Centralizers of Rank One in the First Weyl Algebra

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Abstract. Centralizers of rank one in the first Weyl algebra have genus zero.

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1 Introduction

Take an element a of the first Weyl algebra A_1. *Rank* of the centralizer $C(a)$ of this element is the greatest common divisor of the orders of elements in $C(a)$ (orders as differential operators).
   This note contains a proof of the following

Theorem 1.1. *If the centralizer $C(a)$ of an element $a \in A_1 \setminus K$, where A_1 is the first Weyl algebra defined over a field K of characteristic zero, has rank 1, then $C(a)$ can be embedded into a polynomial ring $K[z]$.*

   The classical works of Burchnall and Chaundy where the systematic research of commuting differential operators was initiated are also devoted primarily to the case of rank 1 but the coefficients of the operators considered by them are analytic functions. Burchnall and Chaundy treated only monic differential operators which doesn't restrict generality if the coefficients are analytic functions. Situation is completely different if the coefficients are polynomial.

2 First Weyl algebra A_1 and its skew field of fractions D_1

Before we proceed with a proof, here is a short refresher on the first Weyl algebra.

Definition 2.1. The first Weyl algebra A_1 is an algebra over a field K generated by two elements (denoted here by x and ∂) which satisfy a relation $\partial x - x \partial = 1$.

   When characteristic of K is zero A_1 has a natural representation over the ring of polynomials $K[x]$ by operators of multiplication by x and the derivative ∂ relative to x. Hence the elements of the Weyl algebra can be thought of as differential operators with polynomial coefficients. They can be written as ordinary polynomials

$$a = \sum c_{i,j} x^i \partial^j, \qquad c_{i,j} \in K$$

with ordinary addition but a more complicated multiplication.
   Algebra A_1 is rather small, its Gelfand–Kirillov dimension is 2, hence it is a two-sided Ore ring. Because of that it can be embedded in a skew field D_1. A detailed discussion of skew fields

related to Weyl algebras and their skew fields of fraction, as well as a definition of Gelfand–Kirillov dimension can be found in a paper [11].

If characteristic of K is zero then the centralizer $C(a)$ of any element $a \in A_1 \setminus K$ is a commutative subalgebra of A_1 of the transcendence degree one. This theorem which was first proved by Issai Schur in 1904 (see [37]) and by Shimshon Amitsur by purely algebraic methods (see [1]) has somewhat entertaining history which is described in [17].

Definition 2.2. The rank of a centralizer is the greatest common divisors of the orders of elements of $C(a)$ considered as differential operators, i.e., of degrees of elements of $C(a)$ relative to ∂.

3   Leading forms of elements of A_1

Given $\rho, \sigma \in \mathbb{Z}$ it is possible to define a weight degree function w on A_1 by

$$w(x) = \rho, \qquad w(\partial) = \sigma, \qquad w\big(x^i \partial^j\big) = \rho i + \sigma j,$$
$$w(a) = \max w\big(x^i \partial^j\big) \mid c_{i,j} \neq 0 \qquad \text{for} \quad a = \sum c_{i,j} x^i \partial^j.$$

Definition 3.1. The leading form $\bar{a}$ of a is

$$\bar{a} = \sum c_{i,j} x^i \partial^j \mid w\big(x^i \partial^j\big) = w(a).$$

One of the nice properties of A_1 which was used by Dixmier in his seminal research of the first Weyl algebra (see [8], Lemma 2.7) is the following property of the leading forms of elements of A_1:

- if $\rho + \sigma > 0$ then $\overline{[a,b]} = \{\bar{a}, \bar{b}\}$ for $a, b \in A_1$, where $[a,b] = ab - ba$ and $\{\bar{a}, \bar{b}\} = \bar{a}_\partial \bar{b}_x - \bar{a}_x \bar{b}_\partial$ is the standard Poisson bracket of $\bar{a}$, $\bar{b}$ as commutative polynomials ($\bar{a}_\partial$ etc. are the corresponding partial derivatives), provided $\{\bar{a}, \bar{b}\} \neq 0$;

- if $\{\bar{a}, \bar{b}\} = 0$ and $w(a) \neq 0$ then $\bar{b}$ is proportional over K to a fractional power of $\bar{a}$.

The main ingredient of the considerations below is this property of the leading forms.

To make considerations clearer the reader may use the Newton polygons of elements of A_1. The Newton polygon of $a \in A_1$ is the convex hull of those points (i, j) on the plane for which $c_{i,j} \neq 0$. The Newton polygons of elements of A_1 are less sensible than the Newton polygons of polynomials in two variables because they depend on the way one chooses to record elements of A_1 but only those edges which are independent of the choice will be used.

4   Proof of the theorem

Case 1: $a = \partial^n + \sum\limits_{i=1}^{n} a_i(x) \partial^{n-i}$

If $a \in K[\partial]$ then $C(a) = K[\partial]$, a ring of polynomials in one variable. Otherwise consider the leading form α of a which contains ∂^n and is not a monomial. This form has a non-zero weight and corresponding ρ, σ satisfy conditions of the Dixmier's lemma (both ρ and σ are positive).

The leading forms of the elements from $C(a)$ are Poisson commutative with α since these elements commute with a. Therefore they are proportional over K to the fractional powers of α (as a commutative polynomial). Because the rank of $C(a)$ is 1 we should have $\alpha = c\big(\partial + c_1 x^k\big)^n$.

A mapping ϕ of A_1 to A_1 defined by

$$x \to x, \qquad \partial \to \partial - c_1 x^k$$

is an automorphism of A_1. It is easy to see that the Newton polygon of $\phi(a)$ belongs to the Newton polygon of a and has a smaller area.

Hence there exists an automorphism

$$\psi: \quad x \to x, \qquad \partial \to \partial + p(x)$$

such that $\psi(a) \in K[\partial]$. Therefore $C(a) = K\big[\psi^{-1}(\partial)\big]$, a polynomial ring in one variable.

