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Another Approach to Juhl's Conformally Covariant Differential Operators from S^n to S^{n-1}

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Abstract. A family $(\mathbf{D}_\lambda)_{\lambda\in\mathbb{C}}$ of differential operators on the sphere S^n is constructed. The operators are conformally covariant for the action of the subgroup of conformal transformations of S^n which preserve the smaller sphere $S^{n-1}\subset S^n$. The family of conformally covariant differential operators from S^n to S^{n-1} introduced by A. Juhl is obtained by composing these operators on S^n and taking restrictions to S^{n-1}.

Key words: conformally covariant differential operators; Juhl's covariant differential operators

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1 Introduction

Let $S=S^n$ be the n-dimensional sphere in $\mathbb{R}^{n+1}$ and let $G=\mathrm{SO}_0(1,n+1)$ be (the neutral component of) the group of conformal transformations of S. Let $S'\simeq S^{n-1}$ be the subspace of points of S with vanishing last coordinate ($x_n=0$ in our notation) and let $G'\simeq\mathrm{SO}_0(1,n)$ be the conformal group of S', viewed as the subgroup of G which stabilizes S'. Let $(\pi_\lambda)_{\lambda\in\mathbb{C}}$ be the scalar principal series of representations of G acting on $C^\infty(S)$. Denote by $\pi_{\lambda|G'}$ its restriction to G'. Let $(\pi'_\mu)_{\mu\in\mathbb{C}}$ be the scalar principal series of G' acting on $C^\infty(S')$.

In [6] A. Juhl has constructed a family $\mathcal{D}_N(\lambda)_{\lambda\in\mathbb{C},N\in\mathbb{N}}$ of differential operators from $C^\infty(S)$ into $C^\infty(S')$, which are intertwining operators between $\pi_{\lambda|G'}$ and $\pi'_{\lambda+N}$.[1] Later, these operators were obtained by T. Kobayashi and B. Speh in [11] as residues of a meromorphic family of *symmetry breaking operators* associated to the restriction problem for the pair (G,G'). A third point of view was proposed by T. Kobayashi and M. Pevzner in [9, 10], based on the F-method. Similar operators were recently constructed for differential forms on spheres [4, 8].

The new approach to Juhl's operators which I present in this article follows a method that I used for similar problems, in the context of the restriction problem for a pair $(G\times G,G')$ where $G'=G$ embedded diagonally in $G\times G$. I was influenced by a reminiscence of the Ω-*process* which yields both the *transvectants* and the *Rankin–Cohen brackets*. These operators may be viewed as covariant bi-differential operators for the group $\mathrm{SL}(2,\mathbb{R})$, or symmetry breaking differential operators from $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ to its diagonal subgroup. For a presentation of these classical results see Section 5 of [2] for a quick overview or [12] for a thorough exposition of the transvectants.

The new method was introduced in a collaboration with R. Beckmann for the conformal group of the sphere (see [1]) and the scalar principal series, then for $G=\mathrm{SL}(2n,\mathbb{R})$ and the degenerate principal series acting on the Grasmmannian $\mathrm{Gr}(n,2n;\mathbb{R})$ (see [2]).

The first step of the method, for the present case, is to introduce the multiplication by x_n, viewed as an operator M on $C^\infty(S)$. The operator M is a "universal" G'-intertwining operator,

[1] Our λ corresponds to $-\lambda$ in Juhl's notation.

in the sense that, for any $\lambda \in \mathbb{C}$, the operator M intertwines $\pi_{\lambda|G'}$ and $\pi_{\lambda-1|G'}$. Next recall the family of Knapp–Stein operators $(I_\lambda)_{\lambda\in\mathbb{C}}$ which are G-intertwining operators with respect to $(\pi_\lambda, \pi_{n-\lambda})$. The operator[2]

$$\mathbf{D}_\lambda = I_{n-\lambda-1} \circ M \circ I_\lambda$$

obtained by twisting M by the appropriate Knapp–Stein intertwining operators is clearly an intertwining operator with respect to $(\pi_{\lambda|G'}, \pi_{\lambda+1|G'})$. Our main result (see Theorem 3.2) is that $\mathbf{D}_\lambda$ is a *differential* operator. The proof is obtained in the non compact realization of the principal series (passing from S^n to $\mathbb{R}^n$ by a conformal map) and uses Euclidean Fourier transform.

The construction of conformally covariant differential operators from S^n to S^{n-1} is now easy. For N a non-negative integer, consider

$$\mathbf{D}_{N,\lambda} = \mathbf{D}_{\lambda+N-1} \circ \cdots \circ \mathbf{D}_{\lambda+1} \circ \mathbf{D}_\lambda \qquad \text{or} \qquad \mathbb{D}_{N,\lambda} = I_{n-\lambda-N} \circ M^N \circ I_\lambda.$$

The two families of differential operators on S (which coincide up to a meromorphic function of λ) are covariant with respect to $(\pi_{\lambda|G'}, \pi_{\lambda+N|G'})$. Finally, let

$$\mathbf{D}_N(\lambda) = \operatorname{res} \circ \mathbf{D}_{N,\lambda},$$

where res is the restriction map from $C^\infty(S)$ to $C^\infty(S')$. The operator $\mathbf{D}_N(\lambda)$ is a differential operator from S to S' which is covariant with respect to $(\pi_{\lambda|G'}, \pi'_{\lambda+N})$. The family $\mathbf{D}_N(\lambda)_{\lambda\in\mathbb{C},N\in\mathbb{N}}$ essentially coincides with Juhl's family.

The operator $\mathbf{D}_\lambda$ has a simple expression in the non compact picture, see (4.6). It is tempting to find a more direct approach to this operator. This is achieved in the last section, by using yet another realization of the principal series, sometimes called the *ambient space* realization. The way the operator is constructed is much simpler, and it is then easy to determine its expression in the non compact picture (recovering the expression of $\mathbf{D}_\lambda$ on $\mathbb{R}^n$, see Proposition 7.8), but also in the compact realization (see Proposition 7.9), that is to say as a G'-conformally covariant differential operator on S. Some generalization of these formulæ in the realm of conformal geometry on a Riemannian manifold seems plausible.

2 The principal series of $\mathrm{SO}_0(1, n+1)$ and the Knapp–Stein intertwining operators

Let E be a Euclidean space of dimension $n+1$, and choose an orthonormal basis $\{e_0, e_1, \dots, e_n\}$. Let $S = S^n$ be the unit sphere of E, i.e.,

$$S = \{x = (x_0, x_1, \dots, x_n),\ x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

Let $\mathbf{E}$ be the vector space $\mathbb{R} \oplus E$, with the Lorentzian quadratic form

$$Q(\mathbf{x}) = [(t,x),(t,x)] = t^2 - |x|^2 \qquad \text{for} \quad \mathbf{x} = (t,x), \quad t \in \mathbb{R}, \quad x \in E.$$

For $\mathbf{x} = (t,x) \in \mathbf{E}$, we let

$$t(\mathbf{x}) = t, \qquad \mathbf{x}_E = x.$$

The space of isotropic lines $\mathcal{S}$ in $\mathbf{E}$ can be identified with S by the map

$$S \ni x \longmapsto d_x = \mathbb{R}(1,x) \in \mathcal{S}, \qquad \mathcal{S} \ni d \longmapsto d \cap \{t = 1\}.$$

[2] For technical reasons, a normalizing factor is introduced, see (3.4).

