

Double Lowering Operators on Polynomials

Paul TERWILLIGER

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA

E-mail: terwilli@math.wisc.edu

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Abstract. Recently Sarah Bockting-Conrad introduced the double lowering operator ψ for a tridiagonal pair. Motivated by ψ we consider the following problem about polynomials. Let \mathbb{F} denote an algebraically closed field. Let x denote an indeterminate, and let $\mathbb{F}[x]$ denote the algebra consisting of the polynomials in x that have all coefficients in \mathbb{F} . Let N denote a positive integer or ∞ . Let $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ denote scalars in \mathbb{F} such that $\sum_{h=0}^{i-1} a_h \neq \sum_{h=0}^{i-1} b_h$ for $1 \leq i \leq N$. For $0 \leq i \leq N$ define polynomials $\tau_i, \eta_i \in \mathbb{F}[x]$ by $\tau_i = \prod_{h=0}^{i-1} (x - a_h)$ and $\eta_i = \prod_{h=0}^{i-1} (x - b_h)$. Let V denote the subspace of $\mathbb{F}[x]$ spanned by $\{x^i\}_{i=0}^N$. An element $\psi \in \text{End}(V)$ is called double lowering whenever $\psi\tau_i \in \mathbb{F}\tau_{i-1}$ and $\psi\eta_i \in \mathbb{F}\eta_{i-1}$ for $0 \leq i \leq N$, where $\tau_{-1} = 0$ and $\eta_{-1} = 0$. We give necessary and sufficient conditions on $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ for there to exist a nonzero double lowering map. There are four families of solutions, which we describe in detail.

Key words: tridiagonal pair; q -exponential function; basic hypergeometric series; q -binomial theorem

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1 Introduction

This paper is mainly about polynomials and special functions, but in order to motivate things we first discuss a topic in linear algebra. The topic has to do with tridiagonal pairs [17] and their associated double lowering operator [7, 8, 9, 10]. A reader unfamiliar with tridiagonal pairs can safely skip to Section 2. Let V denote a vector space with finite positive dimension. A tridiagonal pair on V is an ordered pair of linear maps $A: V \rightarrow V$ and $A^*: V \rightarrow V$ that satisfy the following conditions:

- (i) each of A, A^* is diagonalizable;
- (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d,$$

where $V_{-1} = 0$ and $V_{d+1} = 0$;

- (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq \delta,$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$;

- (iv) there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$.

The tridiagonal pair concept originated in the theory of Q -polynomial distance-regular graphs [4, 11] where it is used to describe how the adjacency matrix is related to each dual adjacency matrix [17, Example 1.4], [29, Section 3]. Since that origin, the tridiagonal pair concept has

found applications to all sorts of topics in special functions (orthogonal polynomials of the Askey-scheme [3, 25, 27, 32, 33], the Askey–Wilson algebra [14, 35, 40, 42]), Lie theory (the \mathfrak{sl}_2 loop algebra [19], the tetrahedron algebra [15, 21]), statistical mechanics (the Onsager algebra [12, 16] and q -Onsager algebra [5, 6, 23, 31, 37, 38]), and quantum groups (the equitable presentation [1, 24, 36], the quantum affine \mathfrak{sl}_2 algebra [2, 18, 20, 22], L -operators [26, 39]). For more information about the above topics, see [28, 34] and the references therein.

Let A, A^* denote a tridiagonal pair on V , as in the above definition. By [17, Lemma 4.5] the integers d and δ from (ii), (iii) are equal; this common value is called the diameter of the pair. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) for the eigenspace V_i (resp. V_i^*). By [17, Theorem 11.1] the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. For this constraint the solutions can be given in closed form [17, Theorem 11.2]. The “most general” solution is called q -Racah, and will be described shortly.

By construction the vector space V has a direct sum decomposition into the eigenspaces $\{V_i\}_{i=0}^d$ of A and the eigenspaces $\{V_i^*\}_{i=0}^d$ of A^* . The vector space V has two other direct sum decompositions of interest, called the first split decomposition $\{U_i\}_{i=0}^d$ and second split decomposition $\{U_i^\Downarrow\}_{i=0}^d$. By [17, Theorem 4.6] the first split decomposition satisfies

$$\begin{aligned} U_0 + U_1 + \cdots + U_i &= V_0^* + V_1^* + \cdots + V_i^*, \\ U_i + U_{i+1} + \cdots + U_d &= V_i + V_{i+1} + \cdots + V_d \end{aligned}$$

for $0 \leq i \leq d$. By [17, Theorem 4.6] the second split decomposition satisfies

$$\begin{aligned} U_0^\Downarrow + U_1^\Downarrow + \cdots + U_i^\Downarrow &= V_0^* + V_1^* + \cdots + V_i^*, \\ U_i^\Downarrow + U_{i+1}^\Downarrow + \cdots + U_d^\Downarrow &= V_0 + V_1 + \cdots + V_{d-i} \end{aligned}$$

for $0 \leq i \leq d$. By [17, Theorem 4.6],

$$\begin{aligned} (A - \theta_i I)U_i &\subseteq U_{i+1}, & (A^* - \theta_i^* I)U_i &\subseteq U_{i-1}, \\ (A - \theta_{d-i} I)U_i^\Downarrow &\subseteq U_{i+1}^\Downarrow, & (A^* - \theta_i^* I)U_i^\Downarrow &\subseteq U_{i-1}^\Downarrow \end{aligned}$$

for $0 \leq i \leq d$, where $U_{-1} = 0$, $U_{d+1} = 0$ and $U_{-1}^\Downarrow = 0$, $U_{d+1}^\Downarrow = 0$.

In [7, Sections 11 and 15] Sarah Bockting-Conrad introduces a linear map $\Psi: V \rightarrow V$ such that

$$\Psi U_i \subseteq U_{i-1}, \quad \Psi U_i^\Downarrow \subseteq U_{i-1}^\Downarrow, \quad 0 \leq i \leq d.$$

This map is called the double lowering operator or Bockting operator. In [7, Sections 9 and 15] Bockting-Conrad introduces an invertible linear map $\Delta: V \rightarrow V$ that commutes with Ψ and sends U_i onto U_i^\Downarrow for $0 \leq i \leq d$. The maps Ψ and Δ are related in the following way. For $0 \leq i \leq d$ define two polynomials

$$\tau_i = (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1}), \quad (1.1)$$

$$\eta_i = (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-i+1}) \quad (1.2)$$

in a variable x . Define the scalars

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}, \quad 1 \leq i \leq d.$$

By [7, Theorem 17.1],

$$\Delta = \sum_{i=0}^d \frac{\eta_i(\theta_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_i} \Psi^i, \quad \Delta^{-1} = \sum_{i=0}^d \frac{\tau_i(\theta_d)}{\vartheta_1 \vartheta_2 \cdots \vartheta_i} \Psi^i \quad (1.3)$$

provided that each of $\vartheta_1, \vartheta_2, \dots, \vartheta_d$ is nonzero.

Shortly we will describe Ψ and Δ in more detail, but first we restrict to the q -Racah case. In this case there exist nonzero scalars a, b, q such that $q^4 \neq 1$ and

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \quad \theta_i^* = bq^{d-2i} + b^{-1}q^{2i-d}$$

for $0 \leq i \leq d$. Define a linear map $K: V \rightarrow V$ such that for $0 \leq i \leq d$, U_i is an eigenspace of K with eigenvalue q^{d-2i} . Define a linear map $B: V \rightarrow V$ such that for $0 \leq i \leq d$, U_i^\downarrow is an eigenspace of B with eigenvalue q^{d-2i} . For notational convenience define $\psi = (q - q^{-1})(q^d - q^{-d})\Psi$. By [9, Lemma 5.3] and [8, Lemma 5.4],

$$B\Delta = \Delta K, \quad K\psi = q^2\psi K, \quad B\psi = q^2\psi B.$$

By [8, Theorem 9.8] the map ψ is equal to each of the following:

$$\begin{aligned} \frac{I - BK^{-1}}{q(aI - a^{-1}BK^{-1})}, & \quad \frac{I - KB^{-1}}{q(a^{-1}I - aKB^{-1})}, \\ \frac{q(I - K^{-1}B)}{aI - a^{-1}K^{-1}B}, & \quad \frac{q(I - B^{-1}K)}{a^{-1}I - aB^{-1}K}. \end{aligned}$$

This result is used in [8, Theorem 9.9] to obtain

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}BK + a^{-1}B^2 = 0.$$

By (1.3) and [9, Theorem 7.2],

$$\Delta = \exp_q \left(\frac{a}{q - q^{-1}} \psi \right) \exp_{q^{-1}} \left(-\frac{a^{-1}}{q - q^{-1}} \psi \right).$$

Motivated by this factorization, in [9, Sections 6,7] Bockting-Conrad introduces an invertible linear map $M: V \rightarrow V$ such that

$$\begin{aligned} K \exp_q \left(\frac{a^{-1}}{q - q^{-1}} \psi \right) &= \exp_q \left(\frac{a^{-1}}{q - q^{-1}} \psi \right) M, \\ B \exp_q \left(\frac{a}{q - q^{-1}} \psi \right) &= \exp_q \left(\frac{a}{q - q^{-1}} \psi \right) M. \end{aligned}$$

By [9, Section 6],

$$M = \frac{aK - a^{-1}B}{a - a^{-1}}, \quad M\psi = q^2\psi M.$$

By [9, Lemma 6.2], M is equal to each of

$$\begin{aligned} (I - a^{-1}q\psi)^{-1}K, & \quad K(I - a^{-1}q^{-1}\psi)^{-1}, \\ (I - aq\psi)^{-1}B, & \quad B(I - aq^{-1}\psi)^{-1}. \end{aligned}$$

By [9, Lemma 6.7],

$$\frac{qM^{-1}K - q^{-1}KM^{-1}}{q - q^{-1}} = I, \quad \frac{qM^{-1}B - q^{-1}BM^{-1}}{q - q^{-1}} = I.$$

We just listed many results about ψ , Δ , K , B , M . In the present paper, we interpret these results in terms of polynomials. The polynomials in question are essentially (1.1), (1.2) although we adopt a more general point of view. In the next section we will describe a problem about polynomials, and for the rest of the paper we will describe the solution. In this description we will encounter analogs of the above results. We hope that the above results are illuminated by our description.

2 Definitions and first steps

We now begin our formal argument. The following assumptions and notational conventions apply throughout the paper. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. Let \mathbb{F} denote an algebraically closed field. Every vector space discussed in this paper is over \mathbb{F} . Every algebra discussed in this paper is associative, over \mathbb{F} , and has a multiplicative identity. Let x denote an indeterminate. Let $\mathbb{F}[x]$ denote the algebra consisting of the polynomials in x that have all coefficients in \mathbb{F} . Throughout the paper we use the following convention: *the symbols N , n refer to an integer or ∞ ; the symbols i , j , k refer to an integer.*

We now describe a problem about polynomials. Let N denote a positive integer or ∞ . We consider an ordered pair of sequences

$$\{a_i\}_{i=0}^{N-1}, \quad \{b_i\}_{i=0}^{N-1} \tag{2.1}$$

such that $a_i, b_i \in \mathbb{F}$ for $0 \leq i \leq N-1$. To avoid degenerate situations, we assume that

$$a_0 + a_1 + \dots + a_{i-1} \neq b_0 + b_1 + \dots + b_{i-1}, \quad 1 \leq i \leq N. \tag{2.2}$$

We call the ordered pair (2.1) the *data*. For $0 \leq i \leq N$ define polynomials $\tau_i, \eta_i \in \mathbb{F}[x]$ by

$$\tau_i = (x - a_0)(x - a_1) \cdots (x - a_{i-1}), \tag{2.3}$$

$$\eta_i = (x - b_0)(x - b_1) \cdots (x - b_{i-1}). \tag{2.4}$$

The polynomials τ_i, η_i are monic of degree i . For notational convenience define $\tau_{-1} = 0$ and $\eta_{-1} = 0$. Let V denote the subspace of $\mathbb{F}[x]$ spanned by $\{x^i\}_{i=0}^N$. Each of $\{\tau_i\}_{i=0}^N, \{\eta_i\}_{i=0}^N$ is a basis for V . Let $\text{End}(V)$ denote the algebra consisting of the \mathbb{F} -linear maps from V to V . Define $\Delta \in \text{End}(V)$ such that

$$\Delta\tau_i = \eta_i, \quad 0 \leq i \leq N. \tag{2.5}$$

Note that Δ is invertible.

Definition 2.1. An element $\psi \in \text{End}(V)$ is called *double lowering* (with respect to the given data) whenever both

$$\psi\tau_i \in \mathbb{F}\tau_{i-1}, \quad \psi\eta_i \in \mathbb{F}\eta_{i-1}$$

for $0 \leq i \leq N$.

Definition 2.2. Define

$$\mathcal{L} = \{\psi \in \text{End}(V) \mid \psi \text{ is double lowering}\}.$$

Note that \mathcal{L} is a subspace of the vector space $\text{End}(V)$. We call \mathcal{L} the *double lowering space* for the given data.

Definition 2.3. The data (2.1) is called *double lowering* whenever the double lowering space $\mathcal{L} \neq 0$.

Problem 2.4. Find necessary and sufficient conditions for the data (2.1) to be double lowering. In this case describe \mathcal{L} and Δ .

The above problem is solved in the present paper. The necessary and sufficient conditions are given in Theorem 12.1. By that theorem, there are four cases. For the first three cases, \mathcal{L} and Δ are described in Section 6. For the fourth case, \mathcal{L} and Δ are described in Section 13. We would like to acknowledge that the above problem was previously solved by R. Vidunas under the assumption that $a_i = \theta_i$ and $b_i = \theta_{N-i}$ for $0 \leq i \leq N-1$, with $\{\theta_i\}_{i=0}^N$ mutually distinct [41].

We have some remarks. The polynomials (2.3), (2.4) satisfy

$$\tau_0 = 1, \quad \tau_1 = x - a_0, \quad \eta_0 = 1, \quad \eta_1 = x - b_0. \quad (2.6)$$

Moreover

$$(x - a_i)\tau_i = \tau_{i+1}, \quad (x - b_i)\eta_i = \eta_{i+1}, \quad 0 \leq i \leq N-1. \quad (2.7)$$

Lemma 2.5. Assume that $\psi \in \text{End}(V)$ is double lowering. Then $\psi 1 = 0$. Moreover

$$\psi\tau_1 = \psi x = \psi\eta_1, \quad (2.8)$$

and this common value is contained in \mathbb{F} .

Proof. Apply ψ to each side of the equations in (2.6), and use Definition 2.1. ■

For $0 \leq n \leq N$ define

$$V_n = \text{Span} \{x^i\}_{i=0}^n. \quad (2.9)$$

We have $V_N = V$. We have $\dim(V_i) = i + 1$ for $0 \leq i \leq N$, and $V_{i-1} \subseteq V_i$ for $1 \leq i \leq N$. For notational convenience define $V_{-1} = 0$.

