

ON THE ALTERNATE DISCRETE PAINLEVÉ EQUATIONS AND RELATED SYSTEMS

by

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Abstract. — We examine the family of discrete Painlevé equations which were introduced under the qualifier of “alternate”. We show that there exists a transformation between the two canonical forms of these equations, and we proceed to link these forms to the contiguity relations of the continuous P_{VI} . We describe the full degeneration cascade of this contiguity, obtaining all related discrete Painlevé equations (among which, one which has never been derived before) as well as mappings which are solvable by linearisation.

Résumé (Sur les équations discrètes alternatives de Painlevé et les systèmes associés)

Nous étudions la famille des équations de Painlevé discrètes dites « alternatives ». Nous exhibons une transformation entre les deux formes canoniques et nous relierions celles-ci aux relations de contiguïté de l'équation de Painlevé continue P_{VI} . Nous décrivons la cascade de dégénérescence complète liée à cette contiguïté ; nous explicitons toutes les équations de Painlevé discrètes correspondantes (dont une inconnue à ce jour) ainsi que des applications résolubles par linéarisation.

1. Introduction

While the discrete Painlevé equations (d- \mathbb{P} s) have properties which mirror those of their continuous counterparts, there exists an aspect where the two families differ drastically: it concerns the abundance of the two sets of equations. The continuous Painlevé equations are traditionally given under six canonical forms [6] and, while the situation is somewhat more complicated than that [2], it remains that their number is restricted and small. The number of the known discrete Painlevé equations, on the other hand, has been steadily increasing resulting to, literally, dozens of various discrete analogues of the discrete Painlevé transcendental equations [4]. To fix the ideas, we remind the reader that the term discrete Painlevé equations is used to designate a nonlinear, nonautonomous, integrable, second order mapping, the continuous limit of

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which is a Painlevé equation. This last feature has been the source of two difficulties. First, as was shown in various works, and proven in a systematic way by Sakai [15], the discrete Painlevé equations may contain up to seven parameters while the number of parameters of the continuous Painlevé equations cannot exceed four (in the case of P_{VI}). Thus, all continuous limits of discrete Painlevé equations, with a number of parameters at least equal to four, are constrained to lead to P_{VI} . Second, even for discrete equations with a number of parameters less than four, one has a profusion of equations with the same continuous limit. What is most unfortunate is that, when the discrete Painlevé equations were first discovered, their naming was based essentially on their continuous limit [13]. Thus, a $d\mathbb{P}$ with continuous limit P_I was called “discrete P_I ” and so on. These difficulties were alleviated later, thanks to the Sakai classification, based on the affine Weyl group that describes the transformations of each $d\mathbb{P}$. But the traditional names of the $d\mathbb{P}$ s, once introduced, turned out to be impossible to eradicate. Among the various ways to deal with the nomenclature difficulty was the introduction of qualifiers like “standard”, “alternate”, “asymmetric” and so on.

Thus, when in [3] we derived the $d\mathbb{P}$:

$$(1.1) \quad \frac{z_{n-1} + z_n}{1 - x_{n-1}x_n} + \frac{z_n + z_{n+1}}{1 - x_nx_{n+1}} = x_n + \frac{1}{x_n} + 2z_n + 2\mu$$

and found that its continuous limit was P_{II} , we dubbed it alternate $d\text{-}P_{II}$ (“alternative” could have been a better choice of adjective), in order to distinguish it from the “standard” $d\text{-}P_{II}$, $x_{n+1} + x_{n-1} = (z_n x_n + a)/(1 - x_n^2)$. In the same paper, the alternate $d\text{-}P_I$ was also obtained:

$$(1.2) \quad \frac{z_{n-1} + z_n}{x_{n-1} + x_n} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = x_n^2 + 1$$

As a matter of fact, (1.2) is the difference equation obtained by Jimbo and Miwa in [7] from the contiguity relations of the solutions of the (continuous) Painlevé II.

