PROPERTIES OF LAMÉ OPERATORS WITH FINITE MONODROMY

by

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Abstract. — This survey paper contains recent developments in the study of Lamé operators having finite monodromy group. We present the approach based on the pull-back theory of Klein, that allowed the description of the projective monodromy groups by Baldassarri ([**Bal81**]), as well as the connection with Grothendieck's theory of dessins d'enfants, that leads to explicit properties and formulae. The results of Beukers and van der Waall ([**BvdW04**]) concerning the full monodromy group are also presented. The last section describes the Lamé operators L_1 with finite monodromy in terms of the values of the Weierstraß zeta function corresponding to the elliptic curve associated to L_1, as well as the connection with the modular forms.

Résumé (**Propriétés des opérateurs de Lamé à monodromie finie**). — Cet article présente quelques développements récents dans l'étude des opérateurs de Lamé à monodromie finie. On décrit l'approche basée sur la théorie des pull-back développée par Klein et utilisée par Baldassarri ([**Bal81**]) pour décrire la monodromie projective. On fait ensuite le lien avec la théorie des dessins d'enfants de Grothendieck, qui amène à des descriptions et à des formules explicites. On revient également sur les résultats de Beukers et van der Waall ([**BvdW04**]) concernant la monodromie. La dernière partie est consacrée à l'étude des opérateurs de Lamé L_1 avec monodromie finie en termes des valeurs de la fonction zéta de Weierstraß correspondant à la courbe elliptique attachée à L_1 et au lien avec les formes modulaires.

1. Introduction

Let C be an algebraic curve defined over $\mathbb{C}$ (smooth, projective and geometrically connected), or, equivalently a compact Riemann surface. We denote $K = K(C)$ the function field of C.

Let D be a nontrivial derivation on K over $\mathbb{C}$ and

$$(1.1) \qquad L = D^m + p_1 D^{m-1} + \cdots + p_{m-1} D + p_m$$

be a linear differential operator of order m on C, where $p_i \in K$ for $i \in \{1, \ldots, m\}$. If $P \in C$ corresponds to the valuation v_P of K and t is a local parameter at P, then locally

$$(1.2) \qquad L = q \left(\frac{d^m}{dt^m} + p_1' \frac{d^{m-1}}{dt^{m-1}} + \cdots + p_{m-1}' \frac{d}{dt} + p_m' \right)$$

where $q, p_i' \in K$, $i \in \{1, \ldots, m\}$. The point P is a regular point for L if $v_P(p_i') \geq 0$ for all $i \in \{1, \ldots, m\}$ and a singular point otherwise. Obviously the set S of singular points of L is finite, let $S = \{P_1, P_2, \ldots, P_r\}$. If $v_P(p_i') \geq -i$ then the singular point P is called regular. At each regular point L has n independent solutions which are holomorphic. We shall suppose that all the singular points of the operators we are dealing with in this paper are regular and, moreover, if P is a regular singular point of an operator L as in (1.1) then L has m independent solutions at P of the form

$$(1.3) \qquad u_i = t^{\alpha_i}(c_0 + c_1 t + \ldots)$$

$i = 0, \ldots, m$, with $\alpha_i \in \mathbb{Q}$. The rational numbers α_i are called the exponents of L at P and they are the roots of a polynomial equation of degree m, the indicial equation. Under these assumptions, if all the exponents are distinct, but differ only by integers, then every solution $y(t)$ is either holomorphic, or can be made so locally around P after a transformation $y = t^\rho y^*$ (see Poole [**Poo60**]).

If $P \in C \setminus S$, analytic continuation of the functions in a basis of solutions in P yields to the monodromy representation

$$(1.4) \qquad \pi_1(C \setminus S) \to GL(m, \mathbb{C})$$

For various points P and different basis of solutions, these representations are conjugated to each other. The image is called the monodromy group of the operator L. It is a subgroup of $GL(m, \mathbb{C})$, well-defined up to conjugation. The monodromy group is in general a subgroup of the differential Galois group attached to the operator L. If the singular points of L are regular, then the differential Galois group and the Zariski closure of the monodromy group coincide.

It is well known that, in general, a differential operator L is parameterised by the set of singular points S, the set E of values of the mr exponents and $v_{g,m}(r)$ accessory parameters: for example (see Ince [**Inc44**] or Dwork [**Dwo90**])

$$(1.5) \qquad v_{0,m}(r) = (m-1)[m(r-2) - 2]/2.$$

Let B be the set of the accessory parameters.

For the rest of this paper we shall consider only second order differential operators. If τ is the ratio of two functions in a basis of the set of solutions of L at an arbitrary

point $P \in C \setminus S$, the analytic continuation of τ along the paths in $\pi_1(P, C \setminus S)$ yields to a map

$$(1.6) \qquad\qquad \pi_1(P, C \setminus S) \to PGL(2, \mathbb{C})$$

The image of this map is called the projective monodromy group of the operator L. Its conjugation class does not depend on P, nor on τ.

