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A REMARK ABOUT THE PAINLEVÉ TRANSCENDENTS

by

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Abstract. **—** The Painlevé equations are Hamiltonian systems that, except for Painlevé I, depend on parameters. For some values of the parameters "classical" particular solutions, either algebraic or transcendent, are known. To such equations the Galoisian method is then relevant: the Hamiltonian system is not completely integrable by means of rational or meromorphic first integrals provided that the identity component of the Galois group of the variational equation along such a solution is non-commutative.

We prove with this method the non-complete integrability by rational (or even, meromorphic at infinity) first integrals of a discrete sub-family of the Painlevé II equations.

Résumé **(Une remarque sur les transcendantes de Painlevé).** **—** À l'exception de l'équation de Painlevé I, les équations de Painlevé sont des systèmes hamiltoniens qui dépendent de paramètres. Pour certaines valeurs de ceux-ci, elles admettent des solutions particulières « classiques » algébriques ou transcendantes. On peut alors leur appliquer la méthode galoisienne : un système hamiltonien n'est pas complètement intégrable en termes d'intégrales premières rationnelles ou méromorphes dès lors que la composante neutre du groupe de Galois différentiel de l'équation variationnelle le long d'une telle solution est non-commutatif.

Nous établissons par cette méthode la non-intégrabilité en termes d'intégrales premières rationnelles (voire même, méromorphes à l'infini) d'une sous-famille discrète des équations de Painlevé II.

1. Introduction

The Painlevé transcendents are the solutions of the six Painlevé's families of equations. This contribution is devoted to prove a non-integrability result for a discrete subfamily of Painlevé II equation:

$$\ddot{x} = 2x^3 + tx + \alpha,$$

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where α is a complex parameter. More concretely, it is known that the Painlevé equations can be written as Hamiltonian systems. We shall prove that for $\alpha \in \mathbf{Z}$, the corresponding Hamiltonian system of Painlevé II is non-integrable by *rational* first integrals. Our method relies on the Galois differential approach to the non-integrability of Hamiltonian systems by means of the variational equation along a particular solution (see [**4**] and [**3**]). So, we shall use a version of a joint result of the author with Jean-Pierre Ramis on a necessary condition of integrability of a Hamiltonian system by means of rational first integrals: when the variational equation along the solution has irregular singular points at the infinity, if the Hamiltonian system is integrable by means of rational first integrals, necessarily the identity component of the Galois group of the variational equation must be a commutative group.

Here we illustrate our approach to non-integrability with Painlevé II but, of course, we believe that similar studies can be done for the others Painlevé families with rational particular solutions.

The idea of this contribution comes from some discussions with J.-A. Weil (see also section 4).

Along this contribution we assume that the reader is familiarized with the main definitions and results of the Galois theory of linear differential equations (see [**7**] for a standard reference, or [**3**], chapter 2, for the main definitions and results useful here).

2. Non-integrability Theorem

Let us consider a complex symplectic analytic manifold of dimension $2n$ and

$$\dot{z} = X_H(z) \tag{1}$$

an analytic Hamiltonian system defined on it. Let Γ the Riemann surface corresponding to an integral curve $z = z(t)$ (which is not an equilibrium point) of X_H. Then we can obtain the variational equations along Γ,

$$\dot{\xi} = X'_H(z(t))\xi. \tag{2}$$

Furthermore the coefficients of the matrix $X'_H(z(t))$ are holomorphic on Γ.

By using the linear first integral $dH(z(t)$ of the variational equation it is possible to reduce it by one degree of freedom, and obtain the so called normal variational equation

$$\dot{\xi} = JS(t)\xi,$$

where, as usual,

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

is the matrix of the symplectic form (of dimension $2(n-1)$). Furthermore the coefficients of the matrix $S(t)$ are holomorphic on Γ.

Now, we shall complete the Riemann surface Γ with some equilibrium points and (possibly) the point at infinity, in such a way, that the coefficients of the matrix $S(t)$ are meromorphic on this extended Riemann surface $\overline{\Gamma} \supset \Gamma$. So, the field of coefficients K of the variational equation (and of the normal variational equation) is the field of meromorphic functions on $\overline{\Gamma}$. To be more precise, $\overline{\Gamma}$ is contained in the Riemann surface defined by the desingularization of the analytical (in general singular) curve C in the phase space given by the integral curve $z = z(t)$ with their adherent equilibrium points, the singularities of the Hamiltonian system and the points at infinity.

