So we can assume that the field extension $L/K(\mathbb{P}^1_k)$ is of the form $\mathbb{C}(y)/\mathbb{C}(y^n)$ for some $y \in L$, and with τ acting as $y \mapsto \omega^{-1}y$. This yields that D is of the form (1.4).

***R* a smooth elliptic curve.** In this case we let y be an arbitrary point of E. We use the notations $B, \tilde{R}, E, l$ in the same way as above. Since $|\tilde{R} \cap E| = 1$ and $E \cong \mathbb{P}^1_l$ there can be no extension of $K(E)$, ramified in only one point. So D is unramified on E. Hence if we view D as an element of $\mathrm{Br}(\mathbb{P}^1_l)$ then it is only ramified on the point of order 2 corresponding to the covering $R = \tilde{R} \to B$. Thus as in Theorem 2.2 $D = L(x, \tau)^H$ with $(L, \tau) = \mathrm{ram}_{\tilde{R}}(D) = \mathrm{ram}_R(D)$, Now it follows from (1.3) that L is the function field of an unramified covering S of degree n of R. The map $R \to B$ is a quotient by an involution of R. We can choose the origin for the group law on R in such a way that this involution is given by $u \mapsto -u$. We can lift this involution to one of the same form on S. This shows that L and hence D have the required form. □

A Some remarks in the case that k is not algebraically closed

This appendix contains some personal communication by Colliot-Thélène concerning the case where k is not algebraically closed. Any errors or inaccuracies are mine. The main result is Theorem A.1 which provides a very partial substitute to (1.3). The insertion of the hypotheses that k is of characteristic zero is due to me. It allowed me to smoothen the proof, but it is very likely unnecessary.

From the previous sections it appears that the most interesting elements of $\mathrm{Br}(K(\mathbb{P}^2_k))$ are those that are ramified along a smooth curve, so we will be concerned with those. Let R be a smooth curve in $\mathbb{P}^2_k$ and let U be its complement. Then we are interested in $\mathrm{Br}(U)$. The ramification of an element of $\mathrm{Br}(U)$ can be viewed as an element of $H^1_{\mathrm{et}}(R, \mathbb{Q}/\mathbb{Z})$ and we want to understand when, conversely, an element of $H^1_{\mathrm{et}}(R, \mathbb{Q}/\mathbb{Z})$ can be lifted to one of $\mathrm{Br}(U)$. To state the main result we need a few notions from the theory of etale cohomology. We state these in the least generality possible. Let C be a smooth projective curve and let D be an effective divisor on C. Then associated to D there is a map ψ_D which is the composition

$$H^1_{\text{et}}(C, \mathbb{Z}/n) \to H^1(D, \mathbb{Z}/n) \to H^1(k, \mathbb{Z}/n)$$

The first arrow is the restriction map (inverse image), and the last arrow is the trace map (direct image) [1]. ψ_D is additive in D [1, XVII.6.3.27]. This yields a pairing

$$\langle\ ,\ \rangle : H^1(C, \mathbb{Z}/n) \times \operatorname{Div}(C)/n \to H^1(k, \mathbb{Z}/n) : (z, D) \to \psi_D(z) \tag{A.1}$$

If $f : C \to C'$ is a finite morphism then it follows from [1, XVII.6.3.19] that the pairing (A.1) satisfies the compatibilities

$$\langle f_* u, E\rangle = \langle u, f^* E\rangle \tag{A.2}$$

$$\langle v, f_* F\rangle = \langle f^* v, F\rangle \tag{A.3}$$

Assume that $D = (f)$ is a principal divisor. Then f defines a map $f : C \to \mathbb{P}^1$ and D is the inverse image under f of $E = (0) - (\infty)$. Applying (A.2) with this E we find $\langle u, D\rangle = \langle f_* u, (0)\rangle - \langle f_* u, (\infty)\rangle$. Now we claim that $\langle f_* u, (p)\rangle$ for a rational point $p \in \mathbb{P}^1$ is independent of p. This shows that ψ_D only depends on the divisor class of D. The claim amounts to proving that if $f, g : \operatorname{Spec} k \to \mathbb{P}^1$ are two embeddings then $f^* = g^*$. It is clear that to prove this we may replace $\mathbb{P}^1$ by $\mathbb{A}^1$. Let $h : \mathbb{A}^1 \to \operatorname{Spec} k$ be the projection. Then $hf = hg$ and hence $f^* h^* = g^* h^*$. However h^* is an isomorphism ("homotopy invariance"). Therefore $f^* = g^*$. Hence (A.1) factors to yield a pairing

$$H^1_{\text{et}}(C, \mathbb{Z}/n) \times \operatorname{Pic}(C)/n \to H^1(k, \mathbb{Z}/n) \tag{A.4}$$