The alternate $d\text{-}P_{II}$ has been the object of a very detailed study [8], where we have presented its Lax pair, Miura transformations, auto-Bäcklund transformations, special solutions, and so on. Moreover, the study of alternate $d\text{-}P_{II}$ has revealed the property of self-duality, which has been crucial for the geometrical description of discrete \mathbb{P} s in terms of affine Weyl groups.

In this paper, we examine the equations of the “alternate” family, from a slightly different point of view. We show in particular how they can be derived from the contiguity relations of the solutions of the continuous P_{VI} equation. While the systematic application of this approach mostly leads to known $d\mathbb{P}$ s, we obtain also one new $d\mathbb{P}$ which has an unusual form. We show how we can, starting from the contiguity relation, obtain also discrete equations which are not $d\mathbb{P}$ s but linearisable mappings.

2. The canonical form of alternate d- \mathbb{P} s

Finding the canonical form of a given equation, be it differential or difference, is a highly nontrivial task. The criteria of “canonicity” are not always explicit and thus, sometimes, the choice of the canonical form is a question of ... choice. For the discrete \mathbb{P} s, both difference- and q -, we have presented in [12] an approach which classified the forms based on the QRT matrices of the mapping that one finds in the autonomous limit of the d- \mathbb{P} . However, this approach concerns only what we have called the “standard” family of d- \mathbb{P} s, and thus does not apply to the alternate forms.

For d- \mathbb{P} s, the only transformations that are allowed in order to bring a given d- \mathbb{P} under canonical form are homographic transformations. For reasons that will become obvious in the next section, we have sought a transformation which would bring the l.h.s. of the alternate d- P_{II} equation to the form of the l.h.s. of the alternate d- P_I . This transformation turns out to be simply:

$$(2.1) \quad y = \frac{x+1}{x-1}$$

Thus, starting from (1.1) we obtain the mapping:

$$(2.2) \quad \frac{z_{n-1} + z_n}{y_{n-1} + y_n} + \frac{z_n + z_{n+1}}{y_n + y_{n+1}} = \frac{4(z_n y_n + \mu)}{y_n^2 - 1} + \frac{4(y_n^2 + 1)}{(y_n^2 - 1)^2}$$

While the l.h.s. of the equation becomes identical to that of alternate d- P_I , the r.h.s. becomes substantially more complicated.

At this point, it is interesting to notice that the transformation (2.1) is an involution, and therefore it transforms the l.h.s. of the alternate d- P_{II} into that of the alternate d- P_I and *vice-versa*. Indeed applying the transformation (2.1) to (1.2) we obtain:

$$(2.3) \quad \frac{z_{n-1} + z_n}{1 - y_{n-1} y_n} + \frac{z_n + z_{n+1}}{1 - y_n y_{n+1}} = \frac{4z}{1 - y_n} + \frac{4y_n(y_n^2 + 1)}{(1 - y_n)^4}$$

Finally, we wish to point out another interesting transformation that exists for equations of the form of alternate d- P_I and alternate d- P_{II} . It consists in simply inverting x . For an equation of the form (1.1) and r.h.s. $R(x)$ we find that, after the transformations that restore the l.h.s. to its initial form, the r.h.s. becomes $R'(x) = 4z - R(\frac{1}{x})$. In particular, for the alternate d- P_{II} we find that the equation is invariant if we invert x provided we change the sign of x and μ . In the alternate d- P_I case, if we start from an equation of the form (1.2) and r.h.s. $R(x)$, we obtain, after the proper manipulations so as to leave the l.h.s. invariant, a new r.h.s. $R'(x) = \frac{4z}{x} - \frac{1}{x^2} R(\frac{1}{x})$.