Case 2: $a = x^m \partial^n + \displaystyle\sum_{i=1}^{n} a_i(x)\partial^{n-i}, \ m > 0$

As above, the leading forms of elements of $C(a)$ are proportional to the fractional powers of the leading form α of a (as a commutative polynomial) as long as α is the leading form of a relative to weights ρ, σ of x and ∂ provided $\rho + \sigma > 0$ and the weight of α is not zero. Because of that and since the rank is assumed to be one, n divides m.

The Newton polygon $\mathcal{N}(a)$ of a has the vertex (m, n). If we picture $\mathcal{N}(a)$ on a plane where the x axis is horizontal and the ∂ axis is vertical then (m, n) belongs to two edges: left an right. It is also possible that these edges coincide, or that $\mathcal{N}(a)$ consists just of the vertex (m, n).

If the left edge exists, i.e., is not just the vertex (m, n), then the ray with the vertex (m, n) containing this edge cannot intersect the y axis above the origin. Indeed, assume that this is the case and the point of intersection is $(0, \mu)$, where μ is a positive rational number, which must be smaller than n.

If we take weights $\rho = \mu - n$, $\sigma = m$ then $\rho + \sigma = \mu - n + m > 0$ since $m \geq n$ and we can apply the Dixmier's lemma to the corresponding leading form of a. But then

$$\bar{a} = \left(x^d \partial + cx^s\right)^n, \qquad \text{where} \quad s = \frac{mn}{\mu - n},$$

which is impossible since $s < 0$.

Therefore a has a non-trivial leading form of zero weight relative to the weights $\rho = -1$, $\sigma = d$, where $d = \frac{m}{n}$. This form can be the monomial $x^m \partial^n$, or a polynomial in $x^d \partial$.

Lemma 4.1. *If a has the leading form of weight zero relative to the weights*

$$\rho < 0 < \sigma, \qquad \rho + \sigma \geq 0$$

then $C(a)$ is a subring of a ring of polynomials in one variable. (Here the rank of $C(a)$ is not essential.)

Proof. Any $b \in C(a)$ has a non-zero leading form $\bar{b}$ of weight zero (relative to ρ, σ). Indeed, if $\rho + \sigma > 0$ and $w(b) \neq 0$ then $\{\bar{a}, \bar{b}\} \neq 0$ by the Dixmier's lemma.

If $\rho + \sigma = 0$ then $\bar{a} \in K[x\partial]$ and only elements of $K[x\partial]$ commute with it (see the remark below).

Hence the restriction map $b \to \bar{b}$ is an isomorphism. An algebra generated by all $\bar{b}$ is a subalgebra of $K[x^\sigma \partial^{-\rho}]$ if we assume that $\rho, \sigma \in \mathbb{Z}$ and relatively prime. $\blacksquare$

Remark 4.2. An equality

$$x^i \partial^i = (t - i + 1)(t - i + 2) \cdots (t - i + i),$$

where $t = x\partial$ is the Euler operator is easy to check since

$$xt = xx\partial = x(\partial x - 1) = tx - x = (t - 1)x$$

and thus $xp(t)\partial = p(t-1)t$ (see similar computations in [4]).

Case 3: $a = a_0(x)\partial^n + \sum_{i=1}^{n} a_i \partial^{n-i}$

In this case $a_0 = \alpha^n$, $\alpha \in K[x]$. We may assume that $\alpha \notin K$ and that $\alpha(0) = 0$ (applying an automorphism $x \to x + c$, $\partial \to \partial$ if necessary). We may also assume that the origin is a vertex of $\mathcal{N}(a)$ since we can replace a by $a + c$, where c is any element of K. Then $\mathcal{N}(a)$ has the horizontal edge with the right vertex (m, n) and the left vertex (m', n) where m' is divisible by n. As above, the edge with vertices (m', n) and $(0, 0)$ corresponds to the leading form of a of zero weight and Lemma 4.1 shows that $C(a)$ is isomorphic to a subring of $K[x^{d'}\partial]$, where $d' = \frac{m'}{n}$.

This finishes a proof of the theorem.

Since the proof of the theorem turned out to be too simple and too short we can complement it by an attempt to describe the rank one centralizers more precisely. In the first case it is already done, the centralizer is isomorphic to $K[z]$, where $z = \partial + p(x)$ for some $p(x) \in K[x]$.

It would be interesting to describe a for which $C(a)$ is not isomorphic to a polynomial ring. The second case described above provides us with examples of this phenomenon.

5   Centralizers in Case 2

Let us call (m, n) the leading vertex of $\mathcal{N}(a)$ and the edges containing this vertex the leading edges. If the extension of the right leading edge intersects the x axis in the point $(\nu, 0)$, where $\nu > m - n$ and we take $\rho = n$, $\sigma = \nu - m$ then $\rho + \sigma = n + \nu - m > 0$ and by the Dixmier's lemma the leading form $\bar{a}$ which corresponds to this weight is $\left(x^d \partial + cx^k\right)^n$, where $k \geq d$. Then, similar to the Case 1 we will make an automorphism

$$x \to x, \qquad \partial \to \partial - cx^{k-d},$$

which will collapse the right leading edge to the leading vertex.

After several steps like that we will obtain $\phi(a)$, where ϕ is an automorphism of A_1, such that the right leading edge of $\mathcal{N}(\phi(a))$ is parallel to the bisectrix of the first quadrant. For this edge $\rho + \sigma = 0$ and we cannot apply the Dixmier's lemma to the corresponding leading form.

Since $C(a)$ and $C(\phi(a))$ are isomorphic we will assume that the right leading edge of $\mathcal{N}(a)$ is parallel to the bisectrix.

If the left leading edge of $\mathcal{N}(a)$ is not parallel to the bisectrix we can consider the centralizer of a as a subalgebra of $K\left[\partial, x, x^{-1}\right]$ and proceed with automorphisms

$$x \to x, \qquad \partial \to \partial - c_1 x^{k-d},$$

where $k - d < -1$ since the Dixmier's lemma will be applicable to the corresponding leading forms.