Lemma 2.6. For $0 \leq n \leq N$, each of

$$\{\tau_i\}_{i=0}^n, \quad \{\eta_i\}_{i=0}^n$$

is a basis for V_n .

Proof. Each of τ_i, η_i has degree i for $0 \leq i \leq N$. ■

Lemma 2.7. For $\psi \in \mathcal{L}$ and $0 \leq i \leq N$,

$$\psi V_i \subseteq V_{i-1}.$$

Moreover $\psi^{i+1} V_i = 0$.

Proof. By Definition 2.1 and Lemma 2.6. ■

For $T \in \text{End}(V)$, T is called *nilpotent* whenever there exists a positive integer j such that $T^j = 0$. The map T is called *locally nilpotent* whenever for all $v \in V$ there exists a positive integer j such that $T^j v = 0$. If T is nilpotent then T is locally nilpotent. For $N \neq \infty$, if T is locally nilpotent then T is nilpotent.

Lemma 2.8. Each element of \mathcal{L} is locally nilpotent. If $N \neq \infty$ then each element of \mathcal{L} is nilpotent.

Proof. By Lemma 2.7 and the comments below it. ■

Lemma 2.9. For $1 \leq i \leq N$,

(i) in τ_i the coefficient of x^{i-1} is

$$-a_0 - a_1 - \cdots - a_{i-1};$$

(ii) in η_i the coefficient of x^{i-1} is

$$-b_0 - b_1 - \cdots - b_{i-1}.$$

Proof. Use (2.3), (2.4). ■

Lemma 2.10. For $1 \leq i \leq N$,

(i) $\eta_i - \tau_i \in V_{i-1}$;

(ii) in $\eta_i - \tau_i$ the coefficient of x^{i-1} is

$$a_0 + a_1 + \cdots + a_{i-1} - b_0 - b_1 - \cdots - b_{i-1}.$$

Proof. (i) Each of τ_i, η_i is monic with degree i . (ii) Use Lemma 2.9. ■

Definition 2.11. Define a map $A: V_{N-1} \rightarrow V_N, v \mapsto xv$. Note that A is \mathbb{F} -linear.

Lemma 2.12. We have

$$AV_{n-1} \subseteq V_n, \quad 1 \leq n \leq N.$$

Proof. By (2.9). ■

Lemma 2.13. For $0 \leq i \leq N-1$,

$$A\tau_i = a_i\tau_i + \tau_{i+1}, \quad A\eta_i = b_i\eta_i + \eta_{i+1}.$$

Proof. By (2.7) and Definition 2.11. ■

We mention an elementary result for later use.

Lemma 2.14. Assume that $T \in \text{End}(V)$ is locally nilpotent. Then $I - T$ is invertible, and $(I - T)^{-1} = \sum_{i=0}^N T^i$.

3 Adjusting the data

In this section we describe how the double lowering space is affected when we adjust the data in an affine way.

Let $\text{GL}_2(\mathbb{F})$ denote the group of invertible 2×2 matrices that have all entries in \mathbb{F} .

Definition 3.1. Let G denote the subgroup of $\text{GL}_2(\mathbb{F})$ consisting of the matrices

$$\begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix}, \quad 0 \neq s \in \mathbb{F}, \quad t \in \mathbb{F}.$$

The above matrix is denoted $g(s, t)$.

Lemma 3.2. With reference to Definition 3.1,

- (i) $g(s, t)g(S, T) = g(sS, T + tS)$;
 (ii) the inverse of $g(s, t)$ is $g(s^{-1}, -s^{-1}t)$.

Proof. Routine matrix multiplication. ■

For an algebra \mathcal{A} , an *automorphism* of \mathcal{A} is an algebra isomorphism $\mathcal{A} \rightarrow \mathcal{A}$.

Lemma 3.3. *The group G acts on the algebra $\mathbb{F}[x]$ as a group of automorphisms in the following way: each element $g(s, t) \in G$ sends $x \mapsto sx + t$.*

Proof. This is routinely checked using Lemma 3.2. ■

Recall from Definition 2.2 the double lowering space \mathcal{L} for the data (2.1). Pick $0 \neq s \in \mathbb{F}$ and $t \in \mathbb{F}$. Let \mathcal{L}' denote the double lowering space for the data

$$\{sa_i + t\}_{i=0}^{N-1}, \quad \{sb_i + t\}_{i=0}^{N-1}.$$

Proposition 3.4. *The following (i)–(iii) hold for the above scalars s, t and $g = g(s, t)$:*

- (i) there exists an \mathbb{F} -linear map $\mathcal{L} \rightarrow \mathcal{L}'$, $\psi \mapsto g^{-1}\psi g$;
 (ii) there exists an \mathbb{F} -linear map $\mathcal{L}' \rightarrow \mathcal{L}$, $\zeta \mapsto g\zeta g^{-1}$;
 (iii) the maps in (i), (ii) above are inverses, and hence bijections.

Proof. (i) For $\psi \in \mathcal{L}$ we show that $g^{-1}\psi g \in \mathcal{L}'$. For $\alpha \in \mathbb{F}$ define $\alpha' = s\alpha + t$. For $0 \leq i \leq N$ define

$$\begin{aligned} \tau'_i &= (x - a'_0)(x - a'_1) \cdots (x - a'_{i-1}), \\ \eta'_i &= (x - b'_0)(x - b'_1) \cdots (x - b'_{i-1}). \end{aligned}$$

For notational convenience define $\tau'_{-1} = 0$ and $\eta'_{-1} = 0$. Pick an integer i , $0 \leq i \leq N$. One checks that g sends

$$\tau'_i \mapsto s^i \tau_i, \quad \eta'_i \mapsto s^i \eta_i.$$

By Definition 2.1, ψ sends τ_i (resp. η_i) to a scalar multiple of τ_{i-1} (resp. η_{i-1}). By these comments $g^{-1}\psi g$ sends τ'_i (resp. η'_i) to a scalar multiple of τ'_{i-1} (resp. η'_{i-1}). Therefore $g^{-1}\psi g \in \mathcal{L}'$. We have shown that there exists a map $\mathcal{L} \rightarrow \mathcal{L}'$, $\psi \mapsto g^{-1}\psi g$. By construction this map is \mathbb{F} -linear.

(ii) Similar to the proof of (i) above. (iii) By construction. ■

Corollary 3.5. *Pick $0 \neq s \in \mathbb{F}$ and $t \in \mathbb{F}$. Then the data (2.1) is double lowering if and only if the data*

$$\{sa_i + t\}_{i=0}^{N-1}, \quad \{sb_i + t\}_{i=0}^{N-1}$$

is double lowering.

Proof. By Definition 2.3 and Proposition 3.4. ■

We have a comment.

Lemma 3.6. *Referring to the data (2.1), for distinct $a'_0, b'_0 \in \mathbb{F}$ define*

$$s = \frac{a'_0 - b'_0}{a_0 - b_0}, \quad t = \frac{a_0 b'_0 - a'_0 b_0}{a_0 - b_0}.$$

Then $s \neq 0$ and

$$sa_0 + t = a'_0, \quad sb_0 + t = b'_0.$$

Proof. Routine. ■

Corollary 3.5 and Lemma 3.6 show that for double lowering data (2.1), the scalars a_0 and b_0 are “free”, with $a_0 \neq b_0$ being the only constraint.

4 The parameters ϑ_i

We continue to discuss the double lowering space \mathcal{L} for the data (2.1). In this section we use \mathcal{L} to define some scalars $\{\vartheta_i\}_{i=0}^N$ that will play a role in our theory.

Definition 4.1. For $0 \leq i \leq N$ define

$$\vartheta_i = \frac{a_0 + a_1 + \cdots + a_{i-1} - b_0 - b_1 - \cdots - b_{i-1}}{a_0 - b_0}.$$

Referring to Definition 4.1, we have $\vartheta_0 = 0$ and $\vartheta_1 = 1$. By (2.2) we have

$$\vartheta_i \neq 0, \quad 1 \leq i \leq N.$$

Proposition 4.2. *The following (i)–(iii) hold for $\psi \in \mathcal{L}$ and $1 \leq i \leq N$:*

- (i) *for the polynomial ψx^i the coefficient of x^{i-1} is $\vartheta_i \psi x$;*
- (ii) $\psi \tau_i = (\vartheta_i \psi x) \tau_{i-1}$;
- (iii) $\psi \eta_i = (\vartheta_i \psi x) \eta_{i-1}$.

Proof. We use induction on i . First assume that $i = 1$, and recall $\vartheta_1 = 1$. Assertion (i) is vacuously true. Assertions (ii), (iii) hold by (2.8) and since $\tau_0 = 1 = \eta_0$. Next assume that $i \geq 2$. By Lemma 2.7, $\psi x^i \in V_{i-1}$. Let the scalar α_i be the coefficient of x^{i-1} in ψx^i . Since $\psi \in \mathcal{L}$, $\psi \tau_i \in \mathbb{F} \tau_{i-1}$. Since each of τ_i, τ_{i-1} is monic,

$$\psi \tau_i = \alpha_i \tau_{i-1}. \tag{4.1}$$

Similarly,

$$\psi \eta_i = \alpha_i \eta_{i-1}. \tag{4.2}$$

We show that $\alpha_i = \vartheta_i \psi x$. Using (4.1), (4.2) we see that

$$(a_0 - b_0) \vartheta_{i-1} x^{i-2} \tag{4.3}$$

times

$$\alpha_i - \vartheta_i \psi x \tag{4.4}$$

is equal to

$$\psi(\eta_i - \tau_i) - (a_0 - b_0) \vartheta_i \psi x^{i-1} \tag{4.5}$$

minus α_i times

$$\eta_{i-1} - \tau_{i-1} - (a_0 - b_0) \vartheta_{i-1} x^{i-2} \tag{4.6}$$

plus $(a_0 - b_0) \vartheta_i$ times

$$\psi x^{i-1} - (\psi x) \vartheta_{i-1} x^{i-2}. \tag{4.7}$$

By Lemma 2.10 and Definition 4.1,

$$\eta_i - \tau_i - (a_0 - b_0) \vartheta_i x^{i-1} \in V_{i-2}. \tag{4.8}$$

In this inclusion, we apply ψ to each side and use Lemma 2.7 to find that (4.5) is contained in V_{i-3} . In (4.8) we replace i by $i-1$, to find that (4.6) is contained in V_{i-3} . By induction (4.7) is contained in V_{i-3} . By these comments the polynomial (4.3) times the scalar (4.4) is contained in V_{i-3} . Consider the factors in the polynomial (4.3). We have $a_0 - b_0 \neq 0$ and $\vartheta_{i-1} \neq 0$ and $x^{i-2} \notin V_{i-3}$. So the polynomial (4.3) is not contained in V_{i-3} . Consequently the scalar (4.4) is zero, so $\alpha_i = \vartheta_i \psi x$. The result follows from this and (4.1), (4.2). \blacksquare

Corollary 4.3. *The map $\mathcal{L} \rightarrow \mathbb{F}$, $\psi \mapsto \psi x$ is injective.*

Proof. For $\psi \in \mathcal{L}$ such that $\psi x = 0$, we show that $\psi = 0$. The vector space V has a basis $\{\tau_i\}_{i=0}^N$. By Definition 2.1, $\psi\tau_0 = 0$. By Proposition 4.2(ii), $\psi\tau_i = 0$ for $1 \leq i \leq N$. By these comments $\psi = 0$. ■

Corollary 4.4. *Assume that $\mathcal{L} \neq 0$. Then the map in Corollary 4.3 is a bijection. Moreover \mathcal{L} has dimension 1.*

Proof. By Corollary 4.3. ■

Definition 4.5. An element $\psi \in \mathcal{L}$ is called *normalized* whenever $\psi x = 1$.

Lemma 4.6. *The double lowering space \mathcal{L} contains a normalized element if and only if $\mathcal{L} \neq 0$; in this case the normalized element is unique.*

Proof. By Corollary 4.4 and Definition 4.5. ■

Lemma 4.7. *Assume that $\mathcal{L} \neq 0$. Then the vector space \mathcal{L} has a basis consisting of its normalized element.*

Proof. The vector space \mathcal{L} has dimension 1, and its normalized element is nonzero. ■

Lemma 4.8. *For $\psi \in \text{End}(V)$ the following are equivalent:*

- (i) $\psi \in \mathcal{L}$ and ψ is normalized;
- (ii) for $0 \leq i \leq N$ both

$$\psi\tau_i = \vartheta_i\tau_{i-1}, \quad \psi\eta_i = \vartheta_i\eta_{i-1}. \quad (4.9)$$

Proof. (i) \Rightarrow (ii) Set $\psi x = 1$ in Proposition 4.2(ii), (iii). (ii) \Rightarrow (i) We have $\psi \in \mathcal{L}$ by Definition 2.2 and (4.9). To see that ψ is normalized, set $i = 1$ in (4.9) and use Lemma 2.5 to obtain $\psi x = 1$. ■

5 Describing \mathcal{L} using Δ

We continue to discuss the double lowering space \mathcal{L} for the data (2.1). In this section we describe \mathcal{L} using the map Δ from (2.5).

Proposition 5.1. *For $\psi \in \text{End}(V)$ the following (i)–(iii) are equivalent:*

- (i) $\Delta\psi = \psi\Delta$ and

$$\psi\tau_i \in \mathbb{F}\tau_{i-1}, \quad 0 \leq i \leq N; \quad (5.1)$$

- (ii) $\Delta\psi = \psi\Delta$ and

$$\psi\eta_i \in \mathbb{F}\eta_{i-1}, \quad 0 \leq i \leq N; \quad (5.2)$$

- (iii) $\psi \in \mathcal{L}$.

