Good Wild Harmonic Bundles and Good Filtered Higgs Bundles

Takuro MOCHIZUKI

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail: *takuro@kurims.kyoto-u.ac.jp*

Received July 16, 2020, in final form June 28, 2021; Published online July 17, 2021
https://doi.org/10.3842/SIGMA.2021.068

Abstract. We prove the Kobayashi–Hitchin correspondence between good wild harmonic bundles and polystable good filtered λ-flat bundles satisfying a vanishing condition. We also study the correspondence for good wild harmonic bundles with the homogeneity with respect to a group action, which is expected to provide another way to construct Frobenius manifolds.

Key words: wild harmonic bundles; Higgs bundles; λ-flat bundles; Kobayashi–Hitchin correspondence

2020 Mathematics Subject Classification: 53C07; 58E15; 14D21; 81T13

Dedicated to Professor Kyoji Saito
on the occasion of his 77th birthday

1 Introduction

Let X be a smooth projective variety with a simple normal crossing hypersurface H. Let L be an ample line bundle on X. We shall prove the following theorem, that is the Kobayashi–Hitchin correspondence for good wild harmonic bundles and good filtered λ-flat bundles.

Theorem 1.1 (Corollary 2.24)**.** *The following objects are equivalent:*

- *Good wild harmonic bundles on* (X, H)*.*
- μ_L*-Polystable filtered* λ*-flat bundles* $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ *on* (X, H) *satisfying*

$$\int c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1} = 0, \qquad \int \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2} = 0.$$

We shall recall the precise definitions of the objects in Section 2.

In [51], we have already proved that good wild harmonic bundles on (X, H) induce μ_L-polystable good filtered λ-flat bundles satisfying the vanishing condition. Note that 0-flat bundles are equivalent to Higgs bundles, and 1-flat bundles are flat bundles in the ordinary sense. Moreover, we studied an analogue of Theorem 1.1 in the case $\lambda = 1$, i.e., the correspondence between good wild harmonic bundles and μ_L-polystable good filtered flat bundles satisfying a similar vanishing condition [51, Theorem 16.1.1]. It was applied to the study of the correspondence between semisimple algebraic holonomic D-modules and pure twistor D-modules.

In this paper, as a complement, we shall explain the proof for all λ. There is no new essential difficulty to prove Theorem 1.1 after our studies [46, 47, 48, 49, 51] on the basis of [62, 63]. Moreover, in some parts of the proof, the arguments can be simplified in the Higgs

This paper is a contribution to the Special Issue on Primitive Forms and Related Topics in honor of Kyoji Saito for his 77th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Saito.html

case. However, because the Higgs case is also particularly important, it would be useful to explain a rather detailed proof. We shall also explain the correspondences in homogeneous cases which would be useful in a generalized Hodge theory.

1.1 Kobayashi–Hitchin correspondences

1.1.1 Kobayashi–Hitchin correspondence for vector bundles

We briefly recall a part of the history of this type of correspondences. (See also [25, 35, 41].) For a holomorphic vector bundle E on a compact Riemann surface C, we set $\mu(E) := \deg(E)/\operatorname{rank}(E)$, which is called the slope of E. A holomorphic bundle E is called stable (resp. semistable) if $\mu(E') < \mu(E)$ (resp. $\mu(E') \leq \mu(E)$) holds for any holomorphic subbundle $E' \subset E$ such that $0 < \operatorname{rank}(E') < \operatorname{rank}(E)$. It is called polystable if it is a direct sum of stable subbundles with the same slope. This stability, semistability and polystability conditions were introduced by Mumford [56] for the construction of the moduli spaces of vector bundles with reasonable properties. Narasimhan and Seshadri [58] established the equivalence between unitary flat bundles and polystable bundles of degree 0 on compact Riemann surfaces.

Let (X, ω) be a compact connected Kähler manifold. For any torsion-free $\mathcal{O}_X$-module $\mathcal{F}$, the slope of $\mathcal{F}$ with respect to ω is defined as

$$\mu_\omega(\mathcal{F}) := \frac{\int_X c_1(\mathcal{F})\omega^{\dim X - 1}}{\operatorname{rank}\mathcal{F}}.$$

If the cohomology class of ω is the first Chern class of an ample line bundle L, then $\mu_\omega(\mathcal{F})$ is also denoted by $\mu_L(\mathcal{F})$. Then, a torsion-free $\mathcal{O}_X$-module $\mathcal{F}$ is called μ_ω-stable if $\mu_\omega(\mathcal{F}') < \mu_\omega(\mathcal{F})$ holds for any saturated coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $0 < \operatorname{rank}(\mathcal{F}') < \operatorname{rank}(\mathcal{F})$. This condition was first studied by Takemoto [71, 72]. It is also called μ_ω-stability, or slope stability. Slope semistability and slope polystability are naturally defined.

Bogomolov [4] introduced the T-stability condition for torsion-free sheaves on connected projective surfaces, and he proved the inequality of the Chern classes $c_2(E) - (r-1)c_1(E)^2/2r \geq 0$ for any T-semistable bundle E of rank r. We do not recall the precise definition of T-stability condition here, but we note that if a holomorphic vector bundle on a complex projective manifold is slope semistable, then it is T-semistable. (See [4, Section 7] for more details.) Gieseker [19] gave a different proof of the inequality for slope semistable bundles. The inequality is called Bogomolov–Gieseker inequality or Bogomolov inequality.

Inspired by these works, Kobayashi [32] introduced the concept of Hermitian–Einstein condition for metrics of holomorphic vector bundles. Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle on a Kähler manifold (X, ω). Let h be a Hermitian metric of E. Let $R(h)$ denote the curvature of the Chern connection $\nabla_h = \overline{\partial}_E + \partial_{E,h}$, associated with h and $\overline{\partial}_E$. Then, h is called Hermitian–Einstein if $\Lambda R(h)^\perp = 0$, where $R(h)^\perp$ denotes the trace-free part of $R(h)$. In particular, he proved in [32] that if a holomorphic vector bundle on a compact Kähler manifold has a Hermitian–Einstein metric, then it is T-semistable. Kobayashi [33, 34] and Lübke [40] proved that a holomorphic vector bundle on a compact connected Kähler manifold satisfies the slope polystability condition if it has a Hermitian–Einstein metric. Moreover, Lübke [39] established the so called Kobayashi–Lübke inequality for the first and the second Chern forms associated with Hermitian–Einstein metrics, which is reduced to the inequality $\operatorname{Tr}\big((R(h)^\perp)^2\big)\omega^{\dim X-2} \geq 0$ in the form level. In particular, it implies the Bogomolov–Gieseker inequality for holomorphic vector bundles $(E, \overline{\partial}_E)$ with a Hermitian–Einstein metric h on compact Kähler manifolds (X, ω). Moreover, if $c_1(E) = 0$ and $\int_X \operatorname{ch}_2(E)\omega^{\dim X-2} = 0$ are satisfied for such $(E, \overline{\partial}_E, h)$, and if we impose $\det(h)$ is flat, then the Kobayashi–Lübke inequality implies that $R(h) = 0$, i.e., ∇_h is flat.

Independently, in [36], Hitchin proposed a problem to ask an equivalence of the stability condition and the existence of a metric h such that $\Lambda R(h) = 0$, under the vanishing of the first Chern class of the bundle. (See [25] for more precise explanation.) It clearly contains the most important essence. He also suggested possible applications of the vanishings. His problem stimulated Donaldson whose work on this topic brought several breakthroughs to whole geometry.

In [14], Donaldson introduced the method of global analysis to reprove the theorem of Narasimhan–Seshadri. In [15], by using the method of the heat flow associated with the Hermitian–Einstein condition, he established the equivalence of the slope polystability condition and the existence of a Hermitian–Einstein metric for holomorphic vector bundles on any complex projective surface. The important concept of Donaldson functional was also introduced in [15].

Eventually, Donaldson [16] and Uhlenbeck–Yau [73] established the equivalence on any dimensional complex projective manifolds. Note that Uhlenbeck–Yau proved it for any compact Kähler manifolds, more generally. The correspondence is called with various names; Kobayashi–Hitchin correspondence, Hitchin–Kobayashi correspondence, Donaldson–Hitchin–Uhlenbeck–Yau correspondence, etc. In this paper, we call it the Kobayashi–Hitchin correspondence.

As a consequence of the Kobayashi–Hitchin correspondence and the Kobayashi–Lübke inequality, we also obtain an equivalence between unitary flat bundles and slope polystable holomorphic vector bundles E satisfying $\mu_\omega(E) = 0$ and $\int_X \mathrm{ch}_2(E)\omega^{\dim X-2} = 0$. Note that Mehta and Ramanathan [43, 44] deduced the equivalence on complex projective manifolds directly from the equivalence in the surface case due to Donaldson [15].

1.1.2 Higgs bundles and λ-flat bundles

Such correspondences have been also studied for vector bundles equipped with some additional structure, which are also called Kobayashi–Hitchin correspondences in this paper. One of the most rich and influential is the case of Higgs bundles, pioneered by Hitchin and Simpson.

Let $(E,\overline{\partial}_E)$ be a holomorphic vector bundle on a compact Riemann surface C. A Higgs field of $(E,\overline{\partial}_E)$ is a holomorphic section θ of $\mathrm{End}(E)\otimes\Omega^1_C$. Let h be a Hermitian metric of E. We obtain the Chern connection $\overline{\partial}_E+\partial_{E,h}$ and its curvature $R(h)$. Let $\theta^\dagger_h$ denote the adjoint of θ. In [24], Hitchin introduced the following equation, called the Hitchin equation,

$$R(h)+\big[\theta,\theta^\dagger_h\big]=0. \tag{1.1}$$

Such $(E,\overline{\partial}_E,\theta,h)$ is called a harmonic bundle. In particular, he studied the case $\mathrm{rank}\,E=2$. Among many deep results in [24], he proved that a Higgs bundle $(E,\overline{\partial}_E,\theta)$ has a Hermitian metric h satisfying (1.1) if and only if it is polystable of degree 0. Here, a Higgs bundle $(E,\overline{\partial}_E,\theta)$ is called stable (resp. semistable) if $\mu(E')<\mu(E)$ (resp. $\mu(E')\leq\mu'(E)$) holds for any holomorphic subbundle $E'\subset E$ such that $\theta(E')\subset E'\otimes\Omega^1_C$ and that $0<\mathrm{rank}(E')<\mathrm{rank}(E)$, and a Higgs bundle is called polystable if it is a direct sum of stable Higgs subbundles with the same slope. By this equivalence and another equivalence due to Donaldson [17] between irreducible flat bundles and twisted harmonic maps, Hitchin obtained that the moduli space of polystable Higgs bundles of degree 0 and the moduli space of semisimple flat bundles are isomorphic. His work revealed that the moduli spaces of Higgs bundles and flat bundles have extremely rich structures.

The higher dimensional case was studied by Simpson [62]. Note that Simpson started his study independently motivated by a new way to construct variations of Hodge structure, which we shall mention later in Section 1.2.1. For a holomorphic vector bundle $(E,\overline{\partial}_E)$ on a complex manifold X with arbitrary dimension, a Higgs field θ is defined to be a holomorphic section of $\mathrm{End}(E)\otimes\Omega^1_X$ satisfying the additional condition $\theta\wedge\theta=0$. Suppose that X has a Kähler form. Let h be a Hermitian metric of E. Let $F(h)$ denote the curvature of the connection $\nabla_h+\theta+\theta^\dagger_h$. A Hermitian metric h of a Higgs bundle $(E,\overline{\partial}_E,\theta)$ is called Hermitian–Einstein if $\Lambda F(h)^\perp=0$. When X is compact, the slope stability, semistability and polystability conditions for Higgs bundles are naturally defined in terms of the slopes of Higgs subsheaves. Simpson established that a Higgs bundle $\big(E,\overline{\partial}_E,\theta\big)$ on a compact Kähler manifold (X,ω) has a Hermitian–Einstein metric if and only if it is slope polystable. Moreover, he generalized the Kobayashi–Lübke inequality for the Chern forms to the context of Higgs bundles, which is reduced to the inequality $\mathrm{Tr}\big(\big(F(h)^{\perp}\big)^2\big)\omega^{\dim X-2}\geq 0$ in the form level for any Hermitian–Einstein metric h of $\big(E,\overline{\partial}_E,\theta\big)$. Here, the condition $\theta\wedge\theta=0$ is essential. In particular, it implies that if $\big(E,\overline{\partial}_E,\theta\big)$ on a compact Kähler manifold (X,ω) satisfies $\mu_\omega(E)=0$ and $\int_X \mathrm{ch}_2(E)\omega^{\dim X-2}=0$, then a Hermitian–Einstein metric h of $\big(E,\overline{\partial}_E,\theta\big)$ is *a pluri-harmonic metric*, i.e., the connection $\nabla_h+\theta+\theta_h^\dagger$ is flat. It is equivalent to the following:

$$\partial_{E,h}\theta=0,\qquad \overline{\partial}\theta_h^\dagger=0,\qquad R(h)+\big[\theta,\theta_h^\dagger\big]=0.$$

A Higgs bundle $\big(E,\overline{\partial}_E,\theta\big)$ with a pluri-harmonic metric is called a harmonic bundle. This equivalence and another important equivalence due to Corlette [11] induce an equivalence between semisimple flat bundles and polystable Higgs bundles $\big(E,\overline{\partial}_E,\theta\big)$ satisfying $\mu_\omega(E)=0$ and $\int_X \mathrm{ch}_2(E)\omega^{\dim X-2}=0$ on any connected compact Kähler manifold.

After the work of Corlette, Donaldson, Hitchin and Simpson, it turned out that the moduli space $\mathcal{M}(X)$ of flat bundles on a complex projective manifold X has a hyper-Kähler metric. In particular, it induces the twistor space of the moduli space $\mathrm{TW}(\mathcal{M}(X))$, which is a complex analytic space with a fibration $\mathrm{TW}(\mathcal{M}(X))\longrightarrow\mathbb{P}^1$, such that the fiber over 1 is the moduli space of flat bundles, and that the fiber over 0 is the moduli space of Higgs bundles with vanishing rational Chern classes. The notion of λ-connections was introduced and developed by Deligne and Simpson [65, 66] for a more complex analytic construction of the twistor space $\mathrm{TW}(\mathcal{M}(X))$. They obtain the family of the moduli spaces $\mathcal{M}^\lambda(X)$ of λ-flat bundles on X, and the family of the moduli spaces $\mathcal{M}^\mu\big(X^\dagger\big)$ of μ-flat bundles on the conjugate $X^\dagger$ of X. They proved that the twistor space $\mathrm{TW}(\mathcal{M}(X))$ can be obtained as the gluing of the two families $\coprod_\lambda\mathcal{M}^\lambda(X)$ and $\coprod_\mu\mathcal{M}^\mu\big(X^\dagger\big)$ by the natural identification of $\mathcal{M}^\lambda(X)=\mathcal{M}^\mu\big(X^\dagger\big)$ for $\lambda\mu=1$.

These correspondences are not only really interesting, but also provide a starting point of the further investigations. Simpson pursued the comparison of flat bundles, Higgs bundles and more generally λ-flat bundles in deeper levels [64, 66], and developed the non-abelian Hodge theory [65]. In particular, he explained that the Kobayashi–Hitchin correspondences for λ-flat bundles can be studied in a unified way [64]. For more recent study on the moduli spaces of λ-connections, see [10, 26, 27, 67], for example.

1.1.3 Filtered case

It is interesting to generalize such correspondences for objects on complex quasi-projective manifolds. We need to impose a kind of boundary condition, that is parabolic structure.

Mehta and Seshadri [45] introduced the concept of parabolic structure of vector bundles on compact Riemann surfaces. Let C be a compact Riemann surface with a finite subset $D\subset C$. Let E be a holomorphic vector bundle on C. A parabolic structure of E at D is a tuple of filtrations $F_\bullet(E_{|P})$ $(P\in D)$ indexed by $]-1,0]$ satisfying $F_a(E_{|P})=\bigcap_{b>a}F_b(E_{|P})$. Set $\mathrm{Gr}^F_a(E_{|P}):=F_a(E)/F_{<a}(E)$, and

$$\deg(E,F):=\deg(E)-\sum_{P\in D}\sum_{-1<a\leq 0}a\dim\mathrm{Gr}^F_a(E_{|P}).$$

We set $\mu(E,F):=\deg(E,F)/\operatorname{rank}(E)$. For any subbundle $E'\subset E$, filtrations $F(E'_{|P})$ on $E'_{|P}$ are induced as $F_a(E'_{|P}):=F_a(E_{|P})\cap E'_{|P}$. Then, (E,F) is called stable if $\mu(E',F)<\mu(E,F)$ for any subbundle $E'\subset E$ with $0<\operatorname{rank}(E')<\operatorname{rank}(E)$. Semistability and polystability conditions are also defined naturally. Then, Mehta and Seshadri proved an equivalence of irreducible unitary flat bundles on $C \setminus D$ and stable parabolic vector bundles (E, F) with $\mu(E, F) = 0$ on (C, D).

For some purposes, it is more convenient to replace parabolic bundles with filtered bundles introduced by Simpson [62, 63]. Let $\mathcal{O}_C(*D)$ denote the sheaf of meromorphic functions on C which may have poles along D. Let $\mathcal{V}$ be a locally free $\mathcal{O}_C(*D)$-module. A filtered bundle $\mathcal{P}_*\mathcal{V}$ over $\mathcal{V}$ is a tuple of lattices $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ $(\boldsymbol{a} = (a_P)_{P\in D} \in \mathbb{R}^D)$ such that (i) $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}(*D) = \mathcal{V}$, (ii) the restriction of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ to a neighbourhood of $P \in D$ depends only on a_P, (iii) $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{n}}\mathcal{V} = \mathcal{P}_{\boldsymbol{a}}\mathcal{V}\big(\sum n_P P\big)$ for any $\boldsymbol{a} \in \mathbb{R}^D$ and $\boldsymbol{n} \in \mathbb{Z}^D$, (iv) for any $\boldsymbol{a} \in \mathbb{R}^D$, there exists $\boldsymbol{\epsilon} \in \mathbb{R}^D_{>0}$ such that $\mathcal{P}_{\boldsymbol{a}}\mathcal{V} = \mathcal{P}_{\boldsymbol{a}+\boldsymbol{\epsilon}}\mathcal{V}$. Let $\boldsymbol{0}$ denote $(0, \dots, 0) \in \mathbb{R}^D$. Then, $\mathcal{P}_{\boldsymbol{0}}\mathcal{V}$ is equipped with the parabolic structure F induced by the images of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P} \longrightarrow \mathcal{P}_{\boldsymbol{0}}\mathcal{V}_{|P}$ $(P \in D)$. It is easy to observe that filtered bundles are equivalent to parabolic bundles. We set $\mu(\mathcal{P}_*\mathcal{V}) := \mu(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, F)$ for the filtered bundle $\mathcal{P}_*\mathcal{V}$.

Simpson [62, 63] generalized the theorem of Mehta-Seshadri to the correspondences of tame harmonic bundles and regular filtered λ-flat bundles on compact Riemann surfaces. A harmonic bundle $\big(E, \overline{\partial}_E, \theta, h\big)$ on $C \setminus D$ is called tame on (C, D) if the closure of the spectral curve of θ in $T^*C(\log D)$ is proper over C. A regular filtered λ-flat bundle consists of a filtered bundle $\mathcal{P}_*\mathcal{V}$ equipped with a flat λ-connection $\mathbb{D}^\lambda \colon \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_C$ such that $\mathbb{D}^\lambda \cdot \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes \Omega^1_C(\log D)$ for any $\boldsymbol{a} \in \mathbb{R}^D$. Stability, semistable and polystable conditions are naturally defined in terms of the slope. Then, Simpson established the equivalence of tame harmonic bundles on (C, D) and polystable regular filtered λ-flat bundles $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ satisfying $\mu(\mathcal{P}_*\mathcal{V}) = 0$. Note that filtered bundles express the growth order of the norms of holomorphic sections with respect to the metrics. We should mention that the study of the asymptotic behaviour of tame harmonic bundles is much harder than that of the asymptotic behaviour of unitary flat bundles. Hence, it is already hard to prove that tame harmonic bundles induce regular filtered λ-flat bundles.

There are several directions to generalize. One is a generalization in the context of tame harmonic bundles on higher dimensional varieties. Let X be a smooth connected projective variety with a simple normal crossing hypersurface H and an ample line bundle L. Then, there should be equivalences of tame harmonic bundles on (X, H) and μ_L-polystable regular filtered λ-flat bundles $(\mathcal{P}_*\mathcal{V}, \theta)$ on (X, H) satisfying $\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1} = 0$ and $\int_X \operatorname{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2} = 0$ for each $\lambda \in \mathbb{C}$. In [2], Biquard studied the case where H is smooth. In [37, 38, 70], Li, Narasimhan, Steer and Wren studied the correspondence for parabolic bundles without flat λ-connections. In [30], Jost and Zuo studied the correspondence between semisimple flat bundles and tame harmonic bundles. In [46, 47, 48, 49], the author obtained the satisfactory equivalences for tame harmonic bundles. Note that Donagi and Pantev recently proposed an attractive application of the Kobayashi–Hitchin correspondence for tame harmonic bundles to the study of geometric Langlands theory [13].