Hence there exists an automorphism

$$x \to x, \qquad \partial \to \partial + q(x)$$

of $K[\partial, x, x^{-1}]$ such that $\psi(a) = x^{m-n}p(t)$, where $t = x\partial$ (here $q(x)$ is a Laurent polynomial while $p(t)$ is a polynomial).

We see that centralizers of elements with the leading vertex (m, n) are isomorphic to centralizers of elements $x^{m-n}p(t)$, $p(t) \in K[t]$, $\deg_t(p(t)) = n$. If $a = x^{m-n}p(t)$ and $m - n = 0$ then $C(a) = K[t]$. Assume now that $m > n$.

If $b \in C(a)$ then we can present b as the sum of forms, homogeneous relative to the weight w given by

$$w(x) = 1, \qquad w(\partial) = -1: \quad b = \sum_i x^i b_i(t).$$

Since

$$[a, b] = \sum_i \left[a, x^i b_i(t) \right] = \sum_i x^{m-n+i} \big(p(t-i)b_i(t) - b_i(t-m+n)p(t) \big) = 0$$

all $x^i b_i(t) \in C(a)$. Hence $C(a)$ is a linear span of elements, homogeneous relative to the weight w.

Lemma 5.1. *If $b \in C(a)$ is a w-homogeneous element then $b^\nu = ca^\mu$ for some relatively prime integers μ and ν, and $c \in K$.*

Proof. The leading vertex of b is $\lambda(m, n)$, where $\lambda = \frac{\mu}{\nu}$, $\mu, \nu \in \mathbb{Z}$, $(\mu, \nu) = 1$ since we can apply the Dixmier's lemma to the leading forms of a and b relative to the weight $w_1(x) = 1$, $w_1(\partial) = 1$. Therefore b^ν and a^μ have the same leading vertex $\mu(m, n)$ and $\deg_x(b^\nu - ca^\mu) < \deg_x(b^\nu)$ with the appropriate choice of $c \in K$. If $b_1 = b^\nu - ca^\mu \neq 0$ then b_1 is a homogeneous element of $C(a)$ and its leading vertex must be proportional to the leading vertex of a. This is impossible since $w(b_1) = \mu w(a) = \mu(m-n)$: if $\xi(m, n)$ is the leading vertex of b_1 then $w(b_1) = \xi(m-n) = \mu(m-n)$ and $\xi = \mu$. ∎

Since the rank of $C(a)$ is 1 we can find two elements

$$b_1 = x^{\beta_1} q_1(t), \qquad b_2 = x^{\beta_2} q_2(t) \qquad \text{such that} \quad \deg_t(b_2) = \deg_t(b_1) + 1.$$

Then $b = b_2 b_1^{-1}$ belongs to D_1 (the skew field of fractions of A_1), and commutes with a. Using a relation $xt = (t-1)x$ we can write that

$$b = x^{\beta_2} q_2(t) \big(x^{\beta_1} q_1(t) \big)^{-1} = x^{\beta_2} q_2(t) q_1(t)^{-1} x^{-\beta_1} = x^{\beta_2 - \beta_1} r(t),$$

where $r(t) \in K(t)$.

The leading vertex of b can be defined as the difference of the leading vertices of b_2 and b_1 and is proportional to the leading vertex of a. Since $\deg_t(r) = \deg_t(b_2) - \deg_t(b_1) = 1$ this vertex in coordinates x, t is $(d-1, 1)$. (Recall that $d = \frac{m}{n}$.)

If r is a polynomial then $C(a) = K\big[x^{d-1} r\big]$; if $d = 1$ then $C(a) = K[t]$; if r is not a polynomial and $d > 1$ then some powers of $x^{d-1} r$ are polynomials: say, $a = cb^n$, $c \in K$ because considerations of Lemma 5.1 are applicable to w homogeneous elements of D_1 commuting with a.

In the last case $r(t) \in K(t)$ but $r(t)r(t+d-1) \cdots r(t+(k-1)(d-1)) \in K[t]$ (observe that $tx = x(t+1)$). We can reduce this to $r(t)r(t+1) \cdots r(t+k-1) \in K[t]$ by rescaling t and r.

By shifting t if necessary we may assume that one of the roots of $r(t)$ is 0 and represent r as a product $r_0 r_1$, where all roots and poles of r_0 are in $\mathbb{Z}$ and all roots and poles of r_1 are not in $\mathbb{Z}$. It is clear that

$$r_0(t)r_0(t+1) \cdots r_0(t+k-1) \in K[t] \qquad \text{and} \qquad r_1(t)r_1(t+1) \cdots r_1(t+k-1) \in K[t].$$

Since $\deg(r) = 1$ and $\deg(r_i) \geq 0$ (because $k \deg(r_i) \geq 0$), degree of one of the r_i is equal to zero and $r_i(t)r_i(t+1) \cdots r_i(t+k-1) \in K$ for this r_i. But then $r_i(t) = r_i(t+k)$ which is impossible for a non-constant rational function. Since $r_0(0) = 0$ and $r \neq 0$ we see that r_1 is a constant and all roots and poles of r are in $\mathbb{Z}$.

We can assume now that 0 is the largest root of r and write

$$r = ts(t), \qquad \text{where} \quad s(t) = \frac{\prod_{i=1}^p (t + \lambda_i)}{\prod_{i=1}^p (t + \mu_i)} \in K(t) \setminus K,$$

$$\lambda_i \in \mathbb{Z}, \qquad \mu_i \in \mathbb{Z}, \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p, \qquad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_p.$$

If $\mu_1 < 0$ then $r(t)r(t+1) \cdots r(t+k-1)$ would have a pole at $t = -\mu_1$. Hence $\mu_1 > 0$ and all poles of $s(t)$ are negative integers while all zeros of $s(t)$ are non-positive integers.

