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Locally Nilpotent Derivations of Free Algebra of Rank Two

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Abstract. In commutative algebra, if δ is a locally nilpotent derivation of the polynomial algebra $K[x_1,\dots,x_d]$ over a field K of characteristic 0 and w is a nonzero element of the kernel of δ, then $\Delta = w\delta$ is also a locally nilpotent derivation with the same kernel as δ. In this paper we prove that the locally nilpotent derivation Δ of the free associative algebra $K\langle X,Y\rangle$ is determined up to a multiplicative constant by its kernel. We show also that the kernel of Δ is a free associative algebra and give an explicit set of its free generators.

Key words: free associative algebras; locally nilpotent derivations; algebras of constants

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To the 80th anniversary of Dmitry Fuchs

1 Introduction

Let K be a field of characteristic 0. Locally nilpotent derivations δ of polynomial algebras $K[x_1,\dots,x_d]$ and their kernels $\ker(\delta)$ are subjects of active investigation. Traditionally, the kernel of a derivation δ of $K[x_1,\dots,x_d]$ is called the algebra of constants of δ and is denoted by $K[x_1,\dots,x_d]^\delta$. The algebras of constants of locally nilpotent derivations play an essential role in the study of the automorphism group of $K[x_1,\dots,x_d]$, including the generation of $\mathrm{Aut}(K[x,y])$ by tame automorphisms, the Jacobian conjecture, in invariant theory, fourteenth Hilbert's problem and other important topics. See the books by Nowicki [18], van den Essen [29], and Freudenburg [10] for details. In particular, using locally nilpotent derivations, Rentschler [20] gave an easy proof of the theorem of Jung–van der Kulk [11, 30] that all automorphisms of $K[x,y]$ are tame. Another natural proof based on locally nilpotent derivations was given by Makar-Limanov [15], see also the book [6]. The most natural way to define the Nagata automorphism [17]

$$(x,y,z)\to\left(x-2\left(xz+y^2\right)y-\left(xz+y^2\right)^2z,y+\left(xz+y^2\right)z,z\right)$$

is also in terms of locally nilpotent derivations, see Bass [1] and Smith [25]. The famous Jacobian conjecture is equivalent to several conjectures stated in the language of locally nilpotent derivations, see [29]. Several nice counterexamples to fourteenth Hilbert's problem are obtained as algebras of constants of locally nilpotent derivations, see the survey and the book by Freudenburg [9, 10] and the survey by Nowicki [19]. On the other hand, the well known theorem of

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Weitzenböck [31] states that if δ is a nilpotent linear operator acting on the d-dimensional vector space $Kx_1 \oplus \cdots \oplus Kx_d$, then the algebra of constants of the locally nilpotent derivation of $K[x_1, \ldots, x_d]$ which extends δ is a finitely generated algebra. A modern proof of the theorem is given by Seshadri [22], with further simplification by Tyc [27], see also [18]. Clearly, the algebra of constants $K[x_1, \ldots, x_d]^\delta$ coincides with the algebra of invariants of the linear operator

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots .$$

If δ is a locally nilpotent derivation of $K[x_1, \ldots, x_d]$ and $0 \neq w \in K[x_1, \ldots, x_d]^\delta$, then $\Delta = w\delta$ is also a locally nilpotent derivation with the same algebra of constants as δ. In particular, starting from the Weitzenböck derivation of $K[x, y, z]$ defined by

$$\delta(x) = -2y, \qquad \delta(y) = z, \qquad \delta(z) = 0,$$

$w = xz + y^2 \in K[x, y, z]^\delta$, and $\Delta = \left(xz + y^2\right)\delta$ one obtains the Nagata automorphism as $\exp(\Delta)$. We would like to mention that Shestakov and Umirbaev [23, 24] proved the Nagata conjecture that the Nagata automorphism is wild with methods of noncommutative algebra.