Let $G = \mathrm{SO}_0(1, n+1)$ be the connected component of the group of isometries of $\mathbf{E}$. Then G acts on $\mathcal{S}$ and this action can be transferred to an action on S. More explicitly, if $x = (x_0, x_1, \dots, x_n) \in S$, and $g \in G$, observe that $t(g(1,x)) > 0$ and define $g(x) \in S$ by

$$\big(1, g(x)\big) = t\big(g.(1,x)\big)^{-1} g.(1,x).$$

Set, for $g \in G$ and $x \in S$

$$\kappa(g,x) = t\big(g.(1,x)\big)^{-1}.$$

Clearly $\kappa(g,x)$ is a smooth, strictly positive function on $G \times S$. Moreover $\kappa(g,x)$ satisfies the *cocycle property*: for any g_1, g_2 and any $x \in S$,

$$\kappa(g_1 g_2, x) = \kappa\big(g_1, g_2(x)\big)\kappa(g_2, x).$$

This action of G on S is known to be *conformal*. For $g \in G$, $x \in S$ and ξ an arbitrary tangent vector to S at x

$$|Dg(x)\xi| = \kappa(g,x)|\xi|,$$

and hence $\kappa(g,x)$ is the *conformal factor* of g at x.

Associated to the action of G on S there is a family of representations on $C^\infty(S)$, which, from the point of view of harmonic analysis is the *scalar principal series* of G. For $\lambda \in \mathbb{C}$, $g \in G$ and $f \in C^\infty(S)$, let

$$\pi_\lambda(g) f(x) = \kappa\big(g^{-1}, x\big)^\lambda f\big(g^{-1}(x)\big).$$

The formula defines a (smooth) representation π_λ of G on $C^\infty(S)$.

The Knapp–Stein intertwining operators are a major tool in harmonic analysis of G (as of any semi-simple Lie group, see, e.g., [7]). For $\lambda \in \mathbb{C}$ and $f \in C^\infty(S)$, let

$$I_\lambda f(x) = \frac{1}{\Gamma(\lambda - \frac{n}{2})} \int_S |x-y|^{-2n+2\lambda} f(y) dy, \tag{2.1}$$

where dy stands for the Lebesgue measure on S induced by the Euclidean structure. For $\operatorname{Re}\lambda > \frac{n}{2}$, this formula defines a continuous operator I_λ on $C^\infty(S)$.

Proposition 2.1.

i) The definition (2.1) can be analytically continued in λ to all of $\mathbb{C}$.

ii) The analytic continuation yields a holomorphic family of operators I_λ on $C^\infty(S)$, which satisfy the intertwining relation

$$\forall\, g \in G, \qquad I_\lambda \circ \pi_\lambda(g) = \pi_{n-\lambda}(g) \circ I_\lambda. \tag{2.2}$$

The following complementary result will be needed later.

Proposition 2.2. *For any $\lambda \in \mathbb{C}$*

$$I_\lambda \circ I_{n-\lambda} = \frac{\pi^n}{\Gamma(\lambda)\Gamma(n-\lambda)} \operatorname{id}. \tag{2.3}$$

The next result corresponds to reducibility points for the scalar principal series. Let $\mathcal{P}(S)$ be the space of restrictions to S of polynomial functions on E, and for $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the space of restrictions to S of polynomials on E of degree $\leq k$. Finally, let $\mathcal{P}_k^\perp$ be the subspace of $C^\infty(S)$ given by

$$\mathcal{P}_k^\perp = \left\{ f \in C^\infty(S),\ \int_S f(x)p(x)dx = 0,\ \text{for any } p \in \mathcal{P}_k \right\}.$$

.

Proposition 2.3.

i) Let $\lambda = n + k, k \in \mathbb{N}$. Then

$$\operatorname{Im}(I_{n+k}) = \mathcal{P}_k, \qquad \operatorname{Ker}(I_{n+k}) = \mathcal{P}_k^\perp. \tag{2.4}$$

ii) Let $\lambda = -k, k \in \mathbb{N}$. Then

$$\operatorname{Ker}(I_{-k}) = \mathcal{P}_k, \qquad \operatorname{Im}(I_{-k}) = \mathcal{P}_k^\perp. \tag{2.5}$$

3 Construction of the family $\widetilde{\mathbf{D}}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \in \mathbb{C}$

Now let $E' = \{x \in E,\, x_n = 0\}$ and $S' = S \cap E'$. Then S' is an $(n-1)$-dimensional sphere. Let G' be the subgroup of elements of G of the form

$$g = \begin{pmatrix} & & & 0 \\ & g' & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \qquad g' \in \mathrm{SO}_0(1, n).$$

Clearly, G' is a subgroup of G, isomorphic to $\mathrm{SO}_0(1, n)$. Elements of G' preserve the hyperplane $\{x_n = 0\}$ in $\mathbf{E}$ and hence the action of G' on S preserves S'.

For $x \in E$, write $x = (x', x_n)$, with $x' \in \mathbb{R}^n$. For $g \in G'$,

$$g(1, x) = g(1, x', x_n) = \big(g'.(1, x'), x_n\big).$$

If $x \in S$, the last equation can be rewritten as

$$\kappa(g, x)^{-1}\big(1, g(x)\big) = \big(g'.(1, x'), x_n\big),$$

so that

$$g(x)_n = \kappa(g, x)x_n. \tag{3.1}$$

In the sequel, the distinction between g and g' in the notation is abandoned, the context providing the correct interpretation.

Let M be the operator defined on $C^\infty(S)$ by

$$Mf(x) = x_n f(x), \qquad f \in C^\infty(S).$$

Proposition 3.1. *The operator M satisfies*

$$\forall\, g \in G' \qquad M \circ \pi_\lambda(g) = \pi_{\lambda-1}(g) \circ M. \tag{3.2}$$

Proof. This is an immediate consequence of (3.1). ■

Next let $\mathbf{D}_\lambda$ be the operator on $C^\infty(S)$ defined by

$$\mathbf{D}_\lambda = I_{n-\lambda-1} \circ M \circ I_\lambda,$$

which corresponds to the following diagram

$$\begin{array}{ccc} C^\infty(S) & \xrightarrow{\mathbf{D}_\lambda} & C^\infty(S) \\ \big\downarrow I_\lambda & & \big\uparrow I_{n-\lambda-1} \\ C^\infty(S) & \xrightarrow{M} & C^\infty(S). \end{array}$$

As a consequence of the intertwining property of the Knapp–Stein operators (2.2) and Proposition 3.1, $\mathbf{D}_\lambda$ satisfies for $g \in G'$

$$\mathbf{D}_\lambda \circ \pi_\lambda(g) = \pi_{\lambda+1}(g) \circ \mathbf{D}_\lambda. \tag{3.3}$$

Otherwise said, the operator $\mathbf{D}_\lambda$ is covariant with respect to $(\pi_{\lambda|G'}, \pi_{\lambda+1|G'})$.

Theorem 3.2. *The operator* $\mathbf{D}_\lambda$ *is a differential operator on* S.

The proof of Theorem 3.2 will be given at the end the next section.

Proposition 3.3. *Let* $\lambda \in (n+\mathbb{N}) \cup (-1-\mathbb{N})$. *Then* $\mathbf{D}_\lambda = 0$.

Proof. Let first $\lambda = n+k$ for some $k \in \mathbb{N}$. Then $I_\lambda = I_{n+k}$, and by (2.4) $\mathrm{Im}(I_\lambda) = \mathcal{P}_k$. Next $\mathrm{Im}(M \circ I_\lambda) \subset \mathcal{P}_{k+1}$. Now $I_{n-\lambda-1} = I_{-k-1}$ and using (2.5), $I_{n-\lambda-1} \circ M \circ I_\lambda = 0$.