Proof. (i) \Rightarrow (ii) We show (5.2). Using (2.5),

$$\psi\eta_i = \psi\Delta\tau_i = \Delta\psi\tau_i \in \mathbb{F}\Delta\tau_{i-1} = \mathbb{F}\eta_{i-1}.$$

(ii) \Rightarrow (i) Similar to the proof of (i) \Rightarrow (ii). (i), (ii) \Rightarrow (iii) By Definitions 2.1, 2.2. (iii) \Rightarrow (i) We have (5.1) by Definition 2.2. We show $\Delta\psi = \psi\Delta$. By Lemma 4.7 we may assume that ψ is normalized. The vector space V has a basis $\{\tau_i\}_{i=0}^N$. By Lemma 4.8 we have

$$\psi\tau_i = \vartheta_i\tau_{i-1}, \quad \psi\eta_i = \vartheta_i\eta_{i-1}, \quad 0 \leq i \leq N.$$

So for $0 \leq i \leq N$,

$$\Delta\psi\tau_i = \vartheta_i\Delta\tau_{i-1} = \vartheta_i\eta_{i-1} = \psi\eta_i = \psi\Delta\tau_i.$$

By these comments $\Delta\psi = \psi\Delta$. ■

We introduce some notation. For $0 \leq i \leq j \leq N$ define

$$\begin{bmatrix} j \\ i \end{bmatrix}_\vartheta = \frac{\vartheta_j\vartheta_{j-1}\cdots\vartheta_{j-i+1}}{\vartheta_1\vartheta_2\cdots\vartheta_i}. \quad (5.3)$$

Note that

$$\begin{bmatrix} j \\ i \end{bmatrix}_\vartheta = \begin{bmatrix} j \\ j-i \end{bmatrix}_\vartheta, \quad 0 \leq i \leq j \leq N.$$

Proposition 5.2. *The following (i)–(iii) are equivalent:*

(i) $\mathcal{L} \neq 0$;

(ii) for $0 \leq j \leq N$,

$$\eta_j = \sum_{i=0}^j \eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_\vartheta \tau_i; \quad (5.4)$$

(iii) for $0 \leq j \leq N$,

$$\tau_j = \sum_{i=0}^j \tau_{j-i}(b_0) \begin{bmatrix} j \\ i \end{bmatrix}_\vartheta \eta_i.$$

Assume that (i)–(iii) hold, and let $\psi \in \mathcal{L}$ be normalized. Then

$$\Delta = \sum_{i=0}^N \frac{\eta_i(a_0)}{\vartheta_1\vartheta_2\cdots\vartheta_i} \psi^i, \quad (5.5)$$

$$\Delta^{-1} = \sum_{i=0}^N \frac{\tau_i(b_0)}{\vartheta_1\vartheta_2\cdots\vartheta_i} \psi^i. \quad (5.6)$$

Proof. (i) \Rightarrow (ii) Let j be given. There exist scalars $\{\alpha_i\}_{i=0}^j$ in \mathbb{F} such that

$$\eta_j = \sum_{i=0}^j \alpha_i \tau_i. \quad (5.7)$$

For $0 \leq i \leq j$ we show

$$\alpha_i = \eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_{\vartheta}. \quad (5.8)$$

Let $\psi \in \mathcal{L}$ be normalized. In (5.7) we apply ψ^i to each side, and evaluate the result using Lemma 4.8. We then set $x = a_0$, and use the fact that $\tau_0 = 1$ and $\tau_k(a_0) = 0$ for $1 \leq k \leq N$. By these comments

$$\eta_{j-i}(a_0)\vartheta_j\vartheta_{j-1}\cdots\vartheta_{j-i+1} = \alpha_i\vartheta_1\vartheta_2\cdots\vartheta_i.$$

From this equation we obtain (5.8). By (5.7), (5.8) we obtain (5.4), so (ii) holds.

(ii) \Rightarrow (i) Define $\psi \in \text{End}(V)$ such that $\psi\tau_i = \vartheta_i\tau_{i-1}$ for $0 \leq i \leq N$. We have $\psi \neq 0$ since $N \geq 1$ and $\vartheta_1 = 1$, $\tau_0 = 1$. We show $\psi \in \mathcal{L}$. To do this, it is convenient to first show that ψ satisfies (5.5). To obtain (5.5), for $0 \leq j \leq N$ we apply each side of (5.5) to τ_j . Concerning the left-hand side of (5.5), we have $\Delta\tau_j = \eta_j$ by (2.5). Concerning the right-hand side of (5.5),

$$\begin{aligned} \sum_{i=0}^j \frac{\eta_i(a_0)}{\vartheta_1\vartheta_2\cdots\vartheta_i} \psi^i \tau_j &= \sum_{i=0}^j \frac{\eta_i(a_0)\vartheta_j\vartheta_{j-1}\cdots\vartheta_{j-i+1}}{\vartheta_1\vartheta_2\cdots\vartheta_i} \tau_{j-i} = \sum_{i=0}^j \eta_i(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_{\vartheta} \tau_{j-i} \\ &= \sum_{i=0}^j \eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_{\vartheta} \tau_i = \eta_j. \end{aligned}$$

We have shown that each side of (5.5) sends $\tau_j \mapsto \eta_j$ for $0 \leq j \leq N$. Therefore (5.5) holds. By (5.5), Δ is a polynomial in ψ . Consequently $\psi\Delta = \Delta\psi$. The map ψ satisfies Proposition 5.1(i), so $\psi \in \mathcal{L}$ by Proposition 5.1(i), (iii). Therefore $\mathcal{L} \neq 0$.

(i) \Leftrightarrow (iii) Interchange the roles of $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ in the proof of (i) \Leftrightarrow (ii).

Now assume that (i)–(iii) hold. We saw in the proof of (ii) \Rightarrow (i) that (5.5) holds. Interchanging the roles of $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ in that proof, we see that (5.6) holds. \blacksquare

Later in the paper, we will obtain necessary and sufficient conditions for the data (2.1) to satisfy conditions (i)–(iii) in Proposition 5.2; our result is Theorem 12.1. In order to motivate this result, we look at some examples of double lowering data. This will be done in the next section.

6 First examples of double lowering data

We continue to discuss the double lowering space \mathcal{L} for the data (2.1). In this section we give three assumptions under which this data is double lowering. Under each assumption we describe the polynomials $\{\tau_i\}_{i=0}^{N-1}$, $\{\eta_i\}_{i=0}^{N-1}$ from (2.3), (2.4), the parameters $\{\vartheta_i\}_{i=0}^N$ from Definition 4.1, and the map Δ from (2.5).

As a warmup, we examine the condition (5.4) for some small values of j .

Lemma 6.1. *The following (i)–(iv) hold.*

- (i) $\eta_0 = \tau_0$.
- (ii) $\eta_1 = \eta_1(a_0)\tau_0 + \tau_1$.
- (iii) For $N \geq 2$,

$$\eta_2 = \eta_2(a_0)\tau_0 + \vartheta_2\eta_1(a_0)\tau_1 + \tau_2.$$

(iv) For $N \geq 3$,

$$\eta_3 = \eta_3(a_0)\tau_0 + \vartheta_3\eta_2(a_0)\tau_1 + \vartheta_3\eta_1(a_0)\tau_2 + \tau_3 + (x - a_0)\varepsilon,$$

where

$$\varepsilon = (b_0 - a_1)(b_2 - a_1) - (a_0 - b_1)(a_2 - b_1).$$

Proof. To verify these equations, evaluate the terms using (2.3), (2.4) and Definition 4.1. ■

Lemma 6.2. Assume that $N \leq 2$. Then $\mathcal{L} \neq 0$.

Proof. By Proposition 5.2(i), (ii) and Lemma 6.1. ■

Lemma 6.3. Assume that $N = 3$. Then $\mathcal{L} \neq 0$ if and only if

$$(a_0 - b_1)(a_2 - b_1) = (b_0 - a_1)(b_2 - a_1).$$

Proof. By Proposition 5.2(i),(ii) and Lemma 6.1. ■

Lemma 6.4. Assume that $a_{i-1} = b_i$ for $1 \leq i \leq N - 1$. Then the following (i)–(v) hold:

(i) $\mathcal{L} \neq 0$;

(ii) $\eta_i = (x - b_0)\tau_{i-1}$ for $1 \leq i \leq N$;

(iii) $\eta_i(a_0) = 0$ for $2 \leq i \leq N$;

(iv) $\vartheta_i = \frac{a_{i-1} - b_0}{a_0 - b_0}$ for $1 \leq i \leq N$;

(v) $\Delta = I + (a_0 - b_0)\psi$, where $\psi \in \mathcal{L}$ is normalized.

Proof. (ii) By (2.3), (2.4). (iii) By (ii) and since $\tau_j(a_0) = 0$ for $1 \leq j \leq N$. (iv) Use Definition 4.1. (i) Apply Proposition 5.2(i), (ii). (v) By (5.5) and (iii) above. ■

Lemma 6.5. Assume that $a_i = b_{i-1}$ for $1 \leq i \leq N - 1$. Then the following (i)–(v) hold:

(i) $\mathcal{L} \neq 0$;

(ii) $\tau_i = (x - a_0)\eta_{i-1}$ for $1 \leq i \leq N$;

(iii) $\tau_i(b_0) = 0$ for $2 \leq i \leq N$;

(iv) $\vartheta_i = \frac{a_0 - b_{i-1}}{a_0 - b_0}$ for $1 \leq i \leq N$;

(v) $\Delta^{-1} = I + (b_0 - a_0)\psi$, where $\psi \in \mathcal{L}$ is normalized.

Proof. Interchange the roles of $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ in Lemma 6.4. ■

For the rest of this section, assume that $N \geq 2$. Also for the rest of this section, fix $\theta \in \mathbb{F}$ and assume

$$a_0 \neq \theta, \quad b_0 \neq \theta; \tag{6.1}$$

$$a_i = \theta, \quad b_i = \theta, \quad 1 \leq i \leq N - 2; \tag{6.2}$$

$$\frac{\theta - a_{N-1}}{\theta - b_0} = \frac{\theta - b_{N-1}}{\theta - a_0} \quad \text{if } N \neq \infty. \tag{6.3}$$

Using Definition 4.1,

$$\vartheta_i = 1, \quad 1 \leq i \leq N - 1$$

and for $N \neq \infty$,

$$a_{N-1} = b_0 + \vartheta_N(\theta - b_0), \quad b_{N-1} = a_0 + \vartheta_N(\theta - a_0).$$

Using (5.3),

$$\left[\begin{matrix} j \\ i \end{matrix} \right]_{\vartheta} = 1, \quad 0 \leq i \leq j \leq N-1$$

and for $N \neq \infty$,

$$\left[\begin{matrix} N \\ i \end{matrix} \right]_{\vartheta} = \vartheta_N, \quad 1 \leq i \leq N-1.$$

For $0 \leq i \leq N$ the polynomials τ_i, η_i are described in the table below:

i	τ_i	η_i
0	1	1
$1 \leq i \leq N-1$	$(x - a_0)(x - \theta)^{i-1}$	$(x - b_0)(x - \theta)^{i-1}$
N	$(x - a_0)(x - \theta)^{N-2}(x - a_{N-1})$	$(x - b_0)(x - \theta)^{N-2}(x - b_{N-1})$

For $1 \leq i \leq N$ the values of $\eta_i - \tau_i$ and $\eta_i(a_0)$ are described in the table below:

i	$\eta_i - \tau_i$	$\eta_i(a_0)$
$1 \leq i \leq N-1$	$(a_0 - b_0)(x - \theta)^{i-1}$	$(a_0 - b_0)(a_0 - \theta)^{i-1}$
N	$\vartheta_N(a_0 - b_0)(x - \theta)^{N-1}$	$\vartheta_N(a_0 - b_0)(a_0 - \theta)^{N-1}$

Lemma 6.6. *Under assumptions (6.1)–(6.3) the following (i)–(iii) hold:*

- (i) $\mathcal{L} \neq 0$;
- (ii) $\Delta = \frac{I + (\theta - b_0)\psi}{I + (\theta - a_0)\psi}$;
- (iii) $\Delta^{-1} = \frac{I + (\theta - a_0)\psi}{I + (\theta - b_0)\psi}$.

In the above lines $\psi \in \mathcal{L}$ is normalized.

Proof. (i) We invoke Proposition 5.2(i), (ii). For $0 \leq j \leq N$ we verify (5.4). We may assume that $2 \leq j \leq N$; otherwise we are done by Lemma 6.1. For $N \neq \infty$ we separate the cases $2 \leq j \leq N-1$ and $j = N$. It suffices to show that

$$\eta_j = \tau_j + \eta_j(a_0) + \sum_{i=1}^{j-1} \eta_{j-i}(a_0)\tau_i, \quad 2 \leq j \leq N-1, \quad (6.4)$$

$$\eta_N = \tau_N + \eta_N(a_0) + \vartheta_N \sum_{i=1}^{N-1} \eta_{N-i}(a_0)\tau_i, \quad \text{if } N \neq \infty. \quad (6.5)$$

For $2 \leq j \leq N$ the values of $\eta_j - \tau_j$ and $\eta_j(a_0)$ are given in the table above the lemma statement. Also for $2 \leq j \leq N$,

$$\begin{aligned} \sum_{i=1}^{j-1} \eta_{j-i}(a_0)\tau_i &= (a_0 - b_0)(x - a_0) \sum_{i=1}^{j-1} (a_0 - \theta)^{j-i-1} (x - \theta)^{i-1} \\ &= (a_0 - b_0)(x - a_0)(a_0 - \theta)^{j-2} \sum_{k=0}^{j-2} \left(\frac{x - \theta}{a_0 - \theta} \right)^k \\ &= (a_0 - b_0)(x - \theta)^{j-1} - (a_0 - b_0)(a_0 - \theta)^{j-1}. \end{aligned}$$

Using the above comments we routinely verify (6.4), (6.5).

(ii) We will verify the equation by showing that the two sides agree on V_j for $2 \leq j \leq N$. Using (5.5) and $\psi^{j+1}V_j = 0$ we see that on V_j ,

$$\begin{aligned} \Delta - I &= \sum_{i=1}^j \frac{\eta_i(a_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_i} \psi^i = \frac{\eta_j(a_0)}{\vartheta_j} \psi^j + \sum_{i=1}^{j-1} \eta_i(a_0) \psi^i = (a_0 - b_0) \psi \sum_{k=0}^{j-1} (a_0 - \theta)^k \psi^k \\ &= (a_0 - b_0) \psi \frac{I - (a_0 - \theta)^j \psi^j}{I - (a_0 - \theta) \psi} = \frac{(a_0 - b_0) \psi}{I - (a_0 - \theta) \psi}. \end{aligned}$$

The result follows. (iii) By (ii) above. ■

We just gave some examples of double lowering data. There is another example that is somewhat more involved; it will be described later in the paper.