The transformations presented in this section do show that there is no reason to prefer an alternate d- P_I form to an alternate d- P_{II} one, and *vice-versa*. Still, they cannot settle the question of finding the canonical form of the alternate d- \mathbb{P} s. In order to provide a satisfactory answer we must go back to the origin of these equations. As we have shown in [3], these d- \mathbb{P} s stem from contiguity relations of the continuous P_{II}

and P_{III} respectively. Thus, the question of canonical forms of the alternate d- \mathbb{P} s can be recast as a question on the canonical form of contiguity relations of continuous \mathbb{P} s. This makes possible to enlarge the scope of the investigations and analyse in more generality the discrete \mathbb{P} s that appear as contiguities.

3. Discrete \mathbb{P} s as contiguity relations of continuous \mathbb{P} s

Since we are going to examine the relations of discrete to continuous \mathbb{P} s through the contiguities of the latter, it is natural to start with the most general continuous \mathbb{P} , namely P_{VI} . The discrete \mathbb{P} s related to P_{VI} have been examined already in [10], and also in [9]. In what follows, we shall present a somewhat different approach which will allow the treatment of the equations involved on the same footing.

The continuous P_{VI} equation is traditionally given in the form

$$(3.1) \quad w'' = \frac{1}{2} \left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t} \right) w'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t} \right) w' + \frac{w(w-1)(w-t)}{2t^2(t-1)^2} \left(\alpha^2 - \frac{\beta^2}{w^2} + \frac{\gamma^2(t-1)}{(w-1)^2} + \frac{(1-\delta^2)t(t-1)}{(w-t)^2} \right)$$

where in this particular parametrisation α , β , γ and δ are exactly the monodromy exponents θ_∞ , θ_0 , θ_1 and θ_t . In this form it is assumed that three of the singular points of P_{VI} are located at the fixed points $\infty, 0, 1$, while the last singular point at t remains movable. While this is an assumption that simplifies substantially the form of P_{VI} , it is by no means necessary. It is in fact interesting to investigate the form of P_{VI} when all four singular points are movable: at $a(t)$, $b(t)$, $c(t)$ and $d(t)$. In this case, we can rewrite P_{VI} as (with the constraint $\frac{(a-d)(b-c)}{(a-c)(b-d)} = t$)

$$(3.2) \quad w'' = \frac{1}{2} \left(\frac{1}{w-a} + \frac{1}{w-b} + \frac{1}{w-c} + \frac{1}{w-d} \right) w'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{a'}{w-a} + \frac{b'}{w-b} + \frac{c'}{w-c} + \frac{d'}{w-d} + e \right) w' + (w-a)(w-b)(w-c)(w-d) \left(\frac{f}{(w-a)^2} + \frac{g}{(w-b)^2} + \frac{h}{(w-c)^2} + \frac{k}{(w-d)^2} \right)$$

where

$$e = \frac{a'-b'}{a-b} + \frac{a'-c'}{a-c} + \frac{(a-d)(b-c)}{(a-b)(a-c)} + \frac{(a-d)a'}{(a-b)(a-c)} + \frac{(a-d)b'}{(a-b)(c-b)} + \frac{(a-d)c'}{(a-c)(b-c)}$$

and f, g, h, k are lengthy expressions which cannot be given in a paper of reasonable length. They are of the form $f = f_0\alpha^2 + f_1$, and similarly for g, h, k , where the derivatives of a, b, c, d appear only in f_1, g_1, h_1, k_1 . In order to recover the ‘‘standard’’ expression (3.1) we take $a \rightarrow \infty$, $b \rightarrow 0$, and $c \rightarrow 1$, whereupon $d \rightarrow t$.

The Miura transformations associated to (3.1) have been derived in the form of a first degree relation by Okamoto [11] and rediscovered in a different approach by Nijhoff and collaborators [9]. Using this Miura transformation one can derive the Schlesinger transformations for P_{VI} and obtain the contiguity relations following the well-established procedure [3]. (We will avoid at this point all comments on the fine distinctions on the “Schlesinger transformation” proper terminology [1]. While these distinctions are necessary when one aims at a rigorous treatment they are beyond the scope of the practical approach adopted here, where our aim is just the derivation of discrete \mathbb{P} s).