Now let $\lambda = -k$, with $k \geq 1$. Then $I_\lambda = I_{-k}$ and by (2.5), $\mathrm{Im}(I_\lambda) = \mathcal{P}_k^\perp$. Next $\mathrm{Im}(M \circ I_\lambda) \subset \mathcal{P}_1 \mathcal{P}_k^\perp \subset \mathcal{P}_{k-1}^\perp$. Now $I_{n-\lambda-1} = I_{n+k-1}$ which using (2.4) implies $I_{n-\lambda-1} \circ M \circ I_\lambda = 0$. ■

To compensate for these zeroes of $\mathbf{D}_\lambda$, introduce

$$\widetilde{\mathbf{D}}_\lambda = \Gamma(\lambda+1)\Gamma(n-\lambda)\mathbf{D}_\lambda \tag{3.4}$$

for $\lambda \notin (n+\mathbb{N}) \cup (-1-\mathbb{N})$ and extend continuously to all of $\mathbb{C}$ to get a holomorphic family $(\widetilde{\mathbf{D}}_\lambda)_{\lambda \in \mathbb{C}}$ of differential operators on S covariant with respect to $(\pi_{\lambda|G'}, \pi_{\lambda+1|G'})$.

4 The expression of $\widetilde{\mathbf{D}}_\lambda$ in the non-compact picture

Consider the point $-\mathbf{1} = (-1, 0, \ldots, 0) \in S$. The stereographic projection with source at $-\mathbf{1}$ provides a diffeomorphism from $S \setminus \{-\mathbf{1}\}$ onto the hyperplane $\{x_n = 1\}$. The inverse map (up to a scaling by a factor 2) $c \colon \mathbb{R}^n \longrightarrow S$ is given by

$$c(\xi) = \begin{pmatrix} \dfrac{1-|\xi|^2}{1+|\xi|^2} \\ \dfrac{2\xi_1}{1+|\xi|^2} \\ \vdots \\ \dfrac{2\xi_n}{1+|\xi|^2} \end{pmatrix}. \tag{4.1}$$

When using this local chart on S, we refer to the *non-compact picture*, as a reference to semisimple harmonic analysis.

Geometric considerations (or an elementary computation) show that, for $\xi, \eta \in \mathbb{R}^n$

$$|c(\xi) - c(\eta)|^2 = \kappa(c, \xi)|\xi - \eta|^2 \kappa(c, \eta),$$

where, for $\xi \in \mathbb{R}^n$, we set

$$\kappa(c, \xi) = 2\big(1 + |\xi|^2\big)^{-1}.$$

There is an infinitesimal version of this result, namely

$$|Dc(\xi)\eta| = \kappa(c, \xi)|\eta|$$

for $\xi, \eta \in \mathbb{R}^n$. This last statement shows that c is conformal from $\mathbb{R}^n$ with its standard Euclidean structure into S.

The action of g on S can be transferred as a (rational) action of G on $\mathbb{R}^n$, namely $c^{-1} \circ g \circ c$. For notational convenience, we still denote this action on $\mathbb{R}^n$ by $(g, \xi) \longmapsto g(\xi)$, $g \in G$, $\xi \in \mathbb{R}^n$. As the map c is conformal, the transferred action of G on $\mathbb{R}^n$ is still conformal. For $g \in G$ defined at $\xi \in \mathbb{R}^n$, we let $\kappa(g, \xi)$ be the corresponding conformal factor of g at ξ.

Let $\lambda \in \mathbb{C}$. For $f \in C^\infty(S)$ let $C_\lambda(f)$ be defined by

$$C_\lambda(f)(\xi) = \kappa(c, \xi)^\lambda f(c(\xi)), \qquad \xi \in \mathbb{R}^n$$

and let $\mathcal{H}_\lambda$ be the image of C_λ. It is easily proved that

$$\mathcal{S}\big(\mathbb{R}^n\big) \subset \mathcal{H}_\lambda \subset \mathcal{S}'\big(\mathbb{R}^n\big),$$

where $\mathcal{S}(\mathbb{R}^n)$ stands for the Schwartz space on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its dual, the space of tempered distributions.

The representation π_λ can be transferred in the non-compact model, using C_λ as intertwining map, i.e., set

$$\rho_\lambda(g) = C_\lambda \circ \pi_\lambda(g) \circ C_\lambda^{-1}.$$

Using the cocycle property of κ, ρ_λ can be realized as

$$\rho_\lambda(g)f(\xi) = \kappa\big(g^{-1}, \xi\big)^\lambda f\big(g^{-1}(\xi)\big),$$

where $f \in \mathcal{H}_\lambda$ and $g \in G$.

Similarly, the Knapp–Stein operators can be transferred to the non-compact picture. For $s \in \mathbb{C}$, consider the expression

$$h_s(\xi) = \frac{1}{\Gamma\big(\frac{n}{2} + \frac{s}{2}\big)}|\xi|^s, \qquad \xi \in \mathbb{R}^n.$$

For $\mathrm{Re}(s) > -n$, h_s is locally summable with moderate growth at infinity, hence defines a tempered distribution. The ($\mathcal{S}'(\mathbb{R}^n)$-valued) function $s \mapsto h_s$ can be extended by analytic continuation to $\mathbb{C}$ and the Γ factor in the definition of h_s is so chosen that it extends as an *entire* function with values in $\mathcal{S}'(\mathbb{R}^n)$ (for more details see, e.g., [5]).

For $\lambda \in \mathbb{C}$, the Knapp–Stein operator J_λ is given by

$$J_\lambda f = h_{-2n+2\lambda} \star f,$$

or more concretely

$$J_\lambda f(\xi) = \frac{1}{\Gamma\big(\lambda - \frac{n}{2}\big)} \int_{\mathbb{R}^n} |\xi - \eta|^{-2n+2\lambda} f(\eta) d\eta.$$

As for any $s \in \mathbb{C}$ h_s is a tempered distribution, J_λ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 4.1. *Let* $\lambda \in \mathbb{C}$*. Then for* $f \in \mathcal{S}(\mathbb{R}^n)$

$$J_\lambda f = \big(C_{n-\lambda} \circ I_\lambda \circ C_\lambda^{-1}\big) f.$$

Proof. As $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{H}_\lambda \subset \mathcal{S}'(\mathbb{R}^n)$, both sides are well-defined and belong to $\mathcal{S}'(\mathbb{R}^n)$. For $\operatorname{Re}\lambda > \frac{n}{2}$, both sides are given by convergent integrals, and the equality is proved by a change of variable. The general case follows by analytic continuation. The intertwining property of the Knapp–Stein operators can be formulated in the following way. ■

Proposition 4.2. *Let* $f \in C_c^\infty(S)$ *and let* $g \in G$ *such that* g^{-1} *is defined on* $\operatorname{Supp}(f)$*. Then*

$$J_\lambda\big(\rho_\lambda(g) f\big) = \rho_{n-\lambda}(g)\big(J_\lambda f\big),$$

where the two sides of the equation are viewed as tempered distributions on $\mathbb{R}^n$*.*

Proof. The condition implies that both f and $\rho_\lambda(g) f$ are contained in $\mathcal{S}(\mathbb{R}^n)$. Hence

$$\begin{aligned} J_\lambda\big(\rho_\lambda(g) f\big) &= \big(C_{n-\lambda} \circ I_\lambda \circ C_\lambda^{-1}\big)\big(\rho_\lambda(g) f\big) \\ &= (C_{n-\lambda} \circ I_\lambda)\big(\pi_\lambda(g) C_\lambda^{-1} f\big) = \big(C_{n-\lambda} \circ \pi_{n-\lambda}(g)\big) \circ \big(I_\lambda \circ C_\lambda^{-1}\big) f \\ &= \rho_{n-\lambda}(g) \circ \big(C_{n-\lambda} \circ I_\lambda \circ C_\lambda^{-1}\big) f = \rho_{n-\lambda}(g)(J_\lambda f). \end{aligned}$$ ■

The following formulæ will be needed in the sequel

$$|\xi|^2 h_s(\xi) = \frac{n+s}{2} h_{s+2}(\xi), \tag{4.2}$$

$$\frac{\partial}{\partial \xi_n} h_s(\xi) = \frac{2s}{n+s-2} \xi_n h_{s-2}(\xi), \tag{4.3}$$

where at $s = -n+2$, the last formula has to be understood by analytic continuation.