Locally nilpotent derivations of free associative algebras $K\langle X_1, \ldots, X_d\rangle$ have not been studied as intensively as in the commutative case. We shall mention the old result of Falk [8] who described the intersection of the kernels of the formal partial derivatives $\partial/\partial X_j$ of $K\langle X_1, \ldots, X_d\rangle$, and the relations of the formal partial derivatives with theory of algebras with polynomial identity due to Specht [26], see also [6] for further development. Drensky and Gupta [7] studied the kernels of Weitzenböck derivations of $K\langle X_1, \ldots, X_d\rangle$ and established that in all nontrivial cases the kernel is not finitely generated. As in the case of polynomial algebras, the candidate for a wild automorphism, the automorphism of Anick [2, p. 343]

$$(X, Y, Z) \to (X + Z(XZ - ZY), Y + (XZ - ZY)Z, Z)$$

can also be expressed as $\exp(\Delta)$ for the locally nilpotent derivation Δ of $K\langle X, Y, Z\rangle$ defined by

$$\Delta(X) = Z(XZ - ZY), \qquad \Delta(Y) = (XZ - ZY)Z, \qquad \Delta(Z) = 0.$$

The wildness of the Anick automorphism was established by Umirbaev [28].

In this paper we study locally nilpotent derivations Δ of the free unitary associative algebra $K\langle X, Y\rangle$ over a field K of characteristic 0. As in the commutative case we shall call the kernel of Δ the algebra of constants of Δ and denote it by $K\langle X, Y\rangle^\Delta$. Our main result is that the locally nilpotent derivations of $K\langle X, Y\rangle$ are determined up to a multiplicative constant by their algebras of constants.

It is easy to see that Δ is of the form $\Delta(U) = 0$, $\Delta(V) = f(U)$, with respect to a suitable system of generators U, V of $K\langle X, Y\rangle$. This follows from the description of Rentschler [20] of the locally nilpotent derivations of $K[x, y]$ and the isomorphism of the automorphism groups of $K[x, y]$ and $K\langle X, Y\rangle$ which is a consequence of the theorem of Jung–van der Kulk [11, 30] and its analogue for the automorphisms of $K\langle X, Y\rangle$ due to Czerniakiewicz [3, 4] and Makar-Limanov [14]. This result is similar to the recent description of locally nilpotent derivations of the free Poisson algebra with two generators given by Makar-Limanov, Turusbekova, and Umirbaev [16].

As a consequence of the result of Lane [13] and Kharchenko [12] the algebra of constants $K\langle X, Y\rangle^\Delta$ of the nontrivial Weitzenböck derivation Δ of $K\langle X, Y\rangle$ is a free associative algebra. A set of free generators of this algebra was given by Drensky and Gupta [7]. We generalize this result and show that the algebra $K\langle X, Y\rangle^\Delta$ is free for any locally nilpotent derivation Δ of $K\langle X, Y\rangle$. As in [7] we give an explicit set of free generators of $K\langle X, Y\rangle^\Delta$. See also [5] where it is shown that $K\langle X, Y\rangle^\Delta$ is a free associative algebra for a nontrivial homogeneous derivation (and from which the freeness in our case can be deduced).

2 Preliminaries

For an algebra R over a field K a linear operator $\delta\colon R\to R$ is called a derivation if it satisfies the Leibniz law $\delta(ab)=\delta(a)b+a\delta(b)$. The kernel of a derivation δ is denoted by R^δ and the elements of the kernel are called δ-constants (or just constants when this is not confusing). A derivation δ is called locally nilpotent if for any $r\in R$ there exists a natural number n (which depends on r) for which $\delta^n(r)=0$. The function

$$\deg(r)=\max\big(d\,|\,\delta^d(r)\neq 0\big),\qquad \deg(0)=-\infty,$$

is a degree function with familiar properties:

$$\begin{aligned}&\deg(r_1r_2)=\deg(r_1)+\deg(r_2),\qquad \deg(r_1+r_2)\leq\max(\deg(r_1),\deg(r_2)),\\&\deg(r_1+r_2)=\max(\deg(r_1),\deg(r_2))\qquad\text{when}\quad \deg(r_1)\neq\deg(r_2),\\&\deg(\delta(r))=\deg(r)-1\qquad\text{if}\quad \delta(r)\neq 0.\end{aligned}$$

The set of all lnds (locally nilpotent derivations) of R is denoted by $\mathrm{LND}(R)$.

The intersection $\bigcap R^\delta$, $\delta\in\mathrm{LND}(R)$, of kernels of all locally nilpotent derivations of R is denoted by $\mathrm{AK}(R)$ (absolute Konstanten of R, sometimes denoted as $\mathrm{ML}(R)$).

If $\delta\in\mathrm{LND}(R)$ and characteristic of K is zero then the linear operator

$$\exp(\delta)=1+\frac{\delta}{1!}+\frac{\delta^2}{2!}+\cdots$$

is an automorphism of R.