In another natural direction of generalization, we should consider more singular objects than regular filtered Higgs or flat bundles. A harmonic bundle $\big(E, \overline{\partial}_E, \theta, h\big)$ on $X \setminus H$ is called wild if the closure of the spectral variety of θ in the projective completion of T^*X is complex analytic. For the analysis, we should impose that the spectral variety of the harmonic bundle satisfies some non-degeneracy condition along H. (See Section 2.7.1.) This is not essential because the condition is always satisfied once we replace X by its appropriate blow up. The notion of regular filtered λ-flat bundle is appropriately generalized to the notion of good filtered λ-flat bundle. The results of Simpson should be generalized to equivalences of good wild harmonic bundles and μ_L-polystable good filtered λ-flat bundles $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ satisfying $\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1} = 0$ and $\int_X \operatorname{ch}_2(\mathcal{P}_*\mathcal{V})c_2(L)^{\dim X-2} = 0$. Sabbah [59] studied the correspondence between semisimple meromorphic flat bundles and wild harmonic bundles in the one dimensional case. Biquard and Boalch [3] obtained generalization for wild harmonic bundles in the one dimensional case. Boalch informed the author that wild generalization in the context of the Higgs case was not expected in those days.

As mentioned, the author studied the wild harmonic bundles on any dimensional varieties in [51]. We obtained that good wild harmonic bundles induce μ_L-polystable good filtered λ-flat bundles satisfying the vanishing conditions. Moreover, we proved that the construction induces an equivalence of good wild harmonic bundles and slope polystable good filtered flat bundles satisfying the vanishing condition. Such an equivalence for meromorphic flat bundles is particularly interesting because we may apply it to prove a conjecture of Kashiwara [31] on semisimple algebraic holonomic $\mathcal{D}$-modules. See a survey paper [54] for more details on this application.

In [51], we did not give a proof of the equivalence for wild harmonic bundles in the case $\lambda \neq 1$ because it is rather obvious that a similar argument can work after [46, 47, 48, 49, 51] on the basis of [62, 63]. But, because the Higgs case is also important, it would be better to have a reference in which a rather detailed proof is explained. It is one reason why the author writes this manuscript. As another reason, in the next subsection, we shall explain an application to the correspondence for good wild harmonic bundles with homogeneity, which is expected to be useful in the generalized Hodge theory.

1.2 Homogeneity with respect to group actions

1.2.1 Variation of Hodge structure

As mentioned, Simpson [62] was motivated by the construction of polarized variation of Hodge structure. Let us recall the definition of polarized complex variation of Hodge structure given in [62], instead of the original definition of polarized variation of Hodge structure due to Griffiths. A complex variation of Hodge structure of weight w is a graded C^∞-vector bundle $V=\bigoplus_{p+q=w}V^{p,q}$ equipped with a flat connection ∇ satisfying the Griffiths transversality condition, i.e., $\nabla^{0,1}(V^{p,q})\subset\Omega^{0,1}\otimes\bigl(V^{p+1,q-1}\oplus V^{p,q}\bigr)$ and $\nabla^{1,0}(V^{p,q})\subset\Omega^{1,0}\otimes\bigl(V^{p-1,q+1}\oplus V^{p,q}\bigr)$, where $\nabla^{p,q}$ denote the (p,q)-part of ∇. A polarization of a complex variation of Hodge structure is a flat Hermitian pairing $\langle\cdot,\cdot\rangle$ satisfying the following conditions: (i) the decomposition $V=\bigoplus V^{p,q}$ is orthogonal with respect to $\langle\cdot,\cdot\rangle$, (ii) $\bigl(\sqrt{-1}\bigr)^{p-q}\langle\cdot,\cdot\rangle$ is positive definite on $V^{p,q}$.

A polarization of pure Hodge structure typically appears when we consider the Gauss–Manin connection associated with a smooth projective morphism $f\colon\mathcal{X}\longrightarrow\mathcal{Y}$. Namely, the family of vector spaces $H^w\bigl(f^{-1}(y)\bigr)$ $(y\in\mathcal{Y})$ naturally induces a flat bundle on $\mathcal{Y}$. With the Hodge decomposition, it is a variation of Hodge structure of weight w. A relatively ample line bundle induces a polarization on the variation of Hodge structure.

Simpson discovered a completely different way to construct a polarized variation of Hodge structure. Let $(V=\bigoplus V^{p,q},\nabla)$ be a complex variation of Hodge structure. Note that $\nabla^{0,1}$ induces holomorphic structures $\overline{\partial}_{V^{p,q}}\colon V^{p,q}\longrightarrow V^{p,q}\otimes\Omega^{0,1}$ of $V^{p,q}$. We set $\overline{\partial}_V:=\bigoplus\overline{\partial}_{V^{p,q}}$. Then, $\bigl(V=\bigoplus V^{p,q},\overline{\partial}_V\bigr)$ is a graded holomorphic vector bundle. We also note that $\nabla^{1,0}$ induces linear maps $V^{p,q}\longrightarrow V^{p-1,q+1}\otimes\Omega^{1,0}$, and hence $\theta\colon V\longrightarrow V\otimes\Omega^{1,0}$. It is easy to check that θ is a Higgs field of $\bigl(V,\overline{\partial}_V\bigr)$. Such a graded holomorphic bundle $V=\bigoplus_{p+q=w}V^{p,q}$ with a Higgs field θ such that $\theta(V^{p,q})\subset V^{p-1,q+1}\otimes\Omega^{1,0}$ is called a Hodge bundle of weight w. In general, we cannot construct a complex variation of Hodge structure from a Hodge bundle. However, Simpson discovered that if a Hodge bundle $(V=\bigoplus V^{p,q},\theta)$ on a compact Kähler manifold satisfies the stability condition and the vanishing condition, then there exists a flat connection ∇ and a flat Hermitian pairing $\langle\cdot,\cdot\rangle$ such that (i) $(V=\bigoplus V^{p,q},\nabla)$ is a complex variation of Hodge structure which induces the Hodge bundle, (ii) $\langle\cdot,\cdot\rangle$ is a polarization of $(V=\bigoplus V^{p,q},\nabla)$. Indeed, according to the equivalence of Simpson between Higgs bundles and harmonic bundles, there exists a pluri-harmonic metric h of (V,θ). It turns out that the flat connection $\nabla_h+\theta+\theta_h^\dagger$ satisfies the Griffiths transversality. Moreover, the decomposition $V=\bigoplus V^{p,q}$ is orthogonal with respect to h, and flat Hermitian paring $\langle\cdot,\cdot\rangle$ is constructed by the relation $\bigl(\sqrt{-1}\bigr)^{p-q}\langle\cdot,\cdot\rangle_{V^{p,q}}=h_{|V^{p,q}}$.

Note that a Hodge bundle is regarded as a Higgs bundle $\big(V, \overline{\partial}_V, \theta\big)$ with an S^1-homogeneity, i.e., $\big(V, \overline{\partial}_V\big)$ is equipped with an S^1-action such that $t \circ \theta \circ t^{-1} = t \cdot \theta$ for any $t \in S^1$. It roughly means that semistable Hodge bundles correspond to the fixed points in the moduli space of semistable Higgs bundles with respect to the natural S^1-action induced by $t\big(E, \overline{\partial}_E, \theta\big) = \big(E, \overline{\partial}_E, t\theta\big)$.

By the deformation $\big(E, \overline{\partial}_E, \alpha\theta\big)$ $(\alpha \in \mathbb{C}^*)$, any semistable Higgs bundles is deformed to an S^1-fixed point in the moduli space, i.e., a semistable Hodge bundle as $\alpha \to 0$. Note that the Higgs field of the limit is not necessarily 0. Hence, by the equivalence between Higgs bundles and flat bundles, it turns out that any flat bundle is deformed to a flat bundle underlying a polarized variation of Hodge structure.

In particular, Simpson [62] applied these ideas to construct uniformizations of some types of projective manifolds. He also applied it to prove that some type of discrete groups cannot be the fundamental group of any projective manifolds in [64].

1.2.2 TE-structure

We recall that a complex variation of Hodge structure on X induces a TE-structure in the sense of Hertling [21], i.e., a holomorphic vector bundle $\mathcal{V}$ on $\mathcal{X} := \mathbb{C}_\lambda \times X$ with a meromorphic flat connection

$$\widetilde{\nabla}\colon\ \mathcal{V} \longrightarrow \mathcal{V} \otimes \mathcal{O}_{\mathcal{X}}\big(\mathcal{X}^0\big) \otimes \Omega^1_{\mathcal{X}}\big(\log \mathcal{X}^0\big),$$

where $\mathcal{X}^0 := \{0\} \times X$. Indeed, for a complex variation of Hodge structure $(V = \bigoplus V^{p,q}, \nabla)$, $F^p(V) := \bigoplus_{p_1 \geq p} V^{p_1,q_1}$ are holomorphic subbundles with respect to $\nabla^{0,1}$. Thus, we obtain a decreasing filtration of holomorphic subbundles $F^p(V)$ $(p \in \mathbb{Z})$ satisfying the Griffiths transversality $\nabla^{1,0} F^p(V) \subset F^{p-1}(V) \otimes \Omega^{1,0}$. Let $p\colon \mathbb{C}^*_\lambda \times X \longrightarrow X$ denote the projection. We obtain the induced flat bundle $(p^*V, p^*\nabla)$. By the Rees construction, p^*V extends to a locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{V}$, on which $\widetilde{\nabla} := p^*\nabla$ is a meromorphic flat connection satisfying the condition $\widetilde{\nabla}\mathcal{V} \subset \mathcal{V} \otimes \mathcal{O}_{\mathcal{X}}\big(\mathcal{X}^0\big) \otimes \Omega^1_{\mathcal{X}}\big(\log \mathcal{X}^0\big)$.

It is recognized that a TE-structure appears as a fundamental piece of interesting structures in various fields of mathematics. For instance, TE-structure is an ingredient of Frobenius manifold, which is important in the theory of primitive forms and flat structures due to K. Saito [61], the topological field theory of Dubrovin [18], the tt^*-geometry of Cecotti–Vafa [7, 8], the Gromov–Witten theory, the theory of Landau–Ginzburg models, etc. For the construction of Frobenius manifolds, it is an important step to obtain TE-structures. Abstractly, TE-structure is also an important ingredient of semi-infinite variation of Hodge structure [1, 9, 28], TERP structure [21, 22, 23], integrable variation of twistor structure [60], etc. (See also [50, 53].)

1.2.3 Homogeneous harmonic bundles

As Simpson applied his Kobayashi–Hitchin correspondence to construct complex variations of Hodge structure, we may apply Theorem 1.1 to construct TE-structures with some additional structure. It is done through harmonic bundles with homogeneity as in the Hodge case.

Let X be a complex manifold equipped with an S^1-action. Let $\big(E, \overline{\partial}_E\big)$ be an S^1-equivariant holomorphic vector bundle. Let θ be a Higgs field of $\big(E, \overline{\partial}_E\big)$, which is homogeneous with respect to the S^1-action, i.e., $t^*\theta = t^m\theta$ for some $m \neq 0$. Let h be an S^1-invariant pluri-harmonic metric of $\big(E, \overline{\partial}_E, \theta\big)$. Then, as studied in [53, Section 3], we naturally obtain a TE-structure. More strongly, it is equipped with a grading in the sense of [9, 28], and it also underlies a polarized integrable variation of pure twistor structure of weight 0 [60]. Moreover, if there exists an S^1-equivariant isomorphism between $\big(E, \overline{\partial}_E, \theta, h\big)$ and its dual, the TE-structure is enhanced to a semi-infinite variation of Hodge structure with a grading [1, 9, 28]. If the S^1-action on X

is trivial, this is the same as the construction of a variation of Hodge structure from a Hodge bundle with a pluri-harmonic metric for which the Hodge decomposition is orthogonal.

Let H be a simple normal crossing hypersurface of X. From an S^1-homogeneous good wild harmonic bundle $\big(E,\overline{\partial}_E,\theta,h\big)$ on (X,H), as mentioned above, we obtain a TE-structure with a grading on $X\setminus H$. Moreover, it extends to a meromorphic TE-structure on (X,H) as studied in [53, Section 3]. We obtain the mixed Hodge structure as the limit objects at the boundary, which is useful for the study of more detailed properties of the TE-structure.

1.2.4 An equivalence

Let X be a complex projective manifold with a simple normal crossing hypersurface H and an ample line bundle L, equipped with a $\mathbb{C}^*$-action. A good filtered Higgs bundle $(\mathcal{P}_*\mathcal{V},\theta)$ is called $\mathbb{C}^*$-homogeneous if $\mathcal{P}_*\mathcal{V}$ is $\mathbb{C}^*$-equivariant and $t^*\theta=t^m\cdot\theta$ for some $m\neq 0$. Then, we obtain the following theorem by using Theorem 1.1. (See Section 8.1.2 for the precise definition of the stability condition in this context.)

Theorem 1.2 (Corollary 8.11)**.** *There exists an equivalence between the following objects:*

- *μ_L-polystable $\mathbb{C}^*$-homogeneous good filtered Higgs bundles $(\mathcal{P}_*\mathcal{V},\theta)$ on (X,H) satisfying*

$$\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1}=\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}=0.$$

- *S^1-homogeneous good wild harmonic bundles on (X,H).*

As mentioned in Section 1.2.3, Theorem 1.2 allows us to obtain a meromorphic TE-structure on (X,H) with a grading from a μ_L-polystable $\mathbb{C}^*$-equivariant good filtered Higgs bundle satisfying the vanishing condition. We already applied it to a classification of solutions of the Toda equations on $\mathbb{C}^*$ [52]. It seems natural to expect that this construction would be another way to obtain Frobenius manifolds.

Although we explained the homogeneity with respect to an S^1-action, Theorem 1.2 is generalized for K-homogeneous good wild harmonic bundles as explained in Section 8, where K is any compact Lie group.

2 Good filtered λ-flat bundles and wild harmonic bundles

2.1 Filtered sheaves and filtered λ-flat sheaves

2.1.1 Filtered sheaves

Let X denote a complex manifold with a simple normal crossing hypersurface H. Let $H=\bigcup_{i\in\Lambda}H_i$ denote a decomposition such that each H_i is smooth. Note that H_i are not necessarily connected. For any $P\in H$, a holomorphic coordinate neighbourhood $(X_P,z_1,\dots,z_n)$ around P is called admissible if $H_P:=H\cap X_P=\bigcup_{i=1}^{\ell(P)}\{z_i=0\}$. For such an admissible coordinate neighbourhood, there exists the map $\rho_P\colon\{1,\dots,\ell(P)\}\longrightarrow\Lambda$ determined by $H_{\rho_P(i)}\cap X_P=\{z_i=0\}$. We obtain the map $\kappa_P\colon\mathbb{R}^\Lambda\longrightarrow\mathbb{R}^{\ell(P)}$ by $\kappa_P(\boldsymbol{a})=\big(a_{\rho(1)},\dots,a_{\rho(\ell(P))}\big)$.

Let $\mathcal{O}_X(*H)$ denote the sheaf of meromorphic functions which may have poles along H. Let $\mathcal{E}$ be any coherent torsion free $\mathcal{O}_X(*H)$-module. A filtered sheaf over $\mathcal{E}$ is defined to be a tuple of coherent $\mathcal{O}_X$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}\subset\mathcal{E}$ ($\boldsymbol{a}\in\mathbb{R}^\Lambda$) satisfying the following conditions:

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}\subset\mathcal{P}_{\boldsymbol{b}}\mathcal{E}$ if $\boldsymbol{a}\leq\boldsymbol{b}$, i.e., $a_i\leq b_i$ for any $i\in\Lambda$.
- $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}(*H)=\mathcal{E}$ for any $\boldsymbol{a}\in\mathbb{R}^\Lambda$.

- $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{n}}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E}\big(\sum_{i\in\Lambda} n_i H_i\big)$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$ and $\boldsymbol{n} \in \mathbb{Z}^\Lambda$.
- For any $\boldsymbol{a} \in \mathbb{R}^\Lambda$ there exists $\boldsymbol{\epsilon} \in \mathbb{R}^\Lambda_{>0}$ such that $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{\epsilon}}\mathcal{E} = \mathcal{P}_{\boldsymbol{a}}\mathcal{E}$.
- For any $P \in H$, we take an admissible coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around P. Then, for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_{|X_P}$ depends only on $\kappa_P(\boldsymbol{a})$.

For any coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{E}' \subset \mathcal{E}$, we obtain a filtered sheaf $\mathcal{P}_*\mathcal{E}'$ over $\mathcal{E}'$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}' := \mathcal{P}_{\boldsymbol{a}}\mathcal{E} \cap \mathcal{E}'$. If $\mathcal{E}'$ is saturated, i.e., $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$ is torsion-free, we obtain a filtered sheaf $\mathcal{P}_*\mathcal{E}''$ over $\mathcal{E}''$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}'' := \operatorname{Im}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{E} \longrightarrow \mathcal{E}''\big)$.

A morphism of filtered sheaves $f\colon \mathcal{P}_*\mathcal{E}_1 \longrightarrow \mathcal{P}_*\mathcal{E}_2$ is defined to be a morphism $f\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ of $\mathcal{O}_X(*H)$-modules such that $f(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_1) \subset \mathcal{P}_{\boldsymbol{a}}\mathcal{E}_2$ for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$.

Remark 2.1. The concept of filtered bundles on curves was introduced by Mehta and Seshadri [45] and Simpson [62, 63]. A higher dimensional version was first studied by Maruyama and Yokogawa [42] for the purpose of the construction of the moduli spaces.

2.1.2 Restriction and gluing

Let $U \subset X$ be any open subset. We set $H_U = H \cap U$. Let $H_U = \bigcup_{j\in\Lambda_U} H_{U,j}$ be the irreducible decomposition. For any $j \in \Lambda_U$, we have $i(j) \in \Lambda$ such that $H_{U,j}$ is a connected component of $H_{i(j)} \cap U$. For any $P \in H_U$, we set $\Lambda_U(P) := \{j \in \Lambda_U \mid P \in H_{U,j}\}$.

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf over $\mathcal{E}$. We shall define a filtered sheaf over the $\mathcal{O}_U(*H_U)$-module $\mathcal{E}_{|U}$. Let $\boldsymbol{b} \in \mathbb{R}^{\Lambda_U}$. For any $P \in H_U$, we choose $\boldsymbol{a}(P,\boldsymbol{b}) \in \mathbb{R}^\Lambda$ such that $a(P,\boldsymbol{b})_{i(j)} = b_j$ for any $j \in \Lambda_U(P)$, and we obtain the following $\mathcal{O}_{U,P}$-submodule of the stalk $(\mathcal{E}_{|U})_P$:

$$\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_{|U})_P := \mathcal{P}_{\boldsymbol{a}(P,\boldsymbol{b})}(\mathcal{E})_P.$$

It is independent of the choice of $\boldsymbol{a}(P,\boldsymbol{b})$ as above. There uniquely exists a coherent $\mathcal{O}_U$-submodule $\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_{|U})$ of $\mathcal{E}_{|U}$ such that (i) $\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_{|U})(*H_U) = \mathcal{E}_{|U}$, and (ii) for any $P \in H_U$, the stalk of $\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_{|U})$ at P is equal to $\mathcal{P}_{\boldsymbol{b}}(\mathcal{E}_{|U})_P$. Thus, we obtain a filtered sheaf $\mathcal{P}_*(\mathcal{E}_{|U})$ over $\mathcal{E}_{|U}$, which is denoted as $\mathcal{P}_*\mathcal{E}_{|U}$.

Let $X = \bigcup_{k\in\Gamma} X^{(k)}$ be an open covering. We set $H^{(k)} = H \cap X^{(k)}$. For any filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$, we obtain filtered sheaves $\mathcal{P}_*\mathcal{E}_{|X^{(k)}}$ over $\mathcal{E}_{|X^{(k)}}$ as the restriction. Conversely, let $\mathcal{P}_*\big(\mathcal{E}_{|X^{(k)}}\big)$ $(k \in \Gamma)$ be filtered sheaves over $\mathcal{E}_{|X^{(k)}}$ such that $\mathcal{P}_*\big(\mathcal{E}_{|X^{(k)}}\big)_{|X^{(k)}\cap X^{(\ell)}} = \mathcal{P}_*\big(\mathcal{E}_{|X^{(\ell)}}\big)_{|X^{(k)}\cap X^{(\ell)}}$ for any $k, \ell \in \Gamma$.

Lemma 2.2. *There uniquely exists a filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$ such that $\mathcal{P}_*\mathcal{E}_{|X^{(k)}} = \mathcal{P}_*\big(\mathcal{E}_{|X^{(k)}}\big)$ for any $k \in \Gamma$.*

Proof. Let $\boldsymbol{a} \in \Lambda$. For any $P \in H$, there exists $k \in \Gamma$ such that $P \in X^{(k)}$. Let $H^{(k)} = \bigcup_{j\in\Lambda^{(k)}} H_j^{(k)}$ be the irreducible decomposition. For any $j \in \Lambda^{(k)}$, we have $i(k,j) \in \Lambda$ such that $H_j^{(k)}$ is a connected component of $H_{i(k,j)} \cap X^{(k)}$. Thus, we obtain a map $\Lambda^{(k)} \longrightarrow \Lambda$. For any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, let $\boldsymbol{a}^{(k)}$ be the image of $\boldsymbol{a}$ by the induced map $\mathbb{R}^\Lambda \longrightarrow \mathbb{R}^{\Lambda^{(k)}}$, and we obtain the following $\mathcal{O}_{X,P}$-submodule of $\mathcal{E}_P$:

$$\mathcal{P}_{\boldsymbol{a}}(\mathcal{E})_P := \mathcal{P}_{\boldsymbol{a}^{(k)}}\big(\mathcal{E}_{|X^{(k)}}\big)_P.$$

There uniquely exists a coherent $\mathcal{O}_X$-submodule $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ of $\mathcal{E}$ such that (i) $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}(*H) = \mathcal{E}$, and (ii) for any $P \in H$, the stalk of $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E})$ at P is equal to $\mathcal{P}_{\boldsymbol{a}}(\mathcal{E})_P$. Thus, we obtain a filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$ with the desired property. The uniqueness is also clear. ∎

2.1.3 Reflexive filtered sheaves

A filtered sheaf $\mathcal{P}_*\mathcal{E}$ on (X,H) is called reflexive if each $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ is a reflexive $\mathcal{O}_X$-module. Note that it is equivalent to the "reflexive and saturated" condition in [46, Definition 3.17] by the following lemma.