As the pole of the stereographic projection has been chosen in S', the map c maps the hyperplane $\{\xi_n = 0\}$ into S'. It allows to transfer the map M to the non-compact picture.

Lemma 4.3. *Let* $g \in G'$ *be defined at* $\xi \in \mathbb{R}^n$*. Then*

$$g(\xi)_n = \kappa(g, \xi) \xi_n. \tag{4.4}$$

Proof. Let $\xi \in \mathbb{R}^n$ and let $x = c(\xi) \in S \setminus \{-\mathbf{1}\}$. Then

$$c(\xi)_n = \kappa(c, \xi) \xi_n, \qquad g(x)_n = \kappa(g, x) x_n, \qquad c^{-1}(x) = \kappa\big(c^{-1}, x\big) x_n$$

the first equality by (4.1), the second by (3.1), and the third also by (4.1). As κ satisfies a cocycle relation, we get

$$\big((c^{-1} \circ g \circ c)(\xi)\big)_n = \kappa\big(c^{-1} \circ g \circ c, \xi\big) \xi_n,$$

which gives (4.4). ■

Lemma 4.4. *Let* $\lambda \in \mathbb{C}$ *and* $f \in C^\infty(S)$*. Then*

$$C_{\lambda-1}(Mf)(\xi) = \xi_n C_\lambda(f)(\xi), \qquad \xi \in \mathbb{R}^n.$$

Proof. Let $\xi = (\xi', \xi_n)$. By (4.1), $c(\xi)_n = \kappa(c, \xi)\xi_n$, so that

$$C_{\lambda-1}(Mf)(\xi) = \kappa(c, \xi)^{\lambda-1} Mf\big(c(\xi)\big) = \kappa(c, \xi)^\lambda \xi_n f\big(c(\xi)\big) = \xi_n C_\lambda(f)(\xi).$$ ■

Abusing notation, M will be used for the operator (on $C^\infty(\mathbb{R}^n)$ say) of multiplication by ξ_n. The operator M maps $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) into $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$), and for any $\lambda \in \mathbb{C}$, the operator M maps $\mathcal{H}_\lambda$ into $\mathcal{H}_{\lambda-1}$ (Lemma 4.4).

Proposition 4.5. *Let $\lambda \in \mathbb{C}$. The operator $M\colon \mathcal{H}_\lambda \longrightarrow \mathcal{H}_{\lambda-1}$ satisfies*

$$\forall\, g \in G', \qquad M \circ \rho_\lambda(g) = \rho_{\lambda-1}(g) \circ M.$$

Otherwise said, the operator M intertwines the representations $\pi_{\lambda|G'}$ and $\pi_{\lambda-1|G'}$.

Proof. Let $f \in \mathcal{H}_\lambda$. Then

$$\begin{aligned}\big(M \circ \rho_\lambda(g)\big)f(\xi) &= \xi_n \kappa\big(g^{-1},\xi\big)^\lambda f\big(g^{-1}(\xi)\big) = \kappa\big(g^{-1},\xi\big)^{\lambda-1}\big(g^{-1}(\xi)\big)_n f\big(g^{-1}(\xi)\big)\\ &= \rho_{\lambda-1}(g)(Mf)\big(g^{-1}(\xi)\big)\end{aligned}$$

and the statement follows. ■

Having introduced the non-compact version of the main ingredients, we observe that the Knapp–Stein operators are convolution operators, whereas M is the multiplication by an elementary polynomial. So the Fourier transform is well-fitted for computations in this context. Define the Fourier transform on $\mathbb{R}^n$ as usual by

$$\widehat{f}(\eta) = \int_{\mathbb{R}^n} e^{i\langle\eta,\xi\rangle} f(\xi)d\xi$$

initially for functions in $\mathcal{S}(\mathbb{R}^n)$ and extend by duality to $\mathcal{S}'(\mathbb{R}^n)$.

The Fourier transform of h_s is given by

$$\widehat{h}_s = 2^{n+s}\pi^{\frac{n}{2}} h_{-n-s}.$$

For this result see, e.g., [5].

Thanks to the above observations, it is possible to define the composition $M \circ J_\lambda$ as an operator from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.

Lemma 4.6. *For $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\big((M \circ J_\lambda)f\big)\widehat{\ }(\eta) = -i\pi^{\frac{n}{2}} 2^{-n+2\lambda}\left(h_{n-2\lambda}(\eta)\frac{\partial \widehat{f}}{\partial \eta_n}(\eta) + \frac{n-2\lambda}{n-\lambda-1}\eta_n h_{n-2-2\lambda}(\eta)\widehat{f}(\eta)\right). \quad (4.5)$$

Proof. As observed earlier, the Knapp–Stein operator J_λ is a convolution operator on $\mathbb{R}^n$, so that

$$(J_\lambda f)\widehat{\ }(\eta) = \widehat{h}_{-2n+2\lambda}(\eta)\widehat{f}(\eta) = 2^{-n+2\lambda}\pi^{\frac{n}{2}} h_{n-2\lambda}(\eta)\widehat{f}(\eta).$$

Next, for any distribution $\varphi \in \mathcal{S}'(\mathbb{R}^n)$

$$\widehat{M\varphi} = -i\frac{\partial}{\partial \eta_n}\widehat{\varphi}$$

and (4.5) follows, using (4.3). ■

The composition $J_{n-\lambda-1} \circ M \circ J_\lambda$ is not well-defined on $\mathcal{S}(\mathbb{R}^n)$. However, a formal computation (using again Fourier transforms) can be made and leads to a differential operator, which is at the origin of the definition (4.6) below. In order to give a rigorous argument, it is necessary to follow an indirect route.

For $\lambda \in \mathbb{C}$, let E_λ be the differential operator on $\mathbb{R}^n$ defined by

$$E_\lambda = (2\lambda - n + 2)\frac{\partial}{\partial \xi_n} + \xi_n \Delta, \tag{4.6}$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial \xi_j^2}$ is the usual Laplacian on $\mathbb{R}^n$. Notice that the operator E_λ maps $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) into $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$), so that we may consider the composition $J_{\lambda+1} \circ E_\lambda$.