7 Extending the data

Throughout this section, we assume that N is an integer at least 2. Recall the data $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ from (2.1), and assume that this data is double lowering. Let $a_N, b_N \in \mathbb{F}$ satisfy

$$a_0 + a_1 + \cdots + a_N \neq b_0 + b_1 + \cdots + b_N,$$

giving data

$$\{a_i\}_{i=0}^N, \quad \{b_i\}_{i=0}^N. \tag{7.1}$$

In this section we obtain necessary and sufficient conditions on a_N, b_N for the data (7.1) to be double lowering. By (2.7),

$$\tau_{N+1} = (x - a_N) \tau_N, \quad \eta_{N+1} = (x - b_N) \eta_N.$$

Lemma 7.1. *The following (i)–(iii) are equivalent:*

- (i) the data (7.1) is double lowering;
- (ii) we have

$$\eta_{N+1} = \sum_{i=0}^{N+1} \eta_{N-i+1}(a_0) \begin{bmatrix} N+1 \\ i \end{bmatrix}_{\vartheta} \tau_i; \tag{7.2}$$

(iii) we have

$$\tau_{N+1} = \sum_{i=0}^{N+1} \tau_{N-i+1}(b_0) \begin{bmatrix} N+1 \\ i \end{bmatrix}_{\vartheta} \eta_i.$$

Proof. By Proposition 5.2. ■

Lemma 7.2. *We have*

$$\eta_{N+1} = \sum_{i=0}^N \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} (a_i - b_N) \tau_i + \sum_{i=1}^{N+1} \eta_{N-i+1}(a_0) \begin{bmatrix} N \\ i-1 \end{bmatrix}_{\vartheta} \tau_i, \tag{7.3}$$

$$\tau_{N+1} = \sum_{i=0}^N \tau_{N-i}(b_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} (b_i - a_N) \eta_i + \sum_{i=1}^{N+1} \tau_{N-i+1}(b_0) \begin{bmatrix} N \\ i-1 \end{bmatrix}_{\vartheta} \eta_i. \tag{7.4}$$

Proof. We show (7.3). Using Proposition 5.2,

$$\begin{aligned}\eta_{N+1} &= (x - b_N)\eta_N = (x - b_N) \sum_{i=0}^N \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} \tau_i \\ &= \sum_{i=0}^N \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} ((a_i - b_N)\tau_i + \tau_{i+1}) \\ &= \sum_{i=0}^N \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} (a_i - b_N)\tau_i + \sum_{i=1}^{N+1} \eta_{N-i+1}(a_0) \begin{bmatrix} N \\ i-1 \end{bmatrix}_{\vartheta} \tau_i.\end{aligned}$$

Line (7.4) is similarly obtained. ■

Proposition 7.3. *The following (i)–(iii) are equivalent:*

(i) *the data (7.1) is double lowering;*

(ii) *for $0 \leq i \leq N-1$ such that $\eta_i(a_0) \neq 0$,*

$$\begin{aligned}(a_0 + \cdots + a_i - b_0 - \cdots - b_i)(a_{N-i} - b_N) \\ = (a_0 - b_i)(a_{N-i} + \cdots + a_N - b_{N-i} - \cdots - b_N);\end{aligned}$$

(iii) *for $0 \leq i \leq N-1$ such that $\tau_i(b_0) \neq 0$,*

$$\begin{aligned}(b_0 + \cdots + b_i - a_0 - \cdots - a_i)(b_{N-i} - a_N) \\ = (b_0 - a_i)(b_{N-i} + \cdots + b_N - a_{N-i} - \cdots - a_N).\end{aligned}$$

Proof. (i) \Leftrightarrow (ii) We invoke Lemma 7.1(i), (ii). Subtract (7.2) from (7.3) to obtain an equation $0 = \sum_{i=1}^N d_i \tau_i$ where

$$d_i = \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} (a_i - b_N) + \eta_{N-i+1}(a_0) \begin{bmatrix} N \\ i-1 \end{bmatrix}_{\vartheta} - \eta_{N-i+1}(a_0) \begin{bmatrix} N+1 \\ i \end{bmatrix}_{\vartheta} \quad (7.5)$$

for $1 \leq i \leq N$. Note that (7.2) holds iff $0 = \sum_{i=1}^N d_i \tau_i$ iff $d_i = 0$ for $1 \leq i \leq N$. For $1 \leq i \leq N$ we simplify (7.5) using

$$\eta_{N-i+1}(a_0) = \eta_{N-i}(a_0)(a_0 - b_{N-i})$$

and

$$\begin{bmatrix} N \\ i-1 \end{bmatrix}_{\vartheta} = \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} \frac{\vartheta_i}{\vartheta_{N-i+1}}, \quad \begin{bmatrix} N+1 \\ i \end{bmatrix}_{\vartheta} = \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta} \frac{\vartheta_{N+1}}{\vartheta_{N-i+1}}.$$

We find that d_i is equal to

$$\frac{\eta_{N-i}(a_0)}{\vartheta_{N-i+1}} \begin{bmatrix} N \\ i \end{bmatrix}_{\vartheta}$$

times

$$(a_i - b_N)\vartheta_{N-i+1} + (a_0 - b_{N-i})(\vartheta_i - \vartheta_{N+1})$$

for $1 \leq i \leq N$. Therefore, (7.2) holds if and only if

$$\eta_{N-i}(a_0) = 0 \quad \text{or} \quad (a_i - b_N)\vartheta_{N-i+1} = (a_0 - b_{N-i})(\vartheta_{N+1} - \vartheta_i)$$

for $1 \leq i \leq N$. Replacing i by $N-i$, we see that (7.2) holds if and only if

$$\eta_i(a_0) = 0 \quad \text{or} \quad (a_{N-i} - b_N)\vartheta_{i+1} = (a_0 - b_i)(\vartheta_{N+1} - \vartheta_{N-i})$$

for $0 \leq i \leq N-1$. The result follows in view of Definition 4.1.

(i) \Leftrightarrow (iii) Similar to the proof of (i) \Leftrightarrow (ii). ■

Our next general goal is to solve the equations in Proposition 7.3(ii), (iii). The main solution will involve a type of sequence, said to be recurrent.

8 Recurrent sequences

Throughout this section let n denote an integer at least 2, or ∞ . let $\{a_i\}_{i=0}^n$ denote scalars in \mathbb{F} .

Definition 8.1 (see [30, Definition 8.2]). Let β, γ, ϱ denote scalars in \mathbb{F} .

- (i) The sequence $\{a_i\}_{i=0}^n$ is said to be (β, γ, ϱ) -recurrent whenever

$$a_{i-1}^2 - \beta a_{i-1} a_i + a_i^2 - \gamma(a_{i-1} + a_i) = \varrho \quad (8.1)$$

for $1 \leq i \leq n$.

- (ii) The sequence $\{a_i\}_{i=0}^n$ is said to be (β, γ) -recurrent whenever

$$a_{i-1} - \beta a_i + a_{i+1} = \gamma \quad (8.2)$$

for $1 \leq i \leq n-1$.

- (iii) The sequence $\{a_i\}_{i=0}^n$ is said to be β -recurrent whenever

$$a_{i-2} - (\beta + 1)a_{i-1} + (\beta + 1)a_i - a_{i+1} \quad (8.3)$$

is zero for $2 \leq i \leq n-1$.

- (iv) The sequence $\{a_i\}_{i=0}^n$ is said to be recurrent whenever there exists $\beta \in \mathbb{F}$ such that $\{a_i\}_{i=0}^n$ is β -recurrent.

Lemma 8.2. *The following are equivalent:*

- (i) *the sequence $\{a_i\}_{i=0}^n$ is recurrent;*
- (ii) *there exists $\beta \in \mathbb{F}$ such that $\{a_i\}_{i=0}^n$ is β -recurrent.*

Proof. By Definition 8.1. ■

Lemma 8.3. *For $\beta \in \mathbb{F}$ the following are equivalent:*

- (i) *the sequence $\{a_i\}_{i=0}^n$ is β -recurrent;*
- (ii) *there exists $\gamma \in \mathbb{F}$ such that $\{a_i\}_{i=0}^n$ is (β, γ) -recurrent.*

Proof. (i) \Rightarrow (ii) For $2 \leq i \leq n-1$, the expression (8.3) is zero by assumption, so

$$a_{i-2} - \beta a_{i-1} + a_i = a_{i-1} - \beta a_i + a_{i+1}.$$

The left-hand side of (8.2) is independent of i , and the result follows.

(ii) \Rightarrow (i) For $2 \leq i \leq n-1$, subtract the equation (8.2) at i from the corresponding equation obtained by replacing i by $i-1$, to find (8.3) is zero. ■

Lemma 8.4. *The following (i), (ii) hold for all $\beta, \gamma \in \mathbb{F}$.*

- (i) *Suppose $\{a_i\}_{i=0}^n$ is (β, γ) -recurrent. Then there exists $\varrho \in \mathbb{F}$ such that $\{a_i\}_{i=0}^n$ is (β, γ, ϱ) -recurrent.*
- (ii) *Suppose $\{a_i\}_{i=0}^n$ is (β, γ, ϱ) -recurrent, and that $a_{i-1} \neq a_{i+1}$ for $1 \leq i \leq n-1$. Then $\{a_i\}_{i=0}^n$ is (β, γ) -recurrent.*

Proof. Let p_i denote the expression on the left in (8.1), and observe

$$p_i - p_{i+1} = (a_{i-1} - a_{i+1})(a_{i-1} - \beta a_i + a_{i+1} - \gamma)$$

for $1 \leq i \leq n-1$. Assertions (i), (ii) are both routine consequences of this. \blacksquare

Definition 8.5. Assume that $\{a_i\}_{i=0}^n$ is recurrent. By a *parameter triple* for $\{a_i\}_{i=0}^n$ we mean a 3-tuple β, γ, ϱ of scalars in \mathbb{F} such that $\{a_i\}_{i=0}^n$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent.

Note that a recurrent sequence has at least one parameter triple.

9 Recurrent sequences in closed form

In this section, we describe the recurrent sequences in closed form. Let n denote an integer at least 2, or ∞ .

Lemma 9.1 (see [30, Lemma 9.2]). *The recurrent sequences $\{a_i\}_{i=0}^n$ are described in the table below:*

case	a_i	comments
I	$\alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}$	$q \notin \{0, 1, -1\}$
II	$\alpha_1 + \alpha_2 i + \alpha_3 \binom{i}{2}$	
III	$\alpha_1 + \alpha_2 (-1)^i + \alpha_3 i (-1)^i$	$\text{char}(\mathbb{F}) \neq 2$

In the above table $q, \alpha_1, \alpha_2, \alpha_3$ are scalars in \mathbb{F} .

Lemma 9.2. *The following scalars β, γ, ϱ give a parameter triple for the recurrent sequence $\{a_i\}_{i=0}^n$ in Lemma 9.1.*

Case I:

$$\beta = q + q^{-1}, \quad \gamma = -\alpha_1(q-1)^2 q^{-1}, \quad \varrho = \alpha_1^2(q-1)^2 q^{-1} - \alpha_2 \alpha_3 (q - q^{-1})^2.$$

Case II:

$$\beta = 2, \quad \gamma = \alpha_3, \quad \varrho = \alpha_2^2 - \alpha_2 \alpha_3 - 2\alpha_1 \alpha_3.$$

Case III:

$$\beta = -2, \quad \gamma = 4\alpha_1, \quad \varrho = \alpha_3^2 - 4\alpha_1^2.$$

Proof. This is routinely checked using Definition 8.5. \blacksquare

Lemma 9.3. *Referring to Lemma 9.1, for $0 \leq i \leq n+1$ the sum $a_0 + a_1 + \cdots + a_{i-1}$ is given in the table below:*

case	$a_0 + a_1 + \cdots + a_{i-1}$
I	$\alpha_1 i + \alpha_2 \frac{1-q^i}{1-q} + \alpha_3 \frac{1-q^{-i}}{1-q^{-1}}$
II	$\alpha_1 i + \alpha_2 \binom{i}{2} + \alpha_3 \binom{i}{3}$
III, i even	$\alpha_1 i - \alpha_3 i/2$
III, i odd	$\alpha_1 i + \alpha_2 + \alpha_3(i-1)/2$

Proof. Use induction on i . \blacksquare

Note 9.4. Referring to Case III of the above table, the subcases i even and i odd can be handled in the following uniform way. For $0 \leq i \leq n+1$,

$$a_0 + a_1 + \cdots + a_{i-1} = \frac{2\alpha_2 - \alpha_3 + 4\alpha_1 i + (\alpha_3 - 2\alpha_2)(-1)^i - 2\alpha_3 i(-1)^i}{4}.$$

10 Twin recurrent sequences

Let n denote an integer at least 2, or ∞ . Let $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ denote scalars in \mathbb{F} .

Definition 10.1. Assume that the sequences $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ are recurrent. These sequences are called *twins* whenever they have a parameter triple in common.

Lemma 10.2. *Assume that the sequences $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ are recurrent. These sequences are twins if and only if they are related in the following way:*

Case I:

$$a_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}, \quad b_i = \alpha_1 + \alpha'_2 q^i + \alpha'_3 q^{-i}, \quad \alpha'_2 \alpha'_3 = \alpha_2 \alpha_3.$$

Case II:

$$a_i = \alpha_1 + \alpha_2 i + \alpha_3 \binom{i}{2}, \quad b_i = \alpha'_1 + \alpha'_2 i + \alpha_3 \binom{i}{2},$$

$$(\alpha_2 - \alpha'_2)(\alpha_2 + \alpha'_2 - \alpha_3) = 2(\alpha_1 - \alpha'_1)\alpha_3.$$

Case III:

$$a_i = \alpha_1 + \alpha_2 (-1)^i + \alpha_3 i (-1)^i, \quad b_i = \alpha_1 + \alpha'_2 (-1)^i + \alpha'_3 i (-1)^i,$$

$$\alpha'_3 = \alpha_3 \quad \text{or} \quad \alpha'_3 = -\alpha_3.$$

Proof. First assume that the sequences $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ are related in the specified way. Then these sequences share the parameter triple β, γ, ϱ from Lemma 9.2. Therefore these sequences are twins. Next assume that the sequences $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ are twins. It follows from Definition 8.1 and Lemma 9.1 that they are related in the specified way. ■

11 A characterization of twin recurrent sequences

In this section we explain what twin recurrent sequences have to do with the equations in Proposition 7.3. Let n denote an integer at least 2, or ∞ . Let $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ denote scalars in \mathbb{F} .

Definition 11.1. For $0 \leq i \leq j \leq n$ let $E(i, j)$ denote the equation

$$(a_0 + \cdots + a_i - b_0 - \cdots - b_i)(a_{j-i} - b_j) = (a_0 - b_i)(a_{j-i} + \cdots + a_j - b_{j-i} - \cdots - b_j).$$

Lemma 11.2. *The equations $E(0, j)$ and $E(j, j)$ hold for $0 \leq j \leq n$.*

Proof. This is routinely checked. ■

Proposition 11.3. *Assume that the sequences $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ are recurrent and twins. Then $E(i, j)$ holds for $0 \leq i \leq j \leq n$.*

Proof. This is routinely verified for each Case I–III in Lemma 10.2. To carry out the verification, use the formulas in Lemma 9.3. ■

In the next two lemmas, we give some additional solutions for the equations $E(i, j)$ in Definition 11.1.

Lemma 11.4. *Assume that $a_i = b_{i-1}$ for $1 \leq i \leq n$. Then $E(i, j)$ holds for $0 \leq i \leq j \leq n$.*

Proof. For $0 \leq i \leq j \leq n$, each side of $E(i, j)$ is equal to $(a_0 - b_i)(a_{j-i} - b_j)$. ■

Lemma 11.5. *Pick $\theta \in \mathbb{F}$ and assume*

$$\begin{aligned} a_i &= \theta, & b_i &= \theta, & 1 \leq i \leq n-1, \\ (\theta - a_0)(\theta - a_n) &= (\theta - b_0)(\theta - b_n) & \text{if } n \neq \infty. \end{aligned} \quad (11.1)$$

Then $E(i, j)$ holds for $0 \leq i \leq j \leq n$.