Below, if A is an abelian group then we denote by ${}_nA$ the subgroup consisting of elements annihilated by n. One now has the following result :

Theorem A.1 — *Assume that k has characteristic zero. Let L be a line in $\mathbb{P}^2_k$ and put $D = R \cap L$ Then there is an exact sequence*

$$0 \to {}_n\operatorname{Br}(k) \to {}_n\operatorname{Br}(U) \to H^1_{\text{et}}(R, \mathbb{Z}/n) \xrightarrow{\psi_D} H^1_{\text{et}}(k, \mathbb{Z}/n) \tag{A.5}$$

Proof. We sketch the proof, leaving some details to the reader. We put $Y = \mathbb{P}^2_k$. All cohomology will be etale cohomology. Consider the commutative diagram given by localization sequences

(A.6)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Pic}(Y)/n & \to & \mathrm{Pic}(U)/n & \to & H^2_R(Y,\mathbb{G}_m)/n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^2(Y,\mu_n) & \to & H^2(U,\mu_n) & \longrightarrow & H^3_R(Y,\mu_n) & \longrightarrow & H^3(Y,\mu_n) \to H^3(U,\mu_n) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_n\mathrm{Br}(Y) & \longrightarrow & {}_n\mathrm{Br}(U) & \longrightarrow & {}_nH^3_R(Y,\mathbb{G}_m) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(this diagram comes from considering the homology of a certain 3×3-square of complexes of injectives, and hence the squares involving two connecting maps are actually only commutative up to sign). The columns and the middle row are clearly exact. Since Y is smooth, the map $\mathrm{Br}(Y) \to \mathrm{Br}(U)$ is injective and hence the lower row in (A.6) is exact since it is obtained from applying $\mathrm{Hom}(\mathbb{Z}/n, -)$ to the exact sequence

$$0 \to \mathrm{Br}(Y) \to \mathrm{Br}(U) \to H^3_R(Y, \mathbb{G}_m)$$

Finally, again because Y is smooth we have that $H^2_R(Y, \mathbb{G}_m) = 0$ [10, (6.5)]. Combining all this, and taking into account that $\mathrm{Br}(Y) = \mathrm{Br}(k)$ we obtain the following long exact sequence

$$0 \to {}_n\,\mathrm{Br}(k) \to {}_n\,\mathrm{Br}(U) \to H^3_R(Y, \mu_n) \to H^3(Y, \mu_n) \to H^3(U, \mu_n) \tag{A.7}$$

By purity we have $H^3_R(Y, \mu_n) = H^1(R, \mathbb{Z}/n)$ and the Leray spectral sequence $E_2^{pq} = H^p(\bar{k}, H^q(Y_{\bar{k}}, \mu_n)) \Rightarrow H^n(Y, \mu_n)$ yields an exact sequence

$$0 \to H^3(k, \mu_n) \to H^3(Y, \mu_n) \to H^1(k, H^2(Y_{\bar{k}}, \mu_n))$$

The part $H^3(k, \mu_n)$ survives in $H^3(U, \mu_n)$ since it even survives in the function field of U (which is rational). Using the fact that $H^2(Y_{\bar{k}}, \mu) = \mathbb{Z}/n$ we now easily obtain an exact sequence like (A.5), where the last map is given by the composition

(A.8)
$$H^1(R, \mathbb{Z}/n) \xrightarrow{\text{purity}} H^3_R(Y, \mu_n) \to H^3(Y, \mu_n) \to H^1(k, H^2(Y_{\bar{k}}, \mu_n)) = H^1(k, \mathbb{Z}/n)$$

and we have to show that this is equal to ψ_D. We now assume that L is not tangent to R. The fact that we can do this follows from the hypotheses that k is of characteristic zero and hence is "big enough". It follows that D is smooth over k.

Using the compatibility with restriction of the isomorphism given by purity and the Leray spectral sequence yields a commutative diagram where the vertical arrows are restriction maps.

(A.9)
$$\begin{array}{ccc} H^1(R,\mathbb{Z}/n) & \longrightarrow & H^1(k,H^2(Y_{\bar{k}},\mu_n)) = H^1(k,\mathbb{Z}/n) \\ \downarrow & & \downarrow \\ H^1(D,\mathbb{Z}/n) & \longrightarrow & H^1(k,H^2(L_{\bar{k}},\mu_n)) = H^1(k,\mathbb{Z}/n) \end{array}$$