Lemma 4.7. *For $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\begin{aligned} &\big((J_{\lambda+1} \circ E_\lambda) f\big)\widehat{\ }(\eta) \\ &= -i 2^{-n+2+2\lambda} \pi^{\frac{n}{2}} \left((\lambda - n + 1) h_{n-2\lambda}(\eta) \frac{\partial \widehat{f}}{\partial \eta_n}(\eta) + (2\lambda - n)\eta_n h_{-2\lambda+n-2}(\eta) \widehat{f}(\eta) \right). \end{aligned} \tag{4.7}$$

Proof. Using (4.2) and (4.3),

$$\begin{aligned} \big(E_\lambda f\big)\widehat{\ }(\eta) &= (-i)(2\lambda - n + 2)\eta_n \widehat{f}(\eta) + (-i)\frac{\partial}{\partial \eta_n}\big(-|\eta|^2 \widehat{f}(\eta)\big) \\ &= (-i)\left((2\lambda - n)\eta_n \widehat{f}(\eta) - |\eta|^2 \frac{\partial \widehat{f}}{\partial \eta_n}(\eta) \right). \end{aligned}$$

Next

$$\begin{aligned} &\big((J_{\lambda+1} \circ E_\lambda) f\big)\widehat{\ }(\eta) = \widehat{h}_{-2n+2\lambda+2}(\eta)(E_\lambda f)\widehat{\ }(\eta) \\ &= 2^{-n+2+2\lambda} \pi^{\frac{n}{2}} (-i) \left((2\lambda - n)\eta_n h_{n-2-2\lambda}(\eta) \widehat{f}(\eta) - (-\lambda + n - 1) h_{n-2\lambda}(\eta) \frac{\partial \widehat{f}}{\partial \eta_n}(\eta) \right). \end{aligned}$$

■

Comparison of (4.5) and (4.7) yields the next result.

Proposition 4.8.

$$M \circ J_\lambda = \frac{1}{4(\lambda - n + 1)} J_{\lambda+1} \circ E_\lambda. \tag{4.8}$$

Remark 4.9. This equality has to be understood as an equality of operators from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. For $\lambda = n - 1$, $J_{\lambda+1} = J_n$ is equal (up to a constant $\neq 0$) to the operator $f \longmapsto \left(\int_{\mathbb{R}^n} f(\xi) d\xi\right) 1$. Now for $f \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} E_\lambda f(\xi) d\xi = 0$ as is easily seen by integration by parts. Hence, $J_{\lambda+1} \circ E_\lambda$ vanishes for $\lambda = n - 1$, so that (4.8) has to be interpreted as a residue formula.

Proposition 4.10. *Let $f \in C_c^\infty(\mathbb{R}^n)$ and assume that $g \in G'$ is such that g^{-1} is defined on* $\mathrm{Supp}(f)$. *Then*

$$\big(E_\lambda \circ \rho_\lambda(g)\big) f = \big(\rho_{\lambda+1}(g) \circ E_\lambda\big) f.$$

Proof. As a consequence of the intertwining property of J_λ (Proposition 4.2) and of M (Proposition 3.1),

$$(M \circ J_\lambda)\big(\rho_\lambda(g) f\big) = \rho_{n-\lambda-1}(g)(M \circ J_\lambda) f.$$

Hence, by (4.8) (assuming for a while that $\lambda \neq n - 1$)

$$\big(J_{\lambda+1} \circ E_\lambda\big)\rho_\lambda(g) f = \rho_{n-\lambda-1}(g)\big((J_{\lambda+1} \circ E_\lambda) f\big).$$

Now $\mathrm{Supp}(E_\lambda f) \subset \mathrm{Supp}(f)$, so that g^{-1} is defined on $\mathrm{Supp}(E_\lambda f)$. Hence, by Proposition 4.2

$$\big(\rho_{n-\lambda-1}(g) \circ J_{\lambda+1}\big) E_\lambda f = \big(J_{\lambda+1} \circ \rho_{\lambda+1}(g)\big) E_\lambda f,$$

so that

$$J_{\lambda+1}\big((E_\lambda \circ \rho_\lambda(g)) f\big) = J_{\lambda+1}\big((\rho_{\lambda+1}(g) \circ E_\lambda) f\big).$$

Now, for λ generic, the operator $J_{\lambda+1}$ is injective on $\mathcal{S}(\mathbb{R}^n)$, hence

$$E_\lambda \circ \rho_\lambda(g) f = \rho_{\lambda+1}(g) \circ E_\lambda f.$$

The general result follows by continuity, as the family E_λ depends holomorphically on λ. ■

Proof of Theorem 3.2. The covariance property of the differential operator E_λ allows to construct a *global* differential operator on S which is expressed in the non-compact picture to E_λ. In fact to fully cover the sphere S, we only need another chart, which can be chosen as the analog of the map c but constructed from the stereographic projection corresponding to the pole $\mathbf{1} = (1, 0, \dots, 0)$ instead of $-\mathbf{1}$. Consider the element s of G given by

$$s = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then s acts on S by

$$s(x) = s \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_0 \\ -x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In particular, s maps $-\mathbf{1}$ to $\mathbf{1}$ and preserves S'. In the non-compact picture, the map s is defined for $\xi \neq 0$ and is given by

$$(\xi_1, \xi_2, \dots, \xi_n) \longmapsto \left(-\frac{\xi_1}{|\xi|^2}, \frac{\xi_2}{|\xi|^2}, \dots, \frac{\xi_n}{|\xi|^2}\right). \tag{4.9}$$

The two charts $\xi \longmapsto c(\xi)$ and $\xi \longmapsto s\big(c(\xi)\big)$ cover S. Their common domain corresponds to $\xi \neq 0$, the change of chart being given by (4.9), which is the local expression in the non-compact picture of the transform s. So Proposition 4.10, when applied to $g = s$ is exactly what is needed to prove that there is a global differential operator $\mathbf{E}_\lambda$ on S which is expressed by E_λ in the non-compact model. Clearly $\mathbf{E}_\lambda$ satisfies

$$\forall\, g \in G', \qquad \mathbf{E}_\lambda \circ \pi_\lambda(g) = \pi_{\lambda+1}(g) \circ \mathbf{E}_\lambda.$$

By (4.8),

$$M \circ I_\lambda = \frac{1}{4(\lambda - n + 1)} I_{\lambda+1}(g) \circ \mathbf{E}_\lambda.$$

Compose both sides with $I_{n-\lambda-1}$ and use (2.3) to get

$$\mathbf{D}_\lambda = \frac{\pi^{-n}}{4(\lambda-n+1)\Gamma(n-\lambda-1)\Gamma(\lambda+1)}\mathbf{E}_\lambda$$

or equivalently

$$\widetilde{\mathbf{D}}_\lambda = -\frac{1}{4\pi^n}\mathbf{E}_\lambda.$$

This relation implies in particular that $\widetilde{\mathbf{D}}_\lambda$ is a differential operator on S. ■

5 The families $\mathbf{D}_{\lambda,N}$, $\widetilde{\mathbf{D}}_{\lambda,N}$ and $\mathbf{D}_N(\lambda)$

For $N \geq 1$, set

$$\widetilde{\mathbf{D}}_{\lambda,N} = \widetilde{\mathbf{D}}_{\lambda+N-1} \circ \cdots \circ \widetilde{\mathbf{D}}_\lambda.$$

Let M^N be the operator on $C^\infty(S)$ given by multiplication by x_n^N. Set

$$\mathbb{D}_{\lambda,N} = I_{n-N-\lambda} \circ M^N \circ I_\lambda.$$

Proposition 5.1.

i) $\widetilde{\mathbf{D}}_{\lambda,N}$ *and* $\mathbb{D}_{\lambda,N}$ *are differential operators on* S *which intertwine* $\pi_{\lambda|G'}$ *and* $\pi_{\lambda+N|G'}$.

ii)

$$\widetilde{\mathbf{D}}_{\lambda,N} = \pi^{n(N-1)}\Gamma(\lambda+N)\Gamma(n-\lambda-N)\mathbb{D}_{\lambda,N}. \tag{5.1}$$

Proof. Repeated uses of (3.2) show that, for any $\mu \in \mathbb{C}$, M^N intertwines $\pi_{\mu|G'}$ and $\pi_{\mu-N|G'}$. Hence $\mathbb{D}_{\lambda,N}$ intertwines $\pi_{\lambda|G'}$ and $\pi_{\lambda+N|G'}$. On the other hand, repeated uses of (3.3) proves that $\widetilde{\mathbf{D}}_{\lambda,N}$ also intertwines $\pi_{\lambda|G'}$ and $\pi_{\lambda+N|G'}$.