If α_1, α_2 are the exponents of the operator L at a point $P \in C$, let $\Delta_{P,L} = |\alpha_1 - \alpha_2|$ be the *exponent difference* of L at P and $\Delta_L = \sum_{P \in \mathbb{P}^1} (\Delta_{P,L} - 1)$. Hereafter, a singular point where the exponent difference is an integer greater than 2 is called a *quasi-apparent singularity.* As in [**BvdW04**], a second order operator L is called *pure* if it has no quasi-apparent singularity.

Definition 1.1. — The couples (C, L), (C', L') are called projectively equivalent if there exists an isomorphism $f : C \to C'$ such that L is a weak pull-back of L' via f.

In this situation, L and L' have the same projective monodromy group and the same exponent differences. Throughout this paper, an abstract operator will be an equivalence class of couples (C, L). Eventually, the curve C may not be mentioned explicitly, if no confusion is possible.

Let now $f : C \to C'$ be a non constant morphism of algebraic curves and L and L' be second order linear differential operators on C and C' respectively. We say that L is a weak pull-back of L' via f if $\tau' \circ f$ is a ratio of independent solutions of L, provided that τ' is a ratio of independent solutions of L'. As we are interested in studying the set of differential operators modulo the projective equivalence, we shall use freely in this paper the notation $f^* L'$ for a weak pull-back of the operator L'. If $L = f^* L'$, it follows immediately that $\Delta_{P,L} = e_P \cdot \Delta_{f(P),L'}$ for any $P \in C$, where e_P is the ramification index of f at P. The Riemann-Hurwitz formula implies (see Baldassarri and Dwork [**BD79**], Lemma 1.5, or Baldassarri [**Bal80**])

$$(1.7) \qquad\qquad \Delta_L + 2 - 2g(C) = \deg f \cdot (\Delta_{L'} + 2 - 2g(C')).$$

2. Second order differential operators with algebraic solutions

The problem we are interested in is the following: which are the conditions that one has to impose on the sets S, E, B for the solutions of the corresponding operator L to be all algebraic over K? A more precise question is the following version of Dwork's accessory parameter problem: *let V be the set of all operators of order 2 on the curve C, with fixed S and E. Let V_1 be the subset of V corresponding to equations with a full set of algebraic solutions. Does V_1 correspond to an algebraic subset of V?*

Remark 2.1. — In this paper, we shall present a global approach to this type of question. Nevertheless, the following connection with the p-adic operators is worth mentioning. Suppose, for simplicity, that $C = \mathbb{P}^1$ and the coefficients of L are in $\overline{\mathbb{Q}}(x)$. One can reduce the coefficients of L modulo almost all primes of the field of definition of L. Also, one can ask about the p-adic behaviour of the solutions near singular points, for various primes p. If a solution of L is algebraic, then for almost all primes the series representing this solution converges and is bounded by unity in the open p-adic disk $D(0, 1^-)$ of radius unity and centre at the origin (where p is the residue characteristic). Dwork formulated the following conjecture in [**Dwo90**]:

Let V be the set of all operators of order n with coefficients in $\overline{\mathbb{Q}}(x)$, with fixed S and E. Let V_1 the subset of V corresponding to equations where solutions converge in $D(t, 1^-)$ for almost all p. Then V_1 corresponds to an algebraic subset of V.

Here, t is a generic point in some transcendental extension of $\mathbb{Q}_p$, $|t|_p = 1$, such that the residue class of t is transcendental over $\mathbb{F}_p$. On the other hand, if an operator L has a full set of algebraic solutions, then for almost all primes the reduced operator has a full set of solutions or, equivalently, its p-curvature is zero. The celebrated p-curvature conjecture of Grothendieck states that the converse is also true: an operator L has a full set of algebraic solutions if and only if the p-curvature of the reduced operator is zero for almost all primes. For more details on p-adic differential operators, see Dwork [**Dwo81**], [**Dwo90**]. For Katz's proof of Grothendieck's conjecture for Picard-Fuchs operators, see Katz [**Kat72**]. We should also mention (see Honda [**Hon81**] and also Katz [**Kat70**]) that nilpotent p-curvature for almost all p implies that the singularities of a linear operator L are regular. Moreover, if this happens for a set of primes of density 1, then the exponents are rational numbers.

If L is a second order differential operator on C, the following properties are equivalent:

1. - L has a full set of algebraic solutions
2. - the monodromy group of L is finite
3. - the projective monodromy group of L is finite and the Wronskian is an algebraic function over K

In this case, the projective monodromy group is conjugated with the Galois group of the extension $K \subset K(\tau)$, where τ is the ratio of two functions in a base of the space of solutions of L.