In the sequel we fix a field K of characteristic 0 and consider the polynomial algebra $K[x,y]$ and the free associative algebra $K\langle X,Y\rangle$. Let

$$\pi\colon\ K\langle X,Y\rangle\to K[x,y]$$

be the natural homomorphism. We denote the elements U, V, etc. of $K\langle X,Y\rangle$ by upper case symbols and their images under π by the same lower case symbols u, v, etc. Let C be the commutator ideal of $K\langle X,Y\rangle$. It is generated by the commutator

$$T_1=[Y,X]=YX-XY.$$

By the theorem of Jung–van der Kulk [11, 30], the automorphisms of $K[x,y]$ are tame, i.e., are compositions of affine automorphisms

$$x\to a_1x+a_2y+a_3,\qquad y\to b_1x+b_2y+b_3,\qquad a_i,b_i\in K,\qquad a_1b_2-a_2b_1\neq 0,$$

and triangular automorphisms

$$x\to x,\qquad y\to y+p(x),\qquad p(x)\in K[x].$$

A similar theorem of Czerniakiewicz [3, 4] and Makar-Limanov [14] states that the automorphisms of $K\langle X,Y\rangle$ are also tame. Therefore

$$\Psi(T_1)=cT_1,\qquad c\in K^*,$$

for any automorphism Ψ of $K\langle X,Y\rangle$ (indeed, just check that this is true for affine and triangular automorphisms).

The structure of the automorphism groups of $K[x,y]$ and $K\langle X,Y\rangle$ is also known, it is a free product of the subgroups of affine and triangular automorphisms with amalgamation along their intersection [21]. So we can think that there is a group H isomorphic to $\mathrm{Aut}\,K[x,y]$ and $\mathrm{Aut}\,K\langle X,Y\rangle$ which acts on $K[x,y]$ and $K\langle X,Y\rangle$.

Any automorphism of $K\langle X,Y\rangle$ induces an automorphism of $K[x,y]$ and, since the structure of the group H insures that this is one to one correspondence, any automorphism of $K[x,y]$ can be uniquely lifted to an automorphism of $K\langle X,Y\rangle$.

We shall use below a lexicographic ordering of monomials of $K\langle X,Y\rangle$ defined by $Y \gg X > 1$ and denote by $\overline{S}$ the leading monomial of $S \in K\langle X,Y\rangle$.

In the sequel we shall show that we can reduce our considerations to the case when the lnd Δ is such that

$$\Delta(X) = 0, \qquad \Delta(Y) = F = f(X),$$

where $0 \neq f(x) \in K[x]$. In this special case we shall define the operator $\boxdot$ on $K\langle X,Y\rangle$ by

$$\boxdot(A) = YAF - FAY, \qquad A \in K\langle X,Y\rangle,$$

and shall fix the sequence $T_1, T_2, \ldots$, starting with $T_1 = YX - XY$ and then inductively

$$T_{i+1} = \boxdot^i(T_1).$$

3 Description of locally nilpotent derivations

Though the lnds of $K\langle X,Y\rangle$ are similar to the lnds of $K[x,y]$ there are also significant differences.

It is quite clear that $\mathrm{AK}(K[x,y]) = K$ (just observe that the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are locally nilpotent) but we shall show later that $\mathrm{AK}(K\langle X,Y\rangle) = K[T_1]$. The following lemma shows that $\mathrm{AK}(K\langle X,Y\rangle) \supseteq K[T_1]$.

Lemma 3.1. *$\delta(T_1) = 0$ for any lnd of $K\langle X,Y\rangle$.*

Proof. If $\delta \in \mathrm{LND}(K\langle X,Y\rangle)$ then $\lambda\delta \in \mathrm{LND}(K\langle X,Y\rangle)$ for any $\lambda \in K$. Take $\Psi_\lambda = \exp(\lambda\delta)$; then $\Psi_\lambda([Y,X]) = c(\lambda)[Y,X]$, where $c(t) \in K[t]$ (recall that δ is an lnd). On the other hand $\Psi_\lambda\Psi_\mu = \Psi_{\lambda+\mu}$, i.e., $c(s)c(t) = c(s+t)$. Since $c(s) \neq 0$ this is possible only if $c(t) = 1$. Hence $\delta([Y,X]) = 0$. ■

Now we shall prove that lnds of $K\langle X,Y\rangle$ are similar to those of $K[x,y]$.