Then, in the above situation, it is proved in [**4**] the following result:

Theorem 1

(I) Assume the points at the infinity of the variational equation (2) are regular singular points. A necessary condition for the existence of n meromorphic first integrals of X_H in involution and independent in a neighborhood of the analytical curve C (not necessarily on C itself) is that the identity component of the Galois groups of the variational equation and of the normal variational equation must be a commutative group.

(II) Assume that between the points at the infinity of the variational equation (2) there are irregular singular points. A necessary condition for the existence of n rational first integrals of X_H in involution and independent in a neighborhood of the analytical curve C (not necessarily on C itself) is that the identity component of the Galois groups of the variational equation and of the normal variational equation must be a commutative group.

(III) Assume, as in (II), that between the points at the infinity of the variational equation (2) there are irregular singular points. A necessary condition for the existence of n germs of meromorphic first integrals of X_H in involution and independent in a neighborhood of these points at infinity of the analytical curve C (not necessarily on C itself) is that the identity component of the (local) Galois groups of the variational equation at these points must be a commutative group.

We recall that when there are n independent first integrals in involution of X_H ones says that the Hamiltonian system (1) is integrable. Hence, the above theorem gives three non-integrability criteria: (I) by meromorphic first integrals, (II) by rational first integrals and (III) by local meromorphic first integrals. See [**3**] for other references and more information.

3. Application to Painlevé II

The second Painlevé transcendent is given by the solutions of the Painlevé II equation

$$\ddot{x} = 2x^3 + tx + \alpha, \tag{3}$$

being α a complex parameter.

Since the work of Malmquist we know that the Painlevé transcendents can be expressed as Hamiltonian systems of $1 + 1/2$ degrees of freedom. For Painlevé II the Hamiltonian is

$$H_0(y, x, t) = \frac{1}{2}y^2 - (x^2 + \frac{1}{2}t)y - (\alpha + \frac{1}{2})x,$$

and the differential equation (3) is equivalent to the Hamiltonian system

$$\dot{x} = \frac{\partial H_0}{\partial y} = y - x^2 - \frac{1}{2}t, \quad \dot{y} = -\frac{\partial H_0}{\partial x} = 2xy + \alpha + \frac{1}{2}$$

([**2, 5**]).

Now, by a standard procedure in Hamiltonian dynamics, from the above non-autonomous Hamiltonian system we can obtain a two degrees of freedom autonomous Hamiltonian system such that the non-autonomous system is included as a subsystem. For the Hamiltonian H_0, it is given by

$$H(y, x, z, e) = H_0(x, y, z) + e.$$

So, the associated Hamiltonian system is

$$\begin{array}{c} \dot{x} = y - x^2 - \frac{1}{2}z, \\ \dot{y} = 2xy + \alpha + \frac{1}{2}, \\ \dot{z} = 1, \\ \dot{e} = \frac{1}{2}y. \end{array} \tag{4}$$

It seems clear that the dynamical system (4) is equivalent to the Painlevé II equation (3), in the sense that from the solutions of one of them we obtain immediately the solutions of the other. In particular, for any reasonable meaning of the word "integrable", the integrability of one of them implies the integrability of the other. We remark that the function $e(t) = \frac{1}{2}\int y(t)dt$ is very related to the τ function of the Painlevé equation (3) ([**6**]).

The variational equation along $\Gamma: \ x = x(t), \ y = y(t), \ z = z(t), \ e = e(t)$ is

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} -2x(t) & 1 & -\frac{1}{2} & 0 \\ 2y(t) & 2x(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}. \tag{5}$$

The normal variational equation is given by

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -2x(t) & 1 \\ 2y(t) & 2x(t) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{6}$$

Given a differential system

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t)) & d(t)) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{7}$$

with coefficients in a differential field K, by an elimination process it is equivalent to the second order equation

$$\ddot{\xi} - (a(t) + d(t) + \frac{\dot{b}(t)}{b(t)})\dot{\xi} - (\dot{a}(t) + b(t)c(t) - a(t)d(t) - \frac{a(t)\dot{b}(t)}{b(t)})\xi = 0, \tag{8}$$

where $\xi := \xi_1$. We remark that the equations (8) and (9) are equivalent in the sense that they represent the same D-module. In particular, the Galois groups of both equations are the same.