Next, $\widetilde{\mathbf{D}}_{\lambda,N}$ as a composition of differential operators on S is a differential operator. So it remains to prove (5.1).

Substitute $\mathbf{D}_{\lambda+j} = I_{n-\lambda-j-1} \circ M \circ I_{\lambda+j}$ for $0 \leq j \leq N-1$ to get

$$\begin{aligned}&\mathbf{D}_{\lambda+N-1} \circ \mathbf{D}_{\lambda+N-2} \circ \cdots \circ \mathbf{D}_\lambda\\ &\quad = I_{-\lambda+n-N} \circ \cdots \circ I_{-\lambda+n-j-1} \circ M \circ I_{\lambda+j} \circ I_{-\lambda+n-j} \circ M \circ I_{\lambda+j-1} \circ \cdots \circ I_\lambda,\end{aligned}$$

and use (2.3) repeatedly for $\lambda+j$ to obtain

$$\begin{aligned}&\mathbf{D}_{\lambda+N-1} \circ \mathbf{D}_{\lambda+N-2} \circ \cdots \circ \mathbf{D}_\lambda\\ &\quad = \pi^{n(N-1)}\left(\prod_{j=1}^{N-1}\Gamma(\lambda+j)\Gamma(n-\lambda-j)\right)^{-1} I_{-\lambda+n-N} \circ M^N \circ I_\lambda.\end{aligned}$$

Multiply by the appropriate Γ factors coming from (3.4) to get the formula. ■

The group G' acts conformally on S'. The scalar principal series for $G' \simeq \mathrm{SO}_0(1,n)$ is defined as follows: for $\mu \in \mathbb{C}$, for $g \in G'$ and $f \in C^\infty(S')$,

$$\pi'_\mu(g)f(x) = \kappa\big(g^{-1},x\big)^\mu f\big(g^{-1}(x)\big), \qquad x \in S'. \tag{5.2}$$

Let res: $C^\infty(S) \longrightarrow C^\infty(S')$ be the restriction map from S to S', defined for $f \in C^\infty(S)$ by $(\operatorname{res} f)(x) = f(x), x \in S'$. The last remark makes clear that for $\lambda \in \mathbb{C}$ and for $g \in G'$,

$$\operatorname{res} \circ \pi_\lambda(g) = \pi'_\lambda(g) \circ \operatorname{res}. \tag{5.3}$$

Define the differential operator $\mathbf{D}_N(\lambda)\colon C^\infty(S) \longrightarrow C^\infty(S')$ by

$$\mathbf{D}_N(\lambda) = \operatorname{res} \circ \widetilde{\mathbf{D}}_{\lambda,N}.$$

Theorem 5.2. $\mathbf{D}_N(\lambda)$ *satisfies*

$$\forall\, g \in G' \qquad \mathbf{D}_N(\lambda) \circ \pi_\lambda(g) = \pi'_{\lambda+N}(g) \circ \mathbf{D}_N(\lambda).$$

The proof follows immediately from the covariance property of $\widetilde{\mathbf{D}}_{\lambda,N}$ and of the restriction map (5.3).

6 The family $E_N(\lambda)$

The previous constructions of differential operators made for S and S' can be made in a similar manner in the non compact picture, i.e., for $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. For $N \in \mathbb{N}$, let $E_{\lambda,N}$ be defined by

$$E_{\lambda,N} = E_{\lambda+N-1} \circ \cdots \circ E_\lambda$$

and

$$E_N(\lambda) = \operatorname{res} \circ E_{\lambda,N},$$

where res is the restriction from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$. Then $E_{\lambda,N}$ is a differential operator on $\mathbb{R}^n$ which is covariant with respect to $(\rho_{\lambda|G'}, \rho_{\lambda+N|G'})$ and $E_N(\lambda)$ is a differential operator from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ which is covariant with respect to $(\rho_{\lambda|G'}, \rho'_{\lambda+N})$.[3]

In this section, for the sake of completeness, we compare $E_N(\lambda)$ with Juhl's operator for the non compact model. For $\xi \in \mathbb{R}^n$, introduce the notation $\xi = (\xi', \xi_n)$ where $\xi' \in \mathbb{R}^{n-1}$. Let $\Delta' = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial \xi_j^2}$.

Proposition 6.1. *Let* $E\colon C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1})$ *be a differential operator and assume that* E *is covariant with respect to* $(\rho_{\lambda|G'}, \rho'_{\lambda+N})$ *for some* $N \in \mathbb{N}$*. Then there exits a family of complex constants* a_j*,* $0 \le j \le [\frac{N}{2}]$ *such that*

$$E = \operatorname{res} \circ \sum_{j=0}^{[\frac{N}{2}]} a_j \left(\frac{\partial}{\partial \xi_n}\right)^{N-2j} \Delta'^j.$$

Proof. By the definition of a differential operator from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$, E can be written as a locally finite sum

$$\sum_{i,J} a_{i,J}(\xi') \operatorname{res} \circ \left(\frac{\partial}{\partial \xi_n}\right)^i \partial^J,$$

where $J = (j_1, j_2, \dots, j_{n-1})$ is a $(n-1)$-tuple, $\partial^J = \prod_{k=1}^{n-1} \left(\frac{\partial}{\partial \xi_k}\right)^{j_k}$ and $a_{i,J}$ is a smooth function of $\xi' \in \mathbb{R}^{n-1}$.

[3]The representation ρ' is the principal series for G' realized in the $\mathbb{R}^{n-1}$, defined in analogy with (5.2).

The invariance by translations forces the $a_{i,J}$ to be constants (and also the sum to be finite). The invariance by $\mathrm{SO}(n-1)$ forces the expression to be of the form

$$\sum_{i,j} a_{i,j} \left(\frac{\partial}{\partial \xi_n}\right)^i (\Delta')^j$$

and finally the covariance under the action of the dilations forces $i + 2j = N$. The statement follows. ■

Notice that the proof uses only the covariance property for the parabolic subgroup of affine conformal diffeomorphisms of $\mathbb{R}^{n-1}$. The full covariance condition implies further conditions on the coefficients $a_{i,j}$, explicitly written by A. Juhl (see [6], condition (5.1.2) for N even and (5.1.22) for N odd), proving in particular that there exists (up to a constant) a unique covariant differential operator. Now let

$$E_N(\lambda) = \sum_{j=0}^{[\frac{N}{2}]} a_j(\lambda, N) \left(\frac{\partial}{\partial \xi_n}\right)^{N-2j} \Delta'^j,$$

where $a_j(\lambda, N)$ are complex numbers.

To find the ratio between $E_N(\lambda)$ and the corresponding Juhl's operator, it is enough to know some coefficient of $E_N(\lambda)$ and to compare it to the corresponding coefficient of Juhl's operator. It turns out that the coefficient $a_0(\lambda, N)$ is rather easy to compute.