A fraction $\frac{t+\lambda_i}{t+\mu_i}$ can be presented as $\frac{f_i(t)}{f_i(t+1)}$ if $\lambda_i < \mu_i$ or as $\frac{f_i(t+1)}{f_i(t)}$ if $\lambda_i > \mu_i$: indeed,

$$\frac{t+d}{t} = \frac{(t+1)(t+2)\cdots(t+d)}{t(t+1)\cdots(t+d-1)} \qquad \text{if} \quad d > 0,$$

take the reciprocal fraction if $d < 0$. Because of that $s(t)$ can be written as

$$\frac{s_1(t)s_2(t+1)}{s_1(t+1)s_2(t)}, \qquad s_i(t) \in K[t].$$

Write $s_1(t) = s_3(t)s_4(t)$, $s_2(t) = s_4(t)s_5(t)$, where $s_4(t)$ is the greatest common divisor of $s_1(t)$ and $s_2(t)$. Then

$$s(t) = \frac{s_3(t)s_4(t)s_4(t+1)s_5(t+1)}{s_3(t+1)s_4(t+1)s_4(t)s_5(t)} = \frac{s_3(t)s_5(t+1)}{s_3(t+1)s_5(t)}.$$

All roots of $s_3(t)$ must be not positive, otherwise the largest positive root of $s_3(t)$ would be a root of $s(t)$ (which doesn't have positive roots) since this root couldn't be canceled by a root of $s_5(t)$ or $s_3(t+1)$.

Now,

$$q(t) = r(t)r(t+1)\cdots r(t+k-1) = t(t+1)\cdots(t+k-1)\frac{s_3(t)s_5(t+k)}{s_3(t+k)s_5(t)}$$

is a polynomial. If $s_3(t) \notin K$ and its smallest root is i, where $i \leq 0$, then the denominator of $q(t)$ has a zero in $i-k$ which is less then $1-k$ and cannot be canceled by a zero in the numerator since $s_3(t+k)$ and $s_5(t+k)$ are relatively prime.

Hence

$$s_3(t) \in K, \qquad s(t) = \frac{s_5(t+1)}{s_5(t)}, \qquad \text{and} \qquad q(t) = t(t+1)\cdots(t+k-1)\frac{s_5(t+k)}{s_5(t)}.$$

We can uniquely write

$$s_5(t) = \prod_{i \in I} \phi_{k,p_i}(t+i), \qquad \text{where} \quad \phi_{k,p}(t) = \prod_{j=0}^{p}(t+jk),$$

and all p_i are maximal possible. Then

$$\frac{s_5(t+k)}{s_5(t)} = \prod_{i \in I}\frac{t+i+p_ik+k}{t+i} \qquad \text{and} \qquad t+i_1 \neq t+i_2+p_{i_2}k+k$$

for all $i_1, i_2 \in I$ because of the maximality of p_i. Hence $I \subset \{1, \ldots, k-1\}$ and each i is used at most once.

As we have seen, all roots of $s_5(t)$ are of multiplicity 1 and since

$$(xr)^N = x^N \prod_{i=0}^{N-1}(t+i)\frac{s_5(t+N)}{s_5(t)}$$

the elements $(xr)^N \in K[x]$ for sufficiently large N. Therefore the rank of $C(xr)$ is one. In fact, we see that if s_5 is any polynomial with only simple roots then the elements of A_1 which commute with $xt\frac{s_5(t+1)}{s_5(t)}$ form a centralizer of the rank one.

Observe that the rank is not stable under automorphisms: the rank of $C(\phi(xr))$, where $\phi(x) = x + t^M$, $\phi(t) = t$ is $M+1$.

Let us return now to $C(a)$, where $a = x^{m-n}p(t)$, $\deg_t(p(t)) = n$. In this case

$$b = x^{d-1}t\frac{s_5(t+d-1)}{s_5(t)}, \qquad b^n = x^{m-n}t(t+d-1)\cdots(t+(n-1)(d-1))\frac{s_5(t+n(d-1))}{s_5(t)}$$

and all roots of $s_5(t)$ belong to $\{1-d, 2(1-d), \ldots, (n-1)(1-d)\}$.

Additionally we can replace t by $t-c$, $c \in K$.

6   Cases 2 and 3

We understood the structure of $C(a)$ when $a = x^{m-n}p(t)$. Are there substantially different examples of centralizers of rank one which are not isomorphic to a polynomial ring?

Consider the case of an order 2 element commuting with an order 3 element which was completely researched in the work [3] of Burchnall and Chaundry for analytic coefficients.[1] They showed (and for this case it is a straightforward computation) that monic commuting operators of orders 2 and 3 can be reduced to

$$A = \partial^2 - 2\psi(x), \qquad B = \partial^3 - 3\psi(x)\partial - \frac{3}{2}\psi'(x),$$

where $\psi''' = 12\psi\psi'$, i.e., $\psi'' = 6\psi^2 + c_1$ and $(\psi')^2 = 4\psi^3 + c_1\psi + c_2$ (a Weierstrass function). The only rational (even algebraic) solution in this case is (up to a substitution) $\psi = x^{-2}$ when $c_1 = c_2 = 0$. (If ψ is a rational function then the curve parameterized by ψ, ψ' has genus zero, so $4\psi^3 + c_1\psi + c_2 = 4(\psi - \lambda)^2(\psi - \mu)$ and ψ is not an algebraic function of x if $\lambda \neq \mu$.) The corresponding operator

$$A = x^{-2}(t-2)(t+1) = \left(x^{-1}\frac{t^2-1}{t}\right)^2$$

is homogeneous.

In our case we have

$$A = f(x)^2\partial^2 + f_1(x)\partial + f_2(x)$$

since the leading form for $w(x) = 0$, $w(\partial) = 1$ must be the square of a polynomial.