Hence the normal variational equation (7) is equivalent to the second order equation

$$\ddot{\xi} - (2y(t) - 2\dot{x}(t) + 4x^2(t))\xi = 0. \tag{9}$$

Now by using the Hamilton equations (4) and taking $z(t) = t$, we obtain

$$\ddot{\xi} - (6x^2(t) + t)\xi = 0. \tag{10}$$

Now we fix $\alpha = 1$. Then it is well-known that the equation (3) has the particular solution (see, for instance, [**1**])

$$x = -\frac{1}{t}$$

and the associated Hamiltonian system (4) has the particular rational solution

$$\Gamma : \ x(t) = -\frac{1}{t}, \ y(t) = \frac{2}{t^2} + \frac{t}{2}, \ z(t) = t, \ e(t) = -\frac{1}{t} + \frac{t^2}{8}. \tag{11}$$

For this particular solution, (10) is given by

$$\ddot{\xi} - (\frac{6}{t^2} + t)\xi = 0. \tag{12}$$

By means of the change of variable $\xi(t) = t^{1/2}\eta(x)$, $x = i\frac{2}{3}t^{3/2}$, it is converted in Bessel's equation

$$x^2\frac{d^2\eta}{dx^2} + x\frac{d\eta}{dx} + (x^2 - n^2)\eta = 0, \tag{13}$$

with $n = 5/3$.

Now it is well-known that when $n \notin \mathbf{Z} + 1/2$ the identity component of Galois group of Bessel's equation is non-commutative, indeed, for these values the Galois group is $SL(2, \mathbf{C})$ (see, [**3**], section 2.8.2, for a simple proof using Stokes matrices). As the point at $z = t = \infty$ is an irregular singular point of the variational equation, by Theorem 1, (II), we have proved the following proposition:

Proposition 1. — *For $\alpha = 1$, the Hamiltonian system (4) associated to the Painlevé II equation is not integrable by means of rational first integrals.*

Furthermore, it a classical fact that not only for $\alpha = 1$, but for any integer α the Painlevé II equation has particular rational solutions (such solution is (5) for $\alpha = 1$) and there are rational changes of variables in the phase variables called Bäcklund (or

canonical) transformations between the members of this discrete family of Hamiltonian systems ([**1, 6**]). Hence if one of them is non-integrable by rational first integrals, any member of this family satisfies the same property. We have proved the following:

Corollary 1. — *For $\alpha \in \mathbf{Z}$, the Hamiltonian system (4) associated to the Painlevé II equation is not integrable by means of rational first integrals.*

4. Remarks

Remark 1. — As pointed out to me by K. Okamoto (and other colleagues in the meeting) if instead of $\alpha = 1$, we take $\alpha = 0$ with particular solution $x = 0$, we obtain Airy's equation in (11) as normal variational equation and the above result follows in a more direct way. In fact, it seems that Airy's equation was obtained as normal variational equation of this element of the Painlevé II family some time ago by J.-A. Weil following a suggestion by P. Clarkson ([**8**]).

Remark 2. — By using Theorem 1 (III) it is possible to obtain some refinement of Proposition 1 and Corollary 1. So, as Bessel's equation (or either Airy's equation if ones prefer to use the *Remark 1* above) has $SL(2\mathbf{C})$ as local Galois group at $x = \infty$, the identity component of the local Galois group of the variational equation at $z = t = \infty$ is not commutative. Hence, for $\alpha \in \mathbf{Z}$ the Hamiltonian system (4) associated Painlevé II equation is not integrable by germs of meromorphic first integrals in a neighborhood of $z = t = \infty$. This remark was indirectly motivated by an observation of the anonymous referee.

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