We shall not go into the details of the derivation of the contiguity relation of P_{VI} . They can essentially be found in [10] and [1]. Adopting the convenient form of the latter (and correcting a factor of 2 misprint) we can rewrite the contiguity relation as

$$(3.3) \quad \frac{z_{n-1} + z_n}{x_n - x_{n-1}} + \frac{z_n + z_{n+1}}{x_n - x_{n+1}} = \frac{z_n + p(-1)^n}{x_n - a} + \frac{z_n + q(-1)^n}{x_n - b} + \frac{z_n + r(-1)^n}{x_n - c} + \frac{z_n + s(-1)^n}{x_n - d}$$

where $z_n = \delta(n - n_0)$ and we have the constraint $p + q + r + s = 0$. Expression (3.3) is the contiguity relation for a general position of the singularities corresponding to equation (3.2). Notice that if one of the singular points, say a , is taken to ∞ , then the r.h.s. has only three terms and no constraint exists between the surviving q, r, s . Bringing the positions of the singularities in (3.3) to the “standard” ones $\infty, 0, 1, t$, involves homographic transformations of the independent variables, which amounts to going backwards from (3.2) to (3.1).

A more interesting transformation one can perform on (3.3) is to treat the even- and odd-index x s in a different way. For example, if we reverse the sign of one x out of two, i.e., $x_n \rightarrow (-1)^n x_n$, then we obtain an equation (which is reminiscent of that of alternate d- P_I):

$$(3.4) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \sum_{i=1,4} \frac{z_n - \alpha_i(-1)^n}{x_n - a_i(-1)^n}$$

Similarly, if we invert one x out of two, i.e. $x_n \rightarrow x_n^{(-1)^n}$, we obtain a form reminiscent of alternate d- P_{II} :

$$(3.5) \quad \frac{z_{n-1} + z_n}{1 - x_n x_{n-1}} + \frac{z_n + z_{n+1}}{1 - x_n x_{n+1}} = \sum_{i=1,4} \frac{z_n - \alpha_i(-1)^n}{1 - x_n a_i^{(-1)^{n+1}}}$$

This mapping is just the one obtained in [10], with specific values for the a_i s, precisely as a contiguity of the solutions of P_{VI} (but, also, initially as a similarity reduction of the discrete mKdV).

Now, that the general framework is set, we turn to the cases obtained from (3.3) or equivalently, (3.4) or (3.5), by degeneration through coalescence of singularities.

For this, it is essential that we work with the general equation involving four a_i s: fixing three of them to $\infty, 0$ and 1 at the outset makes it impossible to construct the proper degeneration cascade. Of course, once we have obtained the degenerate forms we are free to perform homographic transformations on the a_i s and bring them to the “canonical” values.

The first reduction one can construct from the contiguity relation (3.4) is by taking $a_1 = a_2 = a$. We obtain directly the mapping

$$(3.6) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n - (\alpha_1 + \alpha_2)(-1)^n}{x_n - a(-1)^n} + \sum_{i=3,4} \frac{z_n - \alpha_i(-1)^n}{x_n - a_i(-1)^n}$$

and one can simplify further by taking $a \rightarrow \infty$. The mapping, now, has the form

$$(3.7) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{z_n - \alpha_3(-1)^n}{x_n - a_3(-1)^n} + \frac{z_n - \alpha_4(-1)^n}{x_n - a_4(-1)^n}.$$