Proof. By Lemma 11.2 it suffices to verify $E(i, j)$ for $1 \leq i < j \leq n$. Let i, j be given. For $j < n$, each side of $E(i, j)$ is zero. For $n \neq \infty$ and $j = n$, the equation $E(i, j)$ becomes

$$(a_0 - b_0)(\theta - b_n) = (a_0 - \theta)(a_n - b_n)$$

which is a reformulation of (11.1). ■

Proposition 11.6. *Assume that $a_1 \neq b_0$ and $a_1 \neq b_1$. Further assume that $E(i, j)$ holds for $1 \leq i \leq 2$ and $i+1 \leq j \leq n$. Then the sequences $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$ are recurrent and twins.*

Proof. Using $E(1, 2)$,

$$(a_0 - b_1)(a_2 - b_1) = (b_0 - a_1)(b_2 - a_1). \quad (11.2)$$

Since $a_1 \neq b_1$, there exists a unique pair β, γ of scalars in \mathbb{F} such that

$$a_0 - \beta a_1 + a_2 = \gamma, \quad b_0 - \beta b_1 + b_2 = \gamma.$$

Using these equations we eliminate a_2, b_2 in (11.2):

$$(a_0 - b_1)(\gamma + \beta a_1 - a_0 - b_1) = (b_0 - a_1)(\gamma + \beta b_1 - b_0 - a_1).$$

In this equation we rearrange terms to get

$$a_0^2 - \beta a_0 a_1 + a_1^2 - \gamma(a_0 + a_1) = b_0^2 - \beta b_0 b_1 + b_1^2 - \gamma(b_0 + b_1).$$

Let ϱ denote this common value. We show that each of $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent. To do this, we show that for $2 \leq j \leq n$, each of $\{a_i\}_{i=0}^j$, $\{b_i\}_{i=0}^j$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent. We will use induction on j . First assume that $j = 2$. By construction $\{a_i\}_{i=0}^2$ is (β, γ) -recurrent. By construction and Lemma 8.4(i), the sequence $\{a_i\}_{i=0}^2$ is (β, γ, ϱ) -recurrent. Similarly $\{b_i\}_{i=0}^2$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent. We are done for $j = 2$. Next assume that $j \geq 3$. By $E(1, j)$,

$$(a_0 + a_1 - b_0 - b_1)(a_{j-1} - b_j) = (a_0 - b_1)(a_{j-1} + a_j - b_{j-1} - b_j). \quad (11.3)$$

By $E(2, j)$,

$$\begin{aligned} (a_0 + a_1 + a_2 - b_0 - b_1 - b_2)(a_{j-2} - b_j) \\ = (a_0 - b_2)(a_{j-2} + a_{j-1} + a_j - b_{j-2} - b_{j-1} - b_j). \end{aligned} \quad (11.4)$$

The equations (11.3), (11.4) give a linear system in the unknowns a_j, b_j . For this system the coefficient matrix is

$$C = \begin{pmatrix} a_0 - b_1 & a_1 - b_0 \\ a_0 - b_2 & a_1 + a_2 - b_0 - b_1 \end{pmatrix}.$$

We have

$$\det(C) = (a_0 - b_1)(a_1 + a_2 - b_0 - b_1) - (a_0 - b_2)(a_1 - b_0).$$

In the above equation we simplify the right-hand side using (11.2), to obtain

$$\det(C) = (a_1 - b_0)(a_1 - b_1).$$

Therefore $\det(C) \neq 0$, so the system (11.3), (11.4) has a unique solution for a_j, b_j . We now describe the solution. By induction the sequences $\{a_i\}_{i=0}^{j-1}, \{b_i\}_{i=0}^{j-1}$ are (β, γ) -recurrent and (β, γ, ϱ) -recurrent. Define a'_j, b'_j such that

$$a_{j-2} - \beta a_{j-1} + a'_j = \gamma, \quad b_{j-2} - \beta b_{j-1} + b'_j = \gamma.$$

Consider the two sequences

$$a_0, a_1, \dots, a_{j-1}, a'_j; \tag{11.5}$$

$$b_0, b_1, \dots, b_{j-1}, b'_j. \tag{11.6}$$

By construction, each of (11.5), (11.6) is (β, γ) -recurrent. By construction and Lemma 8.4(i), each of (11.5), (11.6) is (β, γ, ϱ) -recurrent. We show that $a_j = a'_j$ and $b_j = b'_j$. The sequences (11.5), (11.6) are recurrent and twins, so by Proposition 11.3 they satisfy $E(1, j)$ and $E(2, j)$. Therefore, the equations (11.3), (11.4) still hold if we replace a_j, b_j by a'_j, b'_j . We mentioned earlier that the system (11.3), (11.4) has a unique solution for a_j, b_j . By these comments $a_j = a'_j$ and $b_j = b'_j$. Consequently each of $\{a_i\}_{i=0}^j, \{b_i\}_{i=0}^j$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent. The above argument shows that each of $\{a_i\}_{i=0}^n, \{b_i\}_{i=0}^n$ is (β, γ) -recurrent and (β, γ, ϱ) -recurrent. ■

Lemma 11.7. *Assume that $a_{i-1} = b_i$ for $1 \leq i \leq n$. Then the equation $E(1, j)$ holds for $1 \leq j \leq n$. However, in general it is not the case that $E(i, j)$ holds for $0 \leq i \leq j \leq n$.*

Proof. For $0 \leq i \leq j \leq n$ the equation $E(i, j)$ becomes

$$(a_i - b_0)(a_{j-i} - b_j) = (a_0 - b_i)(a_j - b_{j-i}). \tag{11.7}$$

If $i = 1$ then each side of (11.7) is zero, so $E(1, j)$ holds. Assume that $n = 3$ and

$$a_0 = 0 = b_1, \quad a_1 = 1 = b_2, \quad a_2 = 0 = b_3, \quad a_3 = 1, \quad b_0 = 0.$$

Then (11.7) fails for $i = 2$ and $j = 3$. ■

12 The classification of the double lowering data

Recall the data $\{a_i\}_{i=0}^{N-1}, \{b_i\}_{i=0}^{N-1}$ from (2.1). In this section we obtain necessary and sufficient conditions for this data to be double lowering. In view of Lemma 6.2 we assume $N \geq 3$.

Theorem 12.1. *Let N denote an integer at least 3, or ∞ . Let*

$$\{a_i\}_{i=0}^{N-1}, \quad \{b_i\}_{i=0}^{N-1} \tag{12.1}$$

denote scalars in \mathbb{F} such that

$$a_0 + a_1 + \dots + a_{i-1} \neq b_0 + b_1 + \dots + b_{i-1}, \quad 1 \leq i \leq N. \tag{12.2}$$

Then the data (12.1) is double lowering if and only if at least one of the following (i)–(iv) holds:

- (i) $a_{i-1} = b_i$ for $1 \leq i \leq N - 1$;
- (ii) $a_i = b_{i-1}$ for $1 \leq i \leq N - 1$;

(iii) there exists $\theta \in \mathbb{F}$ such that

$$\begin{aligned} a_0 \neq \theta, \quad b_0 \neq \theta, \quad a_i = \theta, \quad b_i = \theta, \quad 1 \leq i \leq N-2, \\ \frac{\theta - a_{N-1}}{\theta - b_0} = \frac{\theta - b_{N-1}}{\theta - a_0} \quad \text{if } N \neq \infty. \end{aligned}$$

(iv) the sequences (12.1) are recurrent and twins.

Proof. First assume that at least one of (i)–(iii) holds. Then (12.1) is double lowering, by Lemmas 6.4, 6.5, 6.6. Next assume that (iv) holds and $N \neq \infty$. By Proposition 11.3 (with $n = N - 1$) the equations $E(i, j)$ hold for $0 \leq i \leq j \leq N - 1$. We delete a_{N-1}, b_{N-1} from (12.1) and consider the data

$$\{a_i\}_{i=0}^{N-2}, \quad \{b_i\}_{i=0}^{N-2}. \quad (12.3)$$

By Lemma 6.2 and induction on N , we may assume that the sequences (12.3) are double lowering. By Proposition 7.3(i), (ii) (with N replaced by $N - 1$) we find that (12.1) is double lowering. Next assume that (iv) holds and $N = \infty$. Then for all integers $j \geq 2$ the sequences $\{a_i\}_{i=0}^j, \{b_i\}_{i=0}^j$ are recurrent and twins. Consequently the data $\{a_i\}_{i=0}^j, \{b_i\}_{i=0}^j$ is double lowering. Therefore the data $\{a_i\}_{i=0}^\infty, \{b_i\}_{i=0}^\infty$ is double lowering. We are done in one direction.

We now reverse the direction. Next assume that (12.1) is double lowering. We break the argument into cases.

Case $a_0 = b_1$. We show that (i) holds. We have $\eta_i(a_0) \neq 0$ for $0 \leq i \leq 1$. By assumption the data (12.1) is double lowering, so the data $\{a_i\}_{i=0}^j, \{b_i\}_{i=0}^j$ is double lowering for $1 \leq j \leq N - 1$. Applying Proposition 7.3(i), (ii) repeatedly (with N replaced by $2, 3, \dots, N - 1$) we find that $E(1, j)$ holds for $2 \leq j \leq N - 1$. Using these equations and (12.2) we routinely obtain $a_{j-1} = b_j$ for $2 \leq j \leq N - 1$. This and $a_0 = b_1$ implies (i).

Case $a_1 = b_0$. Interchanging the roles of $\{a_i\}_{i=0}^{N-1}, \{b_i\}_{i=0}^{N-1}$ in the previous case, we find that (ii) holds.

Case $a_0 \neq b_1, a_1 \neq b_0, a_1 = b_1$. We show that (iii) holds. Define $\theta = a_1 = b_1$, and note that $a_0 \neq \theta, b_0 \neq \theta$. We have $\eta_i(a_0) \neq 0$ for $0 \leq i \leq 2$. By assumption the data (12.1) is double lowering, so the data $\{a_i\}_{i=0}^j, \{b_i\}_{i=0}^j$ is double lowering for $1 \leq j \leq N - 1$. Applying Proposition 7.3(i), (ii) repeatedly (with N replaced by $2, 3, \dots, N - 1$) we find that $E(i, j)$ holds for $1 \leq i \leq 2$ and $i + 1 \leq j \leq N - 1$. Next we show that $a_k = b_k = \theta$ for $2 \leq k \leq N - 2$. We will use induction on k . Assume $N \geq 4$; otherwise there is nothing to prove. Using $E(1, 2), E(1, 3)$ and $a_0 \neq b_1$ we obtain

$$a_2 = b_2\zeta + \theta(1 - \zeta), \quad (12.4)$$

$$a_3 = b_3\zeta + b_2(1 - \zeta^2) - \theta\zeta(1 - \zeta), \quad (12.5)$$

where

$$\zeta = \frac{b_0 - \theta}{a_0 - \theta}. \quad (12.6)$$

In the equation $E(2, 3)$ we eliminate a_2, a_3 using (12.4), (12.5). We evaluate the result using (12.2) (with $i = 3$), to obtain $b_2 = \theta$. In (12.4) we set $b_2 = \theta$ to obtain $a_2 = \theta$. Next assume that $3 \leq k \leq N - 2$. By induction each of $a_2, a_3, \dots, a_{k-1}, b_2, b_3, \dots, b_{k-1}$ is equal to θ . Using this we evaluate $E(1, k), E(1, k + 1)$ to obtain

$$a_k = b_k\zeta + \theta(1 - \zeta), \quad (12.7)$$

$$a_{k+1} = b_{k+1}\zeta + b_k(1 - \zeta^2) - \theta\zeta(1 - \zeta). \quad (12.8)$$

Using (12.7), (12.8) we evaluate $E(2, k+1)$ to obtain $b_k = \theta$. In (12.7) we set $b_k = \theta$ to obtain $a_k = \theta$. We have shown that $a_k = b_k = \theta$ for $2 \leq k \leq N-2$. Now for $N \neq \infty$ we use $E(1, N-1)$ to obtain

$$a_{N-1} = b_{N-1}\zeta + \theta(1 - \zeta).$$

Evaluating this using (12.6) we obtain

$$\frac{\theta - a_{N-1}}{\theta - b_0} = \frac{\theta - b_{N-1}}{\theta - a_0}.$$

We have shown that (iii) holds.

Case $a_0 \neq b_1$, $a_1 \neq b_0$, $a_1 \neq b_1$. We show that (iv) holds. Using Proposition 7.3(i), (ii) as in the previous case, we find that $E(i, j)$ holds for $1 \leq i \leq 2$ and $i+1 \leq j \leq N-1$. Now by Proposition 11.6 (with $n = N-1$), the sequences (12.1) are recurrent and twins. We have shown that (iv) holds. ■

13 \mathcal{L} and Δ for twin recurrent data

Our goal for the rest of the paper is to give a comprehensive description of \mathcal{L} and Δ for twin recurrent data. We will focus on Case I in Lemma 10.2, or more precisely, an adjusted version of this case as described in Section 3.

For the rest of this paper we assume that N is an integer at least 2, or ∞ . Recall the double lowering space \mathcal{L} for the data (2.1). For the rest of this paper, fix nonzero $a, b, q \in \mathbb{F}$ and assume

$$a_i = aq^i + a^{-1}q^{-i}, \quad b_i = bq^i + b^{-1}q^{-i} \quad (13.1)$$

for $0 \leq i \leq N-1$. By (2.2) we have $a \neq b$ and

$$q^i \neq 1, \quad abq^{i-1} \neq 1, \quad 1 \leq i \leq N.$$

Note 13.1. The data $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ is unchanged if we replace

$$q \mapsto q^{-1}, \quad a \mapsto a^{-1}, \quad b \mapsto b^{-1}.$$

Lemma 13.2. *The sequences (2.1) are recurrent and twins.*

Proof. By Lemmas 9.1 and 10.2. ■

Corollary 13.3. *The data (2.1) is double lowering.*

Proof. By Lemma 13.2 along with Lemma 6.2 and Theorem 12.1(iv). ■

Our next general goal is to describe the polynomials $\{\tau_i\}_{i=0}^{N-1}$, $\{\eta_i\}_{i=0}^{N-1}$ from (2.3), (2.4), the parameters $\{\vartheta_i\}_{i=0}^N$ from Definition 4.1, and the map Δ from (2.5).

We mention some formulas for later use.