Proposition 3.2. *Let Δ be a locally nilpotent derivation of $K\langle X,Y\rangle$. Then there is a system of generators U, V of $K\langle X,Y\rangle$ and a polynomial $f(U)$ depending on U only, such that $\Delta(U) = 0$, $\Delta(V) = f(U)$.*

Proof. Let Δ be a locally nilpotent derivation of $K\langle X,Y\rangle$. Clearly, Δ induces a locally nilpotent derivation δ of $K[x,y]$. By the theorem of Rentschler [20], $K[x,y]$ has a system of generators u, v such that $\delta(u) = 0$, $\delta(v) = f(u)$ for some $f(u) \in K[u]$.

As was mentioned above this pair of generators can be uniquely lifted to the pair U, V of generators of $K\langle X,Y\rangle$.

Let us consider the automorphisms

$$\Phi = \exp(\Delta) \in \mathrm{Aut}\,K\langle X,Y\rangle = \mathrm{Aut}\,K\langle U,V\rangle$$

and

$$\varphi = \exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots \in \mathrm{Aut}\,K[x,y] = \mathrm{Aut}\,K[u,v].$$

Then

$$\varphi\colon\ u\to u,\qquad \varphi\colon\ v\to v+f(u).$$

From the uniqueness mentioned in Section 2

$$\varphi(u)=u,\qquad \varphi(v)=v+f(u)$$

implies $\Phi(U)=U$, $\Phi(V)=V+f(U)$. Since $\Phi=\exp(\Delta)=1+\Theta$, where

$$\Theta=\frac{\Delta}{1!}+\frac{\Delta^2}{2!}+\cdots$$

and $\Theta^n(S)=0$ for $S\in K\langle X,Y\rangle$ and a sufficiently large n, we have that

$$\Delta=\log(1+\Theta)=\frac{\Theta}{1}-\frac{\Theta^2}{2}+\frac{\Theta^3}{3}-\cdots$$

and Φ determines uniquely the lnd Δ. Hence $\Delta(U)=0$, $\Delta(V)=f(U)$. ■

Another difference between the locally nilpotent derivations of $K[x,y]$ and $K\langle X,Y\rangle$ is that in the latter case they can be distinguished by their algebras of constants.

Theorem 3.3. *Let Δ_1 and Δ_2 be two non-zero locally nilpotent derivations of $K\langle X,Y\rangle$. Then Δ_1 and Δ_2 have the same algebras of constants if and only if $\Delta_2=\alpha\Delta_1$ for a nonzero $\alpha\in K$.*

Proof. Changing the generators of $K\langle X,Y\rangle$, by Proposition 3.2 we may assume that $\Delta_1(X)$=0, $\Delta_1(Y)=f(X)=F$ for some nonzero $F=f(X)\in K\langle X,Y\rangle$. Since $K\langle X,Y\rangle^{\Delta_1}=K\langle X,Y\rangle^{\Delta_2}$ we have that $\Delta_2(X)=0$. By Lemma 3.1

$$\Delta_2(T_1)=[\Delta_2(Y),X]+[Y,\Delta_2(X)]=[\Delta_2(Y),X]=0.$$

Therefore $\Delta_2(Y)=g(X)=G$. A direct computation gives that

$$T_2=YT_1F-FT_1Y\in K\langle X,Y\rangle^{\Delta_1}.$$

Hence $\Delta_2(T_2)=GT_1F-FT_1G=g(X)T_1f(X)-f(X)T_1g(X)=0$ which implies that $g(x)=\alpha f(x)$ for some $\alpha\in K$. Therefore $\Delta_2=\alpha\Delta_1$. Since $\Delta_1,\Delta_2\neq 0$, we obtain that $\alpha\neq 0$. ■

4 Algebras of constants of derivations of $K\langle X,Y\rangle$

By Proposition 3.2, up to a change of the free generators of $K\langle X,Y\rangle$ every nontrivial locally nilpotent derivation Δ of $K\langle X,Y\rangle$ is of the form

$$\Delta(X)=0,\qquad \Delta(Y)=f(X),$$

where $0\neq f(x)\in K[x]$. In the sequel we shall fix $\deg(f)=m\geq 0$ and Δ as defined above.

Proposition 4.1. $AK(K\langle X,Y\rangle)=K[T_1]$.