The problem of determining the linear operators on $\mathbb{P}^1$ with a full set of algebraic solutions, known in the last decades of the XIX-th century as Fuchs' problem, was solved by Schwarz [**Sch72**] for the hypergeometric operators. Those can be written in the following normalised form:

$$(2.8) \qquad H_{\lambda,\mu,\nu} = D^2 + \frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4x(x-1)}$$

where $\lambda + \mu + \nu > 1$. Such an operator has three singular points, 0, 1 and ∞, where the exponent differences $\Delta_{P,H_{\lambda,\mu,\nu}}$ are equal to λ, μ, ν respectively. Using geometric methods and ideas originated in works of Abel and Riemann, Schwarz obtained a table of 15 cases (up to an ordering of λ, μ, ν) when the algebraicity of the solutions is satisfied. He so determined all the second order operators on the projective line, with three singular points and a full set of algebraic solutions.

Schwarz's solution was developed by Klein [**Kle77**], who reduced the list to five essential cases which emphasise the role played by the regular solids. The values of the parameters λ, μ, ν corresponding to hypergeometric operators algebraically integrable, as well as the corresponding projective monodromy groups, are contained in the following table ("the basic Schwarz list"):

(λ, μ, ν)	$\mathbf{G}_{H_{\lambda,\mu,\nu}}$
$(1/n, 1, 1/n)$	C_n, cyclic of order n
$(1/2, 1/n, 1/2)$	D_n, dihedral of order $2n$
$(1/2, 1/3, 1/3)$	$\mathcal{A}_4$, tetrahedral
$(1/2, 1/3, 1/4)$	$\mathcal{S}_4$, octahedral
$(1/2, 1/3, 1/5)$	$\mathcal{A}_5$, icosahedral

Klein also proved that the second order linear differential operators with a full set of algebraic solutions are weak pull-backs, by a rational function, of the hypergeometric operators in the basic Schwarz.

At about the same time, Jordan [**Jor78**] noticed that the algebraicity of all the solutions is equivalent to the finiteness of the monodromy group. He approached Fuchs' problem for second and higher order operators by purely group-theoretic means and he proved that the finite subgroups of $GL(n, \mathbb{C})$ could by classified into a finite number of families, similarly to the case $n = 2$, when there are two infinite families and three other groups (Jordan's finiteness theorem). For a historic survey of Fuchs' problem, the reader may consult Gray [**Gra86**].

It is not due to the lack of interest in the subject that the case of hypergeometric operators remains, up to our days, the only one where the operators with a full set of algebraic solutions are completely determined. A glance to the formula 1.5 tells us that if L is a second order operator on the projective line with three singular points, then there is no accessory parameter. The operator L is rigid, that is, it is completely determined by the singular points and the local exponents, in other words, by the local data. The reader is referred to Katz [**Kat96**] for more details on the rigidity.

If the accessory parameters are present, the problem becomes much more difficult. And this happens as soon as there is a forth singular point. Along with the p-adic machinery and with group theoretic methods, Klein's results have been, in the last

decades, among the main tools in the study of the second order linear operators with finite monodromy.

Baldassarri and Dwork [**BD79**] reconsidered, in a modern language, the hypergeometric case (the basic Schwarz list) and the possibility of obtaining every second order linear operator on the projective line with finite projective monodromy, as a weak bullback of an operator in this list, by a rational function. Baldassarri generalised this property to operators on an arbitrary algebraic curve:

Theorem 2.2 (**[Bal80], Theorem 1.8**). — *Let L be a second order linear differential operator L on an algebraic curve C, with finite projective monodromy group G. Then there exists a unique hypergeometric operator H belonging to the Schwarz list, having the same projective monodromy group G, such that L is a weak pull-back of H via a morphism $f : C \to \mathbb{P}^1$. Moreover, the function f is also unique, modulo Möbius transformations leaving the operator H invariant and permuting its singular points.*

A simple but important remark is the following: suppose that f is ramified over a point $P \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then, there exists a point $Q \in f^{-1}(P)$ where the ramification index of f is greater than 1. As P is a non-singular point of the hypergeometric operator H, it follows that the exponent difference of L at Q is an integer greater than 1. Hence Q is a quasi-apparent singularity of L. So we have:

Proposition 2.3. — *Suppose that the operator L as in Theorem 2.2 is pure. Then the rational function f is ramified at most over 0, 1 and ∞.*

In this case, the local monodromy has distinct eigenvalues at each singular point and the indicial polynomial determines completely the local monodromy (see Katz [**Kat96**]).

A morphism $f : C \to \mathbb{P}^1$ with at most three branching points is called a Belyi function. The celebrated theorem of Belyi [**Bel79**]

states that such a function exists if and only if C is isomorphic to an algebraic curve defined over a number field. For more details concerning the properties of the Belyi functions the reader is referred to Schneps [**Sch94**].