Lemma 13.4. *For $1 \leq i \leq N-1$,*

$$\begin{aligned} qa_i - a_{i-1} &= (q - q^{-1})aq^i, & qb_i - b_{i-1} &= (q - q^{-1})bq^i, \\ a_i - a_{i-1} &= (q-1)(aq^{i-1} - a^{-1}q^{-i}), & b_i - b_{i-1} &= (q-1)(bq^{i-1} - b^{-1}q^{-i}), \\ a_i - qa_{i-1} &= (1 - q^2)a^{-1}q^{-i}, & b_i - qb_{i-1} &= (1 - q^2)b^{-1}q^{-i}. \end{aligned}$$

Proof. Use (13.1). ■

Lemma 13.5. For $0 \leq i \leq N$,

$$a_0 + a_1 + \cdots + a_{i-1} = \frac{1 - q^i}{1 - q} (a + a^{-1}q^{1-i}),$$

$$b_0 + b_1 + \cdots + b_{i-1} = \frac{1 - q^i}{1 - q} (b + b^{-1}q^{1-i}).$$

Proof. Use (13.1). ■

Next we describe $\{\vartheta_i\}_{i=0}^N$. We give two versions.

Lemma 13.6. For $0 \leq i \leq N$,

$$(i) \quad \vartheta_i = \frac{1 - q^i}{1 - q} \frac{1 - abq^{i-1}}{1 - ab} q^{1-i};$$

$$(ii) \quad \vartheta_i = \frac{1 - q^{-i}}{1 - q^{-1}} \frac{1 - a^{-1}b^{-1}q^{1-i}}{1 - a^{-1}b^{-1}} q^{i-1}.$$

Proof. (i) By Definition 4.1 and Lemma 13.5. (ii) By Note 13.1 and (i) above. ■

We mention some formulas for later use.

Lemma 13.7. For $0 \leq i \leq N - 1$,

$$q\vartheta_{i+1} - \vartheta_i = \frac{q + ab - (q + 1)abq^i}{1 - ab},$$

$$\vartheta_{i+1} - \vartheta_i = \frac{q^{-i} - abq^i}{1 - ab},$$

$$\vartheta_{i+1} - q\vartheta_i = \frac{(1 + q)q^{-i} - q - ab}{1 - ab}.$$

Proof. Use Lemma 13.6. ■

We recall some notation. For an element α in any algebra, define

$$(\alpha; q)_i = (1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{i-1}), \quad i \in \mathbb{N}.$$

We interpret $(\alpha; q)_0 = 1$. We remark that for $j \geq i \geq 0$,

$$(q^{-j}; q)_i (q; q)_{j-i} = (-1)^i (q; q)_j q^{\binom{i}{2}} q^{-ij}. \quad (13.2)$$

Lemma 13.8. For $0 \leq i \leq N$,

$$\vartheta_1 \vartheta_2 \cdots \vartheta_i = \frac{(q; q)_i (ab; q)_i q^{-\binom{i}{2}}}{(1 - q)^i (1 - ab)^i}. \quad (13.3)$$

For $0 \leq i \leq j \leq N$,

$$\vartheta_j \vartheta_{j-1} \cdots \vartheta_{j-i+1} = \frac{(q^{-j}; q)_i (a^{-1}b^{-1}q^{1-j}; q)_i q^{i(j-i)} q^{\binom{i}{2}}}{(1 - q^{-1})^i (1 - a^{-1}b^{-1})^i}. \quad (13.4)$$

Proof. To obtain (13.3), use Lemma 13.6(i). To obtain (13.4), use Lemma 13.6(ii). ■

Lemma 13.9. For $0 \leq i \leq j \leq N$,

$$\begin{bmatrix} j \\ i \end{bmatrix}_\vartheta = \frac{(q^{-j}; q)_i (a^{-1}b^{-1}q^{1-j}; q)_i q^{ij} a^i b^i}{(q; q)_i (ab; q)_i}.$$

Proof. Evaluate (5.3) using Lemma 13.8. ■

We comment on the notation. Let y denote an indeterminate. Let $\mathbb{F}[y, y^{-1}]$ denote the algebra consisting of the Laurent polynomials in y that have all coefficients in \mathbb{F} . This algebra has an automorphism that sends $y \mapsto y^{-1}$. An element of $\mathbb{F}[y, y^{-1}]$ that is fixed by the automorphism is called symmetric. The symmetric elements form a subalgebra of $\mathbb{F}[y, y^{-1}]$ called its symmetric part. There exists an injective algebra homomorphism $\iota: \mathbb{F}[x] \rightarrow \mathbb{F}[y, y^{-1}]$ that sends $x \mapsto y + y^{-1}$. The image of $\mathbb{F}[x]$ under ι is the symmetric part of $\mathbb{F}[y, y^{-1}]$. Via ι we identify $\mathbb{F}[x]$ with the symmetric part of $\mathbb{F}[y, y^{-1}]$.

Lemma 13.10. For $0 \leq i \leq N$,

$$(i) \quad \tau_i = (-1)^i a^{-i} q^{-\binom{i}{2}} (ay; q)_i (ay^{-1}; q)_i;$$

$$(ii) \quad \eta_i = (-1)^i b^{-i} q^{-\binom{i}{2}} (by; q)_i (by^{-1}; q)_i.$$

In the above lines $x = y + y^{-1}$.

Proof. (i) We use (2.3) and (13.1). For $0 \leq j \leq i - 1$,

$$x - a_j = y + y^{-1} - aq^j - a^{-1}q^{-j} = -a^{-1}q^{-j}(1 - ayq^j)(1 - ay^{-1}q^j).$$

The result follows. (ii) Similar to the proof of (i) above. ■

Lemma 13.11. For $0 \leq i \leq N$,

$$(i) \quad \tau_i(b_0) = (-1)^i a^{-i} q^{-\binom{i}{2}} (ab; q)_i (ab^{-1}; q)_i;$$

$$(ii) \quad \eta_i(a_0) = (-1)^i b^{-i} q^{-\binom{i}{2}} (ab; q)_i (a^{-1}b; q)_i.$$

Proof. (i) Set $y = b$ in Lemma 13.10(i), and use $b_0 = b + b^{-1}$. (ii) Similar to the proof of (i) above. ■

Our data is double lowering, so $\mathcal{L} \neq 0$. For the rest of the paper, let $\psi \in \mathcal{L}$ be normalized. The maps Δ, ψ are related by (5.5), (5.6). Our next goal is to interpret (5.5), (5.6) using the q -exponential function. This function is defined as follows. For locally nilpotent $T \in \text{End}(V)$,

$$\exp_q(T) = \sum_{i=0}^N \frac{q^{\binom{i}{2}} (1-q)^i T^i}{(q; q)_i}. \quad (13.5)$$

The map $\exp_q(T)$ is invertible; its inverse is

$$\exp_{q^{-1}}(-T) = \sum_{i=0}^N \frac{(-1)^i (1-q)^i T^i}{(q; q)_i}. \quad (13.6)$$

Lemma 13.12. For locally nilpotent $T \in \text{End}(V)$,

$$(1 - (q-1)T) \exp_q(qT) = \exp_q(T).$$

Proof. To verify this equation, for $0 \leq i \leq N$ compare the coefficient of T^i on each side. ■

Proposition 13.13. We have

$$\Delta = \exp_q(a^{-1}\xi\psi) \exp_{q^{-1}}(-b^{-1}\xi\psi), \quad (13.7)$$

where $\xi = 1 - ab$.

Proof. For $0 \leq j \leq N$ we compare the coefficient of ψ^j on each side of (13.7). For the left-hand side these coefficients are obtained from (5.5). We require

$$\frac{\eta_j(a_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_j} = \sum_{i=0}^j \frac{q^{\binom{i}{2}} a^{-i} (1-q)^i \xi^i}{(q; q)_i} \frac{(-1)^{j-i} b^{i-j} (1-q)^{j-i} \xi^{j-i}}{(q; q)_{j-i}}. \quad (13.8)$$

By (13.3) and the construction,

$$\vartheta_1 \vartheta_2 \cdots \vartheta_j = (q; q)_j (ab; q)_j q^{-\binom{j}{2}} (1-q)^{-j} \xi^{-j}.$$

By Lemma 13.11(ii),

$$\eta_j(a_0) = (-1)^j b^{-j} q^{-\binom{j}{2}} (ab; q)_j (a^{-1}b; q)_j.$$

Using these comments and (13.2), the equation (13.8) becomes

$$(zq^{-j}; q)_j = \sum_{i=0}^j \frac{(q^{-j}; q)_i z^i}{(q; q)_i}, \quad (13.9)$$

where $z = a^{-1}bq^j$. Basic hypergeometric series are discussed in [13, 25]. In (13.9) the sum on the right is the basic hypergeometric series

$${}_1\phi_0 \left(\begin{matrix} q^{-j} \\ - \end{matrix} \middle| q; z \right).$$

This observation reveals that (13.9) is an instance of the q -binomial theorem [13, Section 1.3]. The result follows. \blacksquare

Proposition 13.13 gives a factorization of Δ . We now investigate the factors.

Lemma 13.14. For $0 \leq i \leq N$,

$$\exp_{q^{-1}}(-a^{-1}\xi\psi)\eta_i = \exp_{q^{-1}}(-b^{-1}\xi\psi)\tau_i, \quad (13.10)$$

$$\exp_q(b^{-1}\xi\psi)\eta_i = \exp_q(a^{-1}\xi\psi)\tau_i, \quad (13.11)$$

where $\xi = 1 - ab$.

Proof. By (13.7) and the comments above Lemma 13.12,

$$\exp_{q^{-1}}(-a^{-1}\xi\psi)\Delta = \exp_{q^{-1}}(-b^{-1}\xi\psi). \quad (13.12)$$

To obtain (13.10), apply each side of (13.12) to τ_i and evaluate the result using (2.5). For the equation (13.7), the two factors on the right commute; swapping these factors and proceeding as above,

$$\exp_q(b^{-1}\xi\psi)\Delta = \exp_q(a^{-1}\xi\psi). \quad (13.13)$$

To obtain (13.11), apply each side of (13.13) to τ_i and evaluate the result using (2.5). \blacksquare

Definition 13.15. For $0 \leq i \leq N$ let w_i (resp. w'_i) denote the common value of (13.10) (resp. (13.11)). For notational convenience define $w_{-1} = 0$ and $w'_{-1} = 0$.

Lemma 13.16. For $0 \leq i \leq N$,

$$\begin{aligned}\tau_i &= \exp_q(b^{-1}\xi\psi)w_i, & w_i &= \exp_{q^{-1}}(-b^{-1}\xi\psi)\tau_i, \\ \eta_i &= \exp_q(a^{-1}\xi\psi)w_i, & w_i &= \exp_{q^{-1}}(-a^{-1}\xi\psi)\eta_i\end{aligned}$$

and

$$\begin{aligned}\tau_i &= \exp_{q^{-1}}(-a^{-1}\xi\psi)w'_i, & w'_i &= \exp_q(a^{-1}\xi\psi)\tau_i, \\ \eta_i &= \exp_{q^{-1}}(-b^{-1}\xi\psi)w'_i, & w'_i &= \exp_q(b^{-1}\xi\psi)\eta_i.\end{aligned}$$

In the above lines $\xi = 1 - ab$.

Proof. By Definition 13.15 and the comments above Lemma 13.12. ■

Note 13.17. Referring to Definition 13.15, the polynomials $\{w'_i\}_{i=0}^N$ are obtained from the polynomials $\{w_i\}_{i=0}^N$ by replacing $q \mapsto q^{-1}$, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$.

Example 13.18. The following (i)–(iii) hold:

- (i) $w_0 = 1$;
- (ii) w_1 is equal to each of

$$\tau_1 - (1 - ab)b^{-1}\tau_0, \quad \eta_1 - (1 - ab)a^{-1}\eta_0, \quad x - a^{-1} - b^{-1};$$

- (iii) w_2 is equal to each of

$$\begin{aligned}\tau_2 &- (q^{-1} + 1)(1 - abq)b^{-1}\tau_1 + q^{-1}(1 - ab)(1 - abq)b^{-2}\tau_0, \\ \eta_2 &- (q^{-1} + 1)(1 - abq)a^{-1}\eta_1 + q^{-1}(1 - ab)(1 - abq)a^{-2}\eta_0, \\ &(x - a^{-1} - b^{-1})(x - q^{-1}a^{-1} - q^{-1}b^{-1}) + (q^{-1} - 1)(1 - a^{-1}b^{-1}).\end{aligned}$$

To get w'_0, w'_1, w'_2 replace $q \mapsto q^{-1}$, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ in (i)–(iii) above.

Lemma 13.19. The following (i)–(iii) hold:

- (i) for $0 \leq i \leq N$ the polynomials w_i, w'_i are monic with degree i ;
- (ii) for $0 \leq n \leq N$, each of $\{w_i\}_{i=0}^n, \{w'_i\}_{i=0}^n$ is a basis for the vector space V_n ;
- (iii) each of $\{w_i\}_{i=0}^N, \{w'_i\}_{i=0}^N$ is a basis for the vector space V .

Proof. (i) By (4.9) and Definition 13.15. (ii), (iii) By (i) above. ■

Lemma 13.20. For $0 \leq j \leq N$,

$$\begin{aligned}w'_j &= a^{-j}(ab; q)_j \sum_{i=0}^j \frac{(q^{-j}; q)_i (ay; q)_i (ay^{-1}; q)_i q^i}{(ab; q)_i (q; q)_i} \\ &= a^{-j}(ab; q)_j {}_3\phi_2 \left(\begin{matrix} q^{-j}, ay, ay^{-1} \\ ab, 0 \end{matrix} \middle| q; q \right),\end{aligned}$$

where $x = y + y^{-1}$. To get w_j from w'_j , replace $q \mapsto q^{-1}$, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$.

Proof. In the equation $w'_j = \exp_q(a^{-1}\xi\psi)\tau_j$, expand the q -exponential using (13.5), and evaluate the result using the equation on the left in (4.9). This yields $w'_j = \sum_{i=0}^j \alpha_i \tau_i$ where for $0 \leq i \leq j$,

$$\alpha_i = \frac{q^{\binom{j-i}{2}}(1-q)^{j-i} a^{i-j} \xi^{j-i} \vartheta_j \vartheta_{j-1} \cdots \vartheta_{i+1}}{(q; q)_{j-i}}.$$

Evaluating this using

$$\vartheta_j \vartheta_{j-1} \cdots \vartheta_{i+1} = \frac{\vartheta_1 \vartheta_2 \cdots \vartheta_j}{\vartheta_1 \vartheta_2 \cdots \vartheta_i}$$

and (13.2), (13.3) we obtain

$$\alpha_i = \frac{(-1)^i a^{i-j} (ab; q)_j (q^{-j}; q)_i q^{\binom{i}{2}} q^i}{(ab; q)_i (q; q)_i}.$$

The polynomial τ_i is given in Lemma 13.10(i). The result follows. \blacksquare

Note 13.21. The polynomials $\{w_i\}_{i=0}^N$ and $\{w'_i\}_{i=0}^N$ are in the Al-Salam/Chihara family [25, Section 14.8] if $N = \infty$, and the dual q -Krawtchouk family [25, Section 14.17] if $N \neq \infty$. The Al-Salam/Chihara and dual q -Krawtchouk polynomials satisfy a 3-term recurrence; the details will be given in Lemmas 13.67 and 13.68 below.