Proof. Let us consider derivations

$$\delta_m\colon\ \delta_m(X)=0,\qquad \delta_m(Y)=X^m.$$

Suppose $\delta_m(P)=0$ for all m. We may assume that P is homogeneous relative to X and Y. Write $P=XP_0+YP_1$, then

$$0=\delta_m(P)=X\delta_m(P_0)+X^mP_1+Y\delta_m(P_1).$$

Hence $\delta_m(P_1)=0$ and we can assume by induction on $\deg_Y$ that P_1 belongs to the subalgebra $K\langle X,T_1\rangle$ of $K\langle X,Y\rangle$ generated by X and T_1 and write $P_1=XP_{10}+T_1P_{11}$. If $P_{11}\neq 0$ then $\overline{X^mT_1P_{11}}$ cannot be canceled by any monomial of $X\delta_m(P_0)$ if m is sufficiently large. Hence $P_{11}=0$ and $P_{10}\in K\langle X,T_1\rangle$. Therefore

$$P=XP_0+YXP_{10}=XP_0+T_1P_{10}+XYP_{10}=X(P_0+YP_{10})+T_1P_{10}.$$

Then $\delta_m(P_0+YP_{10})=0$ because $T_1P_{10}\in K\langle X,T_1\rangle$ and we can assume by induction on $\deg_X$ that $P_0+YP_{10}\in K\langle X,T_1\rangle$, i.e., $P\in K\langle X,T_1\rangle$. Of course

$$\mathrm{AK}(K\langle X,Y\rangle)\subseteq K\langle X,T_1\rangle\bigcap K\langle Y,T_1\rangle=K[T_1]$$

since we can switch X and Y. ■

Consider the operator $\boxdot$ on $K\langle X,Y\rangle$ defined in Section 2. We shall prove in this section that the algebra of constants of Δ is the minimal algebra R_F which contains $K\langle X,T_1\rangle$ and is closed under this operator. Since $\boxdot\Delta=\Delta\boxdot$ it is clear that $R_F\subseteq K\langle X,Y\rangle^{\Delta}$. It is worth observing that the kernel of $\boxdot$ is $K[Y]$ if $\deg_X(F)=0$ and 0 if $\deg_X(F)>0$ and that $\deg(\boxdot(A))=\deg(A)$ (where deg is the degree function induced by Δ) if $\deg_X(F)>0$. We shall also denote $\boxdot(A)$ by $\{A\}$. This bracketing is a bit unusual since $\boxdot^n(A)$ will be recorded as $\{\{\ldots\{A\}\ldots\}\}$ with the same number n of the left and right brackets and there can be more than two terms inside of a pair of brackets, but as in the ordinary bracketing in a configuration of three brackets like this $\{A_1\{A_2\}$ the first bracket cannot match the third bracket, it should be matched by a bracket $\}$ to the right of the third bracket and second and third brackets are matched.

Theorem 4.2. *Let $L\in K\langle X,Y\rangle$. If $\Delta^n(L)=0$ then L belongs to the linear span R_F^n of elements $A_1YA_2Y\cdots YA_k$, where $k\leq n$ and each A_i, $1\leq i\leq k$, is a monomial from R_F, endowed with an arbitrary number of matching pairs of brackets $\{\}$.*

Proof. We consider two cases separately.

(a) $m=0$ (we can assume that $\Delta(Y)=1$). Consider the sequence of elements $T_1,\ldots,T_i,\ldots$ defined in Section 2 by $T_1=YX-XY$, $T_{i+1}=\boxdot^i(T_1)$. In this case $\overline{T_i}=Y^iX$ and any element $S\in K\langle X,Y\rangle$ can be written as $S=\sum_{j=0}^{k}S_jY^j$ where $S_j\in K\langle X,T_1,\ldots,T_i,\ldots\rangle$. Since $\Delta(S)=\sum_{j=0}^{k}jS_jY^{j-1}$, $\Delta^n(S)=0$, and $\Delta^k(S)\neq 0$ if $S_k\neq 0$ it is clear that $k<n$.

(b) $m>0$. Let us introduce a weight degree function on $K\langle X,Y\rangle$ by $w(X)=1$, $w(Y)=m$. Then the space V_N spanned by monomials of the weight not exceeding N is mapped by the derivation into itself. We proceed by induction on $w(S)$. If $w(S)$ is sufficiently small, say does not exceed m, the claim is obvious. Assume that for the weight less than N the claim is true.