As a corollary of Belyi's theorem, we have

Proposition 2.4 (**see [Liţ02], Corollary 2.3**). — *Let C be an algebraic curve defined over $\mathbb{C}$. If there exists, on C, a second order linear differential operator L, pure and with finite projective monodromy, then the curve C can be defined over a number field. Moreover, L is projectively equivalent with an operator having all the singular points, as well as the accessory parameters defined over a number field.*

If f is a Belyi function, then $f^{-1}([0, 1])$ can be seen as a bipartite graph on the topological model of the curve C, called dessin d'enfants. The two colours correspond to the inverse images of 0 and 1 respectively. A Belyi function f is clean if the ramification index $e_P = 2$ for all $P \in f^{-1}(1)$, and a dessin is clean if the valency of each

vertex marked with one of the two colours is 2. The Grothendieck Correspondence states that we have a bijection between the set of clean Belyi functions and the set of clean dessins d'enfants, both sets being considered modulo obvious equivalence relations. Moreover, there is a correspondence between the ramification data of f and the combinatorial data of the associated dessin - for example, the valency of a vertex is the ramification index of the corresponding element of $f^{-1}(\{0,1\})$. In each cell there is a unique element of $f^{-1}(\infty)$, and the valency of the cell is twice the ramification index of this element (for details, see Schneps [**Sch94**]).

Definition 2.5. — A Belyi function $f : C \to \mathbb{P}^1$ is called a $*$-function if one of the following conditions is satisfied:

  – $C = \mathbb{P}^1$ and $\{0,1,\infty\} \subseteq f^{-1}(\{0,1,\infty\})$;
  – $C = (E,O)$ is an elliptic curve and $O \in f^{-1}(\{0,1,\infty\})$;
  – $g(C) \geq 2$

Following the notation of Beukers and van der Waall [**BvdW04**], let $\mathcal{A}_0$ be the set of pure differential operators, of second order, with finite projective monodromy (where by "differential operator" we understand a couple (C,L), with C an algebraic curve and L a differential operator on C). Let $\sim$ be the equivalence relation defined in Definition 1.1. Theorems 7.1 and 1.2 in [**BvdW04**] can be extended to the case when we consider differential operators on any algebraic curve C, not only on $\mathbb{P}^1$.

Proposition 2.6 (see also [BvdW04], Theorem 7.1). — *Let G be a finite group and $m > 0$. The set $\overline{\mathcal{A}_0}^{G,m} \subset \mathcal{A}_0/\sim$, induced by operators $L \in \mathcal{A}_0$ with fixed projective monodromy G and $\Delta_L < m$, is finite.*

Proof. — Let L be an operator as in the statement of the proposition, defined over an algebraic curve C. Let H_G be the hypergeometric operator in the basic Schwarz list, having the projective monodromy group G, and let $f : C \to \mathbb{P}^1$ be the function that realises L as a weak pull-back of H_G. Formula 1.7 implies that

$$\deg f \leq \frac{\Delta_L + 2}{\Delta_{H_G} + 2} < \frac{M + 2}{\Delta_{H_G} + 2},$$

so $\deg f$ is bounded. On one hand, Proposition 2.3
implies that f is a Belyi function, and on the other hand there are only finitely many isomorphism classes of Belyi functions with bounded degree. The assertion follows. ☐

Remark 2.7. — The hypothesis in [**BvdW04**], Theorem 7.1 (the sum of all the exponent differences not equal to 1 is bounded) implies the hypothesis of Proposition 2.6 (Δ_L is bounded).

The following theorem is a consequence of Proposition 2.6:

Theorem 2.8 (see [BvdW04], Theorem 1.2). — *The set $\mathcal{A}_0/\sim$ is countable.*

3. Lamé operators with algebraic solutions

3.1. Finite projective monodromy for Lamé operators. — We shall describe in this section the second order linear operators on the Riemann scheme, with four singular points and the Riemann scheme

$$(3.9) \qquad \begin{pmatrix} e_1 & e_2 & e_3 & \infty \\ 0 & 0 & 0 & -\frac{n}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} \end{pmatrix} \begin{array}{c} \\ x \\ \end{array} \Bigg)$$

having a full set of algebraic solutions. Here n is a rational number. We shall denote such an operator with L_n and call it a Lamé operator. According to the formula (1.5), any operator L_n depends on an accessory parameter, besides of the degree n and the singular points e_1, e_2, e_3.

We are interested in studying the operators L_n modulo the relation $\sim$ (the degree n is invariant in an equivalence class). In particular, we identify the operators modulo the homographies of $\mathbb{P}^1$, so we can suppose that $e_1 = 0$, $e_2 = 1$ (alternatively, one can take, in a classical manner, $e_1 + e_2 + e_3 = 0$; see Whittaker and Watson [**WW62**] or, more recently, Baldassarri [**Bal81**]). Every equivalence class contains an element of the form

$$(3.10) \qquad L_n = \left(\frac{d}{dx}\right)^2 + \frac{1}{2}\sum_{i=1}^{3}\frac{1}{x - e_i}\frac{d}{dx} - \frac{n(n+1)x + B}{4\prod_{i=1}^{3}(x - e_i)}$$

which is unique modulo homography (see Chiarellotto [**Chi95**], Remark 1.4). Such an operator is classically known as a Lamé operator, and the uniqueness motivates our terminology. Moreover, in [**Chi95**] Chiarellotto gives explicit formulae for determining a normal operator (of the form $\left(\frac{d}{dx}\right)^2 + Q$) from a Lamé operator (3.10).