Lemma 2.3. *Suppose that $\mathcal{P}_*\mathcal{E}$ is reflexive. Let $\boldsymbol{a}\in\mathbb{R}^\Lambda$. We take $a_i-1<b\leq a_i$, and let $\boldsymbol{a}'\in\mathbb{R}^\Lambda$ be determined by $a'_j=a_j$ $(j\neq i)$ and $a'_i=b$. Then, $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}/\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$ is a torsion-free $\mathcal{O}_{H_i}$-module.*

Proof. Let s be a section of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}/\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$ on an open set $U\subset D_i$. There exists an open subset $\widetilde{U}\subset X$ and a section $\widetilde{s}$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{E}$ on $\widetilde{U}$ such that $\widetilde{U}\cap D_i=U$ and that $\widetilde{s}$ induces s. Note that there exists $Z\subset\widetilde{U}$ of codimension 2 such that $\widetilde{s}_{|\widetilde{U}\setminus Z}$ is a section of $\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}_{|\widetilde{U}\setminus Z}$. Because $\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$ is reflexive, there exists a section $\widetilde{s}'$ of $\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$ on $\widetilde{U}$ such that $\widetilde{s}'_{|\widetilde{U}\setminus Z}=\widetilde{s}_{|\widetilde{U}\setminus Z}$. Hence, we obtain that $\widetilde{s}$ is a section of $\mathcal{P}_{\boldsymbol{a}'}\mathcal{E}$, i.e., $s=0$. ∎

The following lemma is clear.

Lemma 2.4. *Let $\mathcal{P}_*\mathcal{E}$ be a reflexive filtered sheaf on (X,H). Then a coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{E}'\subset\mathcal{E}$ is saturated if and only if the induced filtered sheaf $\mathcal{P}_*\mathcal{E}'$ is reflexive.*

2.1.4 Filtered λ-flat sheaves

Let λ be any complex number. Let $\mathcal{E}$ be a coherent torsion-free $\mathcal{O}_X(*H)$-module. A λ-connection $\mathbb{D}^\lambda\colon\mathcal{E}\longrightarrow\Omega^1_X\otimes\mathcal{E}$ is a $\mathbb{C}$-linear morphism of sheaves such that $\mathbb{D}^\lambda(fs)=f\mathbb{D}^\lambda(s)+\lambda df\otimes s$ for any local sections f and s of $\mathcal{O}_X$ and $\mathcal{E}$, respectively. Note that an $\mathcal{O}_X$-morphism $\mathbb{D}^\lambda\circ\mathbb{D}^\lambda\colon\mathcal{E}\longrightarrow\Omega^2_X\otimes\mathcal{E}$ is induced. If $\mathbb{D}^\lambda\circ\mathbb{D}^\lambda=0$, it is called a flat λ-connection. When $\mathcal{E}$ is equipped with a flat λ-connection, a λ-flat subsheaf of $\mathcal{E}$ means a coherent $\mathcal{O}_X$-submodule $\mathcal{E}'\subset\mathcal{E}$ such that $\mathbb{D}^\lambda(\mathcal{E}')\subset\Omega^1_X\otimes\mathcal{E}'$. A pair of a filtered sheaf $\mathcal{P}_*\mathcal{E}$ over $\mathcal{E}$ and a flat λ-connection $\mathbb{D}^\lambda$ of $\mathcal{E}$ is called a filtered λ-flat connection. It is called reflexive if $\mathcal{P}_*\mathcal{E}$ is reflexive.

2.2 μ_L-stability condition for filtered λ-flat sheaves

Let X be a connected projective manifold with a simple normal crossing hypersurface $H=\bigcup_{i\in\Lambda}H_i$. Let L be an ample line bundle.

2.2.1 Slope of filtered sheaves

Let $\mathcal{P}_*\mathcal{E}$ be a filtered sheaf on (X,H). Recall the definition of the parabolic first Chern class $c_1(\mathcal{P}_*\mathcal{E})$. Let η_i be the generic point of H_i. Note that $\mathcal{O}_{X,\eta_i}$-modules $(\mathcal{P}_{\boldsymbol{a}}\mathcal{E})_{\eta_i}$ depends only on a_i, which is denoted by $\mathcal{P}_{a_i}(\mathcal{E}_{\eta_i})$. We obtain $\mathcal{O}_{H_i,\eta_i}$-modules $\mathrm{Gr}^{\mathcal{P}}_a(\mathcal{E}_{\eta_i}):=\mathcal{P}_a(\mathcal{E}_{\eta_i})/\mathcal{P}_{<a}(\mathcal{E}_{\eta_i})$. Then, we set

$$c_1(\mathcal{P}_*\mathcal{E}):=c_1(\mathcal{P}_{\boldsymbol{a}}\mathcal{E})-\sum_{i\in\Lambda}\sum_{a_i-1<a\leq a_i}a\,\mathrm{rank}\,\mathrm{Gr}^{\mathcal{P}}_a(\mathcal{E}_{\eta_i})[H_i]\in H^2(X,\mathbb{R}).\tag{2.1}$$

Here, $[H_i]$ denote the cohomology class induced by H_i. It is easy to see that $c_1(\mathcal{P}_*\mathcal{V})$ is independent of the choice of $\boldsymbol{a}\in\mathbb{R}^\Lambda$. We set

$$\mu_L(\mathcal{P}_*\mathcal{E}):=\frac{1}{\mathrm{rank}\,\mathcal{E}}\int_X c_1(\mathcal{P}_*\mathcal{E})\cdot c_1(L)^{n-1}.$$

It is called the slope of $\mathcal{P}_*\mathcal{E}$ with respect to L. The following is proved in [46, Lemma 3.7].

Lemma 2.5. *Let $f\colon \mathcal{P}_*\mathcal{E}^{(1)} \longrightarrow \mathcal{P}_*\mathcal{E}^{(2)}$ be a morphism of filtered sheaves which is generically an isomorphism, i.e., the induced morphism $\mathcal{E}^{(1)}_{\eta(X)} \longrightarrow \mathcal{E}^{(2)}_{\eta(X)}$ at the generic point of X is an isomorphism. Then, $\mu_L\big(\mathcal{P}_*\mathcal{E}^{(1)}\big) \le \mu_L\big(\mathcal{P}_*\mathcal{E}^{(2)}\big)$ holds. If the equality holds, f is an isomorphism in codimension one, i.e., there exists an algebraic subset $Z \subset X$ such that (i) the codimension of Z is larger than 2, (ii) $f_{|X\setminus Z}\colon \mathcal{P}_*\mathcal{E}^{(1)}_{|X\setminus Z} \longrightarrow \mathcal{P}_*\mathcal{E}^{(2)}_{|X\setminus Z}$ is an isomorphism.*

2.2.2 μ_L-stability condition

A filtered λ-flat sheaf $(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda)$ on (X,H) is called μ_L-stable (resp. μ_L-semistable) if the following holds:

- Let $\mathcal{E}' \subset \mathcal{E}$ be any λ-flat $\mathcal{O}_X(*H)$-submodule such that $0 < \operatorname{rank}(\mathcal{E}') < \operatorname{rank}(\mathcal{E})$. Then, $\mu_L(\mathcal{P}_*\mathcal{E}') < \mu_L(\mathcal{P}_*\mathcal{E})$ (resp. $\mu_L(\mathcal{P}_*\mathcal{E}') \le \mu_L(\mathcal{P}_*\mathcal{E})$) holds.

A filtered λ-flat sheaf $\big(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda\big)$ is called μ_L-polystable if the following holds:

- $\big(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda\big)$ is μ_L-semistable.
- $\big(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda\big) = \bigoplus \big(\mathcal{P}_*\mathcal{E}_i, \mathbb{D}_i^\lambda\big)$, where each $\big(\mathcal{P}_*\mathcal{E}_i, \mathbb{D}_i^\lambda\big)$ is μ_L-stable.

The following is standard. (See [46, Section 3.1.3] and [49, Section 2.1.4].)

Lemma 2.6. *Suppose that $(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda)$ is a μ_L-polystable reflexive filtered λ-flat sheaf. Then, there exists a unique decomposition $\big(\mathcal{P}_*\mathcal{E}, \mathbb{D}^\lambda\big) = \bigoplus_{i=1}^N \big(\mathcal{P}_*\mathcal{E}_i, \mathbb{D}_i^\lambda\big) \otimes \mathbb{C}^{m(i)}$ such that (i) $\big(\mathcal{P}_*\mathcal{E}_i, \mathbb{D}_i^\lambda\big)$ are μ_L-stable, (ii) $\mu_L(\mathcal{P}_*\mathcal{E}_i) = \mu_L(\mathcal{P}_*\mathcal{E})$, (iii) $\big(\mathcal{P}_*\mathcal{E}_i, \mathbb{D}_i^\lambda\big) \not\simeq \big(\mathcal{P}_*\mathcal{E}_j, \mathbb{D}_j^\lambda\big)$ $(i \neq j)$.*

Remark 2.7. In [46, Section 3.1.3], "the inequality $\operatorname{par-deg}_L(\mathcal{E}'_*) < \operatorname{par-deg}_L(\mathcal{E}_*)$" should be corrected to "the inequality $\mu_L(\mathcal{E}'_*) < \mu_L(\mathcal{E}_*)$".

2.3 Filtered bundles

2.3.1 Filtered bundles in the local case

We recall the notion of filtered bundle in the local case. We shall explain it in the global case in Section 2.3.3. Let U be a neighbourhood of $(0,\dots,0)$ in $\mathbb{C}^n$. We set $H_{U,i} := U \cap \{z_i = 0\}$, and $H_U := \bigcup_{i=1}^{\ell} H_{U,i}$ for some $0 \le \ell \le n$. Let $\mathcal{V}$ be a locally free $\mathcal{O}_U(*H_U)$-module. A filtered bundle $\mathcal{P}_*\mathcal{V}$ over $\mathcal{V}$ is a tuple of locally free $\mathcal{O}_U$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ ($\boldsymbol{a} \in \mathbb{R}^\ell$) such that the following holds:

- $\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{P}_{\boldsymbol{b}}\mathcal{V}$ if $\boldsymbol{a} \le \boldsymbol{b}$, i.e., $a_i \le b_i$ for any $i = 1,\dots,\ell$.
- There exists a frame $\boldsymbol{v} = (v_1,\dots,v_r)$ of $\mathcal{V}$ and tuples $\boldsymbol{a}(v_j) \in \mathbb{R}^\ell$ $(j = 1,\dots,r)$ such that

$$\mathcal{P}_{\boldsymbol{b}}\mathcal{V} = \bigoplus_{j=1}^{r} \mathcal{O}_U\bigg(\sum_i \big[b_i - a_i(v_j)\big] H_{U,i}\bigg) \cdot v_j, \tag{2.2}$$

  where we set $[c] := \max\{p \in \mathbb{Z} \mid p \le c\}$ for any $c \in \mathbb{R}$.

Clearly, a filtered bundle over $\mathcal{V}$ is a filtered sheaf over $\mathcal{V}$.

Remark 2.8. We set $\mathcal{R} := \mathbb{C}[\![z_1,\dots,z_n]\!]$ and $\widetilde{\mathcal{R}} := \mathcal{R}\big[z_1^{-1},\dots,z_\ell^{-1}\big]$. For a free $\widetilde{\mathcal{R}}$-module $\widehat{\mathcal{V}}$, a filtered bundle over $\widehat{\mathcal{V}}$ is defined to be a tuple $\mathcal{P}_*\widehat{\mathcal{V}} := \big(\mathcal{P}_{\boldsymbol{a}}\widehat{\mathcal{V}} \mid \boldsymbol{a} \in \mathbb{R}^{\ell(P)}\big)$ of free $\mathcal{R}$-submodules satisfying similar conditions as above.

2.3.2 Pull back, push-forward and descent with respect to ramified coverings in the local case

Let $\varphi\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be given by $\varphi(\zeta_1,\dots,\zeta_n) = \bigl(\zeta_1^{m_1},\dots,\zeta_\ell^{m_\ell},\zeta_{\ell+1},\dots,\zeta_n\bigr)$. We set $U' := \varphi^{-1}(U)$, $H_{U',i} := \varphi^{-1}(H_{U,i})$ and $H_{U'} := \varphi^{-1}(H_U)$. The induced ramified covering $U' \longrightarrow U$ is also denoted by φ.

For any $\boldsymbol{b} \in \mathbb{R}^\ell$, we set $\varphi^*(\boldsymbol{b}) = (m_i b_i) \in \mathbb{R}^\ell$. For any filtered bundle $\mathcal{P}_*\mathcal{V}_1$ on (U,H_U), we define a filtered bundle $\mathcal{P}_*\mathcal{V}_1'$ on $(U',H_{U'})$ as follows:

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_1' = \sum_{\varphi^*(\boldsymbol{b})+\boldsymbol{n}\leq\boldsymbol{a}} \varphi^*\bigl(\mathcal{P}_{\boldsymbol{b}}\mathcal{V}_1\bigr)\Bigl(\sum n_i H_{U',i}\Bigr).$$

We set $\varphi^*(\mathcal{P}_*\mathcal{V}_1) := \mathcal{P}_*\mathcal{V}_1'$. Thus, we obtain the pull back functor φ^* from the category of filtered bundles on (U,H_U) to the category of filtered bundles on $(U',H_{U'})$.

For any $\boldsymbol{b} \in \mathbb{R}^\ell$, we set $\varphi_*(\boldsymbol{b}) = (m_i^{-1}b_i)$. For any filtered bundle $\mathcal{P}_*\mathcal{V}_2$ on $(U',H_{U'})$, we obtain the following filtered bundle

$$\mathcal{P}_{\boldsymbol{b}}\varphi_*(\mathcal{V}_2) := \varphi_*\mathcal{P}_{\varphi_*(\boldsymbol{b})}\mathcal{V}_2.$$

In this way, we obtain a functor φ_* from the category of filtered bundles on $(U',H_{U'})$ to the category of filtered bundles on (U,H_U).

We set $G := \prod_{i=1}^{\ell}\bigl\{\mu_i \in \mathbb{C}^* \mid \mu_i^{m_i} = 1\bigr\}$. We define the action of G on U' by

$$(\mu_1,\dots,\mu_\ell)(\zeta_1,\dots,\zeta_n) = (\mu_1\zeta_1,\dots,\mu_\ell\zeta_\ell,\zeta_{\ell+1},\dots,\zeta_n).$$

We identify G as the Galois group of the ramified covering $U' \longrightarrow U$. Let $\mathcal{P}_*\mathcal{V}_3$ be a G-equivariant filtered bundles on $(U',H_{U'})$. Then, $\mathcal{P}_*\varphi_*\mathcal{V}_3$ is equipped with an induced G-action. We obtain a filtered bundle $(\mathcal{P}_*\varphi_*\mathcal{V}_3)^G$ on (U,H_U) as the G-invariant part of $\mathcal{P}_*\varphi_*\mathcal{V}_3$, which is called the descent of $\mathcal{P}_*\mathcal{V}_3$ with respect to the G-action. In this way, we obtain a functor from the category of G-equivariant filtered bundles on $(U',H_{U'})$ to the category of filtered bundles on (U,H_U).

For a filtered bundle $\mathcal{P}_*\mathcal{V}_1$ on (U,H_U), the pull back $\varphi^*(\mathcal{P}_*\mathcal{V}_1)$ is a G-equivariant filtered bundle on $(U',H_{U'})$, and its descent is naturally isomorphic to $\mathcal{P}_*\mathcal{V}_1$.

2.3.3 Filtered bundles in the global case

We use the notation in Section 2.1.1. Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(*H)$-module. A filtered bundle $\mathcal{P}_*\mathcal{V} = \bigl(\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \mid \boldsymbol{a} \in \mathbb{R}^\Lambda\bigr)$ over $\mathcal{V}$ be a sequence of locally free $\mathcal{O}_X$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ of $\mathcal{V}$ such that the following holds:

- For any $P \in H$, we take an admissible coordinate neighbourhood $(X_P, z_1,\dots,z_n)$ around P. Then, for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|X_P}$ depends only on $\kappa_P(\boldsymbol{a})$, denoted as $\mathcal{P}^{(P)}_{\kappa_P(\boldsymbol{a})}(\mathcal{V}_{|X_P})$.
- The sequence $\bigl(\mathcal{P}^{(P)}_{\boldsymbol{b}}(\mathcal{V}_{|X_P}) \mid \boldsymbol{b} \in \mathbb{R}^{\ell(P)}\bigr)$ is a filtered bundle over $\mathcal{V}_{|X_P}$ in the sense of Section 2.3.1.

In other words, a filtered bundle is a filtered sheaf (see Section 2.1.1) satisfying the condition in Section 2.3.1 locally around any point of H.

Remark 2.9. The higher dimensional version of filtered bundles was introduced in [47, 48] with a different formulation. See also [5, 6]. In this paper, we essentially follow Iyer and Simpson [29].

2.3.4 The induced bundles and filtrations

For any $I \subset \Lambda$, let $\boldsymbol{\delta}_I \in \mathbb{R}^\Lambda$ be the element whose j-th component is 0 ($j \in \Lambda \setminus I$) or 1 ($j \in I$). We also set $H_I := \bigcap_{i\in I} H_i$ and $\partial H_I := H_I \cap \big(\bigcup_{j\in\Lambda\setminus I} H_j\big)$.

Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle on (X, H). Take $i \in \Lambda$. Let $\boldsymbol{a} \in \mathbb{R}^\Lambda$. For any $a_i - 1 \le b \le a_i$, we set $\boldsymbol{a}(b, i) := \boldsymbol{a} + (b - a_i)\boldsymbol{\delta}_i$. We set

$$ {}^iF_b\big(\mathcal{P}_{\boldsymbol{a}}(\mathcal{V})_{|H_i}\big) := \mathcal{P}_{\boldsymbol{a}(b,i)}\mathcal{V}/\mathcal{P}_{\boldsymbol{a}(a_i-1,i)}\mathcal{V}. $$

It is naturally regarded as a locally free $\mathcal{O}_{H_i}$-module. Moreover, it is a subbundle of $\mathcal{P}_{\boldsymbol{a}}(\mathcal{V})_{|H_i}$. In this way, we obtain a filtration ${}^iF$ of $\mathcal{P}_{\boldsymbol{a}}(\mathcal{V})_{|H_i}$ indexed by $]a_i - 1, a_i]$. We shall also denote it as just F if there is no risk of confusion.

We obtain the induced filtrations ${}^iF$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I}$ if $i \in I$. Let $\boldsymbol{a}_I \in \mathbb{R}^I$ denote the image of $\boldsymbol{a}$ by the projection $\mathbb{R}^\Lambda \longrightarrow \mathbb{R}^I$. Set $]\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I] := \prod_{i\in I}]a_i - 1, a_i]$. For any $\boldsymbol{b} \in]\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, we set

$$ {}^IF_{\boldsymbol{b}}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I}\big) := \bigcap_{i\in I} {}^iF_{b_i}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I}\big). $$

By the condition of filtered bundles, the following compatibility condition holds.

- Let P be any point of H_I. There exist a neighbourhood X_P of P in X and a non-canonical decomposition

  $$ \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|X_P\cap H_I} = \bigoplus_{\boldsymbol{b}\in]\boldsymbol{a}_I-\boldsymbol{\delta}_I,\boldsymbol{a}_I]} \mathcal{G}_{P,\boldsymbol{b}} $$

  such that the following holds for any $\boldsymbol{c} \in]\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$:

  $$ {}^IF_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I\cap X_P}) = \bigoplus_{\boldsymbol{b}\le\boldsymbol{c}} \mathcal{G}_{P,\boldsymbol{b}}. \tag{2.3} $$

  Indeed, there exists a frame $\boldsymbol{v} = (v_1, \dots, v_r)$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ around P with tuples $\boldsymbol{a}(v_i) \in \mathbb{R}^{\ell(P)}$ of real numbers satisfying (2.2), where $\boldsymbol{b}$ is replaced with $\boldsymbol{a}$. There exists the bijection $\kappa\colon I \simeq \{1, \dots, \ell(P)\}$ determined by $H_i \cap X_P = \{z_{\kappa(i)} = 0\}$, by which we identify I with $\{1, \dots, \ell(P)\}$. Let $\mathcal{G}_{P,\boldsymbol{b}}$ be the subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|X_P\cap H_I}$ generated by $v_{j|X_P\cap H_I}$ satisfying $\boldsymbol{a}(v_j) = \boldsymbol{b}$. Then, we obtain the decomposition (2.3).

For any $\boldsymbol{c} \in]\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, we obtain the following locally free $\mathcal{O}_{H_I}$-modules:

$$ {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) := \frac{{}^IF_{\boldsymbol{c}}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I}\big)}{\sum_{\boldsymbol{b}\lneq\boldsymbol{c}} {}^IF_{\boldsymbol{b}}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_I}\big)}. $$

Here, $\boldsymbol{b} = (b_i) \lneq \boldsymbol{c} = (c_i)$ means that $b_i \le c_i$ for any i and that $\boldsymbol{b} \ne \boldsymbol{c}$. Clearly, if $\boldsymbol{c} \in]\boldsymbol{a}'_I - \boldsymbol{\delta}_I, \boldsymbol{a}'_I]$ and $\boldsymbol{a}'_{\Lambda\setminus I} = \boldsymbol{a}_{\Lambda\setminus I}$, we obtain ${}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) = {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}'}\mathcal{V})$.