Take an L for which $w(L)=N$ and $L^{(k)}=0$ (here and further on $L^{(k)}$ denotes $\Delta^k(L)$). We can assume that $L(X,0)=0$ and write

$$L=L_mF+\sum_{i=0}^{m-1}L_iYX^i.$$

Then

$$L_m^{(k)}F+k\sum_{i=0}^{m-1}L_i^{(k-1)}X^iF+\sum_{i=0}^{m-1}L_i^{(k)}YX^i=0.$$

Hence $L_i^{(k)} = 0$ for $i < m$ and

$$\left(L'_m + k\sum_{i=0}^{m-1} L_i X^i\right)^{(k-1)} = 0.$$

Therefore $\widehat{L}^{(k)} = 0$ for $\widehat{L} = L_m F + \sum_{i=0}^{m-1} L_i X^i Y$.

It is sufficient to check the claim for $\widehat{L}$ since $L - \widehat{L} = \sum_{i=0}^{m-1} L_i[Y, X^i]$ satisfies the claim by induction ($w(L_i) < N$ and $[Y, X^i] \in R_F$).

Write $\widehat{L} = L_m F + H_0 Y$. Then $H_0^{(k)} = 0$ and $(L'_m + kH_0)^{(k-1)} = 0$. Hence $L_m^{(k+1)} = 0$ and $\widetilde{L}^{(k)} = 0$ for $\widetilde{L} = kL_m F - L'_m Y$. It is sufficient to check the claim for $\widetilde{L}$ since $k\widehat{L} - \widetilde{L} = (kH_0 + L'_m)Y$ and $kH_0 + L'_m$ satisfy the claim by induction.

Since $L_m^{(k+1)} = 0$ and $w(L_m) < N$ we can write

$$L_m = \sum_{\mathbf{j}} \alpha_{j_0} Y \alpha_{j_1} Y \cdots Y \alpha_{j_k} + S,$$

where $\alpha_{j_i} \in R_F$, the summands are endowed with brackets $\{\}$, and S is the sum of terms in which Y appears less than k times. We can omit S since $kSF - S'Y \in R_F^k$.

Take one of the summands $\mu_{\mathbf{j}}$ and consider $\nu_{\mathbf{j}} = k\mu_{\mathbf{j}} F - \mu'_{\mathbf{j}} Y$. Since Δ and $\boxdot$ commute

$$\nu_{\mathbf{j}} = k\mu_{\mathbf{j}} F - \sum_{i=1}^{k} \alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y,$$

where each term $\alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y$ has the same bracketing as $\mu = \mu_{\mathbf{j}}$.

Consider $P_i = \mu F - \alpha_{j_0} Y \alpha_{j_1} Y \cdots \alpha_{j_{i-1}} F \alpha_{j_i} Y \cdots Y \alpha_{j_k} Y$. It is clear that $P_i^{(k)} = 0$ so we should check that P_i can be recorded as a sum of terms containing only $k-1$ entries of Y (we do not count Y's appearing in $\boxdot$).

Write $\mu = V_1 Y U_1$ where Y is the one which is replaced by F in P_i and introduce two operations:

$$\triangledown_{r,U}(V_1 Y U_1) = V_1 Y U_1 U F - V_1 F U_1 U Y \qquad \text{and} \qquad \triangledown_{l,U}(V_1 Y U_1) = F U V_1 Y U_1 - Y U V_1 F U_1.$$

We shall write $\triangledown_r$ and $\triangledown_l$ when $U = 1$, so $P_i = \triangledown_r(V_1 Y U_1)$.

The operator $\boxdot$ is defined on all algebra while the operations $\triangledown_{r,U}$, $\triangledown_{l,U}$ are defined only on specially recorded elements and their extension does not seem to be canonical.

Assume that $V_1 Y U_1 = \boxdot(V_2 Y U_2)$. Then we need to simplify $\triangledown_r(\boxdot(V_2 Y U_2))$. In order to do this let us compute $[\triangledown_r, \boxdot](V_2 Y U_2)$.