Lemma 6.2.

i) *For* $k \in \mathbb{N}$ *and* $\mu \in \mathbb{C}$,

$$E_\mu \xi_n^k = k(2\mu - n + 1 + k)\xi_n^{k-1}.$$

ii) *For* $N \in \mathbb{N}$ *and for* $\lambda \in \mathbb{C}$,

$$E_{\lambda,N}\left(\xi_n^N\right) = N!(2\lambda - n + N + 1)(2\lambda - n + N + 2)\cdots(N + 2\lambda - n + 2N).$$

iii) *The constant* $a_0(\lambda, N)$ *is given by*

$$a_0(\lambda, N) = (2\lambda - n + N + 1)(2\lambda - n + N + 2)\cdots(2\lambda - n + 2N).$$

Proof. Let f be a function on $\mathbb{R}^n$ which depends only on ξ_n. Then $\Delta' f = 0$, and

$$E_\mu f = \left((2\mu - n + 2)\frac{\partial}{\partial \xi_n} + \xi_n \frac{\partial^2}{\partial \xi_n^2}\right) f,$$

so that *i)* and *ii)* are reduced to elementary one variable computations. For *iii)* observe that

$$E_{\lambda,N}\left(\xi_n^N\right) = a_0(\lambda, N)\left(\frac{\partial}{\partial \xi_n}\right)^N \left(\xi_n^N\right) + 0 + \cdots + 0 = N! a_0(\lambda, N),$$

hence $E_N(\lambda)(\xi_n^N) = N! a_0(\lambda, N)$ and *iii)* follows. ■

The comparison with Juhl's operator is then easy. As his normalization depends on the parity of N, one has to examine two cases.

- In the even case, $E_N(\lambda)$ is obtained by multiplying Juhl's operator by

$$\frac{N!}{(\frac{N}{2})!}2^{\frac{N}{2}-1}\prod_{j=1}^{\frac{N}{2}}(2\lambda-n+N+2j).$$

- In the odd case, $E_N(\lambda)$ is obtained by multiplying Juhl's operator by

$$\frac{N!}{(\frac{N-1}{2})!}2^{\frac{N+1}{2}}\prod_{j=0}^{\frac{N-1}{2}}(2\lambda-n+N+1+2j).$$

7 The operator $\mathbf{D}_\lambda$ in the ambient space model

This last section is devoted to another (simpler) construction of (a multiple of) the operator $\mathbf{D}_\lambda$, using the *ambient space* realization of the principal series.

Let Ξ^+ be the positive light cone,

$$\Xi^+=\big\{\mathbf{x}\in\mathbf{E},\, Q(\mathbf{x})=[\mathbf{x},\mathbf{x}]=0,\, t(\mathbf{x})>0\big\}.$$

For $\lambda\in\mathbb{C}$, let

$$\mathcal{H}_\lambda=\big\{F\in C^\infty(\Xi^+),\ F(t\mathbf{x})=t^{-\lambda}F(\mathbf{x}),\ \text{for } t\in\mathbb{R}^+\big\}.$$

The space $\mathcal{H}_\lambda$ is in one-to-one correspondence with the space $C^\infty(S)$ through the map R_λ

$$\mathcal{H}_\lambda\ni F\longmapsto R_\lambda F\in C^\infty(S),\qquad R_\lambda F(x)=F\big((1,x)\big).$$

The space $\mathcal{H}_\lambda$ inherits the corresponding topology. For $g\in G$, and $F\in\mathcal{H}_\lambda$, let

$$\Pi_\lambda(g)F=F\circ g^{-1}.$$

Then Π_λ defines a representation of G on $\mathcal{H}_\lambda$ and it is easily verified that

$$R_\lambda\circ\Pi_\lambda(g)=\pi_\lambda(g)\circ R_\lambda, \tag{7.1}$$

so that Π_λ is yet another model for the representation π_λ of G.

Let $\Box=\frac{\partial^2}{\partial t^2}-\sum_{j=0}^{n}\frac{\partial^2}{\partial x_j^2}$ be the d'Alembertian on $\mathbf{E}$. It satisfies, for any $g\in G$ and F a smooth function on $\mathbf{E}$

$$\Box(F\circ g)=(\Box F)\circ g. \tag{7.2}$$

The following lemma, which I learnt from [3] is a key result for what follows.

Lemma 7.1. *Let F_1, F_2 be two smooth functions defined in a neighborhood of Ξ^+, positively homogeneous of degree $-\frac{n}{2}+1$ and which coincide on Ξ^+. Then $\Box F_1$ and $\Box F_2$ coincide on Ξ^+.*

Proof. The function F_1-F_2 vanishes on Ξ^+. Notice that $dQ(\mathbf{x})\neq 0$ for any $\mathbf{x}\in\Xi^+$. Hence, there exists a smooth function G defined on a neighborhood of Ξ^+ such that

$$F_1(\mathbf{x})-F_2(\mathbf{x})=Q(\mathbf{x})G(\mathbf{x}).$$

Moreover, G is positively homogeneous of degree $-\frac{n}{2}-1$. Now, for any smooth function H on $\mathbf{E}$

$$\Box(QH) = 2(n+2)H + 4EH + Q\Box H,$$

where $E = t\frac{\partial}{\partial t} + \sum_{j=0}^{n} x_j \frac{\partial}{\partial x_j}$ is the Euler operator. As G is homogeneous of degree $-\frac{n}{2}-1$,

$$EG(\mathbf{x}) = \left(-\frac{n}{2}-1\right)G(\mathbf{x}),$$

and hence $\Box(QG)(x) = 0$ for $x \in \Xi^+$. The lemma follows. ∎

The next result is a reformulation of the previous lemma.

Lemma 7.2. *Let $F \in \mathcal{H}_{\frac{n}{2}-1}$. Extend F smoothly to a positively homogeneous function of degree $-\frac{n}{2}+1$ to neighborhood of Ξ^+. Then the restriction to Ξ^+ of $\Box F$ does not depend on the extension.*

The operator $\Box$ induces a map from $\mathcal{H}_{\frac{n}{2}-1}$ to $\mathcal{H}_{\frac{n}{2}+1}$ and intertwines the action of G. Let Δ_S be the operator defined on $C^\infty(S)$ by

$$\Delta_S = R_{\frac{n}{2}+1} \circ \Box \circ R^{-1}_{\frac{n}{2}-1}.$$

The invariance of $\Box$ (see (7.2)) and the covariance of R_λ (see (7.1)) imply the following proposition.