Here are computations for this case

$$A = (f\partial)^2 - ff'\partial + f_1(x)\partial + f_2(x)$$
$$= \left[f\partial + \frac{1}{2}\left(\frac{f_1}{f} - f'\right)\right]^2 - \frac{1}{2}\left(\frac{f_1}{f} - f'\right)'f - \frac{1}{4}\left(\frac{f_1}{f} - f'\right)^2 + f_2.$$

Denote $f\partial + \frac{1}{2}\left(\frac{f_1}{f} - f'\right)$ by D. Then

$$A = D^2 - 2\phi(x), \qquad \text{where} \quad \phi = \frac{1}{4}\left(\frac{f_1}{f} - f'\right)'f + \frac{1}{8}\left(\frac{f_1}{f} - f'\right)^2 - \frac{1}{2}f_2 \in \mathbb{C}(x).$$

Analogously to Burchnall and Chaundry, if there is an operator of order 3 commuting with A then it can be written as

$$B = D^3 - 3\phi D - \frac{3}{2}\phi'f$$

(this follows from [3] but will be clear from the condition $[A, B] = 0$ as well). In order to find an equation for ϕ we should compute $[A, B]$. Observe that

$$[D, g(x)] = g'f, \qquad [D^2, g] = 2g'fD + (g'f)'f,$$
$$[D^3, g] = 3g'fD^2 + 3(g'f)'fD + ((g'f)'f)'f.$$

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[1]To be on a more familiar ground in this section the field K is the filed $\mathbb{C}$ of complex numbers.

Hence

$$
\begin{aligned}
[A, B] &= -3\left[D^2, \phi D + \frac{1}{2}\phi' f\right] + 2[D^3 - 3\phi D, \phi] \\
&= -3\left[(2\phi' f D + (\phi' f)' f)D + (\phi' f)' f D + \frac{1}{2}((\phi' f)' f)' f\right] \\
&\quad + 2[3\phi' f D^2 + 3(\phi' f)' f D + ((\phi' f)' f)' f] - 6\phi\phi' f \\
&= (-6\phi' f + 6\phi' f)D^2 + (-6(\phi' f)' f + 6(\phi' f)' f)D - \frac{3}{2}((\phi' f)' f)' f + 2((\phi' f)' f)' f - 6\phi\phi' f \\
&= \frac{1}{2}((\phi' f)' f)' f - 6\phi\phi' f.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
((\phi' f)' f)' &= 12\phi\phi', & (\phi' f)' f &= 6\phi^2 + c_1, \\
(\phi' f)' \phi' f &= 6\phi^2 \phi' + c_1 \phi', & (\phi' f)^2 &= 4\phi^3 + 2c_1 \phi + c_2
\end{aligned}
$$

and we have a parameterization of an elliptic curve. Since $f, \phi \in \mathbb{C}(x)$ this curve must have genus 0, i.e.,

$$
4\phi^3 + 2c_1\phi + c_2 = 4(\phi - \lambda)^2(\phi - \mu) \qquad \text{and} \qquad (\phi' f)^2 = 4(\phi - \lambda)^2(\phi - \mu).
$$

Take $z = \frac{\phi' f}{2(\phi - \lambda)}$. Then $\phi - \mu = z^2$ and $\phi' f = 2z(z^2 - \delta^2)$, where $\delta^2 = \lambda - \mu$. Hence $\phi' = 2zz'$, $2zz' f = 2z(z^2 - \delta^2)$ and $z' f = z^2 - \delta^2$.

Assume that $\delta \neq 0$. Since we can re-scale f and z as $f \to 2\delta f$, $z \to \delta z$, let us further assume that $\delta^2 = 1$. Then

$$
z' f = z^2 - 1 \qquad \text{and} \qquad \int \frac{\mathrm{d}z}{z^2 - 1} = \int \frac{\mathrm{d}x}{f}.
$$

Recall that $f \in \mathbb{C}[x]$. Since

$$
2\int \frac{\mathrm{d}z}{z^2 - 1} = \ln \frac{z - 1}{z + 1}
$$

all zeros of f have multiplicity 1 and

$$
\int \frac{\mathrm{d}x}{f} = \ln\left(\prod_i (x - \nu_i)^{c_i}\right),
$$

where $\{\nu_i\}$ are the roots of f and $c_i = (f'(\nu_i))^{-1}$. Therefore

$$
\frac{z - 1}{z + 1} = c\prod_i (x - \nu_i)^{2c_i}, \qquad \text{where} \quad c \neq 0 \quad \text{and} \quad z = \frac{1 + c\prod_i(x - \nu_i)^{2c_i}}{1 - c\prod_i(x - \nu_i)^{2c_i}}.
$$

Now it is time to recall that

$$
\phi = \frac{1}{4}\left(\frac{f_1}{f} - f'\right)' f + \frac{1}{8}\left(\frac{f_1}{f} - f'\right)^2 - \frac{1}{2}f_2,
$$

where $f, f_1, f_2 \in \mathbb{C}[x]$ and thus $f^2\phi = f^2(z^2 + \mu) \in \mathbb{C}[x]$. Because of that

$$
zf = c_1\frac{1 + c\prod_i(x - \nu_i)^{2c_i}}{1 - c\prod_i(x - \nu_i)^{2c_i}}\prod_i(x - \nu_i) \in \mathbb{C}[x],
$$

which is possible only if the rational function $1 - c\prod_i(x - \nu_i)^{2c_i}$ doesn't have zeros.

We can write $\prod_i (x - \nu_i)^{2c_i}$ as $\frac{\prod_j (x-\nu_j)^{2c_j}}{\prod_k (x-\nu_k)^{2c_k}}$, where $2c_j, 2c_k \in \mathbb{Z}^+$. Then

$$\prod_k (x - \nu_k)^{2c_k} - c \prod_j (x - \nu_j)^{2c_j} \in \mathbb{C},$$

which is possible only if $c = 1$.