This is a bit tedious but not difficult:

$$\begin{aligned}
\triangledown_r(\boxdot(V_2 Y U_2)) &= [Y(V_2 Y U_2)F - F(V_2 Y U_2)Y]F - [Y(V_2 F U_2)F - F(V_2 F U_2)Y]Y,\\
\boxdot(\triangledown_r(V_2 Y U_2)) &= Y[(V_2 Y U_2)F - (V_2 F U_2)Y]F - F[(V_2 Y U_2)F - (V_2 F U_2)Y]Y.
\end{aligned}$$

Hence

$$\begin{aligned}
[\triangledown_r, \boxdot](V_2 Y U_2) &= -F(V_2 Y U_2)YF + F(V_2 Y U_2)FY - Y(V_2 F U_2)FY + Y(V_2 F U_2)YF\\
&= [Y(V_2 F U_2) - F(V_2 Y U_2)][Y, F] = -\triangledown_l(V_2 Y U_2)[Y, F].
\end{aligned}$$

Therefore

$$\bigtriangledown_r(\boxdot(V_2YU_2)) = \boxdot(\bigtriangledown_r(V_2YU_2)) - \bigtriangledown_l(V_2YU_2)[Y,F].$$

Since $w(V_2YU_2) < w(V_1YU_1)$ we can apply induction.

Assume now that either $\mu = V \boxdot (V_1YU_1)$ or $\mu = \boxdot(V_1YU_1)U$. If $\mu = V \boxdot (V_1YU_1)$ then $\bigtriangledown_r(V \boxdot (V_1YU_1)) = V \bigtriangledown_r (\boxdot(V_1YU_1))$. If $\mu = \boxdot(V_1YU_1)U$ then $\bigtriangledown_r(\mu) = \bigtriangledown_{r,U}(\boxdot(V_1YU_1))$. Now,

$$[\bigtriangledown_{r,U}, \boxdot](V_1YU_1) = \boxdot[\bigtriangledown_r(V_1YU_1)U - \bigtriangledown_{r,U}(V_1YU_1)] - \bigtriangledown_l(V_1YU_1) \boxdot (U)$$

and induction can be applied in these cases as well.

The last case is when Y does not belong to a bracketed subword. Then $\mu = V_1YU_1$ and $\bigtriangledown_r(\mu) = V_1 \boxdot (U_1)$.

The proof is completed. ■

Corollary 4.3. *The algebra of constants $K\langle X,Y\rangle^\Delta$ coincides with the algebra R_F.*

Proof. As we already mentioned $R_F \subseteq K\langle X,Y\rangle^\Delta$ and it is sufficient to show that if $\Delta(L) = 0$ for $L \in K\langle X,Y\rangle$, then L belongs to R_F. But this is a direct consequence of the case $n = 1$ in Theorem 4.2. ■

Now we are able to establish one of the main properties of the algebra of constants $K\langle X,Y\rangle^\Delta$.

Theorem 4.4. *The algebra of constants $K\langle X,Y\rangle^\Delta$ is a free algebra.*

Proof. By Corollary 4.3 we may work with the algebra R_F instead with $K\langle X,Y\rangle^\Delta$. When $m = 0$ we have seen (in the proof of Theorem 4.2) that R_1 is generated by $X, T_1, T_2, \ldots$. Since $\overline{T_i} = Y^iX$ these elements freely generate R_1. For $m > 0$ producing a generating set is more involved but the freeness can be deduced from a theorem of de W. Jooste [5]. It follows from his theorem that the kernel of the derivation $\overline{\Delta}(X) = 0$, $\overline{\Delta}(Y) = X^m$ is a free algebra. For this derivation any w-homogeneous component (recall that $w(X) = 1$, $w(Y) = m$) of a constant is also a constant, hence there is a homogeneous free generating set $F_1, F_2, \ldots$ of R_{X^m}. There is a bijection π between the elements of R_{X^m} and R_F obtained by replacing X^m in each bracket of an element of R_{X^m} by $F = f(X)$. Therefore $\pi(F_1), \pi(F_2), \ldots$ is a generating set of R_F which is free since $w(\pi(F_i) - F_i) < w(F_i)$. ■

It remains to produce a homogeneous set freely generating R_{X^m}.

Lemma 4.5. *The algebra R_{X^m} is generated by X and bracketed words*

$$T_1^{i_1}X^{j_1}\cdots X^{j_{k-1}}T_1^{i_k},$$

where $i_1, i_2, \ldots, i_k > 0$, $j_1, j_2, \ldots, j_{k-1} < m$, and where the right brackets $\}$ are preceded by T_1 (i.e., there are no configurations $X\}$).

Proof. Denote by D the subalgebra of R_{X^m} which is generated by words described in the lemma. Any element of R_{X^m} can be written as a linear combination of bracketed words $\mu = X^{j_0}T_1^{i_1}X^{j_1}\cdots T_1^{i_k}X^{j_k}$. We shall find an element $B \in D$ with the same leading monomial $\overline{B}$ as the leading monomial $\overline{\mu}$ of μ in the lexicographic order defined by $Y \gg X > 1$. Clearly this is sufficient for the proof of the lemma.