We start by translating x_n to $x_n + (-1)^n(a_3 + a_4)/2$ in which case the two denominators of the r.h.s. become $x_n \pm (-1)^n(a_4 - a_3)/2$ while those of the l.h.s. are unchanged. Rescaling x through $x \rightarrow 2x/(a_4 - a_3)$, we obtain at the r.h.s. the denominators $x_n \pm (-1)^n$. By considering the even and odd n s we can show that the mapping has the form

$$(3.8) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n x_n + \alpha(-1)^n}{x_n^2 - 1}$$

where $\alpha = 2(\alpha_3 + \alpha_4)/(a_4 - a_3)$ and we have translated z_n to $z_n + (-1)^n(\alpha_3 + \alpha_4)/2$. This mapping was first obtained in [14] where we have shown that it is not a discrete \mathbb{P} but rather a mapping that can be reduced to a linear one. Moreover, from the results of [14], one can show that z_n entering (3.8) is an *arbitrary* function of n , and not just a linear one: the mapping is linearisable for any $z(n)$.

However, it is possible to obtain a discrete \mathbb{P} at this level of degeneration. For this, we start by taking $a_1 = -a_2 = a$. Then, from (3.4) we have

$$(3.9) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n x_n - x(\alpha_1 + \alpha_2)(-1)^n + (\alpha_1 - \alpha_2)a}{x_n^2 - a^2} + \sum_{i=3,4} \frac{z_n - \alpha_i(-1)^n}{x_n - a_i(-1)^n}$$

Next, we take $a \rightarrow \infty$ and at the same time we require that $(\alpha_1 - \alpha_2)/a$ be finite, say $-\kappa$. This is perfectly possible while keeping $(\alpha_1 + \alpha_2)$ finite. (This last constraint is essential. Given that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, we cannot have a diverging $(\alpha_1 + \alpha_2)$ unless $(\alpha_3 + \alpha_4)$ also diverges. This would lead to a diverging r.h.s. of (3.9) unless one considers further degeneration of a_3, a_4). Thus, when $a \rightarrow \infty$, the only contribution that survives from the first term in the r.h.s. of (3.9) is the constant κ . Applying the

same homographic transformations to the remaining mapping as for (3.8) we obtain finally

$$(3.10) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \kappa + \frac{(2z_n - \beta(-1)^n)x_n + \alpha(-1)^n}{x_n^2 - 1}$$

with $\alpha = 2(\alpha_3 + \alpha_4)/(a_4 - a_3)$ and $\beta = (\alpha_3 + \alpha_4)$. This mapping was first identified in [14] and is indeed a discrete Painlevé equation.

In order to pursue the degeneration, one must explore the possibility either to send a third singularity to ∞ or collapse the two remaining singularities to a common value. We first examine the degenerations of the linearisable mapping (3.7). Taking $a_1 = a_2 = a_3 = a$, and letting $a \rightarrow \infty$, we obtain the mapping:

$$(3.11) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{z_n}{x_n}$$

where we have translated x and z so as to bring a_4 and α_4 to zero. This mapping is, quite expectedly, a linearisable one and z_n is a free function of n .

Taking $a_3 = a_4 \rightarrow 0$ in (3.7) leads to another linearisable mapping:

$$(3.12) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n}{x_n}$$

Again we have translated z so as to bring $\alpha_3 + \alpha_4$ to zero and in fact z_n is a free function of n . The linearisation of both (3.11) and (3.12) is straightforward. Putting $y_n = x_n/x_{n-1}$ one obtains a homographic mapping for y . In fact this linearisation would work even for totally arbitrary functions of n in all three numerators of (3.11) and (3.12).

Next, we take $a_3 = -a_4 = a$ in (3.7) and find:

$$(3.13) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n x_n - x_n(\alpha_3 + \alpha_4)(-1)^n + (\alpha_3 - \alpha_4)a}{x_n^2 - a^2}$$

At the limit $a \rightarrow 0$, we find again an expression like (3.12), unless we let $\alpha_3 - \alpha_4$ diverge in such a way that $(\alpha_3 - \alpha_4)a$ remains finite with value, say, λ . The end result, after a translation of z , is:

$$(3.14) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{2z_n}{x_n} + \frac{\lambda}{x_n^2}$$

This is, again, a linearisable mapping, first discovered in [14], and z_n can be a free function of n .