2.3.5 The induced filtered bundles

For $\boldsymbol{c} \in \mathbb{R}^I$, we choose $\boldsymbol{a} \in \mathbb{R}^\Lambda$ such that $\boldsymbol{c} \in]\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, and we obtain the following $\mathcal{O}_{H_I}(*\partial H_I)$-module:

$$ {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V}) := {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})(*\partial H_I). $$

It is independent of the choice of $\boldsymbol{a}$ as above. We obtain the irreducible decomposition $\partial H_I = \bigcup_{i\in\Lambda(I)} H_{I,i}$. For any $i \in \Lambda(I)$, there exists $j(i) \in \Lambda \setminus I$ such that $H_{I,i}$ is a connected component of $H_I \cap H_{j(i)}$. Let $\boldsymbol{d} \in \mathbb{R}^{\Lambda(I)}$. For $P \in \partial H_I$, there exists $\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{d}, P) \in \mathbb{R}^\Lambda$ such that (i) $a(\boldsymbol{c}, \boldsymbol{d}, P)_j = c_j$ $(j \in I)$, (ii) $a(\boldsymbol{c}, \boldsymbol{d}, P)_{j(i)} = d_i$ $(i \in \Lambda(I),\ P \in H_{I,i})$. We obtain an $\mathcal{O}_{X,P}$-submodule

$$\mathcal{P}_{\boldsymbol{d}}\big({}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})\big)_P := {}^I\mathrm{Gr}^F_{\boldsymbol{c}}\big(\mathcal{P}_{\boldsymbol{a}(\boldsymbol{c},\boldsymbol{d},P)}\mathcal{V}\big)_P \subset {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})_P.$$

Note that $\mathcal{P}_{\boldsymbol{d}}\big({}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})\big)_P$ is independent of the choice of $\boldsymbol{a}(P)$. There uniquely exists an $\mathcal{O}_{H_I}$-submodule $\mathcal{P}_{\boldsymbol{d}}\big({}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})\big) \subset {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})$ whose stalk at P $(P \in \partial H_I)$ are equal to $\mathcal{P}_{\boldsymbol{d}}\big({}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})\big)_P$. Thus, we obtain the following filtered bundle over ${}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})$ on $(H_I, \partial H_I)$:

$${}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_*\mathcal{V}) := \big(\mathcal{P}_{\boldsymbol{d}}\big({}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{V})\big) \mid \boldsymbol{d} \in \mathbb{R}^{\Lambda(I)}\big).$$

2.3.6 First and second Chern characters for filtered bundles

Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over (X, H). Take any $\boldsymbol{a} \in \mathbb{R}^\Lambda$. As recalled in Section 2.2.1, we obtain the parabolic first Chern class:

$$c_1(\mathcal{P}_*\mathcal{V}) = c_1(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) - \sum_{i\in\Lambda}\sum_{a_i-1<b\le a_i} a_i \operatorname{rank} {}^i\mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{E}_{|H_i}) \cdot [H_i] \in H^2(X,\mathbb{R}).$$

To explain the second parabolic Chern character in $H^4(X,\mathbb{R})$, let us introduce some notation. Let $\mathrm{Irr}(H_i \cap H_j)$ be the set of the irreducible components of $H_i \cap H_j$. For $C \in \mathrm{Irr}(H_I)$, let $[C] \in H^4(X,\mathbb{R})$ denotes the induced cohomology class, and let ${}^C\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ denote the restriction of ${}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ to C. Moreover, $\iota_{i*}\colon H^2(H_i,\mathbb{R}) \longrightarrow H^4(X,\mathbb{R})$ denotes the Gysin map induced by $\iota_i\colon H_i \longrightarrow X$. Then, the second parabolic Chern character is given as follows.

$$\begin{aligned}\mathrm{ch}_2(\mathcal{P}_*\mathcal{V}) :=\ & \mathrm{ch}_2(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) - \sum_{i\in\Lambda}\sum_{a_i-1<b\le a_i} b\cdot\iota_{i*}\big(c_1\big({}^i\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)\big)\big)\\ &+\frac{1}{2}\sum_{i\in\Lambda}\sum_{a_i-1<b\le a_i} b^2 \operatorname{rank}\big({}^i\mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})\big)\cdot[H_i]^2\\ &+\frac{1}{2}\sum_{\substack{(i,j)\in\Lambda^2\\ i\neq j}}\sum_{C\in\mathrm{Irr}(H_i\cap H_j)}\sum_{\substack{a_i-1<c_i\le a_i\\ a_j-1<c_j\le a_j}} c_i\cdot c_j \operatorname{rank} {}^C\mathrm{Gr}^F_{(c_i,c_j)}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})\cdot[C].\end{aligned}$$

Remark 2.10. The higher Chern character for filtered sheaves was defined by Iyer and Simpson [29] in a systematic way. In this paper, we adopt the definition of $\mathrm{ch}_2(\mathcal{P}_*\mathcal{V})$ in [46].

2.4 Good filtered $\boldsymbol{\lambda}$-flat bundles

Let X be a complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i\in\Lambda} H_i$.

2.4.1 Good set of irregular values at $\boldsymbol{P}$

Let P be any point of H. We take an admissible holomorphic coordinate neighbourhood $(X_P, z_1, \dots, z_n)$ around P. Let $f \in \mathcal{O}_X(*H)_P$. If $f \in \mathcal{O}_{X,P}$, we set $\mathrm{ord}(f) := (0,\dots,0) \in \mathbb{R}^{\ell(P)}$. If there exists $\boldsymbol{n} \in \mathbb{Z}^{\ell(P)}_{\le 0} \setminus \{(0,\dots,0)\}$ such that (i) $g := f\prod z_i^{-n_i} \in \mathcal{O}_{X,P}$, (ii) $g(P) \neq 0$, then we set $\mathrm{ord}(f) := \boldsymbol{n}$. Otherwise, $\mathrm{ord}(f)$ is not defined.

For any $\mathfrak{a} \in \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$, we take a lift $\widetilde{\mathfrak{a}} \in \mathcal{O}_X(*H)_P$. If $\mathrm{ord}(\widetilde{\mathfrak{a}})$ is defined, we set $\mathrm{ord}(\mathfrak{a}) := \mathrm{ord}(\widetilde{\mathfrak{a}})$. Otherwise, $\mathrm{ord}(\mathfrak{a})$ is not defined. Note that it is independent of the choice of a lift $\widetilde{\mathfrak{a}}$.

Let $\mathcal{I}_P \subset \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$ be a finite subset. We say that $\mathcal{I}_P$ is a good set of irregular values if the following conditions are satisfied:

- $\mathrm{ord}(\mathfrak{a})$ is defined for any $\mathfrak{a} \in \mathcal{I}_P$.
- $\mathrm{ord}(\mathfrak{a} - \mathfrak{b})$ is defined for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P$.
- $\{\mathrm{ord}(\mathfrak{a} - \mathfrak{b}) \mid \mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P\}$ is totally ordered with respect to the order $\leq_{\mathbb{Z}^{\ell(P)}}$. Here, we define $\boldsymbol{n} \leq_{\mathbb{Z}^{\ell(P)}} \boldsymbol{n}'$ if $n_i \leq n_i'$ for any i.

2.4.2 Good filtered λ-flat bundles

Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(*H)$-module with a flat λ-connection. Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over $\mathcal{V}$. For any $P \in X$, let $\mathcal{O}_{X,\widehat{P}}$ denote the completion of the local ring $\mathcal{O}_{X,P}$ with respect to the maximal ideal. Note that Remark 2.8 has a natural generalization to filtered λ-flat bundles. We say that $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ is unramifiedly good at P if the following holds:

- There exist a good set of irregular values $\mathcal{I}_P \subset \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$ and a decomposition of filtered λ-flat bundles
$$\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big) \otimes \mathcal{O}_{X,\widehat{P}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}_P} \big(\mathcal{P}_*\mathcal{V}_\mathfrak{a}, \mathbb{D}^\lambda_\mathfrak{a}\big) \tag{2.4}$$
such that $\mathbb{D}^\lambda_\mathfrak{a} - d\widetilde{\mathfrak{a}}\,\mathrm{id}_{\mathcal{V}_\mathfrak{a}}$ are logarithmic with respect to the lattices $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_\mathfrak{a}$ for any $\boldsymbol{a} \in \mathbb{R}^{\ell(P)}$ and $\mathfrak{a} \in \mathcal{I}_P$, i.e.,
$$\big(\mathbb{D}^\lambda_\mathfrak{a} - d\widetilde{\mathfrak{a}}\,\mathrm{id}_{\mathcal{V}_\mathfrak{a}}\big)\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_\mathfrak{a} \subset \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_\mathfrak{a} \otimes \Omega^1_X(\log H). \tag{2.5}$$
Here, $\widetilde{\mathfrak{a}}$ denote lifts of $\mathfrak{a}$ to $\mathcal{O}_X(*H)_P$.

We say that $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ is good at P if the following holds:

- There exist a neighbourhood X_P of P in X and a covering map $\varphi_P\colon X_P' \longrightarrow X_P$ ramified over $H_P = H \cap X_P$ such that $\varphi_P^*(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ is unramifiedly good at $\varphi_P^{-1}(P)$. (See Section 2.3.2 for the pull back of filtered bundles.)

We say that $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ is good (resp. unramifiedly good) if it is good (resp. unramifiedly good) at any point of H.

2.5 Prolongation of holomorphic vector bundles with a Hermitian metric

Let X be any complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i \in \Lambda} H_i$. Let $(E, \overline{\partial}_E)$ be a holomorphic vector bundle on $X \setminus H$ with a Hermitian metric h. Let us recall the construction of $\mathcal{O}_X(*H)$-module $\mathcal{P}^h E$ and $\mathcal{O}_X$-modules $\mathcal{P}^h_{\boldsymbol{a}} E$ ($\boldsymbol{a} \in \mathbb{R}^\Lambda$).

Let $\boldsymbol{a} \in \mathbb{R}^\Lambda$. For any open subset $\mathcal{U} \subset X$, let $\mathcal{P}^h_{\boldsymbol{a}} E(\mathcal{U})$ be the space of holomorphic sections s of $E_{|\mathcal{U} \setminus H}$ satisfying the following condition:

- For any point P of $\mathcal{U} \cap H$, let $(X_P, z_1, \dots, z_n)$ be an admissible holomorphic coordinate neighbourhood around P such that X_P is relatively compact in $\mathcal{U}$. Set $\boldsymbol{c} = \kappa_P(\boldsymbol{a})$. (See Section 2.1.1.) Then,
$$|s|_h = O\left(\prod_{i=1}^{\ell(P)} |z_i|^{-c_i - \epsilon}\right)$$
holds on $X_P \setminus H$ for any $\epsilon > 0$.

We obtain an $\mathcal{O}_X$-module $\mathcal{P}^h_{\boldsymbol{a}} E$. We set $\mathcal{P}^h E := \bigcup_{\boldsymbol{a} \in \mathbb{R}^\Lambda} \mathcal{P}^h_{\boldsymbol{a}} E$ which is an $\mathcal{O}_X(*H)$-module. Note that in general, $\mathcal{P}^h_{\boldsymbol{a}} E$ are not necessarily coherent $\mathcal{O}_X$-modules.

Definition 2.11. Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle over (X, H). Let $(E, \overline{\partial}_E)$ be the holomorphic vector bundle obtained as the restriction of $\mathcal{V}$ to $X \setminus H$. A Hermitian metric h is called adapted to $\mathcal{P}_*\mathcal{V}$ if $\mathcal{P}^h_* E = \mathcal{P}_*\mathcal{V}$ in $\iota_*(E) = \iota_*(\mathcal{V}_{|X \setminus H})$, where $\iota\colon X \setminus H \longrightarrow X$ denotes the inclusion.

2.5.1 A sufficient condition

We mention a useful sufficient condition for $\mathcal{P}^h_*E$ to be a filtered bundle, although we do not use it in this paper. Let $g_{X\setminus H}$ be a Kähler metric satisfying the following condition [12]:

- For any $P\in H$, take an admissible holomorphic coordinate neighbourhood $(X_P,z_1,\dots,z_n)$ around P such that X_P is isomorphic to $\prod_{i=1}^n\{|z_i|<1\}$ by the coordinate system. Set $X'_P:=\prod_{i=1}^n\{|z_i|<1/2\}$. Then, $g_{|X'_P\setminus H}$ is mutually bounded with the restriction of the Poincaré metric
$$\sum_{i=1}^{\ell(P)}\frac{\mathrm{d}z_i\,\mathrm{d}\overline{z}_i}{|z_i|^2(\log|z_i|^2)^2}+\sum_{i=\ell(P)+1}^{n}\mathrm{d}z_i\,\mathrm{d}\overline{z}_i.$$

A Hermitian metric h of $\bigl(E,\overline{\partial}_E\bigr)$ is called acceptable if the curvature of the Chern connection is bounded with respect to h and $g_{X\setminus H}$. The following theorem is proved in [51, Theorem 21.3.1].

Theorem 2.12. *If h is acceptable, then $\mathcal{P}^h_*E$ is a filtered bundle, and $\mathcal{P}^hE$ is a locally free $\mathcal{O}_X(*H)$-module.*

2.6 Harmonic bundles

2.6.1 Pluri-harmonic metrics for $\boldsymbol{\lambda}$-flat bundles

Let Y be any complex manifold. Let E be a C^∞-vector bundle on Y. Let $A^{p,q}(E)$ denote the space of C^∞-sections of $\Omega^{p,q}\otimes E$. We set $A^\ell(E):=\bigoplus_{p+q=\ell}A^{p,q}(E)$. In this context, a λ-connection of E is a differential operator $\mathbb{D}^\lambda\colon A^0(E)\longrightarrow A^1(E)$ such that $\mathbb{D}^\lambda(fs)=f\mathbb{D}^\lambda(s)+\bigl(\lambda\partial_Y+\overline{\partial}_Y\bigr)f\otimes s$ for any $f\in C^\infty(Y)$ and $s\in A^0(E)$. We obtain a section $\mathbb{D}^\lambda\circ\mathbb{D}^\lambda\in A^2(\mathrm{End}(E))$. A λ-connection is called flat if $\mathbb{D}^\lambda\circ\mathbb{D}^\lambda=0$.

Let $\bigl(E,\mathbb{D}^\lambda\bigr)$ be a λ-flat bundle on Y. We decompose $\mathbb{D}^\lambda=\mathrm{d}''_E+\mathrm{d}'_E$ into the $(0,1)$-part and the $(1,0)$-part. Then, $\bigl(E,\mathrm{d}''_E\bigr)$ is a holomorphic vector bundle. Let h be a Hermitian metric of E. From h and d''_E, we obtain the differential operator $\delta'_{E,h}$ such that $\mathrm{d}''_E+\delta'_{E,h}$ is a Chern connection. From d'_E and h, we obtain the $(0,1)$-operator $\delta''_{E,h}$ determined by $\lambda\partial h(u,v)=h(\mathrm{d}'_Eu,v)+h(u,\delta''_{E,h}v)$. As in [49, Section 2.2.1], we obtain the operators
$$\overline{\partial}_{E,h}:=\frac{1}{1+|\lambda|^2}\bigl(\mathrm{d}''_E+\lambda\delta'_{E,h}\bigr),\qquad \partial_{E,h}:=\frac{1}{1+|\lambda|^2}\bigl(\overline{\lambda}\mathrm{d}'_E+\delta'_{E,h}\bigr),$$
$$\theta^\dagger_{E,h}:=\frac{1}{1+|\lambda|^2}\bigl(\overline{\lambda}\mathrm{d}''_E-\delta''_{E,h}\bigr),\qquad \theta_{E,h}:=\frac{1}{1+|\lambda|^2}\bigl(\mathrm{d}'_E-\lambda\partial'_{E,h}\bigr).$$
Note that $\mathbb{D}^\lambda=\overline{\partial}_{E,h}+\theta_{E,h}+\lambda\bigl(\partial_{E,h}+\theta^\dagger_{E,h}\bigr)$. We set $\mathbb{D}^{\lambda\star}_{E,h}:=\delta'_{E,h}-\delta''_{E,h}=\partial_{E,h}+\theta^\dagger_{E,h}-\overline{\lambda}\bigl(\overline{\partial}_{E,h}+\theta_{E,h}\bigr)$, and $G(h):=\bigl[\mathbb{D}^\lambda,\mathbb{D}^{\lambda\star}_{E,h}\bigr]$.

Definition 2.13. h is called a pluri-harmonic metric of $\bigl(E,\mathbb{D}^\lambda\bigr)$ if $G(h)=0$. Such a tuple $\bigl(E,\mathbb{D}^\lambda,h\bigr)$ is called a harmonic bundle.

If $\lambda\neq0$, because $\bigl(1+|\lambda|^2\bigr)\bigl(\overline{\partial}_{E,h}+\theta_{E,h}\bigr)=\mathbb{D}^\lambda-\lambda\mathbb{D}^{\lambda\star}_{E,h}$, and $\bigl(\mathbb{D}^\lambda\bigr)^2=\bigl(\mathbb{D}^{\lambda\star}_{E,h}\bigr)^2=0$, we obtain
$$G(h)=-\frac{(1+|\lambda|^2)^2}{\lambda}\bigl(\overline{\partial}_{E,h}+\theta_{E,h}\bigr)^2=-\frac{(1+|\lambda|^2)^2}{\lambda}\bigl(\overline{\partial}^2_{E,h}+\overline{\partial}_{E,h}\theta_{E,h}+\theta^2_{E,h}\bigr).$$
Hence, $G(h)=0$ implies that $\bigl(E,\overline{\partial}_{E,h},\theta_{E,h}\bigr)$ is a Higgs bundle. The metric h is a pluri-harmonic metric for $\bigl(E,\overline{\partial}_{E,h},\theta_{E,h}\bigr)$. Conversely, if h is a pluri-harmonic metric for a Higgs bundle $\bigl(E,\overline{\partial}_E,\theta\bigr)$, we obtain the Chern connection $\overline{\partial}_E+\partial_{E,h}$ associated with $\overline{\partial}_E$ and h, and the adjoint $\theta^\dagger_h$ of θ with respect to h. We obtain a flat λ-connection $\mathbb{D}^\lambda_h=\overline{\partial}_E+\lambda\theta_h+\lambda\partial_{E,h}+\theta$. The metric h is a pluri-harmonic metric for $(E,\mathbb{D}^\lambda_h)$.

Remark 2.14. If $\lambda = 0$, a flat 0-connection is equivalent to a Higgs bundle $\big(E, \overline{\partial}_E, \theta\big)$ by the relation $\mathbb{D}^1 = \overline{\partial}_E + \theta$. In this case, we obtain $G(h) = \big[\overline{\partial}_E + \theta, \partial_{E,h} + \theta_h^\dagger\big]$, and hence $2G(h)$ is equal to the curvature $F(h)$ of the connection $\mathbb{D}_h^1 = \overline{\partial}_E + \theta_h^\dagger + \partial_{E,h} + \theta$.

2.6.2 The case $\lambda \neq 0$

Let $G(h) = G(h)^{2,0} + G(h)^{1,1} + G(h)^{0,2}$ denote the decomposition into (p,q)-parts. If $G(h) = 0$, we clearly obtain $G(h)^{1,1} = 0$. If $\lambda \neq 0$, we obtain the converse.

Proposition 2.15. *Suppose $\lambda \neq 0$. If $G(h)^{1,1} = 0$, we obtain $G(h) = 0$, i.e., h is a pluriharmonic metric of $\big(E, \mathbb{D}^\lambda\big)$.*

Proof. As in [49, Lemma 2.28], the following holds:

$$\overline{\lambda}^{-1}\partial_{E,h}^2 + \lambda^{-1}\theta_{E,h}^2 = 0, \qquad \lambda^{-1}\overline{\partial}_{E,h}^2 + \overline{\lambda}^{-1}\big(\theta_{E,h}^\dagger\big)^2 = 0. \tag{2.6}$$

It is easy to check that $\overline{\partial}_{E,h}^2 = -(\partial_{E,h}^2)^\dagger$, $\big(\theta_{E,h}^\dagger\big)^2 = -(\theta_{E,h}^2)^\dagger$ and $\big(\overline{\partial}_{E,h}\theta_{E,h}\big)^\dagger = \partial_{E,h}\theta_{E,h}^\dagger$.

From the flatness $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$, we obtain

$$(\lambda\partial_{E,h} + \theta_{E,h})^2 = \lambda^2\partial_{E,h}^2 + \lambda\partial_{E,h}\theta_{E,h} + \theta_{E,h}^2 = 0, \tag{2.7}$$

$$\begin{aligned}&\big[\overline{\partial}_{E,h} + \lambda\theta_{E,h}^\dagger, \lambda\partial_{E,h} + \theta_{E,h}\big]\\ &\qquad = \lambda\big(\big[\overline{\partial}_{E,h}, \partial_{E,h}\big] + \big[\theta_{E,h}\theta_{E,h}^\dagger\big]\big) + \overline{\partial}_{E,h}\theta_{E,h} + \lambda^2\partial_{E,h}\theta_{E,h}^\dagger = 0.\end{aligned} \tag{2.8}$$

From (2.6) and (2.7), we obtain

$$\partial_{E,h}\theta_{E,h} = -\lambda^{-1}\big(1 - |\lambda|^2\big)\theta_{E,h}^2.$$

Because $(\overline{\partial}_{E,h}\theta_{E,h}^\dagger)^\dagger = \partial_{E,h}\theta_{E,h}$, we obtain

$$\overline{\partial}_{E,h}\theta_{E,h}^\dagger = \overline{\lambda}^{-1}\big(1 - |\lambda|^2\big)\big(\theta_{E,h}^\dagger\big)^2. \tag{2.9}$$

Note that $G(h)^{1,1} = 0$ is equivalent to $\overline{\partial}_{E,h}\theta_{E,h} = 0$ and $\partial_{E,h}\theta_{E,h}^\dagger = 0$. To obtain $G(h) = 0$, it is enough to prove $\operatorname{Tr}\big(\theta_{E,h}^2\big(\theta_{E,h}^\dagger\big)^2\big) = 0$. Indeed, there exists $C \neq 0$ depending on $\dim Y$ such that for any Kähler form ω of Y we obtain $\operatorname{Tr}\big(\theta_{E,h}^2\big(\theta_{E,h}^\dagger\big)^2\big)\omega^{\dim Y - 2} = C\big|\theta_{E,h}^2\big|_{h,\omega}^2\omega^{\dim Y}$. Hence, the vanishing $\operatorname{Tr}\big(\theta_{E,h}^2\big(\theta_{E,h}^\dagger\big)^2\big) = 0$ implies $\theta_{E,h}^2 = \big(\theta_{E,h}^\dagger\big)^2 = 0$ and $\partial_{E,h}^2 = \overline{\partial}_{E,h}^2 = 0$.