To find the leading monomial $\overline{\mu}$ of a bracketed word μ we should replace all left brackets $\{$ by Y and all right brackets $\}$ by X^m.

If $\overline{\mu}$ starts with X then $\mu = X\mu_1$ (as an element of $K\langle X,Y\rangle$) where $\mu_1 \in R_{X^m}$ and we can use induction on weight to claim that there is an element $B_1 \in D$ such that $\overline{\mu_1} = \overline{B_1}$ (or even that $\mu_1 \in B$).

If μ cannot be written as $\boxdot(\nu)$ then $\mu = (\mu_1)(\mu_2)$ where brackets () separate elements of R_{X^m} and $w(\mu_i) < w(\mu)$. Hence we can use induction to claim that $\overline{\mu_1} = \overline{B_1}$, $\overline{\mu_2} = \overline{B_2}$ where $B_i \in D$.

If $\mu = \boxdot(\nu)$ then $w(\mu) = w(\nu) + 2m$ and we may assume that $\overline{\nu} = \overline{B}$ where $B \in D$. Since $B \in D$ we can write $B = \big(X^{j_0}\big)(V_1)\big(X^{j_1}\big)\cdots(V_k)\big(X^{j_k}\big)$ where $V_i \in D$ and $(X^j) = X^j$ and $\overline{\mu} = YX^{j_0}\overline{(V_1)\big(X^{j_1}\big)\cdots(V_k)}X^{j_k+m}$. Inasmuch as $V_i \in D$ we may assume that the first and the last letters in all V_i (as bracketed words) are T_1.

If $j_0 > 0$ then $\overline{T_1}\big(X^{j_0-1}\big)\overline{(V_1)\big(X^{j_1}\big)\cdots(V_k)}\big(X^{j_k+m}\big) = \overline{\mu}$.

If $j_0 = 0$, $j_s \geq m$ where s is the smallest possible then

$$\overline{\big\{(V_1)\big(X^{j_1}\big)\cdots(V_s)\big\}\big(X^{j_s-m}\big)\cdots(V_k)}\big(X^{j_k+m}\big) = \overline{\mu}.$$

If all $j_s < m$ then $\mu \in D$. ■

Theorem 4.6. *The algebra $D = R_{X^m}$, $m > 0$, is freely generated by X, T_1 and words $\boxdot\big(T_1^{i_1}X^{j_1}\cdots X^{j_{k-1}}T_1^{i_k}\big)$, where $i_1, i_2, \dots, i_k > 0$, $j_1, j_2, \dots, j_{k-1} < m$, and $T_1^{i_1}X^{j_1}\cdots X^{j_{k-1}}T_1^{i_k}$ are bracketed words described in Lemma 4.5 (we shall refer to these words as permissible and to $T_1^{i_1}X^{j_1}\cdots X^{j_{k-1}}T_1^{i_k}$ without brackets as the root of the corresponding word).*

Proof. It is sufficient to check that the leading monomial $\overline{\mu}$ of a permissible word cannot be presented as a product of the leading monomials of permissible words of a smaller weight.

To check this consider the leading monomial $\overline{\mu} = Y^{b_1}\cdots X^{a_{s-1}}Y^{b_s}X^{a_s}$ of a permissible μ. (Observe that $b_1 > 0$, $a_s = m+1$ since $\overline{\boxdot(V)} = Y\overline{V}X^m$.)

The number of T_1 in the bracketed representation of $\mu \in D$ must be equal to s since in the leading monomial of any word from D a subword YX can appear only as $\overline{T_1}$. So the number of brackets { in μ is $\deg_Y(\overline{\mu}) - s$. Of course the number of brackets } is the same.

A subword $Y^{b_i}X^{a_i}$ can appear in $\overline{\mu}$ only as $\{\dots\{T_1\}\dots\}X^{d_i}$ where the number of left brackets is $b_i - 1$, the number of right brackets is the integral part of $\frac{a_i-1}{m}$ and $0 \leq d_i < m$ is the remainder of the division of $a_i - 1$ by m. Therefore the root and the bracketing of μ are uniquely determined by $\overline{\mu}$. But we would have two different bracketings if $\overline{\mu} = (\overline{\nu_1})(\overline{\nu_2})$. This finishes a proof of the theorem. ■

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