We now turn to mapping (3.9) with $a \rightarrow \infty$ and $(\alpha_1 - \alpha_2)a \rightarrow -\kappa$. The first degeneration is obtained by the coalescence $a_3 = a_4 = b$ (and a subsequent translation which brings b to zero). We obtain, after a translation of z , the linearisable mapping:

$$(3.15) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \kappa + \frac{2z_n}{x_n}$$

One can again introduce a free function of n , provided one replaces the $2z_n$ in the r.h.s. of (3.15) by $z_{n-1} + z_{n+1}$. It was first derived in [14] where it was shown that it can be

linearised, with z_n a free function of n . In fact (3.15) is related to (3.14) by the simple transformation $x \rightarrow 1/x$. For a generic (nonlinear) function $z(n)$, the r.h.s. of (3.15) is transformed by the transformation $R'(x_n) = (2z_n + z_{n+1} + z_{n-1})/x_n - R(1/x_n)/x_n^2$ and we have $\kappa = -\lambda$.

A different degeneration does also exist and is, in fact, more interesting. It is obtained if instead of coalescing a_3 and a_4 we send one of them to ∞ and the other to zero. We find in this case:

$$(3.16) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \kappa + \frac{z_n}{x_n}$$

This is a linearisable mapping that has not been derived before. Again z_n can be a free function of n . In order to linearise (3.16) we introduce the Miura $x_n = z_n/(y_{n+1} + y_n + \kappa)$ and find that y_n satisfies the homographic mapping $y_{n+1}y_n z_n C(-1)^n = y_{n+1} + y_n + \kappa$, where C is an integration constant.

A non-linearisable limit can be obtained if we take $a_3 = -a_4 = b$ and subsequently we assume that $b \rightarrow 0$ and $(\alpha_3 - \alpha_4)b$ goes over to a finite value λ . The resulting mapping reads:

$$(3.17) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \kappa + \frac{2(z_n + \mu(-1)^n)}{x_n} + \frac{\lambda}{x_n^2}$$

where $\mu = (\alpha_3 + \alpha_4)/2$. Equation (3.17) is nothing but the alternate d-P_{II} we have presented already in section 2. Indeed, start by scaling x and z so as to have $\kappa = \lambda = -1$ (and μ is to be understood as a rescaled one). Now, it suffices to invert one x out of two, and examining separately the even and odd indices we find that the resulting equation is precisely (1.1).

Finally, a still more interesting degeneracy of contiguities exists. We obtain it by letting three singularities coalesce to infinity. We take $a_1 = a, a_2 = aj, a_3 = aj^2$, (with $j^3 = 1$) and, then, let $a \rightarrow \infty$. From (3.4), we find

$$(3.18) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \frac{z_n - \alpha_4(-1)^n}{x_n - \alpha_4(-1)^n} + \left[3z_n - (\alpha_1 + \alpha_2 + \alpha_3)(-1)^n - ax_n(\alpha_1 + j\alpha_2 + j^2\alpha_3) - a^2(\alpha_1 + j^2\alpha_2 + j\alpha_3)(-1)^n \right] / [x_n^3 - a^3(-1)^n]$$

Due to the constraint $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$, the quantity $\alpha_1 + \alpha_2 + \alpha_3$ must be finite, lest a_4 also diverge. The two other quantities however, namely $(\alpha_1 + j\alpha_2 + j^2\alpha_3)$ and $(\alpha_1 + j^2\alpha_2 + j\alpha_3)$, may both diverge as a^2 and a respectively. Thus, at the limit $a \rightarrow \infty$ we have

$$(3.19) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \lambda(-1)^n x_n + \kappa + \frac{z_n - \alpha_4(-1)^n}{x_n}$$

where we have translated x so as to bring a_4 to zero. This is a new discrete \mathbb{P} , which has not been derived before. It is related to the continuous P_{IV} and the geometry of its transformations can be described by the affine Weyl group $A_2^{(1)}$.