We claim that the restriction map $\mathbb{Z}/n = H^2(Y_{\bar{k}},\mu_n) \to H^2(L_{\bar{k}},\mu_n) = \mathbb{Z}/n$ is an isomorphism. This can be seen for example by taking a point p outside L and putting $V = Y - p$. Then one has $V = L \times \mathbb{A}^1$. By the localization sequence there is an isomorphism $H^2(Y_{\bar{k}},\mu_n) = H^2(V_{\bar{k}},\mu_n)$. By the Kunneth theorem the projection $V \to L$ yields an isomorphism $H^2(L_{\bar{k}},\mu_n) \to H^2(V_{\bar{k}},\mu_n)$. Since the composition of the inclusion $L \to V$ and the projection $V \to L$ is an isomorphism we are through. Hence we now have to show that the bottom arrow of (A.9), is given by the trace map. This arrow is the composition of the two upper horizontal maps and the rightmost vertical map of the diagram

$$\begin{array}{ccccc} H^1(D,\mathbb{Z}/n) & \longrightarrow & H^3_D(L,\mu_n) & \longrightarrow & H^3(L,\mu_n) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(k,H^0(D_{\bar{k}},\mathbb{Z}/n)) & \longrightarrow & H^1(k,H^2_{D_{\bar{k}}}(L_{\bar{k}},\mu_n)) & \longrightarrow & H^1(k,H^2(L_{\bar{k}},\mu_n)) \end{array}$$

Here the vertical maps are obtained from the Leray spectral sequence. Hence we have to show that $H^1(k,-)$ applied to the composition

(A.10)
$$H^0(D_{\bar{k}},\mathbb{Z}/n) \xrightarrow{\text{purity}} H^2_{D_{\bar{k}}}(L_{\bar{k}},\mu_n) \to H^2(L_{\bar{k}},\mu_n) = \mathbb{Z}/n$$

is given by the trace map. $D_{\bar{k}}$ is a finite number of distinct points, equipped with a Galois action. Say $D_{\bar{k}} = \{p_1,\ldots,p_l\}$. Then (A.10) becomes the composition

$$\bigoplus_i H^0(p_i,\mathbb{Z}/n) \xrightarrow{\text{purity}} \bigoplus_i H^2_{p_i}(L_{\bar{k}},\mu_n) \to H^2(L_{\bar{k}},\mu_n)$$

The localization sequence shows that

$$H^2_{p_i}(L_{\bar{k}},\mu_n) \to H^2(L_{\bar{k}},\mu_n)$$

is an isomorphism. Hence (A.10) becomes as, Galois modules,

$$\bigoplus_i (\mathbb{Z}/n)p_i \xrightarrow{=} \bigoplus_i (\mathbb{Z}/n)p_i \to \mathbb{Z}/n$$

where the last map is the sum map. It is now standard that $H^*(k, -)$ applied to the sum map yields the trace map. □

Remark A.2. Presumably the restriction that k has characteristic zero is unnecessary in the previous theorem. Assuming that n is prime to the characteristic should be enough.

Corollary A.3 — *We use the notations of the previous theorem. Assume that R has degree m and that $f : R' \to R$ is an unramified cover of degree n, representing an element z of $H^1(R, \mathbb{Z}/n)$. Assume that the divisor class of D is in the image of $f_* : \mathrm{Pic}(R') \to \mathrm{Pic}(R)$. Then $\psi_D(z) = 0$ and hence z lifts to an element of $\mathrm{Br}(U)$.*

Proof. Assume that $[D] = f_*[E]$ for some divisor E on R'. By construction $f^*z = 0$. Hence according (A.3) we have

$$\psi_D(z) = \langle z, D\rangle = \langle f^*z, E\rangle = 0$$

which shows what we want. □

Example A.4. Assume that we have a triple $(R', \tau, \mathscr{L}')$ where R' is a smooth projective curve of genus one, τ is a translation of order n and $\mathscr{L}'$ is a line bundle of degree 3 on R'. With the help of Corollary A.3 we will construct a division algebra $D(R', \tau, \mathscr{L}')$ with center a rational field of transcendence degree two, which is presumably the same as the one which can be obtained taking the function field of a three dimensional Sklyanin algebra [3, 4, 12, 13] associated to the data $(R', \tau, \mathscr{L}')$. In this way we obtain a construction using Brauer group theory (at least in char. zero) of these division algebras (which are very interesting for ring theory). Put $R = R'/\langle\tau\rangle$ and let $f : R' \to R$ be the quotient map. Then the pair (R', τ) defines an element z of $H^1(R, \mathbb{Z}/n\mathbb{Z})$. Let $\mathscr{L}$ be the norm of $\mathscr{L}'$ and use $\mathscr{L}'$ to embed R in $\mathbb{P}^2_k$. As before let U be the complement of R. Then by Corollary A.3 we can lift z to an element A of $\mathrm{Br}(U)$. The generic fiber of A is of the form $M_t(D)$. Then we define $D(R', \tau, \mathscr{L}') = D$.

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