From (2.9) and $\partial_{E,h}\theta_{E,h}^\dagger = 0$, we obtain $\partial_{E,h}\overline{\partial}_{E,h}\theta_{E,h}^\dagger = \overline{\lambda}^{-1}\big(1 - |\lambda|^2\big)\partial_{E,h}\big((\theta_{E,h}^\dagger)^2\big) = 0$. We also have $\overline{\partial}_{E,h}\partial_{E,h}\theta_{E,h}^\dagger = 0$. Hence, we obtain the following equality:

$$0 = \operatorname{Tr}\big(\theta_{E,h}\partial_{E,h}\overline{\partial}_{E,h}\theta_{E,h}^\dagger\big) = \operatorname{Tr}\big(\theta_{E,h} \cdot \big[\overline{\partial}_{E,h}\partial_{E,h} + \partial_{E,h}\overline{\partial}_{E,h}, \theta_{E,h}^\dagger\big]\big). \tag{2.10}$$

From (2.8), $\overline{\partial}_{E,h}\theta_{E,h} = 0$ and $\partial_{E,h}\theta_{E,h}^\dagger = 0$, we obtain

$$\big[\overline{\partial}_{E,h}, \partial_{E,h}\big] + \big[\theta_{E,h}, \theta_{E,h}^\dagger\big] = 0.$$

Hence, we obtain the following:

$$\begin{aligned}\operatorname{Tr}\big(\theta_{E,h} \cdot \big[\overline{\partial}_{E,h}\partial_{E,h} + \partial_{E,h}\overline{\partial}_{E,h}, \theta_{E,h}^\dagger\big]\big) &= \operatorname{Tr}\big(\theta_{E,h} \cdot \big[-\big[\theta_{E,h}, \theta_{E,h}^\dagger\big], \theta_{E,h}^\dagger\big]\big)\\ &= -2\operatorname{Tr}\big(\theta_{E,h}^2\big(\theta_{E,h}^\dagger\big)^2\big).\end{aligned} \tag{2.11}$$

We obtain the claim of the proposition from (2.10) and (2.11). ■

By using Proposition 2.15 we can improve [49, Corollary 2.30] as follows.

Corollary 2.16. *If $\lambda \neq 0$, the pluri-harmonicity of the metric h is equivalent to the vanishing $G(h)^{1,1} = 0$, i.e., $\overline{\partial}_{E,h}\theta_{E,h} = 0$.*

Remark 2.17. In [49, Lemma 2.29], the claim $\big[\overline{\partial}_{V,h}, \partial_{V,h}\big] + \big[\theta_{V,h}, \theta^{\dagger}_{V,h}\big] = 0$ is incorrect, in general. The author thanks Pengfei Huang for pointing out it.

Remark 2.18. If $\lambda = 1$, the claim of Proposition 2.15 also follows from a Bochner type formula [48, Proposition 21.39], which originally goes back to the study of Simpson [64] in the context of harmonic bundles, the study of Corlette [11] in the context of harmonic metrics for flat bundles on Riemannian manifolds, and the study of Siu [69] in the context of harmonic maps.

2.7 Wild harmonic bundles

2.7.1 Higgs case

Let X be a complex manifold with a simple normal crossing hypersurface $H = \bigcup_{i\in\Lambda} H_i$. Let $\big(E, \overline{\partial}_E, \theta, h\big)$ be a harmonic bundle on $X \setminus H$. It is called wild on (X, H) if the following holds:

- Let $\Sigma_\theta \subset T^*(X \setminus H)$ denote the spectral cover of θ, i.e., Σ_θ denotes the support of the coherent $\mathcal{O}_{T^*(X\setminus H)}$-module induced by $\big(E, \overline{\partial}_E, \theta\big)$. Then, the closure of Σ_θ in the relatively projective completion of T^*X with respect to X is complex analytic.

A wild harmonic bundle $\big(E, \overline{\partial}_E, \theta, h\big)$ is called unramifiedly good at $P \in H$ if the following holds:

- There exists a good set of irregular values $\mathcal{I}_P \subset \mathcal{O}_X(*H)_P/\mathcal{O}_{X,P}$, a neighbourhood X_P, and a decomposition

  $$\big(E, \overline{\partial}_E, \theta\big)_{|X_P\setminus H} = \bigoplus_{\mathfrak{a}\in\mathcal{I}_P} \big(E_{\mathfrak{a}}, \overline{\partial}_{E_{\mathfrak{a}}}, \theta_{\mathfrak{a}}\big)$$

  such that the closure of the spectral cover $\Sigma_{\mathfrak{a}}$ of $\theta_{\mathfrak{a}} - \mathrm{d}\widetilde{\mathfrak{a}}\,\mathrm{id}_{E_{\mathfrak{a}}}$ in $T^*X_P(\log(X_P \cap H))$ is proper over X_P, where $\widetilde{\mathfrak{a}}$ denote lifts of $\mathfrak{a}$ to $\mathcal{O}_X(*H)_P$.

A wild harmonic bundle $\big(E, \overline{\partial}_E, \theta, h\big)$ is called good at $P \in H$ if the following holds:

- There exist a neighbourhood X_P and a covering $\varphi_P\colon X'_P \longrightarrow X_P$ ramified along H'_P such that the pull back $\varphi_P^{-1}\big(E, \overline{\partial}_E, \theta, h\big)_{|X_P}$ is unramifiedly good wild at any point of $\varphi_P^{-1}(H)$.

We say that $\big(E, \overline{\partial}_E, \theta, h\big)$ is good wild (resp. unramifiedly good wild) on (X, H) if it is good wild (resp. unramifiedly good wild) at any point of H.

Note that not every wild harmonic bundle on (X, H) is necessarily good on (X, H). But, the following is known [55, Corollary 15.2.8].

Theorem 2.19. *Let $\big(E, \overline{\partial}_E, \theta, h\big)$ be a wild harmonic bundle on (X, H). Then, there exists a proper birational morphism $\varphi\colon X' \longrightarrow X$ of complex manifolds such that (i) $H' := \varphi^{-1}(H)$ is simple normal crossing, (ii) $X' \setminus H' \simeq X \setminus H$, (iii) $\varphi^{-1}\big(E, \overline{\partial}_E, \theta, h\big)$ is good wild on (X', H').*

2.7.2 The case of λ-flat bundles

A λ-flat bundle $\big(E, \mathbb{D}^\lambda\big)$ with a pluri-harmonic metric h on $X \setminus H$ is called (good, unramifiedly good) wild if the associated Higgs bundle with a pluri-harmonic metric $\big(E, \overline{\partial}_{E,h}, \theta_{E,h}, h\big)$ is a (good, unramifiedly good) wild harmonic bundle.

2.7.3 Prolongation of good wild harmonic bundles to good filtered λ-flat bundles

The following is one of the fundamental theorems in the study of wild harmonic bundles [51, Theorem 7.4.3].

Theorem 2.20. *If $(E,\mathbb{D}^\lambda,h)$ is a good wild harmonic bundle on (X,H), then $(\mathcal{P}^h_*E,\mathbb{D}^\lambda)$ is a good filtered λ-flat bundle on (X,H).*

The following is a consequence of the norm estimate for good wild harmonic bundles [51, Theorem 11.7.2].

Theorem 2.21. *Let $(E,\mathbb{D}^\lambda,h_i)$ $(i=1,2)$ be good wild harmonic bundles on $X\setminus H$ such that $\mathcal{P}^{h_1}_*E=\mathcal{P}^{h_2}_*E$. Then, h_i are mutually bounded around any point of H.*

2.7.4 Prolongation of good wild harmonic bundles in the projective case

Suppose that X is projective and connected. Let L be any ample line bundle on X. The following is proved in [51, Propositions 13.6.1 and 13.6.4].

Proposition 2.22. *Let $(E,\mathbb{D}^\lambda,h)$ be a good wild harmonic bundle on (X,H).*

- *$(\mathcal{P}^h_*E,\mathbb{D}^\lambda)$ is μ_L-polystable with $\mu_L(\mathcal{P}^h_*E)=0$.*
- *We obtain $c_1(\mathcal{P}_*E)=0$ and $\int_X \mathrm{ch}_2(\mathcal{P}_*E)c_1(L)^{\dim X-2}=0$.*
- *Let h' be another pluri-harmonic metric of $(E,\mathbb{D}^\lambda,h)$ such that $\mathcal{P}^{h'}_*E=\mathcal{P}^h_*E$. Then, there exists a decomposition of the λ-flat bundle $(E,\mathbb{D}^\lambda)=\bigoplus(E_j,\mathbb{D}^\lambda_j)$ such that (i) the decomposition is orthogonal with respect to both h and h', (ii) $h_{|E_i}=a_i\cdot h'_{|E_i}$ for some $a_i>0$.*
- *Let $(\mathcal{P}_*\mathcal{V}_1,\mathbb{D}^\lambda_1)$ be any direct summand of $(\mathcal{P}^h_*E,\mathbb{D}^\lambda)$. Let $(E_1,\mathbb{D}^\lambda_1)$ be the λ-flat bundle on $X\setminus H$ obtained as the restriction of $(\mathcal{V}_1,\mathbb{D}^\lambda_1)$, and let h_1 be the metric of E_1 induced by h. Then, $(E_1,\mathbb{D}^\lambda_1,h_1)$ is a harmonic bundle. In particular, we obtain $c_1(\mathcal{P}_*\mathcal{V}_1)=0$ and $\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_1)c_1(L)^{\dim X-2}=0$.*

2.8 Main existence theorem in this paper

Let X be a smooth connected projective complex manifold with a simple normal crossing hypersurface H. Let L be any ample line bundle on X. Let $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X,H). Let $(E,\overline{\partial}_E,\mathbb{D}^\lambda)$ be the λ-flat bundle obtained as the restriction of $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ to $X\setminus H$.

Theorem 2.23. *Suppose that $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ is μ_L-polystable, and that the following vanishing holds:*

$$\mu_L(\mathcal{P}_*\mathcal{V})=0,\qquad \int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}=0. \tag{2.12}$$

Then, there exists a pluri-harmonic metric h of $(E,\overline{\partial}_E,\mathbb{D}^\lambda)$ such that $(\mathcal{V},\mathbb{D}^\lambda)_{|X\setminus H}\simeq(E,\mathbb{D}^\lambda)$ extends to $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)\simeq(\mathcal{P}^h_*E,\mathbb{D}^\lambda)$.

We proved the claim of the theorem in the case $\lambda=1$ in [51, Theorem 16.1.1]. We shall explain the proof in Sections 3–7. Note that the one dimensional case is due to Biquard–Boalch [3].

Corollary 2.24. *There exists the equivalence of the following objects for each λ:*

- *Good wild harmonic bundles on (X,H).*
- *μ_L-polystable good filtered λ-flat bundles $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ satisfying the condition (2.12).*

Remark 2.25. One of the referees raised the following interesting question. Let L be a big and nef line bundle on X such that there exists a positive current T representing $c_1(L)$ whose restriction to $X \setminus H$ is a smooth Kähler form with at most Poincaré growth near H. We do not assume that L is ample. We can define the slope $\mu_L(\mathcal{P}_*\mathcal{V})$ for a filtered sheaves, by using which we can introduce a stability condition for good filtered λ-flat bundles. Then, we may ask whether a statement similar to Theorem 2.23 holds. This question might also be related with a generalization of Kobayashi–Hitchin correspondence to the context of $\mathcal{D}$-modules.

2.8.1 Outline of the proof Theorem 2.23

Let us explain a rough outline of the proof. We shall omit some technical details. Let $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ be a μ_L-stable good filtered λ-flat bundle on (X, H) such that $\mathrm{ch}_i(\mathcal{P}_*\mathcal{V}) = 0$ $(i = 1, 2)$. Let $(V, \mathbb{D}^\lambda)$ be the λ-flat bundle on $X \setminus H$ obtained as the restriction of $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ to $X \setminus H$.

In the case $\dim X = 1$, we shall apply the argument in [63] as follows. For each $P \in H$, we take a holomorphic coordinate neighbourhood (X_P, z_P) around P. We take a Kähler metric $g_{X\setminus H}$ such that $g_{X\setminus H|X_P\setminus\{P\}}$ $(P \in H)$ are mutually bounded with $|z_P|^{2\eta-2}\mathrm{d}z_P\,\mathrm{d}\overline{z}_P$ for some $\eta > 0$. If η is sufficiently small, there exists a Hermitian metric h_{in} of V such that (i) $\mathcal{P}_*^{h_{\mathrm{in}}}V = \mathcal{P}_*\mathcal{V}$, and (ii) $G(h_{\mathrm{in}})$ is bounded with respect to h_{in} and $g_{X\setminus H}$, (iii) $\det(h_{\mathrm{in}})$ is flat. (See Corollary 3.28. Though we state it as a corollary of Proposition 3.27, which also deals with a perturbation, it is easy to deduce it directly from the estimate in the tame case [63].) Moreover for any filtered λ-flat subsheaf $\mathcal{P}_*\mathcal{V}' \subset \mathcal{P}_*\mathcal{V}$, $\deg(\mathcal{P}_*\mathcal{V}')$ is equal to the analytic degree of $(\mathcal{V}', \mathbb{D}^\lambda)_{|X\setminus H}$ with respect to h_{in} and $g_{X\setminus H}$. Then, by [62, Theorem 1], if $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ is stable of degree 0, there exists a harmonic metric h of $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ such that h and h_{in} are mutually bounded (Theorem 4.1). Let us note that the proof allows us to obtain the inequality for the Donaldson functional $M(h_{\mathrm{in}}, h) \leq 0$ (Proposition 4.4). This inequality is useful for the study of the continuity of the family of harmonic metrics of some family of good filtered λ-flat bundles (Proposition 4.5).

For the higher dimensional case, we use the same strategy in [46, 49] and [51]. It is a key step to study the case $\dim X = 2$. There are two naive ideas which are not available as they are.

One is to apply [62, Theorem 1] by constructing a Hermitian metric h_{in} of V such that (i) $\mathcal{P}_*^{h_{\mathrm{in}}}V = \mathcal{P}_*\mathcal{V}$, (ii) $G(h_{\mathrm{in}})$ is dominated in an appropriate way, (iii) $\det(h_{\mathrm{in}})$ is flat. For the construction of such a Hermitian metric h_{in}, a compatibility condition seems necessary between the nilpotent parts of the induced endomorphisms $\mathrm{Res}_i(\mathbb{D}^\lambda)$ and $\mathrm{Res}_j(\mathbb{D}^\lambda)$ on ${}^{i,j}\mathrm{Gr}^F(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$. (See Section 3.5.3 for the endomorphisms $\mathrm{Res}_i(\mathbb{D}^\lambda)$.) Once we prove the existence of a pluri-harmonic, it turns out that such a compatibility condition is satisfied. However, before proving the existence, it is not clear whether such a compatibility condition is satisfied. As a result, it is difficult to construct a Hermitian metric h_{in} with the desired property, in general.

The other is to use Mehta–Ramanathan type theorem (Proposition 3.8), according to which there exists $m > 0$ such that for the 0-set $Y \subset X$ of a generic section of $H^0(X, L^{\otimes m})$, the restriction $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)_{|Y}$ is also stable. Hence, if we fix a flat metric $h_{\det(\mathcal{V})}$ of $\det(\mathcal{V})_{|X\setminus H}$ adapted to $\det(\mathcal{P}_*\mathcal{V})$, there exists a harmonic metric h_Y of $(V, \mathbb{D}^\lambda)_{|Y\setminus H}$ adapted to $\mathcal{P}_*\mathcal{V}_{|Y}$ such that $\det(h_Y) = h_{\det(\mathcal{V})|Y\setminus H}$. If we can prove that there exists a Hermitian metric h of V such that $h_{|Y} = h_Y$ for such generic hypersurfaces Y, then h should be the desired pluri-harmonic metric for $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$. But, the existence of such h is not clear.

Roughly speaking, we combine these two ideas as follows. For any small $\epsilon > 0$, there exists a filtered bundle $\mathcal{P}_*^{(\epsilon)}\mathcal{V}$ over $\mathcal{V}$ such that (i) $(\mathcal{P}_*^{(\epsilon)}\mathcal{V}, \mathbb{D}^\lambda)$ is a μ_L-stable good filtered λ-flat bundle, (ii) $\mathrm{Res}_i(\mathbb{D}^\lambda)$ are semisimple for $\mathcal{P}_*^{(\epsilon)}\mathcal{V}$, (iii) $\det(\mathcal{P}_*^{(\epsilon)}\mathcal{V}) = \det(\mathcal{P}_*\mathcal{V})$, (iv) the difference of $\mathcal{P}_*^{(\epsilon)}\mathcal{V}$ and $\mathcal{P}_*\mathcal{V}$ are dominated by ϵ. (See Section 3.7.2 for more precise conditions.) The last condition implies that $\lim_{\epsilon\to 0}\int \mathrm{ch}_2(\mathcal{P}_*^{(\epsilon)}) = 0$. For $(\mathcal{P}_*^{(\epsilon)}\mathcal{V}, \mathbb{D}^\lambda)$, we can construct $h_{\mathrm{in}}^{(\epsilon)}$ such that (*i*) $\mathcal{P}_*^{h_{\rm in}^{(\epsilon)}}V = \mathcal{P}_*\mathcal{V}$, (*ii*) $G\big(h_{\rm in}^{(\epsilon)}\big)$ is dominated in an appropriate way, (*iii*) $\det\big(h_{\rm in}^{(\epsilon)}\big) = h_{\det(\mathcal{V})}$. By [62, Theorem 1], there exists a Hermitian–Einstein metric $h_{\rm HE}^{(\epsilon)}$ of $\big(V,\mathbb{D}^\lambda\big)$ such that $h_{\rm HE}^{(\epsilon)}$ and $h_{\rm in}^{(\epsilon)}$ are mutually bounded, and that $\det\big(h_{\rm HE}^{(\epsilon)}\big) = h_{\det(\mathcal{V})}$. (See Section 3.1 for Hermitian–Einstein metrics of Higgs bundles.) Moreover, $G\big(h_{\rm HE}^{(\epsilon)}\big) \to 0$ in L^2 as $\epsilon \to 0$. Hence, we would like to construct the desired pluri-harmonic metric as $\lim_{\epsilon\to 0} h_{\rm HE}^{(\epsilon)}$. If ϵ is sufficiently small, $\big(\mathcal{P}_*^{(\epsilon)}\mathcal{V},\mathbb{D}^\lambda\big)_{|Y}$ is also stable for the 0-set Y of a generic section of $H^0\big(X,L^{\otimes m}\big)$, and hence $\big(\mathcal{P}_*^{(\epsilon)}\mathcal{V},\mathbb{D}^\lambda\big)_{|Y}$ has a harmonic metric $h_Y^{(\epsilon)}$ such that $\det\big(h_Y^{(\epsilon)}\big) = h_{\det(\mathcal{V})|Y\setminus H}$. By the continuity of a family of harmonic metrics mentioned above, the sequence $h_Y^{(\epsilon)}$ is convergent to h_Y as $\epsilon \to 0$ (Proposition 4.5). Because $h_{{\rm HE}\,|Y}^{(\epsilon)}$ is not necessarily a harmonic metric of $\big(\mathcal{P}_*^{(\epsilon)}\mathcal{V},\mathbb{D}^\lambda\big)_{|Y}$, it is not necessarily equal to $h_Y^{(\epsilon)}$. But, because the L^2-norm of $G\big(h_{{\rm HE}\,|Y}^{(\epsilon)}\big)$ is dominated by ϵ, we can deduce the convergence of the sequence $h_{{\rm HE}\,|Y}^{(\epsilon)}$ to h_Y as $\epsilon \to 0$ (Proposition 4.8). Hence, we obtain the convergence of $h_{\rm HE}^{(\epsilon)}$ almost everywhere, and the limit satisfies $h_{|Y} = h_Y$ for the 0-set of generic section s of $H^0\big(X,L^{\otimes m}\big)$. Thus, we can prove the theorem in the case $\dim X = 2$. (See Section 7.2 for a more precise argument.)

In the case $\dim X \geq 3$, we use an induction on $\dim X$. By the Mehta–Ramanathan type theorem, there exists $m > 0$ such that for the 0-sets Y_i $(i = 1,2)$ of generic sections of $H^0\big(X,L^{\otimes m}\big)$, $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|Y_i}$ and $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|Y_1\cap Y_2}$ are μ_L-stable. By fixing a flat metric $h_{\det(\mathcal{V})}$ for $\det\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$, there exist pluri-harmonic metric h_{Y_i} of $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|Y_i}$ such that $\det(h_{Y_i}) = h_{\det(\mathcal{V})|Y_i\setminus H}$. Because $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|Y_1\cap Y_2}$ is also μ_L-stable, we obtain that $h_{Y_1|(Y_1\cap Y_2)\setminus H} = h_{Y_2|(Y_1\cap Y_2)\setminus H}$. Hence, there exists a Hermitian metric h of $V_{|X\setminus(H\cup W)}$ for a finite subset W, such that $h_{|Y\setminus H} = h_Y$ for the 0-set Y of a generic section of $H^0\big(X,L^{\otimes m}\big)$. It is easy to see that h is the desired pluri-harmonic metric. (See Section 7.3 for a more precise argument.)