Since

$$z = \frac{\prod_k (x - \nu_k)^{2c_k} + \prod_j (x - \nu_j)^{2c_j}}{\prod_k (x - \nu_k)^{2c_k} - \prod_j (x - \nu_j)^{2c_j}}$$

we see that $z \in \mathbb{C}[x]$. So to produce a $2, 3$ commuting pair we should find a polynomial solution to $f = \frac{z^2-1}{z'}$. If f, z are given then

$$A = (f\partial + \psi)^2 - 2(z^2 - \mu), \qquad B = (f\partial + \psi)^3 - 3(z^2 - \mu)(f\partial + \psi) - 3zz'f$$

is a commuting pair for any $\psi \in \mathbb{C}[x]$ (indeed, $f\psi \in \mathbb{C}[x]$ and $\psi^2 + f\psi' \in \mathbb{C}[x]$, hence $\psi \in \mathbb{C}[x]$). Constant $\mu = -\frac{2}{3}$ since we assumed that $\lambda - \mu = 1$ and $2\lambda + \mu = 0$ because the equation is $(\phi'f)^2 = 4\phi^3 + 2c_1\phi + c_2$.

Here is a series of examples:

$$z = 1 + x^n, \qquad f = \frac{x}{n}(2 + x^n), \qquad \phi = (1 + x^n)^2 - \frac{2}{3} = x^n(2 + x^n) + \frac{1}{3},$$

which correspond to

$$A = \left[\frac{x}{n}(2 + x^n)\partial + \psi\right]^2 - 2\left(x^n(2 + x^n) + \frac{1}{3}\right).$$

Even the simplest one,

$$A = [x(2 + x)\partial]^2 - 2\left[x(2 + x) + \frac{1}{3}\right]$$

cannot be made homogeneous.

It seems that a complete classification of $(2, 3)$ pairs is a daunting task. Our condition on z is that z assumes values ± 1 when $z' = 0$. Let us call such a polynomial admissible. We can look only at reduced monic polynomials $z(x) = x^n + a_2 x^{n-2} + \cdots$ because a substitution $x \to ax + b$ preserves admissibility. Also $\lambda^n z(\lambda^{-1}x)$ preserves admissibility if $\deg(z) = n$ and $\lambda^n = 1$.

Examples above are just one value case. Say, an admissible cubic polynomial is $x^3 - 3 \cdot 2^{-\frac{2}{3}} x$. If $z = (x - \nu)^i (x + \nu)^j + 1$ then it is admissible when

$$\nu^{i+j} = (-1)^{i-1} 2^{1-i-j} \frac{(i + j)^{i+j}}{i^i j^j}.$$

If a composition $h(g(x))$ is admissible then $g' = 0$ and $h' = 0$ should imply that $h(g(x)) = \pm 1$. Hence $h(x)$ should be an admissible function. As far as g is concerned $g' = 0$ should imply that the value of g belongs to the preimage of ± 1 for h which is less restrictive if this preimage is large. Because of that it is hard to imagine a reasonable classification of all admissible polynomials.

On the other hand

$$z^2 \equiv 1 \pmod{z'} \qquad \text{for} \quad z = x^n + a_2 x^{n-2} + \cdots + a_n$$

leads to $n - 1$ equations on $n - 1$ variables with apparently finite number of solutions for each n. Say, for $n = 4$ all admissible polynomials are

$$x^4 \pm 1; \qquad x^4 + ax^2 + \frac{1}{8}a^2, \quad a^4 = 64; \qquad x^4 - 3a^2x^2 + 2\sqrt{2}a^3x + \frac{21}{8}a^4, \quad 337a^8 = 64.$$

Remark 6.1. The number of admissible polynomials of a given degree is finite. Indeed consider first $n - 2$ homogeneous equations on the coefficients $a_2, \ldots, a_n$. They are satisfied if $z^2 \equiv c$ (mod z'), where $c \in \mathbb{C}$. If one of the components of the variety defined by these equations is more than one-dimensional then (by affine dimension theorem) its intersection with the hypersurface given by the last homogeneous equation will be at least one-dimensional while condition $z^2 \equiv 0$ (mod z') is satisfied only by $z = x^n$ (recall that we are considering only reduced monic polynomials).

If $\delta = 0$ then $(\phi' f)^2 = 4(\phi - \lambda)^3$ and $(\phi - \lambda)^{-1/2} = \pm \int \frac{dx}{f}$ is a rational function.

Lemma 6.2. *If $f \in \mathbb{C}[x]$ and $\int \frac{dx}{f}$ is a rational function then f is a monomial, i.e., $f = a(x-b)^d$.*[2]

Proof. If $g' = \frac{1}{f}$ for $g \in \mathbb{C}(x)$ then $g = \frac{h}{f}$, $h \in \mathbb{C}[x]$ since the poles of g are the zeros of f and if the multiplicity of a zero of f is d then the corresponding pole of g has the multiplicity $d - 1$. An equality $g' = \frac{1}{f}$ can be rewritten as $h'f - hf' = f$. If $\deg(h) > 1$ then $\deg(h'f) > \deg(f)$. Hence the leading coefficients of polynomials $h'f$ and hf' are the same. This is possible only when $\deg(h) = \deg(f)$. Therefore there exists a $c \in \mathbb{C}$ for which $\deg(h - cf) < \deg(f)$. Since $(h - cf)'f - (h - cf)f' = f$ we can conclude that $\deg(h_1) = 1$ for $h_1 = h - cf$. Changing the variable we may assume that $h_1 = c_1 x$ and then $c_1(f - xf') = f$ which is possible only if $f = ax^d$. ∎

Hence when $\delta = 0$ we may assume that $f = x^d$. If $d = 0$ then this is the first case and A is a homogeneous operator up to an automorphism. If $d > 0$ then this is the second case and A is a homogeneous operator up to an automorphism of $\mathbb{C}[x^{-1}, x, \partial]$.

These computations show that a description of the structure of centralizers of rank one in A_1 is sufficiently challenging. Can the ring of regular functions of a genus zero curve with one place at infinity be realised as a centralizer of an element of A_1? Here is a more approachable relevant question: is there an element of $A \in D_1 \setminus A_1$ for which $p(A) \in A_1$ for a given a polynomial $p(x) \in \mathbb{C}[x]$?