Going forward we focus on $\{w_i\}_{i=0}^N$; similar results hold for $\{w'_i\}_{i=0}^N$.

Lemma 13.22. *We have*

$$\psi w_i = \vartheta_i w_{i-1}, \quad 0 \leq i \leq N.$$

Proof. By Definition 13.15 and since $\psi \tau_i = \vartheta_i \tau_{i-1}$. \blacksquare

Our next general goal is to describe in more detail how the bases $\{\tau_i\}_{i=0}^N$, $\{\eta_i\}_{i=0}^N$, $\{w_i\}_{i=0}^N$ are related. To this end, we introduce some maps $K, B, M \in \text{End}(V)$.

Definition 13.23. Define $K, B, M \in \text{End}(V)$ such that for $0 \leq i \leq N$,

$$K\tau_i = q^{-i}\tau_i, \quad B\eta_i = q^{-i}\eta_i, \quad Mw_i = q^{-i}w_i.$$

Each of K, B, M is invertible.

Lemma 13.24. *The following (i)–(iii) hold:*

- (i) $K\psi = q\psi K$;
- (ii) $B\psi = q\psi B$;
- (iii) $M\psi = q\psi M$.

Proof. (i) The vectors $\{\tau_i\}_{i=0}^N$ form a basis for V . For $0 \leq i \leq N$,

$$K\psi\tau_i = \vartheta_i K\tau_{i-1} = \vartheta_i q^{1-i}\tau_{i-1}, \quad q\psi K\tau_i = q^{1-i}\psi\tau_i = \vartheta_i q^{1-i}\tau_{i-1}.$$

Therefore $K\psi = q\psi K$. (ii), (iii) Similar to the proof of (i) above. \blacksquare

Lemma 13.25. *The following (i)–(iii) hold:*

- (i) $B\Delta = \Delta K$;

- (ii) $K \exp_q(b^{-1}\xi\psi) = \exp_q(b^{-1}\xi\psi)M$;
 (iii) $B \exp_q(a^{-1}\xi\psi) = \exp_q(a^{-1}\xi\psi)M$.

Proof. (i) For $0 \leq i \leq N$, apply each side to τ_i and use (2.5) along with Definition 13.23.

(ii), (iii) For $0 \leq i \leq N$, apply each side to w_i and use Lemma 13.16 along with Definition 13.23. ■

Proposition 13.26. *The following (i)–(iv) hold:*

- (i) $KM^{-1} = I + (q-1)(a-b^{-1})\psi$;
 (ii) $M^{-1}K = I + (q^{-1}-1)(b^{-1}-a)\psi$;
 (iii) $BM^{-1} = I + (q-1)(b-a^{-1})\psi$;
 (iv) $M^{-1}B = I + (q^{-1}-1)(a^{-1}-b)\psi$.

Proof. (i) The map $T = b^{-1}\xi\psi$ is locally nilpotent. We have $KTK^{-1} = qT$ by Lemma 13.24(i), and $K \exp_q(T) = \exp_q(T)M$ by Lemma 13.25(ii). By these comments and Lemma 13.12,

$$\begin{aligned} \exp_q(T) &= (I - (q-1)T) \exp_q(qT) = (I - (q-1)T) \exp_q(KTK^{-1}) \\ &= (I - (q-1)T)K \exp_q(T)K^{-1} = (I - (q-1)T) \exp_q(T)MK^{-1} \\ &= \exp_q(T)(I - (q-1)T)MK^{-1}. \end{aligned}$$

By this and since $\exp_q(T)$ is invertible,

$$I = (I - (q-1)T)MK^{-1}.$$

The result follows from this and $\xi = 1 - ab$.

(ii) By (i) above and $M\psi = q\psi M$. (iii), (iv) Similar to the proof of (i), (ii) above. ■

Corollary 13.27. *We have*

$$\psi = \frac{bq}{1-ab} \frac{M^{-1}K - KM^{-1}}{(q-1)^2}, \quad (13.14)$$

$$\psi = \frac{aq}{1-ab} \frac{M^{-1}B - BM^{-1}}{(q-1)^2}. \quad (13.15)$$

Proof. To get (13.14) use Proposition 13.26(i), (ii).

To get (13.15) use Proposition 13.26(iii), (iv). ■

Proposition 13.28. *We have*

$$M = \frac{bK - aB}{b-a}.$$

Proof. Compute b times Proposition 13.26(i) minus a times Proposition 13.26(iii). ■

Lemma 13.29. *Each of the following is invertible:*

$$aI - bB^{-1}K, \quad bI - aK^{-1}B, \quad a^{-1}I - b^{-1}BK^{-1}, \quad b^{-1}I - a^{-1}KB^{-1}.$$

Proof. By Proposition 13.28 and since M is invertible. ■

Our next goal is to describe how K , B are related. We will use the following result.

Lemma 13.30. *We have*

$$\psi M = \frac{1}{1-q} \frac{ab}{1-ab} \frac{K-B}{a-b}, \quad (13.16)$$

$$M\psi = \frac{q}{1-q} \frac{ab}{1-ab} \frac{K-B}{a-b}. \quad (13.17)$$

Proof. To get (13.16), subtract Proposition 13.26(iii) from Proposition 13.26(i). To get (13.17) from (13.16), use $M\psi = q\psi M$. ■

Proposition 13.31. *We have*

$$(bK - aB)(K - B) = q(K - B)(bK - aB).$$

Proof. We have $M\psi = q\psi M$ so $M(\psi M) = q(\psi M)M$. Evaluate this using Proposition 13.28 and (13.16). ■

We mention some reformulations of Proposition 13.31.

Corollary 13.32. *We have*

$$0 = aB^2 - \frac{bq-a}{q-1}BK - \frac{aq-b}{q-1}KB + bK^2$$

and

$$\begin{aligned} (bI - aK^{-1}B)(I - KB^{-1}) &= q(I - K^{-1}B)(aI - bKB^{-1}), \\ (aI - bB^{-1}K)(I - KB^{-1}) &= q(I - B^{-1}K)(aI - bKB^{-1}), \\ (aI - bB^{-1}K)(I - BK^{-1}) &= q(I - B^{-1}K)(bI - aBK^{-1}), \\ (bI - aK^{-1}B)(I - BK^{-1}) &= q(I - K^{-1}B)(bI - aBK^{-1}). \end{aligned}$$

Proposition 13.33. *We have*

$$KB^{-1} = \frac{I + (q-1)(a-b^{-1})\psi}{I + (q-1)(b-a^{-1})\psi}, \quad (13.18)$$

$$BK^{-1} = \frac{I + (q-1)(b-a^{-1})\psi}{I + (q-1)(a-b^{-1})\psi}, \quad (13.19)$$

$$K^{-1}B = \frac{I + (q^{-1}-1)(a^{-1}-b)\psi}{I + (q^{-1}-1)(b^{-1}-a)\psi}, \quad (13.20)$$

$$B^{-1}K = \frac{I + (q^{-1}-1)(b^{-1}-a)\psi}{I + (q^{-1}-1)(a^{-1}-b)\psi}. \quad (13.21)$$

In the above fractions the denominator is invertible since ψ is locally nilpotent.

Proof. To get (13.18), equate the two expressions for M^{-1} obtained from Proposition 13.26(i) and (iii). To get (13.19) from (13.18), compute the inverse of each side. To get (13.20), equate the two expressions for M^{-1} obtained from Proposition 13.26(ii) and (iv). To get (13.21) from (13.20), compute the inverse of each side. ■

Lemma 13.34. *The following mutually commute:*

$$\psi, \quad KB^{-1}, \quad BK^{-1}, \quad K^{-1}B, \quad B^{-1}K.$$

Proof. By Proposition 13.33. ■

Proposition 13.35. *We have*

$$\begin{aligned}\psi &= \frac{1}{q-1} \frac{1}{1-ab} \frac{I - KB^{-1}}{b^{-1}I - a^{-1}KB^{-1}}, \\ \psi &= \frac{1}{q-1} \frac{1}{1-ab} \frac{I - BK^{-1}}{a^{-1}I - b^{-1}BK^{-1}}, \\ \psi &= \frac{1}{q^{-1}-1} \frac{1}{1-a^{-1}b^{-1}} \frac{I - K^{-1}B}{bI - aK^{-1}B}, \\ \psi &= \frac{1}{q^{-1}-1} \frac{1}{1-a^{-1}b^{-1}} \frac{I - B^{-1}K}{aI - bB^{-1}K}.\end{aligned}$$

In the above fractions the denominator is invertible by Lemma 13.29.

Proof. In Proposition 13.33 solve for ψ . ■

Proposition 13.36. *We have*

$$\frac{qM^{-1}K - KM^{-1}}{q-1} = I, \tag{13.22}$$

$$\frac{qM^{-1}B - BM^{-1}}{q-1} = I. \tag{13.23}$$

Proof. To get (13.22) use Proposition 13.26(i) and (ii). To get (13.23) use Proposition 13.26(iii) and (iv). ■

We have a comment.

Lemma 13.37. *The relations in Proposition 13.31 and Corollary 13.32 still hold if we replace*

$$q \mapsto q^{-1}, \quad a \mapsto a^{-1}, \quad b \mapsto b^{-1}, \quad K \mapsto K^{-1}, \quad B \mapsto B^{-1}.$$

Proof. Use Proposition 13.31 and Lemma 13.34. ■

We recall some notation. Let $\text{Mat}_{N+1}(\mathbb{F})$ denote the set of $N+1$ by $N+1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, N$. Let $\{v_i\}_{i=0}^N$ denote a basis for V . For $M \in \text{Mat}_{N+1}(\mathbb{F})$ and $T \in \text{End}(V)$, we say that M represents T with respect to $\{v_i\}_{i=0}^N$ whenever $Tv_j = \sum_{i=0}^N M_{i,j}v_i$ for $0 \leq j \leq N$.

Our next goal is to display the matrices that represent ψ , $K^{\pm 1}$, $M^{\pm 1}$, $B^{\pm 1}$ with respect to the bases $\{\tau_i\}_{i=0}^N$, $\{w_i\}_{i=0}^N$, $\{\eta_i\}_{i=0}^N$ of V .

Definition 13.38. Let $\widehat{\psi}$ denote the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that has $(i-1, i)$ -entry ϑ_i for $1 \leq i \leq N$, and all other entries 0. Thus

$$\widehat{\psi} = \begin{pmatrix} 0 & \vartheta_1 & & & \mathbf{0} \\ & 0 & \vartheta_2 & & \\ & & 0 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \vartheta_N \\ \mathbf{0} & & & & & 0 \end{pmatrix}.$$

Lemma 13.39. *The matrix $\widehat{\psi}$ represents ψ with respect to $\{\tau_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ and $\{\eta_i\}_{i=0}^N$.*

Proof. By (4.9) and Lemma 13.22. ■

Lemma 13.40. *The matrix $\text{diag}(1, q^{-1}, q^{-2}, \dots, q^{-N})$ represents K (resp. M) (resp. B) with respect to $\{\tau_i\}_{i=0}^N$ (resp. $\{w_i\}_{i=0}^N$) (resp. $\{\eta_i\}_{i=0}^N$).*

Proof. By Definition 13.23. ■

Lemma 13.41. *We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents K with respect to $\{w_i\}_{i=0}^N$. The (i, i) -entry is q^{-i} for $0 \leq i \leq N$. The $(i-1, i)$ -entry is $(1 - q^{-i})(a - b^{-1}q^{1-i})$ for $1 \leq i \leq N$. All other entries are 0.*

Proof. Use $KM^{-1} = I + (q-1)(a - b^{-1})\psi$ and Lemmas 13.6(ii), 13.39, 13.40. ■

Lemma 13.42. *We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents B with respect to $\{w_i\}_{i=0}^N$. The (i, i) -entry is q^{-i} for $0 \leq i \leq N$. The $(i-1, i)$ -entry is $(1 - q^{-i})(b - a^{-1}q^{1-i})$ for $1 \leq i \leq N$. All other entries are 0.*

Proof. Use $BM^{-1} = I + (q-1)(b - a^{-1})\psi$ and Lemmas 13.6(ii), 13.39, 13.40. ■

Lemma 13.43. *We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents M^{-1} with respect to $\{\tau_i\}_{i=0}^N$. The (i, i) -entry is q^i for $0 \leq i \leq N$. The $(i-1, i)$ -entry is $(q^i - 1)(aq^{i-1} - b^{-1})$ for $1 \leq i \leq N$. All other entries are 0.*

Proof. Use $M^{-1}K = I + (q^{-1} - 1)(b^{-1} - a)\psi$ and Lemmas 13.6(i), 13.39, 13.40. ■

Lemma 13.44. *We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents M^{-1} with respect to $\{\eta_i\}_{i=0}^N$. The (i, i) -entry is q^i for $0 \leq i \leq N$. The $(i-1, i)$ -entry is $(q^i - 1)(bq^{i-1} - a^{-1})$ for $1 \leq i \leq N$. All other entries are 0.*

Proof. Use $M^{-1}B = I + (q^{-1} - 1)(a^{-1} - b)\psi$ and Lemmas 13.6(i), 13.39, 13.40. ■

We give a variation on Proposition 13.26.

Lemma 13.45. *We have*

$$MK^{-1} = \sum_{i=0}^N (-1)^i (1-q)^i (1-ab)^i b^{-i} \psi^i, \quad (13.24)$$

$$K^{-1}M = \sum_{i=0}^N (-1)^i (1-q)^i q^{-i} (1-ab)^i b^{-i} \psi^i, \quad (13.25)$$

$$MB^{-1} = \sum_{i=0}^N (-1)^i (1-q)^i (1-ab)^i a^{-i} \psi^i, \quad (13.26)$$

$$B^{-1}M = \sum_{i=0}^N (-1)^i (1-q)^i q^{-i} (1-ab)^i a^{-i} \psi^i. \quad (13.27)$$

Proof. For each equation in Proposition 13.26, take the inverse of each side and evaluate the result using Lemma 2.14. ■

Lemma 13.46. *We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents M with respect to $\{\tau_i\}_{i=0}^N$. The (i, j) -entry is*

$$\frac{(-1)^{j-i} b^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} - j}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.24) and Lemmas 13.39, 13.40. ■

Lemma 13.47. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents K^{-1} with respect to $\{w_i\}_{i=0}^N$. The (i, j) -entry is

$$\frac{(-1)^{j-i} b^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} + i}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.25) and Lemmas 13.39, 13.40. ■

Lemma 13.48. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents M with respect to $\{\eta_i\}_{i=0}^N$. The (i, j) -entry is

$$\frac{(-1)^{j-i} a^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} - j}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.26) and Lemmas 13.39, 13.40. ■

Lemma 13.49. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents B^{-1} with respect to $\{w_i\}_{i=0}^N$. The (i, j) -entry is

$$\frac{(-1)^{j-i} a^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} + i}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.27) and Lemmas 13.39, 13.40. ■

We give a variation on Proposition 13.33.