3 Preliminaries

3.1 Hermitian–Einstein metrics of λ-flat bundles

Let Y be a Kähler manifold with a Kähler form ω. Let $\big(E,\mathbb{D}^\lambda\big)$ be a λ-flat bundle on Y with a Hermitian metric. Recall that h is called a Hermitian–Einstein metric of the λ-flat bundle if $\Lambda_\omega G(h)^\perp = 0$, where $G(h)^\perp$ denote the trace-free part of $G(h)$, and Λ_ω denote the adjoint of the multiplication by ω (see [35, Section 3.2]). The following is a generalization of Kobayashi–Lübke inequality to the context of λ-flat bundles due to Simpson [62, Proposition 3.4].

Proposition 3.1 (Simpson)**.** *If h is a Hermitian–Einstein metric, there exists $C > 0$ depending only on $n = \dim Y$ such that the following holds:*

$$\operatorname{Tr}\big(\big(G(h)^\perp\big)^2\big)\omega^{n-2} = C\big|G(h)^\perp\big|^2_{h,\omega}\omega^n.$$

As a result, if $\operatorname{Tr}\big(\big(G(h)^\perp\big)^2\big)\omega^{n-2} = 0$*, then we obtain* $G(h)^\perp = 0$.

3.2 Rank one case

Let X be an n dimensional smooth connected projective variety with a simple normal crossing hypersurface H. Let ω be a Kähler form. Let $H = \bigcup_{i\in\Lambda} H_i$ be the irreducible decomposition. Let g_i be a C^∞-Hermitian metric of the line bundle $\mathcal{O}(H_i)$. Let σ_i denote the section of $\mathcal{O}_X(H_i)$ induced by the inclusion $\mathcal{O}_X \longrightarrow \mathcal{O}_X(H_i)$.

Let $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ be a good filtered λ-flat bundle on (X, H) of rank one. For each $i \in \Lambda$, there uniquely exists $a_i \in]-1, 0]$ such that ${}^i\mathrm{Gr}^F_{a_i}(\mathcal{P}_*\mathcal{V}) \neq 0$. Let A be the constant determined by

$$A \int_X \omega^n = 2\pi n\bigl(1 + |\lambda|^2\bigr) \int_X c_1(\mathcal{P}_*\mathcal{V})\omega^{n-1}.$$

The following proposition is standard.

Proposition 3.2. *There exists a Hermitian metric h of the line bundle $E := \mathcal{V}_{|X\setminus D}$ such that (i) $\sqrt{-1}\Lambda_\omega G(h) = A$, (ii) $h\prod_{i\in\Lambda} |\sigma_i|^{2a_i}_{g_i}$ is a Hermitian metric of $\mathcal{P}_{\boldsymbol{a}}(\mathcal{V})$ of C^∞-class. Such a metric is unique up to the multiplication by a positive constant. Moreover, if $c_1(\mathcal{P}_*\mathcal{V}) = 0$, then $R(h) = 0$ holds, and hence h is a pluri-harmonic metric of $(E, \mathbb{D}^\lambda)$.*

Proof. Note that $G(h) = \bigl(1 + |\lambda|^2\bigr)R(h)$ holds in the rank one case. (See [49, Lemma 2.31].) Let h_0' be a C^∞-metric of $\mathcal{P}_{\boldsymbol{a}}E$. We obtain the metric $h_0 := h_0' \cdot \prod_{i\in\Lambda} |\sigma_i|^{-2a_i}_{g_i}$ of E on $X \setminus H$. It is well known that $\frac{\sqrt{-1}}{2\pi}R(h_0)$ naturally extends to a closed $(1,1)$-form on X of C^∞-class which represents $c_1(\mathcal{P}_*E)$. By the condition of A, we obtain $\int_X \bigl(\sqrt{-1}\Lambda_\omega R(h_0) - \bigl(1+|\lambda|^2\bigr)^{-1}A\bigr)\omega^n = 0$. Note that $\sqrt{-1}\Lambda_\omega R(h_0 e^\varphi) = \sqrt{-1}\Lambda_\omega R(h_0) + \sqrt{-1}\Lambda_\omega \overline{\partial}\partial\varphi$. Hence, there exists an $\mathbb{R}$-valued C^∞-function φ_0 such that $\sqrt{-1}\Lambda_\omega R(h_0 e^{\varphi_0}) - \bigl(1 + |\lambda|^2\bigr)^{-1}A = 0$. The metric $h = h_0 e^{\varphi_0}$ has the desired property. The uniqueness is clear.

Suppose that $c_1(\mathcal{P}_*E) = 0$. In the rank one case, a Hermitian metric of E is a pluri-harmonic metric of $(E, \mathbb{D}^\lambda)$, if and only if $R(h) = 0$. Because the cohomology class of $R(h_0)$ is 0, there exists an $\mathbb{R}$-valued C^∞-function φ_0 such that $R(h_0 e^{\varphi_0}) = 0$ by the standard $\partial\overline{\partial}$-lemma. By the uniqueness, we obtain the second claim of the lemma. ■

For the metric h in Proposition 3.2, $\frac{\sqrt{-1}}{2\pi}R(h)$ induces a closed $(1,1)$-form on X of C^∞-class which represents $c_1(\mathcal{P}_*E)$.

3.3 β-subobject and socle for reflexive filtered λ-flat sheaves

Let X and H be as in Section 3.2. Let L be an ample line bundle L on X. For any coherent $\mathcal{O}_X$-module $\mathcal{M}$, we set $\deg_L(\mathcal{M}) := \int_X c_1(\mathcal{M})c_1(L)^{\dim X - 1}$.

3.3.1 β-subobjects

Let $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ be a reflexive filtered λ-flat sheaf on (X, H). For any $A \in \mathbb{R}$, let $\mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$ denote the family of saturated coherent subsheaves $\mathcal{F}$ of $\mathcal{P}_{\boldsymbol{0}}\mathcal{V}$ such that $\deg_L(\mathcal{F}) \geq -A$ and that $\mathcal{F}(*H)$ is a λ-flat subsheaf of $\mathcal{V}$. Any $\mathcal{F} \in \mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$ induces a reflexive filtered sheaf $\mathcal{P}_*(\mathcal{F}(*H))$ by $\mathcal{P}_{\boldsymbol{c}}(\mathcal{F}(*H)) := \mathcal{P}_{\boldsymbol{c}}\mathcal{V} \cap \mathcal{F}(*H)$ for any $\boldsymbol{c} \in \mathbb{R}^\Lambda$. We set $f_A(\mathcal{F}) := \mu_L(\mathcal{P}_*(\mathcal{F}(*H)))$. Thus, we obtain a function f_A on $\mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$.

Lemma 3.3. *The image $f_A\bigl(\mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)\bigr)$ is a finite subset of $\mathbb{R}$. In particular, f_A has the maximum.*

Proof. According to [20, Lemma 2.5], the family $\mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$ is bounded. Hence, by using the flattening stratifications [57, Section 8], it is easy to see that there exists a finite decomposition $\mathcal{S}(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A) = \coprod_{i=1}^N \mathcal{S}_i(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$ such that f_A is constant on each $\mathcal{S}_i(\mathcal{P}_{\boldsymbol{0}}\mathcal{V}, A)$. ■

It is standard that any reflexive filtered λ-flat sheaf has a β-subobject, i.e., the following holds.

Proposition 3.4. *For any reflexive filtered λ-flat sheaf $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$, there uniquely exists a non-zero λ-flat subsheaf $\mathcal{V}_0 \subset \mathcal{V}$ such that the following holds for any non-zero reflexive λ-flat subsheaf $\mathcal{V}' \subset \mathcal{V}$:*

- $\mu_L(\mathcal{P}_*\mathcal{V}') \leq \mu_L(\mathcal{P}_*\mathcal{V}_0)$ *holds.*
- *If* $\mu_L(\mathcal{P}_*\mathcal{V}') = \mu_L(\mathcal{P}_*\mathcal{V}_0)$ *holds, then we obtain* $\mathcal{V}' \subset \mathcal{V}_0$.

Proof. By the formula (2.1), there exists $N > 0$ such that the following holds for any saturated subsheaf $\mathcal{F} \subset \mathcal{P}_{\mathbf{0}}\mathcal{V}$:

$$|\deg_L(\mathcal{F}) - \operatorname{rank}(\mathcal{F})\mu_L(\mathcal{P}_*\mathcal{F}(*H))| < N.$$

We set $A_0 := |\deg_L(\mathcal{P}_{\mathbf{0}}\mathcal{V})| + 10N$. Let B_0 denote the maximum of f_{A_0}. Then, it is easy to see that $\mu_L(\mathcal{P}_*\mathcal{V}') \leq B_0$ for any saturated λ-flat subsheaf $\mathcal{V}' \subset \mathcal{V}$. Moreover, if $\mu_L(\mathcal{P}_*\mathcal{V}') = B_0$, then $\big(\mathcal{P}_*\mathcal{V}', \mathbb{D}^\lambda_{\mathcal{V}'}\big)$ is μ_L-semistable, where $\mathbb{D}^\lambda_{\mathcal{V}'}$ denote the flat λ-connection induced by $\mathbb{D}^\lambda$.

Suppose that the λ-flat subsheaves $\mathcal{V}_i \subset \mathcal{V}$ $(i = 1, 2)$ satisfy $\mu_L(\mathcal{P}_*\mathcal{V}_i) = B_0$. We obtain the subsheaf $\mathcal{V}_1 + \mathcal{V}_2 \subset \mathcal{V}$. Because $\mathcal{V}_1 + \mathcal{V}_2$ is a quotient of $\mathcal{V}_1 \oplus \mathcal{V}_2$, we obtain a filtered sheaf $\mathcal{P}_*(\mathcal{V}_1 + \mathcal{V}_2)$ over $\mathcal{V}_1 + \mathcal{V}_2$. induced by $\mathcal{P}_*\mathcal{V}_1 \oplus \mathcal{P}_*\mathcal{V}_2$. Then, by the μ_L-semistability of $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}^\lambda_i\big)$, we obtain that $B_0 = \mu_L(\mathcal{P}_*\mathcal{V}_1 \oplus \mathcal{P}_*\mathcal{V}_2) \leq \mu_L(\mathcal{P}_*(\mathcal{V}_1 + \mathcal{V}_2))$. Let $\mathcal{V}_3$ denote the saturated subsheaf of $\mathcal{V}$ generated by $\mathcal{V}_1 + \mathcal{V}_2$. We obtain a filtered sheaf $\mathcal{P}_*\mathcal{V}_3$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_3 = \mathcal{P}_{\boldsymbol{a}}(\mathcal{V}) \cap \mathcal{V}_3$. Because the natural morphism $\mathcal{P}_*(\mathcal{V}_1 + \mathcal{V}_2) \longrightarrow \mathcal{P}_*\mathcal{V}_3$ is generically an isomorphism, we obtain $\mu_L(\mathcal{P}_*(\mathcal{V}_1 + \mathcal{V}_2)) \leq \mu_L(\mathcal{P}_*\mathcal{V}_3) \leq B_0$ by Lemma 2.5. Hence, we obtain $\mu_L(\mathcal{P}_*\mathcal{V}_3) = B_0$. Then, the claim of the lemma is clear. ■

3.3.2 Socle

Let $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ be a μ_L-semistable reflexive filtered λ-flat sheaf on (X, H). Let $\mathcal{T}$ denote the family of saturated λ-flat subsheaves $\mathcal{V}' \subset \mathcal{V}$ such that the induced filtered λ-flat sheaf $\big(\mathcal{P}_*\mathcal{V}', \mathbb{D}^\lambda_{\mathcal{V}'}\big)$ is μ_L-stable with $\mu_L(\mathcal{P}_*\mathcal{V}') = \mu_L(\mathcal{P}_*\mathcal{V})$. Let $\mathcal{V}_1$ be the saturated $\mathcal{O}_X(*H)$-submodule of $\mathcal{V}$ generated by $\sum_{\mathcal{V}' \in \mathcal{T}} \mathcal{V}'$. It is a λ-flat subsheaf of $\mathcal{V}$.

Proposition 3.5. $\big(\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_{\mathcal{V}_1}\big)$ *is equal to the direct sum* $\bigoplus_{k=1}^{\ell} \big(\mathcal{P}_*\mathcal{V}^{(k)}, \mathbb{D}^\lambda_{\mathcal{V}^{(k)}}\big)$ *of a tuple of* μ_L*-stable filtered* λ*-flat subsheaves of* $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$*. In particular,* $\big(\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_1\big)$ *is* μ_L*-polystable. The filtered* λ*-flat subsheaf* $\big(\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_{\mathcal{V}_1}\big)$ *is called the socle of* $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$.

Proof. Let $\mathcal{V}^{(i)}$ $(i = 1, 2)$ be saturated λ-flat subsheaves of $\mathcal{V}$ such that (i) $\mu_L\big(\mathcal{P}_*\mathcal{V}^{(i)}\big) = \mu_L(\mathcal{P}_*\mathcal{V})$, (ii) $\big(\mathcal{P}_*\mathcal{V}^{(1)}, \mathbb{D}^\lambda_{\mathcal{V}^{(1)}}\big)$ is μ_L-semistable, (iii) $\big(\mathcal{P}_*\mathcal{V}^{(2)}, \mathbb{D}^\lambda_{\mathcal{V}^{(2)}}\big)$ is μ_L-stable.

Lemma 3.6. *Either* $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$ *or* $\mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} = 0$ *holds.*

Proof. Let us consider the morphism $\iota_1 - \iota_2 \colon \mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)} \longrightarrow \mathcal{V}$, where $\iota_i \colon \mathcal{V}^{(i)} \longrightarrow \mathcal{V}$ denote the inclusions. Let $\mathcal{K}$ denote the kernel. We obtain a filtered sheaf $\mathcal{P}_*\mathcal{K}$ over $\mathcal{K}$ by $\mathcal{P}_{\boldsymbol{a}}\mathcal{K} := \mathcal{K} \cap \mathcal{P}_{\boldsymbol{a}}(\mathcal{V}_1 \oplus \mathcal{V}_2)$. The projection $\mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)} \longrightarrow \mathcal{V}^{(2)}$ induces $\mathcal{K} \simeq \mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} =: \mathcal{I}$. It induces a morphism of filtered λ-flat sheaves $g \colon \big(\mathcal{P}_*\mathcal{K}, \mathbb{D}^\lambda_{\mathcal{K}}\big) \longrightarrow \big(\mathcal{P}_*\mathcal{V}^{(2)}, \mathbb{D}^\lambda_{\mathcal{V}^{(2)}}\big)$. We set $\mu_0 := \mu_L(\mathcal{P}_*\mathcal{V})$. Because $\bigoplus_{i=1,2} \big(\mathcal{P}_*\mathcal{V}^{(i)}, \mathbb{D}^\lambda_{\mathcal{V}^{(i)}}\big)$ and $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ are μ_L-semistable with the same slope μ_0, we obtain that $\big(\mathcal{P}_*\mathcal{K}, \mathbb{D}^\lambda_{\mathcal{K}}\big)$ is also μ_L-semistable with $\mu_L(\mathcal{P}_*\mathcal{K}) = \mu_0$.

Suppose that $\mathcal{K} \neq 0$, i.e., $\mathcal{I} \neq 0$. Because $\mathcal{I}$ is a subsheaf of $\mathcal{V}^{(2)}$, we also obtain a filtered sheaf $\mathcal{P}_*\mathcal{I}$ induced by $\mathcal{P}_*\mathcal{V}^{(2)}$. Because $\mathcal{I} \simeq \mathcal{K}$, we obtain a filtered sheaf $\mathcal{P}'_*\mathcal{I}$ over $\mathcal{I}$ induced by $\mathcal{P}_*\mathcal{K}$. Then, we obtain

$$\mu_0 = \mu_L(\mathcal{P}_*\mathcal{K}) = \mu_L(\mathcal{P}'_*\mathcal{I}) \leq \mu_L(\mathcal{P}_*\mathcal{I}) \leq \mu_L\big(\mathcal{P}_*\mathcal{V}^{(2)}\big) = \mu_0.$$

Because $\big(\mathcal{P}_*\mathcal{V}^{(2)}, \mathbb{D}^\lambda_{\mathcal{V}^{(2)}}\big)$ is μ_L-stable and because $\mathcal{I} \neq 0$, we obtain that $\operatorname{rank}(\mathcal{I}) = \operatorname{rank}\mathcal{V}^{(2)}$, i.e., $\mathcal{I}$ and $\mathcal{V}^{(2)}$ are generically isomorphic. Because $\mu_L(\mathcal{P}_*\mathcal{I}) = \mu_L\big(\mathcal{P}_*\mathcal{V}^{(2)}\big)$, Lemma 2.5 implies that $\mathcal{P}_*\mathcal{I} \longrightarrow \mathcal{P}_*\mathcal{V}^{(2)}$ is an isomorphism in codimension 1. Hence, there exists a closed algebraic subset $Z \subset X$ such that (i) the codimension of Z is larger than 2, (ii) $\mathcal{V}^{(2)}_{|X\setminus Z} \subset \mathcal{V}^{(1)}_{|X\setminus Z}$. Because $\mathcal{V}^{(1)}$ is reflexive we obtain that $\mathcal{V}^{(2)} \subset \mathcal{V}^{(1)}$. ■

Let us study the case where $\mathcal{V}^{(1)} \cap \mathcal{V}^{(2)} = 0$. Let $\mathcal{V}^{(3)}$ denote the saturated λ-flat subsheaf of $\mathcal{V}$ generated by $\mathcal{V}^{(1)} + \mathcal{V}^{(2)}$. Let $\mathcal{P}_*\mathcal{V}^{(3)}$ denote the filtered sheaf over $\mathcal{V}^{(3)}$ induced by $\mathcal{P}_*\mathcal{V}$.

Lemma 3.7. *$\big(\mathcal{P}_*\mathcal{V}^{(3)}, \mathbb{D}^\lambda_{\mathcal{V}^{(3)}}\big)$ is μ_L-semistable, and the induced morphism $g\colon \mathcal{P}_*\mathcal{V}^{(1)} \oplus \mathcal{P}_*\mathcal{V}^{(2)} \longrightarrow \mathcal{P}_*\mathcal{V}^{(3)}$ is an isomorphism in codimension one.*

Proof. We obtain $\mu_0 = \mu_L\big(\mathcal{P}_*\big(\mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)}\big)\big) \le \mu_L\big(\mathcal{P}_*\mathcal{V}^{(3)}\big) \le \mu_L(\mathcal{P}_*\mathcal{V}) = \mu_0$. Hence, we obtain that $\mu_L\big(\mathcal{P}_*\mathcal{V}^{(3)}\big) = \mu_0$ and that $\big(\mathcal{P}_*\mathcal{V}^{(3)}, \mathbb{D}^\lambda_{\mathcal{V}^{(3)}}\big)$ is μ_L-semistable. Because $g\colon \mathcal{P}_*\mathcal{V}^{(1)} \oplus \mathcal{P}_*\mathcal{V}^{(2)} \longrightarrow \mathcal{P}_*\mathcal{V}^{(3)}$ is generically an isomorphism, and because they have the same slope, g is an isomorphism in codimension one by Lemma 2.5. ■

By Lemma 3.7, it is easy to observe that there exists a finite sequence of reflexive λ-flat subsheaves $\mathcal{V}'_j$ $(j = 1, \dots, m)$ such that (i) the induced filtered λ-flat sheaves $\big(\mathcal{P}_*\mathcal{V}'_j, \mathbb{D}^\lambda_{\mathcal{V}'_j}\big)$ are μ_L-stable, (ii) the image of the induced morphism $g\colon \widetilde{\mathcal{V}} := \bigoplus \mathcal{V}'_j \longrightarrow \mathcal{V}_1$ is generically an isomorphism. Because $\mu_0 = \mu_L\big(\mathcal{P}_*\widetilde{\mathcal{V}}\big) \le \mu_L(\mathcal{P}_*\mathcal{V}_1) \le \mu_L(\mathcal{P}_*\mathcal{V}) = \mu_0$, we obtain that $\mu_L\big(\mathcal{P}_*\widetilde{\mathcal{V}}\big) = \mu_L(\mathcal{P}_*\mathcal{V}_1) = \mu_L(\mathcal{P}_*\mathcal{V})$. Hence, g is an isomorphism in codimension one by Lemma 2.5. Because both $\mathcal{P}_*\widetilde{\mathcal{V}}$ and $\mathcal{P}_*\mathcal{V}_1$ are reflexive, we obtain that $\mathcal{P}_*\widetilde{\mathcal{V}} \simeq \mathcal{P}_*\mathcal{V}_1$. Thus, we obtain Proposition 3.5. ■

3.4 Mehta–Ramanathan type theorems

Let X be a smooth connected n-dimensional projective variety with a simple normal crossing hypersurface H. Let $H = \bigcup_{i\in\Lambda} H_i$ be the irreducible decomposition. Let L be an ample line bundle on X.

3.4.1 Restriction to general curves

Let $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ be a reflexive filtered λ-flat sheaf on (X, H). There exists a Zariski closed subset $W \subset X$ with $\dim W < \dim H$ such that (i) the singular locus of H is contained in W, (ii) $\mathcal{P}_*\mathcal{V}_{|X\setminus W}$ is a filtered bundle on $(X \setminus W, H \setminus W)$.