Proposition 7.3. *The operator Δ_S (conformal Laplacian or Yamabe operator on S) is a differential operator on S which is covariant with respect to $(\pi_{\frac{n}{2}-1}, \pi_{\frac{n}{2}+1})$.*

Let $\mathbf{B}_\mu$ be the differential operator on $\mathbf{E}$ defined by

$$\mathbf{B}_\mu F(\mathbf{x}) = x_n \Box F(\mathbf{x}) - 2\mu \frac{\partial F}{\partial x_n}.$$

Lemma 7.4. *Let $\mu \in \mathbb{C}$. Let F be a smooth function on $\mathbf{E}$. Then, on $\{x_n \neq 0\}$,*

$$\mathbf{B}_\mu F(\mathbf{x}) = x_n |x_n|^{-\mu} \Box\big(|x_n|^\mu F\big)(\mathbf{x}) + \mu(\mu-1)\frac{1}{x_n} F(\mathbf{x}). \tag{7.3}$$

Proof. By an elementary calculation,

$$\Box\big(|x_n|^\mu F\big)(\mathbf{x}) = |x_n|^\mu \Box F(\mathbf{x}) - 2\mu \operatorname{sgn}(x_n)|x_n|^{\mu-1}\frac{\partial F}{\partial x_n}(\mathbf{x}) - \mu(\mu-1)|x_n|^{\mu-2}F(\mathbf{x}),$$

so that

$$\Box\big(|x_n|^\mu F\big) + \mu(\mu-1)|x_n|^{\mu-2}F = \operatorname{sgn}(x_n)|x_n|^{\mu-1}\mathbf{B}_\mu F.$$

The conclusion follows, by noticing that $x_n = \operatorname{sgn}(x_n)|x_n|$. ∎

Proposition 7.5. *Let $g \in G'$. Then for F a smooth function on $\mathbf{E}$,*

$$\mathbf{B}_\mu(F \circ g) = (\mathbf{B}_\mu F) \circ g.$$

Proof. As $g \in G'$, the coordinate x_n is unchanged by the action of g, and the action of g commutes with $\frac{\partial}{\partial x_n}$ and with $\Box$. The result follows. ∎

Proposition 7.6. *Let $F \in \mathcal{H}_\lambda$. Extend F smoothly to a neighborhood of Ξ^+ as a positively homogeneous function of degree $-\lambda$. Then the restriction to Ξ^+ of $\mathbf{B}_{\lambda-\frac{n}{2}+1}F$ does not depend on the extension.*

Proof. The function $|x_n|^{\lambda-\frac{n}{2}+1}F(\mathbf{x})$ is homogenous of degree $-\frac{n}{2}+1$, and hence, by Lemma 7.2, for $x \in \Xi^+$, $\Box\big(|x_n|^{\lambda-\frac{n}{2}+1}F\big)(\mathbf{x})$ only depend on the values of F on Ξ^+. Hence, by (7.3), for $\mathbf{x}$ in Ξ^+, $x_n \neq 0$, $\mathbf{B}_{\lambda-\frac{n}{2}+1}F(\mathbf{x})$ does not depend on the extension of F. The result follows by continuity. ■

Proposition 7.7. *The differential operator $\mathbf{B}_{\lambda-\frac{n}{2}+1}$ induces a map from $\mathcal{H}_\lambda$ into $\mathcal{H}_{\lambda+1}$, which commutes with the action of G'.*

Proof. The invariance follows from Proposition 7.5. ■

Having constructed a covariant operator in the ambient space model, it is possible to express it both in the non-compact and in the compact picture.

Proposition 7.8. *The local expression of the operator $\mathbf{B}_{\lambda-\frac{n}{2}+1}$ in the non compact picture is equal to $-E_\lambda$.*

Proof. Let f be a smooth function on $\mathbb{R}^n$. Recall the map c (cf. (4.1)) which realizes the passage from $\mathbb{R}^n$ to S. Its inverse is given by

$$S \setminus \{-\mathbf{1}\} \ni (x_0, x_1, \dots, x_n) \longmapsto \left(\frac{x_1}{1+x_0}, \dots, \frac{x_n}{1+x_0}\right).$$

So map f to a function on S by

$$C_\lambda^{-1}f(x) = (1+x_0)^{-\lambda} f\left(\frac{x_1}{1+x_0}, \dots, \frac{x_n}{1+x_0}\right).$$

Consider the function F on $\mathbf{E}$ defined by

$$F(\mathbf{x}) = (t+x_0)^{-\lambda} f\left(\frac{x_1}{t+x_0}, \dots, \frac{x_n}{t+x_0}\right).$$

Then F is homogenous of degree $-\lambda$ and coincide on S with $C_\lambda^{-1}f$. To compute $\mathbf{B}_{\lambda-\frac{n}{2}+1}F$, first observe that

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x_0}, \qquad \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x_0^2},$$

so that

$$\Box F = -\sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_j^2}.$$

Hence

$$\begin{aligned}\mathbf{B}_{\lambda-\frac{n}{2}+1}F(\mathbf{x}) = &-(t+x_0)^{-\lambda-2}x_n(\Delta f)\left(\frac{x_1}{t+x_0}, \dots, \frac{x_n}{t+x_0}\right)\\ &-2\left(\lambda-\frac{n}{2}+1\right)(t+x_0)^{-\lambda}(t+x_0)^{-1}\frac{\partial f}{\partial \xi_n}\left(\frac{x_1}{t+x_0}, \dots, \frac{x_n}{t+x_0}\right).\end{aligned}$$

Now letting $\mathbf{x} = (1, c(\xi))$,

$$\mathbf{B}_{\lambda-\frac{n}{2}+1}F(1, c(\xi)) = -\xi_n \Delta f(\xi) - (2\lambda - n + 2)\frac{\partial f}{\partial \xi_n}(\xi).$$

A comparison with (4.6) implies the result. ■

Proposition 7.9. *The expression of the operator* $\mathbf{B}_{\lambda-\frac{n}{2}+1}$ *on* S *is given by*

$$x_n|x_n|^{-\lambda+\frac{n}{2}-1}\Delta_S\circ|x_n|^{\lambda-\frac{n}{2}+1}+\left(\lambda-\frac{n}{2}+1\right)\left(\lambda-\frac{n}{2}\right)\frac{1}{x_n}.$$

The expression, a priori defined on $x_n\neq 0$ *can be continued continuously to all of* S*.*

Proof. Let $f\in C^\infty(S)$. Then

$$F(\mathbf{x})=\left(x_0^2+\cdots+x_n^2\right)^{-\lambda}f\left(\frac{x_0}{\sqrt{x_0^2+\cdots+x_n^2}},\dots,\frac{x_n}{\sqrt{x_0^2+\cdots+x_n^2}}\right)$$

is a function defined on $\mathbf{E}\setminus\{0\}$ which is positively homogeneous of degree $-\lambda$ and such that for $x\in S$,

$$F(1,x)=f(x).$$

By (7.3) with $\mu=\lambda-\frac{n}{2}+1$ and for $\mathbf{x}\neq 0$, $x_n\neq 0$,

$$\mathbf{B}_{\lambda-\frac{n}{2}+1}F(\mathbf{x})=x_n|x_n|^{-\lambda+\frac{n}{2}+1}\Box\left(|x_n|^{\lambda-\frac{n}{2}+1}F\right)(\mathbf{x})+\left(\lambda-\frac{n}{2}+1\right)\left(\lambda-\frac{n}{2}\right)\frac{1}{x_n}F(\mathbf{x}).$$

The function $|x_n|^{\lambda-\frac{n}{2}+1}F(\mathbf{x})$ is positively homogeneous of degree $-\frac{n}{2}+1$. Thus, by Lemma 7.2 and the definition of the Yamabe operator Δ_S, for $x\in S$,

$$\mathbf{B}_{\lambda-\frac{n}{2}+1}F(1,x)=x_n|x_n|^{-\lambda+\frac{n}{2}-1}\Delta_S\left(|x_n|^{\lambda-\frac{n}{2}+1}f\right)(x)+\left(\lambda-\frac{n}{2}+1\right)\left(\lambda-\frac{n}{2}\right)\frac{1}{x_n}f(x),$$

from which the statement follows, at least for $x_n\neq 0$. As $\mathbf{B}_{\lambda-\frac{n}{2}+1}$ induces a smooth differential operator on S, the formula determines the operator on all of S by continuity. ■

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