Lemma 13.50. We have

$$KB^{-1} = I + \frac{b-a}{b} \sum_{i=1}^N (-1)^i (1-q)^i (1-ab)^i a^{-i} \psi^i, \quad (13.28)$$

$$BK^{-1} = I + \frac{a-b}{a} \sum_{i=1}^N (-1)^i (1-q)^i (1-ab)^i b^{-i} \psi^i, \quad (13.29)$$

$$K^{-1}B = I + \frac{a-b}{a} \sum_{i=1}^N (-1)^i (1-q)^i q^{-i} (1-ab)^i b^{-i} \psi^i, \quad (13.30)$$

$$B^{-1}K = I + \frac{b-a}{b} \sum_{i=1}^N (-1)^i (1-q)^i q^{-i} (1-ab)^i a^{-i} \psi^i. \quad (13.31)$$

Proof. Evaluate each equation in Proposition 13.33 using Lemma 2.14. ■

Lemma 13.51. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents K with respect to $\{\eta_i\}_{i=0}^N$. The (i, i) -entry is q^{-i} for $0 \leq i \leq N$. The (i, j) -entry is

$$\frac{b-a}{b} \frac{(-1)^{j-i} a^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} - j}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i < j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.28) and Lemmas 13.39, 13.40. ■

Lemma 13.52. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents B with respect to $\{\tau_i\}_{i=0}^N$. The (i, i) -entry is q^{-i} for $0 \leq i \leq N$. The (i, j) -entry is

$$\frac{a-b}{a} \frac{(-1)^{j-i} b^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} - j}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i < j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.29) and Lemmas 13.39, 13.40. ■

Lemma 13.53. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents K^{-1} with respect to $\{\eta_i\}_{i=0}^N$. The (i, i) -entry is q^i for $0 \leq i \leq N$. The (i, j) -entry is

$$\frac{a-b}{a} \frac{(-1)^{j-i} b^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} + i}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i < j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.30) and Lemmas 13.39, 13.40. ■

Lemma 13.54. We give the matrix in $\text{Mat}_{N+1}(\mathbb{F})$ that represents B^{-1} with respect to $\{\tau_i\}_{i=0}^N$. The (i, i) -entry is q^i for $0 \leq i \leq N$. The (i, j) -entry is

$$\frac{b-a}{b} \frac{(-1)^{j-i} a^{i-j} (ab; q)_j (q; q)_j q^{\binom{i}{2} - \binom{j}{2} + i}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i < j \leq N$. All other entries are 0.

Proof. Use (13.3), (13.31) and Lemmas 13.39, 13.40. ■

We recall some notation. Let $\{u_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ denote bases for V . By the *transition matrix* from $\{u_i\}_{i=0}^N$ to $\{v_i\}_{i=0}^N$ we mean the matrix $T \in \text{Mat}_{N+1}(\mathbb{F})$ such that $v_j = \sum_{i=0}^N T_{i,j} u_i$ for $0 \leq j \leq N$.

Our next goal is to display the transition matrices between the bases $\{\tau_i\}_{i=0}^N$, $\{w_i\}_{i=0}^N$, $\{\eta_i\}_{i=0}^N$. Recall the notation $\xi = 1 - ab$.

Consider the following matrices:

$$\exp_q(a^{-1} \xi \widehat{\psi}), \quad \exp_q(b^{-1} \xi \widehat{\psi}). \tag{13.32}$$

Their inverses are

$$\exp_{q^{-1}}(-a^{-1} \xi \widehat{\psi}), \quad \exp_{q^{-1}}(-b^{-1} \xi \widehat{\psi}). \tag{13.33}$$

The matrices (13.32), (13.33) are upper triangular. We now give their entries.

Lemma 13.55. For $0 \neq z \in \mathbb{F}$ the matrix $\exp_q(z \xi \widehat{\psi})$ is upper triangular. Its (i, j) -entry is

$$\frac{(-1)^i z^{j-i} (ab; q)_j (q^{-j}; q)_i q^{i + \binom{i}{2}}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$. The matrix $\exp_{q^{-1}}(-z \xi \widehat{\psi})$ is upper triangular. Its (i, j) -entry is

$$\frac{(-1)^j z^{j-i} (ab; q)_j (q^{-j}; q)_i q^{i + \binom{j}{2}}}{(ab; q)_i (q; q)_i}$$

for $0 \leq i \leq j \leq N$.

Proof. Use (13.2), (13.3), (13.5), (13.6) and Definition 13.38. \blacksquare

Lemma 13.56. *The transition matrices between the basis $\{w_i\}_{i=0}^N$ and the bases $\{\tau_i\}_{i=0}^N$, $\{\eta_i\}_{i=0}^N$ are given in the table below:*

from	to	transition matrix
$\{\tau_i\}_{i=0}^N$	$\{w_i\}_{i=0}^N$	$\exp_{q^{-1}}(-b^{-1}\xi\widehat{\psi})$
$\{w_i\}_{i=0}^N$	$\{\tau_i\}_{i=0}^N$	$\exp_q(b^{-1}\xi\widehat{\psi})$
$\{\eta_i\}_{i=0}^N$	$\{w_i\}_{i=0}^N$	$\exp_{q^{-1}}(-a^{-1}\xi\widehat{\psi})$
$\{w_i\}_{i=0}^N$	$\{\eta_i\}_{i=0}^N$	$\exp_q(a^{-1}\xi\widehat{\psi})$

Proof. By Lemmas 13.16, 13.39. \blacksquare

Next we consider the product

$$\exp_q(a^{-1}\xi\widehat{\psi}) \exp_{q^{-1}}(-b^{-1}\xi\widehat{\psi}). \quad (13.34)$$

The inverse of (13.34) is

$$\exp_q(b^{-1}\xi\widehat{\psi}) \exp_{q^{-1}}(-a^{-1}\xi\widehat{\psi}). \quad (13.35)$$

The matrices (13.34), (13.35) are upper triangular. Shortly we will give their entries.

Lemma 13.57. *The matrix (13.34) is the transition matrix from the basis $\{\tau_i\}_{i=0}^N$ to the basis $\{\eta_i\}_{i=0}^N$. The matrix (13.35) is the transition matrix from the basis $\{\eta_i\}_{i=0}^N$ to the basis $\{\tau_i\}_{i=0}^N$.*

Proof. By Lemma 13.56. \blacksquare

Lemma 13.58. *The matrix (13.34) represents Δ with respect to $\{\tau_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ and $\{\eta_i\}_{i=0}^N$. The matrix (13.35) represents Δ^{-1} with respect to $\{\tau_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ and $\{\eta_i\}_{i=0}^N$.*

Proof. By Proposition 13.13 and Lemma 13.39. \blacksquare

Lemma 13.59. *The matrix (13.34) is upper triangular, with (i, j) -entry*

$$\eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_{\vartheta} \quad (13.36)$$

for $0 \leq i \leq j \leq N$. The matrix (13.35) is upper triangular, with (i, j) -entry

$$\tau_{j-i}(b_0) \begin{bmatrix} j \\ i \end{bmatrix}_{\vartheta} \quad (13.37)$$

for $0 \leq i \leq j \leq N$. To express (13.36), (13.37) in terms of a , b , q use Lemmas 13.9, 13.11.

Proof. By Lemma 13.57, the matrix (13.34) is the transition matrix from the basis $\{\tau_i\}_{i=0}^N$ to the basis $\{\eta_i\}_{i=0}^N$. The entries of this matrix are obtained from Proposition 5.2(ii). The entries of the matrix (13.35) are similarly obtained. \blacksquare

Our next goal is to show how the map A from Definition 2.11 is related to ψ , K , B , M .

Lemma 13.60. *On V_{N-1} ,*

$$\frac{qKA - AK}{q-1} = a^{-1}K^2 + aI, \quad (13.38)$$

$$\frac{qBA - AB}{q-1} = b^{-1}B^2 + bI. \quad (13.39)$$

Proof. We first show (13.38). For $0 \leq i \leq N-1$ apply each side of (13.38) to τ_i , and evaluate the result using Lemma 2.13 along with (13.1) and Definition 13.23. We have

$$\begin{aligned} KA\tau_i &= K(a_i\tau_i + \tau_{i+1}) = q^{-i}a_i\tau_i + q^{-i-1}\tau_{i+1}, \\ AK\tau_i &= q^{-i}A\tau_i = q^{-i}(a_i\tau_i + \tau_{i+1}), \\ (a^{-1}K^2 + aI)\tau_i &= (a^{-1}q^{-2i} + a)\tau_i = q^{-i}a_i\tau_i. \end{aligned}$$

By these comments we obtain (13.38). Equation (13.39) is similarly obtained. \blacksquare

Lemma 13.61. On V_{N-1} ,

$$q\psi A - A\psi = \frac{(q+1)abM^{-1} - (q+ab)I}{ab-1}. \quad (13.40)$$

Proof. For $0 \leq i \leq N-1$ apply each side of (13.40) to τ_i . Using Lemma 2.13 and (4.9),

$$\begin{aligned} \psi A\tau_i &= \psi(a_i\tau_i + \tau_{i+1}) = a_i\vartheta_i\tau_{i-1} + \vartheta_{i+1}\tau_i, \\ A\psi\tau_i &= \vartheta_i A\tau_{i-1} = \vartheta_i(a_{i-1}\tau_{i-1} + \tau_i). \end{aligned}$$

Using Lemma 13.6(i) and Lemma 13.43,

$$M^{-1}\tau_i = q^i\tau_i + (q-1)(a-b^{-1})q^{i-1}\vartheta_i\tau_{i-1}.$$

By the above comments and Lemmas 13.4, 13.7 we get the result. \blacksquare

Lemma 13.62. On V_{N-1} ,

$$\frac{qAM^{-1} - M^{-1}A}{q-1} = (a^{-1} + b^{-1})I + (q - q^{-1})(1 - a^{-1}b^{-1})\psi. \quad (13.41)$$

Proof. For $0 \leq i \leq N-1$ apply each side of (13.41) to τ_i . Evaluate the result using (4.9) and Lemmas 2.13, 13.43 along with Lemmas 13.4, 13.7. \blacksquare

Proposition 13.63. On V_{N-2} ,

$$A^2\psi - (q + q^{-1})A\psi A + \psi A^2 + (q - q^{-1})^2\psi = \frac{(1-q)(1+q^{-1}ab)}{1-ab}A + \frac{(q-q^{-1})(a+b)}{1-ab}I.$$

Proof. Let X denote the expression on either side of (13.40). Compute

$$\frac{qAX - XA}{q-1}$$

and evaluate the result using Lemma 13.62. \blacksquare

Proposition 13.64. On V_{N-1} ,

$$\psi^2 A - (q + q^{-1})\psi A\psi + A\psi^2 = \frac{(1-q)(1+q^{-1}ab)}{1-ab}\psi.$$

Proof. Let X denote the expression on either side of (13.40). Compute $qX\psi - \psi X$ and evaluate the result using $qM^{-1}\psi = \psi M^{-1}$. \blacksquare

Proposition 13.65. On V_{N-2} ,

$$\begin{aligned} A^2M^{-1} - (q + q^{-1})AM^{-1}A + M^{-1}A^2 + (q - q^{-1})^2M^{-1} \\ = (q-1)(q - q^{-1})(q^{-1} + a^{-1}b^{-1})I - q^{-1}(q-1)^2(a^{-1} + b^{-1})A. \end{aligned}$$

Proof. Let Y denote the expression on either side of (13.41). Compute $qYA - AY$ and evaluate the result using Lemma 13.61. ■

Proposition 13.66. *On V_{N-1} ,*

$$M^{-2}A - (q + q^{-1})M^{-1}AM^{-1} + AM^{-2} = (q - 1)(q^{-1} - 1)(a^{-1} + b^{-1})M^{-1}.$$

Proof. Let Y denote the expression on either side of (13.41). Compute $qM^{-1}Y - YM^{-1}$ and evaluate the result using $qM^{-1}\psi = \psi M^{-1}$. ■

In Note 13.21 we mentioned a 3-term recurrence satisfied by the polynomials $\{w_i\}_{i=0}^N$ and $\{w'_i\}_{i=0}^N$. Our next goal is to describe this recurrence.

Lemma 13.67. *We have*

$$xw_i = w_{i+1} + q^{-i}(a^{-1} + b^{-1})w_i + (1 - q^{-i})(1 - q^{1-i}a^{-1}b^{-1})w_{i-1}$$

for $0 \leq i \leq N - 1$, where $w_0 = 1$ and $w_{-1} = 0$.

Proof. The result holds for $i = 0$, since $w_1 = x - a^{-1} - b^{-1}$ by Example 13.18(ii). Assume that $i \geq 1$. By Lemma 13.19(i) there exist scalars $\{\alpha_k\}_{k=0}^{i+1}$ in \mathbb{F} such that $xw_i = \sum_{k=0}^{i+1} \alpha_k w_k$ and $\alpha_{k+1} = 1$. To obtain $\{\alpha_k\}_{k=0}^i$, apply each side of (13.41) to w_i and evaluate the result using Lemma 13.22 along with $M^{-1}w_j = q^j w_j$ for $0 \leq j \leq i$. After a brief calculation this yields $\alpha_i = q^{-i}(a^{-1} + b^{-1})$ and $\alpha_{i-1} = (1 - q^{-i})(1 - q^{1-i}a^{-1}b^{-1})$ and $\alpha_k = 0$ for $0 \leq k \leq i - 2$. The result follows. ■

Lemma 13.68. *We have*

$$xw'_i = w'_{i+1} + q^i(a + b)w'_i + (1 - q^i)(1 - q^{i-1}ab)w'_{i-1}$$

$0 \leq i \leq N - 1$, where $w'_0 = 1$ and $w'_{-1} = 0$.

Proof. In Proposition 13.67 replace $q \mapsto q^{-1}$, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$ and use Note 13.17. ■

This completes our description of \mathcal{L} and Δ for the twin recurrent data from Case I of Lemma 10.2. In this description we encountered analogs of the results from Section 1 about the double lowering operator of a tridiagonal pair. This connection suggests that double lowering operators on polynomials can be used to further develop the theory of tridiagonal pairs; we hope to pursue this in the future.

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