One obviously has $L_n = L_{-n-1}$. The following theorem allows us to suppose $n > 0$:

Theorem 3.1 ([vdW02], Theorem 6.8.9). — *If L_n has finite monodromy group, then $n \notin [-1, 0]$.*

One can see easily that the Wronskian of a "classical" Lamé operator (3.10) is an algebraic function over $\mathbb{C}(x)$, so the algebraicity of the solutions is equivalent to the finiteness of the projective monodromy group.

It is clear from the Riemann scheme that an operator L_n has a quasi-apparent singularity (that may occur only at ∞) if and only if $n \in \mathbb{Z} + \frac{1}{2}$. In this situation we have:

Theorem 3.2 ([Poo60]). — *Suppose $n \in \mathbb{Z} + \frac{1}{2}$. An operator L_n has a full set of algebraic solutions if and only if its projective monodromy group is the Klein four-group. Moreover, this happens if and only if the accessory parameter and the finite singular points satisfy a polynomial equation $f(B, e_1, e_2, e_3) = 0$, $f \in \mathbb{Q}[X, Y, Z, T]$.*

If $n \notin \mathbb{Z} + \frac{1}{2}$, then we have:

Theorem 3.3 ([Liţ04c], Theorem 2.2; see also [Liţ02], Theorem 4.1)

Fix $n \notin \mathbb{Z} + \frac{1}{2}$ and G a finite group. The set of abstract Lamé operators L_n with projective monodromy group G is finite.

Proof. — The assertion is a consequence of Proposition 2.6.

We can see this more explicitly. According to formula (1.7), the degree of a rational function f that realises a Lamé operator with algebraic solutions as a pull-back of a hypergeometric operator with the same projective monodromy group G is:

G	C_N	D_N $(N > 2)$	$\mathcal{A}_4$	$\mathcal{S}_4$	$\mathcal{A}_5$
$\deg f$	$\frac{nN}{2}$	nN	$6n$	$10n$	$30n$

But f is a $*$-function, and the finiteness of the set of $*$-functions of bounded degree ([**Liţ04b**], Corollary 3.3) implies the assertion. □

In fact, the projective monodromy group of a Lamé operator is never cyclic, nor tetrahedral (Baldassarri [**Bal81**], see also Beukers and van der Waall [**BvdW04**], Theorem 4.1). The latter case follows from the fact that G is generated by elements of order two, corresponding to the local monodromy matrices at the finite singularities, while the tetrahedral group cannot be generated by order two elements. The cyclic case is again a consequence of the fact that the monodromy group is generated by elements of order two, and moreover it is abelian. Hence one has a basis of solutions of the form $\sqrt{p_1}$, $\sqrt{p_2}$, with p_1, p_2 polynomials, as the exponents at the finite points are positive. But this situation is in contradiction with the local exponents at infinity.

Alternatively, one can deduce that the projective monodromy group of L_n can not be cyclic or tetrahedral from combinatorial arguments, after describing the ramification data of a rational function f that would realise L_n as a pull-back of a hypergeometric operator with cyclic or tetrahedral projective monodromy (see [**Liţ04a**]). The same type of arguments can be used for determining the possible values of the degree n for each finite projective monodromy group:

Theorem 3.4 ([Liţ04a], Theorem 3.4; see also [Bal81])

1. *There is no Lamé operator with cyclic projective monodromy group.*
2. *There is no Lamé operator with tetrahedral projective monodromy group.*
3. *If the projective monodromy group of the Lamé operator L_n is octahedral, then $n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{3}(\mathbb{Z} + \frac{1}{2})$.*
4. *If the projective monodromy group of the Lamé operator L_n is icosahedral, then $n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2})$.*
5. *If the projective monodromy group of the Lamé operator L_n is dihedral, then $n \in \mathbb{Z}$. If $n \in \mathbb{Z}$ and the projective monodromy group is finite, then this group is dihedral of order at least 6.*

Baldassarri's approach has been recently reconsidered by Maier [**Mai04**].

Using the combinatorics of the dessins d'enfants associated to the Belyi covers provides evidence that for each possible value of the degree n appearing in Theorem 3.4, there exists a Lamé operator L_n with a full set of algebraic solutions. Indeed, in a recent paper Nakanishi [**Nak**] has constructed inductively, for each such value of n, a dessin corresponding to a Belyi function that realises a Lamé operator L_n with predicted octahedral or icosahedral monodromy, as a pull-back of a hypergeometric operator. The dihedral case was dealt with in [**Liţ02**] for $n = 1$ in [**Dah**] for arbitrary $n \in \mathbb{Z}^*$ (see the next section).

Moreover, one could get information on the field of definition of such a Belyi function, by considering the number of dessins with the same combinatorial data. Therefore, one could estimate the degree of the fields of definition of the accessory parameter and of the singular points.