Let Y be a smooth curve in X such that (i) $Y \cap W = \varnothing$, (ii) Y intersects with the smooth part of H transversally. Set $H_Y := H \cap Y$. We obtain a locally free $\mathcal{O}_Y(*H_Y)$-module $\mathcal{V}_{|Y}$. It is equipped with the induced flat λ-connection $\mathbb{D}^\lambda_{|Y}$. Let $\boldsymbol{b} \in \mathbb{R}^{H_Y}$. For any $P \in H_Y$, there exists $i \in \Lambda$ such that $P \in H_i$. We choose $\boldsymbol{a}(P, \boldsymbol{b}) \in \mathbb{R}^\Lambda$ such that $a(P, \boldsymbol{b})_i = b(P)$, and we obtain an $\mathcal{O}_{Y,P}$-submodule $\mathcal{P}_{\boldsymbol{b}}\big(\mathcal{V}_{|Y}\big)_P := \mathcal{P}_{\boldsymbol{a}(P,\boldsymbol{b})}(\mathcal{V})_P$ of $\big(\mathcal{V}_{|Y}\big)_P$, which is independent of the choice of $\boldsymbol{a}(P, \boldsymbol{b})$ as above. There exists a locally free $\mathcal{O}_Y$-module $\mathcal{P}_{\boldsymbol{b}}\big(\mathcal{V}_{|Y}\big) \subset \mathcal{V}_{|Y}$ whose stalk at P is $\mathcal{P}_{\boldsymbol{b}}\big(\mathcal{V}_{|Y}\big)_P$. Thus, we obtain a filtered λ-flat bundle $\big(\mathcal{P}_*\big(\mathcal{V}_{|Y}\big), \mathbb{D}^\lambda_{|Y}\big)$ which is denoted by $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|Y}$.

3.4.2 The stability condition

Proposition 3.8. *A reflexive filtered λ-flat sheaf $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ on (X, H) is μ_L-stable (resp. μ_L-semistable) if and only if the following holds:*

- *For any $m_1 > 0$, there exists $m > m_1$ such that $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|Y}$ is μ_L-stable (resp. μ_L-semistable), where Y denotes a generic 1-dimensional complete intersection of hypersurfaces of $L^{\otimes m}$.*

Proof. The case $\lambda = 1$ is already studied in [51, Section 13.2]. The case $\lambda \neq 0$ is reduced to the case $\lambda = 1$. As for the case $\lambda = 0$, we can prove the claim of the proposition by the argument in [46, Section 3.4], which closely follows the arguments of Mehta–Ramanathan [43, 44] and

Simpson [64]. We use $\mathcal{W}=\Omega^1(ND)$ for a large N instead of $\Omega^1(\log D)$ in [46, Section 3.4]. (See also [51, Section 13.2].) ■

3.4.3 Restrictions of morphisms and the polystability condition

Let us give a complement on the restriction of morphisms of reflexive filtered λ-flat sheaves to generic complete intersection curves, which is a variant of [64, Lemma 3.9]. Let $(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda)$ $(i=1,2)$ be reflexive filtered λ-flat sheaves on (X,H). Let $\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1,\mathbb{D}^\lambda_1),(\mathcal{P}_*\mathcal{V}_2,\mathbb{D}^\lambda_2)\big)$ denote the vector space of morphisms of filtered λ-flat sheaves $(\mathcal{P}_*\mathcal{V}_1,\mathbb{D}^\lambda_1)\longrightarrow(\mathcal{P}_*\mathcal{V}_2,\mathbb{D}^\lambda_2)$. We shall prove a refined claim (Proposition 3.16) of the following proposition in Sections 3.4.4–3.4.6.

Proposition 3.9. *There exists a positive integer $m_0>0$ such that the restriction*

$$\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1,\mathbb{D}^\lambda_1),(\mathcal{P}_*\mathcal{V}_2,\mathbb{D}^\lambda_2)\big)\longrightarrow\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1,\mathbb{D}^\lambda_1)_{|Y},(\mathcal{P}_*\mathcal{V}_2,\mathbb{D}^\lambda_2)_{|Y}\big)$$

is an isomorphism for a generic 1-dimensional complete intersection Y of hypersurfaces of $L^{\otimes m}$ $(m\geq m_0)$.

Before going to the proof of Proposition 3.9, we state a variant of Proposition 3.8 on the μ_L-polystability condition.

Corollary 3.10. *A reflexive filtered λ-flat sheaf $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ on (X,H) is μ_L-polystable if and only if the following holds:*

- *For any $m_1>0$, there exists $m>m_1$ such that $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}$ is μ_L-polystable, where Y denotes the 1-dimensional complete intersection of generic hypersurfaces of $L^{\otimes m}$.*

Proof. If $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ is μ_L-polystable, we obtain a decomposition $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)=\bigoplus(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda_i)$ into μ_L-stable filtered λ-flat sheaves. Applying Proposition 3.8 to each stable component, we obtain the "only if" claim.

Let m_1 be an integer larger than m_0 in Proposition 3.9 for $\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda),(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)\big)$. Suppose that there exists $m>m_1$ such that $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}$ is μ_L-polystable for a generic 1-dimensional complete intersection Y of hypersurfaces of $L^{\otimes m}$. We obtain the decomposition

$$(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}=\bigoplus_{i=1}^{\ell}(\mathcal{P}_*\mathcal{V}_{Y,i},\mathbb{D}^\lambda_{Y,i}) \tag{3.1}$$

into stable filtered Higgs bundles. Let $\pi_{Y,i}$ denote the endomorphisms of $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}$ obtained by composing the projection $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}\longrightarrow(\mathcal{P}_*\mathcal{V}_{Y,i},\mathbb{D}^\lambda_{Y,i})$ with respect to the decomposition (3.1), with the inclusion $(\mathcal{P}_*\mathcal{V}_{Y,i},\mathbb{D}^\lambda_{Y,i})\longrightarrow(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)_{|Y}$. Note that they satisfy $\pi_{Y,i}\circ\pi_{Y,i}=\pi_{Y,i}$, $\pi_{Y,i}\circ\pi_{Y,j}=0$ $(i\neq j)$ and $\sum\pi_{Y,i}=\mathrm{id}$. By Proposition 3.9, there uniquely exist the endomorphisms π_i of $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ such that $\pi_{i|Y}=\pi_{Y,i}$. By Proposition 3.9 again, they satisfy $\pi_i\circ\pi_i=\pi_i$, $\pi_i\circ\pi_j=0$ $(i\neq j)$ and $\sum\pi_i=\mathrm{id}$. Let $\mathcal{V}_i\subset\mathcal{V}$ denote the image of π_i. We define $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_i=\mathcal{V}_i\cap\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ for any $\boldsymbol{a}\in\mathbb{R}^\Lambda$. Because π_i are compatible with $\mathbb{D}^\lambda$ and the filtration $\mathcal{P}_*\mathcal{V}$, we obtain the decomposition $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)=\bigoplus(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda_i)$. By the construction, $(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda_i)_{|Y}=(\mathcal{P}_*\mathcal{V}_{Y,i},\mathbb{D}^\lambda_{Y,i})$ are stable. Hence, $(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda_i)$ are μ_L-stable with $\mu_L(\mathcal{P}_*\mathcal{V}_i,\mathbb{D}^\lambda_i)=\mu_L(\mathcal{P}_*\mathcal{V}_j,\mathbb{D}^\lambda_j)$ $(i\neq j)$, i.e., $(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda)$ is μ_L-polystable. ■

3.4.4 General Enriques–Severi lemma due to Mehta–Ramanathan

To prove Proposition 3.9, we recall the general Enriques–Severi lemma in [43]. Recall $n=\dim X$. For a positive integer m, let S_m denote the projective space of lines in $H^0\big(X,L^{\otimes m}\big)$. For sequences $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t_{>0}$ with $t\leq n-1$, we set $S_{\boldsymbol{m}}:=\prod S_{m_i}$. There exists the correspondence variety $Z_{\boldsymbol{m}}\subset X\times S_{\boldsymbol{m}}$, i.e., $Z_{\boldsymbol{m}}=\big\{(x,s_1,\dots,s_t)\in X\times S_{\boldsymbol{m}}\mid s_i(x)=0,\,1\leq i\leq t\big\}$. For any $s\in S_{\boldsymbol{m}}$, we set $X_s:=Z_{\boldsymbol{m}}\times_{S_{\boldsymbol{m}}}\times\{s\}$.

Let F be a coherent reflexive $\mathcal{O}_X$-module on X. For any $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t_{>0}$ with $t\leq n-1$, and for any $s\in S_{\boldsymbol{m}}$, we set $F_s:=F\otimes_{\mathcal{O}_X}\mathcal{O}_{X_s}$. For any integer m, let $F_s(m)=F_s\otimes_{\mathcal{O}_X}L^{\otimes m}$.

According to [43, Proposition 1.5], there exists a non-empty Zariski open subset $\widetilde{S}_{\boldsymbol{m}}\subset S_{\boldsymbol{m}}$ such that the following holds.

- $\widetilde{S}_{\boldsymbol{m}}\times_{S_{\boldsymbol{m}}}Z_{\boldsymbol{m}}\longrightarrow\widetilde{S}_{\boldsymbol{m}}$ is smooth.
- For any $s\in S_{\boldsymbol{m}}$, F_s is a reflexive $\mathcal{O}_{X_s}$-module.

In the proof of [43, Proposition 3.2], the following proposition is proved.

Proposition 3.11. *Let $t\leq n-2$. There exists a positive integer m_0 depending only on F such that the following holds:*

- *For any $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t_{>0}$ with $m_i\geq m_0$, there exists a non-empty Zariski open subset $U\subset\widetilde{S}_{\boldsymbol{m}}$ such that $H^1(X_s,F_s(-\ell))=0$ for any $s\in U$ and any $\ell\geq m_0$.*

Corollary 3.12. *Let $t\leq n-1$. There exists a positive integer m_0 depending only on F such that the following holds:*

- *For any $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t_{>0}$ with $m_i\geq m_0$, there exists a non-empty Zariski open subset $U\subset\widetilde{S}_{\boldsymbol{m}}$ such that $H^0(X_s,F_s(-\ell))=0$ for any $s\in U$ and any $\ell\geq m_0$.*

Proof. Let m_0 be a positive integer as in Proposition 3.11. We also assume $H^0(X,F(-\ell))=0$ for any $\ell\geq m_0$. We use an induction on t. For $\boldsymbol{m}=(m_1,\dots,m_t)$ with $m_i\geq m_0$, we set $\boldsymbol{m}'=(m_1,\dots,m_{t-1})$. By the assumption of the induction and Proposition 3.11, there exists a non-empty Zariski open subset $U_1'\subset\widetilde{S}_{\boldsymbol{m}'}$ such that $H^i(X_{s'},F_{s'}(-\ell))=0$ $(i=0,1)$ for any $s'\in U_1'$ and any $\ell\geq m_0$.

For any $s\in S_{\boldsymbol{m}}$, let s' denote the image s in $S_{\boldsymbol{m}'}$ by the projection $S_{\boldsymbol{m}}\longrightarrow S_{\boldsymbol{m}'}$. There exists the exact sequence

$$0\longrightarrow\mathcal{O}_{X_{s'}}(-m_t)\longrightarrow\mathcal{O}_{X_{s'}}\longrightarrow\mathcal{O}_{X_s}\longrightarrow 0.\tag{3.2}$$

By [43, Proposition 1.5], there exists a Zariski open subset $U_1\subset\widetilde{S}_{\boldsymbol{m}}$ such that if $s\in U_1$ then we obtain the following exact sequence from (3.2) by taking the tensor product with F:

$$0\longrightarrow F_{s'}(-m_t)\longrightarrow F_{s'}\longrightarrow F_s\longrightarrow 0.$$

We shrink U_1 so that U_1' contains the image of U_1 by the projection $S_{\boldsymbol{m}}\longrightarrow S_{\boldsymbol{m}'}$. Let $\ell\geq m_0$. For any $s\in U_1$, we obtain $H^0(X_{s'},F_{s'}(-\ell))=0$ and $H^1(X_{s'},F_{s'}(-\ell-m_t))=0$ because $s'\in U_1'$. Hence, we obtain $H^0(X_s,F_s(-\ell))=0$. ∎

Corollary 3.13. *Let $t\leq n-1$. There exists m_0 depending only on F such that the following holds:*

- *For any $\boldsymbol{m}=(m_1,\dots,m_t)\in\mathbb{Z}^t_{>0}$ with $m_i\geq m_0$, there exists a non-empty Zariski open subset $U\subset\widetilde{S}_{\boldsymbol{m}}$ such that the natural morphism $H^0(X,F)\longrightarrow H^0(X_s,F_s)$ is an isomorphism for any $s\in U$.*

Proof. It is enough to apply the argument in the first paragraph of the proof of [43, Proposition 3.2] with Proposition 3.11 and Corollary 3.12. (This is essentially pointed out in [64, Lemma 3.9].) ■

Let $T^*_{X_s}X$ denote the conormal bundle of X_s in X for any $s \in \widetilde{S}_{\boldsymbol{m}}$.

Corollary 3.14. *Let $t \leq n-1$. There exists m_0 depending only on F such that the following holds:*

- *For any $\boldsymbol{m} = (m_1, \dots, m_t) \in \mathbb{Z}^t_{>0}$ with $m_i \geq m_0$, there exists a non-empty Zariski open subset $U \subset \widetilde{S}_{\boldsymbol{m}}$ such that $H^0\big(X_s, T^*_{X_s}X \otimes F_s\big) = 0$ for any $s \in U$.*

Proof. Because $T^*_{X_s}X \simeq \bigoplus_{j=1}^t \mathcal{O}_{X_s}(-m_j)$, the claim follows from Corollary 3.12. ■

3.4.5 Flat sections of reflexive filtered $\boldsymbol{\lambda}$-flat sheaves

Let H be a simple normal crossing hypersurface of X with the irreducible decomposition $H = \bigcup_{i\in\Lambda} H_i$. Let $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ be a reflexive filtered λ-flat sheaf on (X, H). Note that there exists a Zariski closed subset $W \subset X$ with $\dim W < \dim H$ such that (i) W contains the singular locus of H, (ii) $\mathcal{P}_*\mathcal{V}_{|X\setminus W}$ is a filtered bundle on $(X \setminus W, H \setminus W)$. For any $\boldsymbol{m} \in \mathbb{Z}^{n-1}$ and for any $s \in S_{\boldsymbol{m}}$, we set $H_s := X_s \times_X H$.

According to [43, Proposition 1.5], there exists a non-empty Zariski open subset $S^\circ_{\boldsymbol{m}} \subset S_{\boldsymbol{m}}$ such that the following holds:

- $Z_{\boldsymbol{m}} \times_{S_{\boldsymbol{m}}} S^\circ_{\boldsymbol{m}} \longrightarrow S^\circ_{\boldsymbol{m}}$ is smooth.
- For any $s \in S^\circ_{\boldsymbol{m}}$, $X_s \cap W = \varnothing$ holds, and X_s intersects with H in $H \setminus W$ transversally. Moreover, $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_s := \mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|X_s}$ $(\boldsymbol{a} \in \mathbb{R}^\Lambda)$ are locally free $\mathcal{O}_{X_s}$-modules.

There exists a non-negative integer N such that $\mathbb{D}^\lambda$ induces a morphism of sheaves $\mathbb{D}^\lambda\colon \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \longrightarrow \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes \Omega^1_X(\log H) \otimes \mathcal{O}_X(NH)$. For $j = 0, 1, \dots, n$, we set

$$\mathcal{C}^j_N(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) = \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes \Omega^j_X(\log H) \otimes \mathcal{O}_X(jNH).$$

The flat λ-connection $\mathbb{D}^\lambda$ induces $\mathbb{D}^\lambda\colon \mathcal{C}^j_N(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) \longrightarrow \mathcal{C}^{j+1}_N(\mathcal{P}_{\boldsymbol{a}}\mathcal{V})$ such that $\mathbb{D}^\lambda \wedge \mathbb{D}^\lambda = 0$. Thus, we obtain a complex of sheaves $\mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}, \mathbb{D}^\lambda\big)$ on X. Clearly, the following holds:

$$\mathbb{H}^0\big(X, \mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}, \mathbb{D}^\lambda\big)\big) = \operatorname{Ker}\Big(H^0(X, \mathcal{P}_{\boldsymbol{a}}\mathcal{V}) \xrightarrow{\mathbb{D}^\lambda} H^0\big(X, \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes \Omega^1_X(\log H) \otimes \mathcal{O}_X(NH)\big)\Big).$$

For any $s \in S^\circ_{\boldsymbol{m}}$, we obtain the filtered λ-flat bundle $\big(\mathcal{P}_*\mathcal{V}_s, \mathbb{D}^\lambda_s\big) := \big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|X_s}$. Let $\boldsymbol{a}(s)$ denote the image of $\mathbb{R}^\Lambda \longrightarrow \mathbb{R}^{H_s}$ induced by the natural map $H_s \longrightarrow \Lambda$. Let $\iota_s\colon X_s \longrightarrow X$ denote the inclusion. We obtain the natural morphism of complexes of sheaves $\mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}, \mathbb{D}^\lambda\big) \longrightarrow \iota_{s*}\mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s, \mathbb{D}^\lambda_s\big)$, which induces

$$\mathbb{H}^0\big(X, \mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}, \mathbb{D}^\lambda\big)\big) \longrightarrow \mathbb{H}^0\big(X_s, \mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s, \mathbb{D}^\lambda_s\big)\big). \tag{3.3}$$

The following proposition is essentially [64, Lemma 3.9].

Proposition 3.15. *There exists a positive integer $m_0 > 0$ such that the following claim holds for any $\boldsymbol{m} = (m_1, \dots, m_{n-1})$ with $m_i \geq m_0$ and a non-empty Zariski open subset $U \subset S^\circ_{\boldsymbol{m}}$.*

- *For any $s \in U$, the natural morphism (3.3) is an isomorphism.*

Proof. According to Corollary 3.13, if m_0 is sufficiently large, there exists a non-empty Zariski open subset $U_1 \subset S^{\circ}_{\boldsymbol{m}}$ such that the following natural morphisms are isomorphisms for any $s \in U$:

$$H^0(X, \mathcal{P}_{\boldsymbol{a}}\mathcal{V}) \longrightarrow H^0(X_s, \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s),$$
$$H^0\big(X, \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X(\log H) \otimes \mathcal{O}_X(NH)\big) \longrightarrow H^0\big(X_s, \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes_{\mathcal{O}_{X_s}} \big(\Omega^1_X(\log H) \otimes \mathcal{O}_{X_s}(NH_s)\big)\big).$$

There exists the following exact sequence:

$$0 \longrightarrow T^*_{X_s}X \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes \mathcal{O}_{X_s}(NH_s) \longrightarrow \big(\Omega^1_X(\log H) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}(NH_s)\big) \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \longrightarrow \Omega^1_{X_s}(\log H_s) \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes \mathcal{O}_{X_s}(NH_s) \longrightarrow 0.$$

According to Corollary 3.14, if m_0 is sufficiently large, there exists a non-empty Zariski open subset $U_2 \subset U_1$ such that the following holds for any $s \in U_2$:

$$H^0\big(X_s, T^*_{X_s}X \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes \mathcal{O}_{X_s}(NH_s)\big) = 0.$$

Hence, the natural morphism

$$H^0\big(X_s, (\Omega^1_X(\log H) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}(NH_s)) \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s\big) \longrightarrow H^0\big(X_s, \Omega^1_{X_s}(\log H_s) \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes \mathcal{O}_{X_s}(NH_s)\big)$$

is injective for any $s \in U_2$. We obtain the injectivity of the following natural morphism for any $s \in U_2$:

$$H^0\big(X, \Omega^1_X(\log H) \otimes \mathcal{P}_{\boldsymbol{a}}\mathcal{V} \otimes \mathcal{O}_X(NH)\big) \longrightarrow H^0\big(X_s, \Omega^1_{X_s}(\log H_s) \otimes \mathcal{P}_{\boldsymbol{a}(s)}\mathcal{V}_s \otimes \mathcal{O}_{X_s}(NH_s)\big).$$

Then, we obtain the claim of the proposition. ■

3.4.6 Morphisms of reflexive filtered λ-flat sheaves

Let $\mathcal{P}_*\mathcal{V}_i$ $(i = 1, 2)$ be reflexive filtered sheaves with meromorphic flat λ-connection $\mathbb{D}^\lambda_i$ on (X, H). Let $\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_1), (\mathcal{P}_*\mathcal{V}_2, \mathbb{D}^\lambda_2)\big)$ denote the vector space of morphisms of filtered λ-flat sheaves $\big(\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_1\big) \longrightarrow \big(\mathcal{P}_*\mathcal{V}_2, \mathbb{D}^\lambda_2\big)$.

Proposition 3.16. *There exists a positive integer $m_0 > 0$ such that the following claim holds for any $\boldsymbol{m} = (m_1, \ldots, m_{n-1})$ with $m_i \geq m_0$ and for a non-empty Zariski open subset $U \subset S_{\boldsymbol{m}}$.*

- *For any $s \in U$, let $\big(\mathcal{P}_*\mathcal{V}_{i,s}, \mathbb{D}^\lambda_{i,s}\big)$ denote the induced filtered λ-flat bundles on (X_s, H_s). Then, the natural morphism*

  $$\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_1), (\mathcal{P}_*\mathcal{V}_2, \mathbb{D}^\lambda_2)\big) \longrightarrow \mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_{1,s}, \mathbb{D}^\lambda_{1,s}), (\mathcal{P}_*\mathcal{V}_{2,s}, \mathbb{D}^\lambda_{2,s})\big)$$

  is an isomorphism.