7 Historical remarks

Apparently the first work which was devoted to the research of commuting differential operators is the work of Georg Wallenberg "Über die Vertauschbarkeit homogener linearer Differentialausdrücke" (see [40]), in which he studied the classification problem of pairs of commuting ordinary differential operators. He didn't work with the Weyl algebra though. Differential operators he was working with have coefficients which are "abstract" differentiable functions.

He mentioned that this problem did not seem to be studied before, even in the fundamental work of Gaston Floquet Sur la théorie des équations différentielles linéaires (see [10]). He credited Floquet with the case of two operators of order one.

Wallenberg started from this point. Then he gave a complete description of commuting operators P and Q when $\operatorname{ord} P = \operatorname{ord} Q = 2$ and when $\operatorname{ord} P = 1$, $\operatorname{ord} Q = n$. So far everything is easy. Then he studied the case of $\operatorname{ord} P = 2$ and $\operatorname{ord} Q = 3$, and noticed that a Weierstrass elliptic function appears in the coefficients of these operators. He dealt with a few more examples such as order 2 and 5, but did not obtain any general theorems.

Issai Schur read the paper of Wallenberg and published in 1904 paper "Über vertauschbare lineare Differentialausdrücke", which was already mentioned. He proved that centralizers of differential operators with differentiable coefficients are commutative by introducing pseudo-differential operators approximately fifty years before the notion appeared under this name.

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[2]I was unable to find a published proof for this observation. This proof is a result of discussions with J. Bernstein and A. Volberg.

The first general results toward the classification of commuting pairs of differential operators which were reported to the London Mathematical Society on June 8, 1922, were established about 20 years later by Joseph Burchnall and Theodore Chaundy (see [3]).

Burchnall and Chaundy published two more papers [4] and [5] devoted to this topic. Curiously enough they didn't know about the Wallenberg's paper and rediscovered his classification of 2, 3 commuting pairs and the fact that the ring of operators commuting with this pair of operators is isomorphic to a ring of regular functions of an elliptic curve.

Arguably the most important fact obtained by them is that an algebraic curve can be related to a pair of commuting operators.

After these works the question about commuting pairs of operators didn't attract much attention until the work of Jacques Dixmier [8] which appeared in 1968. He found elements in A_1 with centralizers which are also isomorphic to the ring of regular functions of an elliptic curve. Unlike examples by Wallenberg and Burchnall and Chaundy, where it was a question of rather straightforward computations, Dixmier's example required ingenuity.

After another gap of approximately ten years the question on pairs of commuting operators was raised in the context of solutions of some important partial differential equations. It seems that Igor Krichever was the first to write on this topic in [14] which appeared in 1976. He also rediscovered that an algebraic curve can be associated to a pair of commuting operators (and attributes to A. Shabat first observation of this kind) and mentions that operators commuting with an operator commute with each other.

Then Vladimir Drinfeld in [9] gave algebro-geometric interpretation of Krichever's results. This approach was further elucidated in a report of David Mumford [30] (D. Kajdan mentioned at the end of the report is David Kazhdan).

Later Krichever wrote an important survey [15] devoted to application of algebraic geometry to solutions of nonlinear PDE.

These works were primarily concerned with centralizers of rank one.

In [16] Krichever considered centralizers of an arbitrary rank and proved that for any centralizer A of a (non-constant) differential operator there exists a marked algebraic curve (γ, P) such that A is isomorphic to a ring of meromorphic functions on γ with poles in P. He remarked that γ is non-singular for a "general position" centralizer.

Motohico Mulase in [29] generalized the results of Krichever in [14], Drinfeld, and Mumford to the case of arbitrary rank. His main theorem is similar to the theorem of Krichever cited above. Apparently he didn't know about [16].

It remains to mention the works devoted to the centralizers of elements of the first Weyl algebra or its skew field of fractions. They primarily provide constructions of examples of centralizers which correspond to the curves of high genus.

Here is a partial list: [6, 7, 12, 13, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 31, 32, 33, 34, 36, 44, 45].

Interested reader will also benefit by looking at lectures by Emma Previato [35] and at the paper [43] by George Wilson as well as papers [38, 39, 42], and [2].

Lastly, it seems that the definition of rank of a centralizer as the greatest common divisor of the orders of its elements first appeared in a work of Wilson [41]. A similar definition of rank formulated slightly differently can be found in the work [9] of Drinfeld.

Acknowledgements

References

[1] Amitsur S.A., Commutative linear differential operators, *Pacific J. Math.* **8** (1958), 1–10.

[2] Burban I., Zheglov A., Fourier–Mukai transform on Weierstrass cubics and commuting differential operators, *Internat. J. Math.* **29** (2018), 1850064, 46 pages, arXiv:1602.08694.

[3] Burchnall J.L., Chaundy T.W., Commutative ordinary differential operators, *Proc. London Math. Soc.* **21** (1923), 420–440.

[4] Burchnall J.L., Chaundy T.W., Commutative ordinary differential operators, *Proc. Roy. Soc. London A* **118** (1928), 557–583.

[5] Burchnall J.L., Chaundy T.W., Commutative ordinary differential operators II. The identity $P^n = Q^m$, *Proc. Roy. Soc. London A* **134** (1931), 471–485.

[6] Davletshina V.N., Mironov A.E., On commuting ordinary differential operators with polynomial coefficients corresponding to spectral curves of genus two, *Bull. Korean Math. Soc.* **54** (2017), 1669–1675, arXiv:1606.01346.

[7] Dehornoy P., Opérateurs différentiels et courbes elliptiques, *Compositio Math.* **43** (1981), 71–99.

[8] Dixmier J., Sur les algèbres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.

[9] Drinfel'd V.G., Commutative subrings of certain noncommutative rings, *Funct. Anal. Appl.* **11** (1977), 9–12.

[10] Floquet G., Sur la théorie des équations différentielles linéaires, *Ann. Sci. École Norm. Sup. (2)* **8** (1879), 3–132.

[11] Gelfand I.M., Kirillov A.A., Sur les corps liés aux algèbres enveloppantes des algèbres de Lie, *Inst. Hautes Études Sci. Publ. Math.* (1966), 5–19.