Theorems 3.3 and 3.4 have the following immediate corollaries, mentioned as open problems by Baldassarri in [**Bal81**], [**Bal87**]:

Corollary 3.5. — *For fixed $n \notin \frac{1}{2}\mathbb{Z}$, there are finitely many Lamé operators L_n with a full set of algebraic solutions.*

Corollary 3.6. — *For fixed n and N, there are finitely many Lamé operators L_n with the dihedral group D_N as projective monodromy group.*

It seems that Dwork proved the second statement and we suppose that he used Klein's theory of pull-backs and p-adic arguments like in [**Dwo90**]. Though, his argument was never published (see Baldassarri [**Bal87**], Singer [**Sin93**] or Morales Ruiz [**MR99**]).

3.2. The dihedral case. — According to Theorem 3.4, the projective monodromy group of a Lamé operator L_n is dihedral of order greater than 6 if and only if $n \in \mathbb{Z}^*$ (we already saw that we can suppose $n > 0$, see also [**vdW02**], Corollary 6.7.5). In this case the Belyi function $f : \mathbb{P}^1 \to \mathbb{P}^1$ that realises L_n as a pull-back of a hypergeometric operator with the same projective monodromy group has the ramification data consistent with one of the following tables (where the first column contains the branching points and the first row the ramification points of f, the other entries representing the distribution of these points in the ramified fibres and the multiplicities) (see Chiarellotto [**Chi95**]):

		0	1	λ	∞	
(Ia)	0					$+nN/2$ points with multiplicity 2
	1					$+n$ points with multiplicity N
	∞	1	1	1	$2n+1$	$+1/2(nN - 2n - 4)$ points with multiplicity 2

(Ib)

	0	1	λ	∞	
0			1		$+1/2(nN-1)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞	1	1		$2n+1$	$+1/2(nN-2n-3)$ points with multiplicity 2

(Ic)

	0	1	λ	∞	
0		1	1		$+1/2(nN-2)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞	1			$2n+1$	$+1/2(nN-2n-2)$ points with multiplicity 2

(Id)

	0	1	λ	∞	
0	1	1	1		$+1/2(nN-3)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞				$2n+1$	$+1/2(nN-2n-1)$ points with multiplicity 2

(II)

	0	1	λ	∞	
0					$+nN/2$ points with multiplicity 2
1		$N/2$	$N/2$		$+n-1$ points with multiplicity N
∞	1			$2n+1$	$+1/2(nN-2n-2)$ points with multiplicity 2

By generalising the arguments in [**Liţ02**], Dahmen [**Dah**] explained how to draw the dessins associated to these covers and how to count the non-isomorphic ones. He obtained the following theorem:

Theorem 3.7 (**[Dah], Theorem 1**). — *Let $\mathcal{C}(n,N)$ be the number of non-homographic covers $\mathbb{P}^1 \to \mathbb{P}^1$ which transform by pull-back a hypergeometric operator H_{D_N} in the basic Schwarz list into a Lamé operator L_n and $\mathcal{L}(n,N)$ the number of non-homographic Lamé operators L_n with finite dihedral monodromy group of order $2N$. We have*

$$(3.11) \qquad \mathcal{C}(n,N) \;=\; \frac{n(n+1)(N-1)(N-2)}{12} + \frac{2}{3}\varepsilon(n,N)$$

$$(3.12) \qquad \mathcal{L}(n,N) \;=\; \sum_{N'|N} \mathcal{C}(n,N')\mu\left(\frac{N}{N'}\right)$$

where μ is the Möbius function and $\varepsilon(n,N)$ is 1 if $3|N$ and $n \equiv 1(\mod 3)$, and 0 otherwise.

As the case $n=1$, solved in [**Liţ02**], is relevant for the Section 5, we shall give the argument hereafter. A different approach to this case is used by Chiarellotto [**Chi95**].

We shall consider the situations described in the tables I, (a)-(d) and II, for $n=1$. Let f be a function with the ramification data as in table II. After composing f with

the automorphism of $\mathbb{P}^1$ that switches 1 and ∞ and leaves 0 fixed ($x \mapsto \frac{x}{x-1}$), one gets a Belyi cover with the following ramification data:

	0	1	λ	∞	
0					$+N/2$ points with multiplicity 2
∞		N/2	N/2		
1	1			3	$+(N-4)/2$ points with multiplicity 2

It follows that the associated dessin has two cells with the same valency, $N/2$. So both should contain the same number of interior edges. But there is only one vertex with valency 1, so there is only one cell containing interior edges. The contradiction implies that the case II is impossible.

Let f be a Belyi function with the ramification data described in one of the tables Ia-Id. If we compose f with the same automorphism as before, we obtain a Belyi function totally ramified above ∞. Hence the associated dessin is a tree. It has one vertex with valency 3 (the "root"), three vertexes with valency 1 (the "leafs") and $N - 3$ vertexes with valency 2, so it has the following shape:

The difference between the situations (a)-(d) consists in the possible colours of the leafs. In all the cases, the sum of the lenghts of the three branches if N, the degree of the function f. So the number of such graphs is the number of triples (a, b, c) with $1 \leq a \leq b \leq c \leq N$ such that $a + b + c = N$. A simple combinatorial computation gives:

$$(3.13) \qquad \mathcal{C}(1, N) = \frac{(N-1)(N-2)}{6} + \frac{2\varepsilon}{3}$$

where $\varepsilon = 1$ if $3|N$ and $\varepsilon = 0$ if not.