A final coalescence corresponds to all four singularities going to ∞ . We put $a_1 = a, a_2 = ia, a_3 = -a, a_4 = -ia$. We obtain, in this case,

$$(3.20) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \left[4z_n x_n^3 - ax_n^2(\alpha_1 + i\alpha_2 - \alpha_3 - i\alpha_4) - a^2 x_n(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)(-1)^n - a^3(\alpha_1 - i\alpha_2 + \alpha_3 - i\alpha_4) \right] / [x_n^4 - a^4]$$

Next, we take $a \rightarrow \infty$ and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ diverge in such a way as to make all three combinations $(\alpha_1 + i\alpha_2 - \alpha_3 - i\alpha_4)/a^3, (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)/a^2, (\alpha_1 - i\alpha_2 + \alpha_3 - i\alpha_4)/a$ finite. We obtain, finally,

$$(3.21) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \mu x_n^2 + \lambda(-1)^n x_n + \kappa$$

which is nothing but the alternate d- P_I equation. Indeed, translating x by $\lambda(-1)^n/(2\mu)$ does not modify the l.h.s. and brings (3.21) to the form (1.2). However, this is possible only if $\mu \neq 0$. On the other hand, if we take $\mu = 0$ in (3.21) we obtain the mapping

$$(3.22) \quad \frac{z_{n-1} + z_n}{x_n + x_{n-1}} + \frac{z_n + z_{n+1}}{x_n + x_{n+1}} = \lambda(-1)^n x_n + \kappa$$

which is not a d- \mathbb{P} but rather a (new) linearisable system and this for z_n a free function of n . In order to linearise it we compute its discrete derivative by upshifting (3.22) and subtracting, eliminating κ . Next we introduce the auxiliary variable $y_n = (z_n + z_{n+1})/(x_n + x_{n+1})$ and find $y_{n-1} - y_{n+1} = \lambda(-1)^n(x_n + x_{n+1})$.

Using the definition of y and introducing $u_n = y_n y_{n-1}$ we find that the latter satisfies the linear equation $u_n - u_{n+1} = \lambda(-1)^n(z_n + z_{n+1})$, with solution $u_n = C + \lambda(-1)^n z_n$. Having u one gets y from a (very simple) homographic relation and finally x though the solution of another linear equation.

4. Conclusion

In this paper, we have examined the discrete Painlevé equations which were introduced (more than 10 years ago) under the qualifier of “alternate”. This term was used for the description of the discrete forms of d- P_I and d- P_{II} obtained as contiguities of the (solutions of) the continuous P_{II} and P_{III} respectively. As we have shown, there does not exist a unique canonical form for these “alternate” equations since the l.h.s. of (1.1) can be transformed to that of (1.2) and vice-versa, and the choice for the final form is a question of taste.

Since these two equations are contiguities of the solutions of continuous Painlevé equations we have addressed the question of systematically deriving the d- \mathbb{P} s that appear as contiguities starting from the most general continuous Painlevé equations, namely P_{VI} . We have chosen to organize our findings along the degeneration cascade, where one starts from the “higher” equation, and obtains the “lower” ones by coalescence of the singularities. Applying this approach to the contiguities of P_{VI} we have obtained essentially known d- \mathbb{P} s with one exception, namely equation (3.19), which has never been derived before. This d- \mathbb{P} , just like all those obtained as contiguities of continuous \mathbb{P} s, is integrable by construction. Its Lax pair can be obtained from the Lax pair of the continuous Painlevé equation (namely P_{IV}), combined with its Schlesinger transformation, as we explained in [5].

Moreover, in the degeneration process, we can obtain not only discrete \mathbb{P} s but also mappings that can be reduced to linear ones, including two that have never been derived before. Thus, our approach offers a unified picture of discrete (difference) \mathbb{P} s and linearisable mappings, makes possible the connection to already existing results, while allowing the discovery of new integrable systems.

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