[12] Grinevich P.G., Rational solutions for the equation of commutation of differential operators, *Funct. Anal. Appl.* **16** (1982), 15–19.

[13] Grünbaum F.A., Commuting pairs of linear ordinary differential operators of orders four and six, *Phys. D* **31** (1988), 424–433.

[14] Krichever I.M., An algebraic-geometric construction of the Zakharov–Shabat equations and their periodic solution, *Sov. Math. Dokl.* **17** (1976), 394–397.

[15] Krichever I.M., Methods of algebraic geometry in the theory of non-linear equations, *Russian Math. Surveys* **32** (1977), no. 6, 185–213.

[16] Krichever I.M., Commutative rings of ordinary linear differential operators, *Funct. Anal. Appl.* **12** (1978), 175–185.

[17] Makar-Limanov L., Centralizers in the quantum plane algebra, in Studies in Lie Theory, *Progr. Math.*, Vol. 243, Birkhäuser Boston, Boston, MA, 2006, 411–416.

[18] Mironov A.E., A ring of commuting differential operators of rank 2 corresponding to a curve of genus 2, *Sb. Math.* **195** (2004), 103–114.

[19] Mironov A.E., Commuting rank 2 differential operators corresponding to a curve of genus 2, *Funct. Anal. Appl.* **39** (2005), 240–243.

[20] Mironov A.E., On commuting differential operators of rank 2, *Sib. Electr. Math. Rep.* **6** (2009), 533–536.

[21] Mironov A.E., Commuting higher rank ordinary differential operators, in European Congress of Mathematics, Eur. Math. Soc., Zürich, 2013, 459–473, arXiv:1204.2092.

[22] Mironov A.E., Self-adjoint commuting ordinary differential operators, *Invent. Math.* **197** (2014), 417–431.

[23] Mironov A.E., Self-adjoint commuting differential operators of rank two, *Russian Math. Surveys* **71** (2016), 751–779.

[24] Mironov A.E., Zheglov A.B., Commuting ordinary differential operators with polynomial coefficients and automorphisms of the first Weyl algebra, *Int. Math. Res. Not.* **2016** (2016), 2974–2993, arXiv:1503.00485.

[25] Mokhov O.I., Commuting ordinary differential operators of rank 3 corresponding to an elliptic curve, *Russian Math. Surveys* **37** (1982), no. 4, 129–130.

[26] Mokhov O.I., Commuting differential operators of rank 3 and nonlinear equations, *Math. USSR-Izv.* **53** (1989), 629–655.

[27] Mokhov O.I., On commutative subalgebras of the Weyl algebra related to commuting operators of arbitrary rank and genus, *Math. Notes* **94** (2013), 298–300, arXiv:1201.5979.

[28] Mokhov O.I., Commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients, in Topology, Geometry, Integrable Systems, and Mathematical Physics, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 234, Amer. Math. Soc., Providence, RI, 2014, 323–336, arXiv:1303.4263.

[29] Mulase M., Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians, *Internat. J. Math.* **1** (1990), 293–342.

[30] Mumford D., An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg–de Vries equation and related nonlinear equation, in Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, 115–153.

[31] Oganesyan V.S., Commuting differential operators of rank 2 and arbitrary genus g with polynomial coefficients, *Russian Math. Surveys* **70** (2015), 165–167.

[32] Oganesyan V.S., Commuting differential operators of rank 2 with polynomial coefficients, *Funct. Anal. Appl.* **50** (2016), 54–61, arXiv:1409.4058.

[33] Oganesyan V.S., An alternative proof of Mironov's results on self-adjoint commuting operators of rank 2, *Sib. Math. J.* **59** (2018), 102–106.

[34] Oganesyan V.S., Commuting differential operators of rank 2 with rational coefficients, *Funct. Anal. Appl.* **52** (2018), 203–213, arXiv:1608.05146.

[35] Previato E., Seventy years of spectral curves: 1923–1993, in Integrable Systems and Quantum Groups (Montecatini Terme, 1993), *Lecture Notes in Math.*, Vol. 1620, Springer, Berlin, 1996, 419–481.

[36] Previato E., Rueda S.L., Zurro M.A., Commuting ordinary differential operators and the Dixmier test, *SIGMA* **15** (2019), 101, 23 pages, arXiv:1902.01361.

[37] Schur I., Über vertauschbare lineare Differentialausdrücke, *Sitzungsber. Berl. Math. Ges.* (1905), 2–8.

[38] Segal G., Wilson G., Loop groups and equations of KdV type, *Inst. Hautes Études Sci. Publ. Math.* (1985), 5–65.

[39] Verdier J.-L., Équations différentielles algébriques, in Séminaire Bourbaki, 30e année (1977/78), *Lecture Notes in Math.*, Vol. 710, Springer, Berlin, 1979, Exp. No. 512, 101–122.

[40] Wallenberg G., Über die Vertauschbarkeit homogener linearer Differentialausdrücke, *Arch. Math. Phys.* **3** (1903), 252–268.

[41] Wilson G., Algebraic curves and soliton equations, in Geometry Today (Rome, 1984), *Progr. Math.*, Vol. 60, Birkhäuser Boston, Boston, MA, 1985, 303–329.

[42] Wilson G., Bispectral commutative ordinary differential operators, *J. Reine Angew. Math.* **442** (1993), 177–204.

[43] Wilson G., Collisions of Calogero–Moser particles and an adelic Grassmannian (with an appendix by I.G. Macdonald), *Invent. Math.* **133** (1998), 1–41.

[44] Zheglov A.B., Mironov A.E., On commuting differential operators with polynomial coefficients corresponding to spectral curves of genus one, *Dokl. Math.* **91** (2015), 281–282.

[45] Zheglov A.B., Mironov A.E., Saparbaeva B.T., Commuting Krichever–Novikov differential operators with polynomial coefficients, *Sib. Math. J.* **57** (2016), 819–823.