The pull-back of a hypergeometric operator H_{D_N} in the basic Schwarz list, via a rational function f as above, is a Lamé operator L_1 with dihedral projective monodromy $D_{N'}$, where $N' \mid N$. Using the uniqueness property in Theorem 2.2, and [**Liţ02**] Proposition 3.1 (describing rational functions that transform H_{D_N} in $H_{D_{N'}}$ via pull-back), we obtain:

$$(3.14) \qquad \mathcal{C}(1, N) = \sum_{N'|N \ , \ N' \neq 2} \mathcal{L}(1, N')$$

hence

$$(3.15) \qquad \mathcal{L}(1, N) = \sum_{N'|N} \mathcal{C}(1, N')\mu\left(\frac{N}{N'}\right)$$

4. The full monodromy group

Beukers and van der Waall studied in [**BvdW04**] the full monodromy group of a Lamé operator with algebraic solutions. This paragraph describes the main result of their work.

As the local monodromy matrices at the finite singularities have eigenvalues ± 1 and they generate the monodromy group M, it follows that M is a reflection group. Using the properties and the classification of the finite reflection groups, Beukers and van der Waall prove the following theorem, that can be regarded as the non-projective version of Theorem 3.4:

Theorem 4.1 ([**BvdW04**], **Theorem 4.4**). — *Suppose that the Lamé operator L_n has finite monodromy group M. Then M must be one of the following*

1) $M = G(4, 2, 2)$, *and then* $n \in \mathbb{Z} + \frac{1}{2}$.
2) $M = G(N, N, 2)$ $(N \geq 3, N \neq 4)$, *and then* $n \in \mathbb{Z}$.
3) $M = G_{12}$, *and then* $n \in \frac{1}{2}(\mathbb{Z} + \frac{1}{2})$.
4) $M = G_{13}$, *and then* $n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2})$.
5) $M = G_{22}$, *and then* $n \in \frac{1}{3}(\mathbb{Z} + \frac{1}{2}) \cup \frac{1}{5}(\mathbb{Z} + \frac{1}{2})$.

Here, $G(4, 2, 2)$ is the group of order 16 generated by

$$\begin{pmatrix} e^{\frac{\pi}{2}} & 0 \\ 0 & e^{-\frac{\pi}{2}} \end{pmatrix} \quad , \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The projective group of $G(4, 2, 2)$ is D_2, the Klein four-group.

The group $G(N, N, 2)$ is dihedral of order $2N$, with the projective group D_N if N is odd or $D_{N/2}$ if N is even.

The group G_{12} is of order 48 and is generated by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} \quad , \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

and the projective group is isomorphic to the octahedral group.

The group G_{13} is generated by the matrices in G_{12} together with $i \cdot Id$. It is of order 96 and the projective group is again isomorphic to the octahedral group.

The group G_{22} is of order 120 and is generated by

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad , \quad \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5 - \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5^4 - \zeta_5 \end{pmatrix} \quad , \quad \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^3 - \zeta_5 & 1 - \zeta_5 \\ \zeta_5^4 - 1 & \zeta_5^2 - \zeta_5^4 \end{pmatrix}$$

where ζ_5 is a primitive 5-th root of unity. The projective group is isomorphic to the icosahedral group.

It is easy to see that Theorem 4.1 agrees with and in fact implies Theorem 3.4.

In [**vdW02**], Chapter 6, van der Waall gives, for each group M in Theorem 4.1, an algorithm that produces for every n the list of Lamé operators L_n with monodromy group M. Some examples are also presented in [**BvdW04**].

5. Lamé operators, elliptic curves and Hecke modular forms

In this last section, we show how the study of Lamé operators with dihedral projective monodromy is related to modular curves. We mainly follow the ideas of [**BvdW04, Zap97**].

5.1. Elliptic curves, Baldassarri's criterion. — In the following, we adopt the notation of §3, especially for the definition of the Lamé operators 3.10. From now on, we restrict to the case $n = 1$. As in [**Zap04**] we start by showing how these operators naturally correspond to pairs (E, P) where E is an elliptic curve over $\mathbb{C}$ and P is a point on it, different from the origin 0_E: starting from an operator L_1, one considers the elliptic curve E defined by the affine Weierstraß equation

$$E : y^2 = 4h(x)$$

where $h(x) = \prod_{i=1}^3 (x - e_i)$, equipped with its canonical degree two cover

$$\pi : E \to \mathbb{P}^1$$

which sends (x, y) to x. With this notation, the point P is one of the two elements of E for which $\pi(P) = B$. Conversely, given a pair (E, P), just choose a Weierstraß equation for E as above and then consider the Lamé operator L_1 associated to h and $B = \pi(P)$. Clearly, both of these constructions are not well-defined, but they induce a bijection between the set of equivalence classes of Lamé operators L_1 and the isomorphism classes of pairs (E, P).

We now give an existence criterion due to F. Baldassarri which characterises Lamé operators with dihedral projective monodromy in terms of properties of the associated elliptic curve.

Proposition 5.1 (see [Bal87], §2). — *Let (E, P) be a pair associated to a Lamé operator L_1. The following conditions are equivalent:*

1. *L_1 has finite (projective) monodromy.*
2. *There exists a rational function g on E having a unique pole at P, a unique zero at $-P$ and such that dg has a double zero at 0_E.*

Remark 5.2. — Let g be as in Proposition 5.1, of minimal degree. Setting $N = \deg(g)$ then P is a torsion point of E. Its order M is equal either to N or $2N$ and coincides with half of the order of the full monodromy of the Lamé operator. In any case, the order of the projective monodromy is equal to $2N$. In particular, the full monodromy is equal to the projective monodromy if and only if N is odd.

Remark 5.3. — Suppose that the Lamé operator L_1 has finite monodromy and let f be the Belyi function associated to it (cf. §3). Then the rational function g is the composition of f with the cover π. In terms of dessins d'enfants, g corresponds to a clean dessin, as described in the following picture, which is the preimage under π of the tree associated to f. The 2-torsion points of the curve are mapped to the central

vertex and the three ends of the tree (up to automorphism, we can always assume that the origin of E is mapped to the central vertex). Following [**Zap97**] we can attach a torsion point on E to such a dessin d'enfant and it turns out that it coincides with the torsion point in Baldassarri's criterion.

5.2. Elliptic functions. — The aim of this paragraph is to translate Baldassarri's criterion in terms of elliptic functions. Suppose that $E = \mathbb{C}/\Lambda$ is an elliptic curve, obtained as the quotient of $\mathbb{C}$ with respect to a lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, with $\tau \in \mathcal{H}$ (here $\mathcal{H}$ denotes the upper half plane). The curve E can be given by the affine equation

$$E : y^2 = 4h(x) = 4x^3 - g_2 x - g_3$$

where $g_2 = g_2(\tau)$ and $g_3 = g_3(\tau)$ denote the usual Eisentein series. This identification is obtained via the Weierstraß $\wp$ function

$$\mathbb{C} \to \mathbb{P}^2$$
$$z \mapsto [\wp(z), \wp'(z), 1]$$

The above map is just the universal cover $\rho : \mathbb{C} \to E$. Let $P \in E \setminus 0_E$ and consider the Lamé operator L_1 associated to the pair (E, P). Following Remark 5.2, in order to have finite monodromy, the point P must be a torsion point of E. If we denote by M its order then the $\mathrm{PSL}_2(\mathbb{Z})$-action on $\mathcal{H}$ allows us to reduce to the case $P = 1/M \in \mathbb{C}$. In this case, up to a multiplicative constant, there exists a unique function g having a unique zero at P and a unique pole at $-P$, both of order M. In terms of the Weierstraß σ function, its pull-back to $\mathbb{C}$ can be expressed as

$$(5.16) \qquad\qquad g(z) = \frac{\sigma(z - 1/M)^M}{\sigma(z + 1/M)^M} e^{4\eta_1 z}$$

where $\eta_1 = \zeta(1/2)$ is the quasi-period (we refer to [**Sil94**] for a complete review on elliptic functions). By taking logarithmic derivatives, we then deduce that dg has a zero at 0_E (which automatically implies that it is a double zero) if and only if

$$(5.17) \qquad\qquad M\zeta(1/M) = 2\zeta(1/2)$$

where ζ is the Weierstraß zeta function associated to τ. Summarising, Baldassarri's criterion can be translated as follows:

Proposition 5.4. — *The notation and hypothesis being as above, the operator L_1 has dihedral monodromy of order $2M$ if and only if the identity 5.17 holds.*

5.3. Modular forms. — We finally translate Proposition 5.4 in term of modular forms. Taking advantage of the classical identity

$$\zeta(z) = \zeta(z, \tau) = \frac{1}{z} + \sum_{m>0} G_{2m+2}(\tau) z^{2m+1}$$

one can easily prove that the function

$$\varphi_M(\tau) = M\zeta(1/M, \tau) - 2\zeta(1/2, \tau)$$

defined on the whole upper half plane $\mathcal{H}$ is a weight one modular form for the congruence subgroup

$$\Gamma_1(M) = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod M \right\}$$

Now, the modular curve

$$Y_1(M) = \mathcal{H}/\Gamma_1(M)$$

naturally parameterises pairs (E, P) where $E/\mathbb{C}$ is an elliptic curve and P is a point on it of exact order M. Moreover, the square of the modular function φ_M defines a regular differential form ω_M on $Y_1(M)$ and Proposition 5.4 leads to the following result:

Theorem 5.5 (see [Zap97], §2.5.5). — *There is a bijection between the set of equivalence classes of Lamé operators with dihedral monodromy of order $2M$ and the set of zeroes of the differential form ω_M on $Y_1(M)$.*

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