Proof. For any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, let $\mathcal{P}_{\boldsymbol{a}}\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$ denote the subsheaf of the $\mathcal{O}_X(*H)$-module $\mathcal{H}om_{\mathcal{O}_X(*H)}(\mathcal{V}_1, \mathcal{V}_2)$ determined as follows for any open subset $U \subset X$:

$$H^0\big(U, \mathcal{P}_{\boldsymbol{a}}(\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2))\big) = \big\{f \in H^0(U, \mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)) \mid f\big(\mathcal{P}_{\boldsymbol{b}}\mathcal{V}_{1|U}\big) \subset \mathcal{P}_{\boldsymbol{a}+\boldsymbol{b}}\big(\mathcal{V}_{2|U}\big)\ \forall \boldsymbol{b} \in \mathbb{R}^\Lambda\big\}.$$

It is easy to see that $\mathcal{P}_{\boldsymbol{a}}\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$ are reflexive $\mathcal{O}_X$-modules. Thus, we obtain a reflexive filtered sheaf $\mathcal{P}_*\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$ with the induced flat λ-connection $\widetilde{\mathbb{D}}^\lambda$. We can easily observe that

$$\mathrm{Hom}\big((\mathcal{P}_*\mathcal{V}_1, \mathbb{D}^\lambda_1), (\mathcal{P}_*\mathcal{V}_2, \mathbb{D}^\lambda_2)\big) = \mathbb{H}^0\big(X, \mathcal{C}^\bullet_N\big(\mathcal{P}_{\boldsymbol{0}}\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2), \widetilde{\mathbb{D}}^\lambda\big)\big)$$

for any large N, where $\boldsymbol{0} = (0, \ldots, 0) \in \mathbb{R}^\Lambda$. Then, the claim follows from Proposition 3.15. ■

3.5 Good filtered λ-flat bundles and ramified coverings

3.5.1 Pull back

Let X be any complex manifold with a simple normal crossing hypersurface H. Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ be a good filtered λ-flat bundle. We set $e:=\operatorname{rank}(\mathcal{V})!$.

For any point $P\in X$, let $(X_P,z_1,\dots,z_n)$ denote an admissible holomorphic coordinate neighbourhood around P. We set $H_P:=X_P\cap H$ and $H_{P,i}:=H_P\cap\{z_i=0\}$. We set $\big(\mathcal{P}_*\mathcal{V}_P,\mathbb{D}^\lambda_P\big):=\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|X_P}$.

By using the coordinate system, we may regard X_P as an open subset of $\mathbb{C}^n$. Let $\varphi_P\colon \mathbb{C}^n\longrightarrow\mathbb{C}^n$ be given by $\varphi_P(\zeta_1,\dots,\zeta_n)=\big(\zeta_1^e,\dots,\zeta_{\ell(P)}^e,\zeta_{\ell(P)+1},\dots,\zeta_n\big)$. We set $\widetilde{X}_P:=\varphi_P^{-1}(X_P)$, $\widetilde{H}_P:=\varphi_P^{-1}(H_P)$ and $\widetilde{H}_{P,i}:=\varphi_P^{-1}(H_{P,i})$. We set $G_P:=\prod_{i=1}^{\ell(P)}\{\alpha_i\in\mathbb{C}\mid\alpha_i^e=1\}$. It is identified with the Galois group of the ramified covering φ_P by the action as in Section 2.3.2.

We obtain the G_P-equivariant good filtered λ-flat bundle $\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P,\widetilde{\mathbb{D}}^\lambda_P\big):=\varphi_P^*\big(\mathcal{P}_*\mathcal{V}_P,\mathbb{D}^\lambda_P\big)$ on $\big(\widetilde{X}_P,\widetilde{H}_P\big)$.

Lemma 3.17. *$\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P,\widetilde{\mathbb{D}}^\lambda_P\big)$ is unramifiedly good.*

Proof. See [51, Lemma 2.2.7]. ■

3.5.2 The associated graded bundles

We obtain the G_P-equivariant filtered bundles ${}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$ $(c\in\mathbb{R})$ on $\big(\widetilde{H}_{P,1},\partial\widetilde{H}_{P,1}\big)$. There exists the G_P-equivariant decomposition

$$
{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)=\bigoplus_{\ell=0}^{e-1}\mathbb{G}_\ell{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big),
$$

where $(\alpha,1,\dots,1)$ acts on $\mathbb{G}_\ell{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$ as the multiplication by α^ℓ.

Lemma 3.18. *The pull back naturally induces the isomorphism $(\varphi_{|\widetilde{H}_{P,1}})^*\big({}^1\mathrm{Gr}^F_{c/e}(\mathcal{P}_*\mathcal{V}_P)\big)\simeq\mathbb{G}_0{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$. As a result, ${}^1\mathrm{Gr}^F_{c/e}(\mathcal{P}_*\mathcal{V}_P)$ is the descent of $\mathbb{G}_0{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$. More generally, the pull back and the multiplication by ζ_1^ℓ induces an isomorphism $(\varphi_{|\widetilde{H}_{P,1}})^*\big({}^1\mathrm{Gr}^F_{(c+\ell)/e}(\mathcal{P}_*\mathcal{V}_P)\big)\simeq\mathbb{G}_\ell{}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$.*

Clearly, there exist a similar decomposition ${}^i\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)=\bigoplus_{\ell=0}^{e-1}\mathbb{G}_\ell{}^i\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$ and isomorphisms for any $i=1,\dots,\ell(P)$.

3.5.3 Residues

Let us recall that we obtain the endomorphisms $\mathrm{Res}_j\big(\mathbb{D}^\lambda\big)$ $(j\in\Lambda)$ on ${}^j\mathrm{Gr}^F_c(\mathcal{P}_*\mathcal{V})$ by using Lemma 3.18. (See [51, Section 2.5.2] for more detailed explanations.)

Let P be any point of H. First, let us construct the residues $\mathrm{Res}_1\big(\mathbb{D}^\lambda_P\big)$ on ${}^1\mathrm{Gr}^F(\mathcal{P}_*\mathcal{V}_P)$. At any $Q\in\widetilde{H}_{P,1}$, we obtain the formal decomposition $\big(\mathcal{P}_{\boldsymbol{a}}\widetilde{\mathcal{V}}_P,\mathbb{D}^\lambda_P\big)\otimes\mathcal{O}_{\widehat{\widetilde{X}_P,Q}}=\bigoplus\big(\mathcal{P}_{\boldsymbol{a}}\widetilde{\mathcal{V}}_{\mathfrak{a}},\widetilde{\mathbb{D}}^\lambda_{\mathfrak{a}}\big)$ as in (2.4). For $a_1-1<c\le a_1$, we obtain the endomorphisms $\mathrm{Res}_1\big(\widetilde{\mathbb{D}}^\lambda_P\big)_Q$ of ${}^1\mathrm{Gr}^F_c\big(\mathcal{P}_{\boldsymbol{a}}\widetilde{\mathcal{V}}\big)_{|Q}$ as the residue of $\bigoplus\big(\widetilde{\mathbb{D}}^\lambda_{\mathfrak{a}}-\mathrm{d}\widetilde{\mathfrak{a}}\,\mathrm{id}_{\widetilde{\mathcal{V}}_{\mathfrak{a}}}\big)$ at Q. According to [51, Lemma 2.5.2], by varying $Q\in\widetilde{H}_{P,1}$, we obtain the endomorphism $\mathrm{Res}_1\big(\widetilde{\mathbb{D}}^\lambda_P\big)$ of the filtered bundle ${}^1\mathrm{Gr}^F_c\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$. It is G_P-equivariant. Hence, we obtain $\mathrm{Res}_1\big(\mathbb{D}^\lambda_P\big)$ on ${}^1\mathrm{Gr}^F_c(\mathcal{P}_*\mathcal{V}_P)$ as the descent of $\frac{1}{e}\mathrm{Res}_1(\widetilde{\mathbb{D}}^\lambda_P)$ on $\mathbb{G}_0{}^1\mathrm{Gr}^F_{ec}\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P\big)$. The factor $\frac{1}{e}$ comes from the relation $e\,\mathrm{d}\zeta_1/\zeta_1=\mathrm{d}z_1/z_1$. Similarly, we obtain $\mathrm{Res}_i\big(\mathbb{D}^\lambda_P\big)$ ${}^i\mathrm{Gr}^F_c(\mathcal{P}_*\mathcal{V}_P)$ for $i=1,\dots,\ell(P)$.

It is easy to see that there exists a globally defined endomorphism $\mathrm{Res}_j\big(\mathbb{D}^\lambda\big)$ on ${}^j\mathrm{Gr}^F_c(\mathcal{P}_*\mathcal{V})$ which is equal to the endomorphisms constructed locally around $P\in H_j$ as above.

3.5.4 Parabolic weights

We introduce some notation. We set $\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},j) := \{b \in \mathbb{R} \mid {}^j\mathrm{Gr}^F_b(\mathcal{P}_*\mathcal{V}) \neq 0\}$ for $j \in \Lambda$. Because we shall often use the pull back by a ramified covering as in Section 3.5.1, for a fixed e, it is convenient to consider

$$\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},j) := \{c + m/e \mid c \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},j),\, m \in \mathbb{Z}\}.$$

Note that $\mathcal{P}\mathrm{ar}\big(\mathcal{P}_*\widetilde{\mathcal{V}}_P,j\big) = \big\{eb \mid b \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}_P,j)\big\}$ for $P \in H$. (See Lemma 3.18.) We define

$$\widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V},j) := \min\big\{|b_1 - b_2| \mid b_1, b_2 \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},j),\, b_1 \neq b_2\big\}.$$

If Λ is finite, we also set $\widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V}) := \min_{j\in\Lambda} \widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V},j)$.

For each $\boldsymbol{a} \in \mathbb{R}^\Lambda$, we set

$$\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},i) := \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},i) \cap]a_i - 1, a_i], \qquad \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},i) := \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},i) \cap]a_i - 1, a_i].$$

We remark the following obvious lemma.

Lemma 3.19. *For each j, there exists $a_j \in \mathbb{R}$ such that $|a_j - b| > (4e\,\mathrm{rank}\,\mathcal{V})^{-1}$ for any $b \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},j)$.*

3.6 Approximation by model filtered $\boldsymbol{\lambda}$-flat bundles

3.6.1 Model filtered $\boldsymbol{\lambda}$-flat bundles

Let Z be a complex manifold. Let Y be a neighbourhood of $\{0\} \times Z$ in $\mathbb{C} \times Z$. We set $H_Y := \{0\} \times Z$. Let e be a positive integer. Let ζ be the standard complex coordinate of $\mathbb{C}$. Consider $\varphi\colon \mathbb{C} \times Z \longrightarrow \mathbb{C} \times Z$ induced by $\zeta \longmapsto \zeta^e$. We set $\widetilde{Y} := \varphi^{-1}(Y)$ and $\widetilde{H} := \varphi^{-1}(H)$. The induced morphism $\widetilde{Y} \longrightarrow Y$ is also denoted by φ. Let G denote the group of the e-th roots of 1, which is naturally identified with the Galois group of the ramified covering φ.

Let $\mathcal{I}$ be a finite subset of $H^0\big(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}(*\widetilde{H})\big)$ which is preserved by the G-action. Let S_1 and S_2 be finite subsets of $]-1,0]$ and $\mathbb{C}$, respectively. Let $V_{\mathfrak{a},a,\alpha}$ ($(\mathfrak{a},a,\alpha) \in \mathcal{I} \times S_1 \times S_2$) be finite dimensional $\mathbb{C}$-vector spaces equipped with a nilpotent endomorphism $f_{\mathfrak{a},a,\alpha}$. Note that $V_{\mathfrak{a},a,\alpha}$ may be 0. We suppose that $\bigoplus_{\mathfrak{a},a,\alpha} V_{\mathfrak{a},a,\alpha}$ is a G-representation such that (i) it is G-equivariant as a vector bundle over $\mathcal{I}$, (ii) $\bigoplus f_{\mathfrak{a},a,\alpha}$ commutes with the G-action.

We set $\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha} := \mathcal{O}_{\widetilde{Y}}(*\widetilde{H}) \otimes V_{\mathfrak{a},a,\alpha}$. We define the filtered bundle $\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}$ over $\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}$ by setting

$$\mathcal{P}_b\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha} := \mathcal{O}_{\widetilde{Y}}\big([b-a]\widetilde{H}\big) \otimes V_{\mathfrak{a},a,\alpha}$$

for any $b \in \mathbb{R}$, where $[b-a] := \max\{n \in \mathbb{Z} \mid n \leq b-a\}$. We define the flat λ-connection $\widetilde{\mathbb{D}}^\lambda_{\mathfrak{a},a,\alpha}$ on $\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}$ by setting

$$\widetilde{\mathbb{D}}^\lambda_{\mathfrak{a},a,\alpha}(v) = \mathrm{d}\mathfrak{a} \cdot v + (\alpha v + f_{\mathfrak{a},a,\alpha}(v))\,\mathrm{d}\zeta/\zeta$$

for any $v \in V_{\mathfrak{a},a,\alpha}$, which we regard as a section of $\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}$ in a natural way. Thus, we obtain a G-equivariant filtered λ-flat bundle $\bigoplus_{\mathfrak{a},a,\alpha}\big(\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}, \widetilde{\mathbb{D}}^\lambda_{\mathfrak{a},a,\alpha}\big)$, called a model filtered λ-flat bundle. If $\mathcal{I}$ induces a good set of irregular values in $\mathcal{O}_{\widetilde{Y}}\big(*\widetilde{H}\big)_Q/\mathcal{O}_{\widetilde{Y},Q}$ at each $Q \in \widetilde{H}$, then $\bigoplus_{\mathfrak{a},a,\alpha}\big(\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}, \widetilde{\mathbb{D}}^\lambda_{\mathfrak{a},a,\alpha}\big)$ is an unramifiedly good filtered λ-flat bundle. It induces a filtered λ-flat bundle on (Y,H) as the descent, which is also called a model filtered λ-flat bundle.

3.6.2 Approximation of good filtered λ-flat bundles

Let $(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$ be any good filtered λ-flat bundle on (Y, H). Assume the following condition:

Condition 3.20. *For each $a \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V})$, the conjugacy classes of $\operatorname{Res}(\mathbb{D}^\lambda)$ on $\operatorname{Gr}^F_a(\mathcal{P}_*\mathcal{V})_{|P}$ are independent of $P \in H$. Note that this condition is trivially satisfied if $\lambda \neq 0$.*

We set $e := \operatorname{rank}(\mathcal{V})!$. Let φ and $(\widetilde{Y}, \widetilde{H})$ be as in Section 3.6.1. We set $(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda) := \varphi^*(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda)$. For each $Q \in \widetilde{H}$, there exists a decomposition

$$(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda) \otimes \mathcal{O}_{\widetilde{Y}, \widehat{Q}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}(Q)} (\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a}}, \widetilde{\mathbb{D}}^\lambda_{\mathfrak{a}})$$

as in (2.5). We obtain the vector spaces $\operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}})_{|Q}$ $(-1 < a \leq 0)$ equipped with the endomorphisms $\operatorname{Res}(\widetilde{\mathbb{D}}^\lambda)$. Condition 3.20 is equivalent to the following.

- The conjugacy classes of $\operatorname{Res}(\widetilde{\mathbb{D}}^\lambda)$ on $\operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}})_{|Q}$ are independent of $Q \in \widetilde{H}$ for any $-1 < a \leq 0$.

In particular, the condition implies that there exists a decomposition

$$\operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}}) = \bigoplus_{\alpha \in \mathbb{C}} \mathbb{E}_\alpha \operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}})$$

on $\widetilde{H}$, where $\operatorname{Res}(\widetilde{\mathbb{D}}^\lambda) - \alpha\,\mathrm{id}$ are nilpotent on $\mathbb{E}_\alpha \operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}})$.

Fix $P \in H$. Let $\widetilde{P} \in \widetilde{H}$ be determined by $\varphi(\widetilde{P}) = P$. We set $V_{\mathfrak{a},a,\alpha} := \mathbb{E}_\alpha \operatorname{Gr}^F_a(\mathcal{P}_0\widetilde{\mathcal{V}}_{\mathfrak{a}})_{|\widetilde{P}}$. Let $f_{\mathfrak{a},a,\alpha}$ be the nilpotent part of $\operatorname{Res}(\widetilde{\mathbb{D}}^\lambda)$ on $V_{\mathfrak{a},a,\alpha}$. For a neighbourhood Y_P of P in Y, we set $H_P := Y_P \cap H$, $\widetilde{Y}_P := \varphi^{-1}(Y_P)$ and $\widetilde{H}_P := \varphi^{-1}(H_P)$. We may assume that any $\mathfrak{a} \in \mathcal{I}(\widetilde{P})$ has a lift $\widetilde{\mathfrak{a}}$ in $H^0(\widetilde{Y}_P, \mathcal{O}_{\widetilde{Y}_P}(*\widetilde{H}_P))$. From the set $\{(\mathfrak{a}, a, \alpha)\} \subset \mathcal{I}(\widetilde{P}) \times]-1, 0] \times \mathbb{C}$ and the tuples $(V_{\mathfrak{a},a,\alpha}, f_{\mathfrak{a},a,\alpha})$, we obtain a model filtered λ-flat bundle

$$(\mathcal{P}_*\widetilde{\mathcal{V}}_0, \widetilde{\mathbb{D}}^\lambda_0) := \bigoplus_{\mathfrak{a},a,\alpha} (\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}, \widetilde{\mathbb{D}}^\lambda_{\mathfrak{a},a,\alpha})$$

on $(\widetilde{Y}_P, \widetilde{H}_P)$. It is unramifiedly good and naturally G-equivariant. As the descent, we obtain a good filtered λ-flat bundle $(\mathcal{P}_*\mathcal{V}_0, \mathbb{D}^\lambda_0)$ on (Y_P, H_P).

Lemma 3.21 (assume Condition 3.20)**.** *For any positive integer m, there exist a neighbourhood Y_P of P in Y and an isomorphism of filtered bundles $\Phi_m \colon \mathcal{P}_*\mathcal{V}_0 \simeq \mathcal{P}_*\mathcal{V}_{|Y_P}$ such that the following holds:*

- *We set $\widetilde{\Phi}_m := \varphi^*(\Phi_m)$ and $A := (\widetilde{\Phi}_m)^*(\widetilde{\mathbb{D}}^\lambda) - \widetilde{\mathbb{D}}^\lambda_0$ on $\widetilde{Y}_P$. Let $A = \sum A_{(\mathfrak{b},b,\beta),(\mathfrak{a},a,\alpha)}$ be the decomposition such that*

$$A_{(\mathfrak{b},b,\beta),(\mathfrak{a},a,\alpha)} \in \operatorname{Hom}(\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}, \widetilde{\mathcal{V}}_{\mathfrak{b},b,\beta}) \otimes \Omega^1_{\widetilde{Y}_P}.$$

Then, we obtain the following for any $c \in \mathbb{R}$:

$$A_{(\mathfrak{b},b,\beta),(\mathfrak{a},a,\alpha)} \cdot \mathcal{P}_c\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha} \subset \mathcal{P}_{c-10m}\widetilde{\mathcal{V}}_{\mathfrak{b},b,\beta} \otimes \Omega^1_{\widetilde{Y}_P} \qquad (\mathfrak{a} \neq \mathfrak{b}),$$

$$A_{(\mathfrak{a},b,\beta),(\mathfrak{a},a,\alpha)} \cdot \mathcal{P}_c\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha} \subset \mathcal{P}_c\widetilde{\mathcal{V}}_{\mathfrak{a},b,\beta} \otimes \Omega^1_{\widetilde{Y}_P}(\log \widetilde{H}_P), \qquad (a, \alpha) \neq (b, \beta),$$

$$\operatorname{Res} A_{(\mathfrak{a},a,\alpha),(\mathfrak{a},a,\alpha)}(\mathcal{P}_c\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}) \subset \mathcal{P}_{<c}\widetilde{\mathcal{V}}_{\mathfrak{a},a,\alpha}.$$

Here, $\mathcal{P}_{<c}\widetilde{\mathcal{V}}_{\mathfrak{b},b,\beta} = \bigcup_{d<c} \mathcal{P}_d\widetilde{\mathcal{V}}_{\mathfrak{b},b,\beta}$.

Proof. By [51, Proposition 2.4.4], for any large inter N, there exists a G-equivariant decomposition of filtered bundles

$$\mathcal{P}_*\widetilde{\mathcal{V}}_{|\widetilde{Y}_P} = \bigoplus_{\mathfrak{a}\in\mathcal{I}(\widetilde{P})} \mathcal{P}_*\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}$$

on $\big(\widetilde{Y}_P, \widetilde{H}_P\big)$ such that the following holds.

- Let $\pi^{(N)}_{\mathfrak{a}}$ denote the projection of $\mathcal{P}_*\widetilde{\mathcal{V}}_{|\widetilde{Y}_P}$ onto $\mathcal{P}_*\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}$, and let $\iota^{(N)}_{\mathfrak{a}}$ denote the inclusion of $\mathcal{P}_*\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}$ into $\mathcal{P}_*\widetilde{\mathcal{V}}_{|\widetilde{Y}_P}$. Then, for any $\mathfrak{a}\neq\mathfrak{b}$ and for any $c\in\mathbb{R}$, we obtain

$$\pi^{(N)}_{\mathfrak{b}} \circ \widetilde{\mathbb{D}}^{\lambda} \circ \iota^{(N)}_{\mathfrak{a}}\big(\mathcal{P}_c\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}\big) \subset \mathcal{P}_{c-2N}\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{b}} \otimes \Omega^1_{\widetilde{Y}_P}.$$

- For $\mathfrak{a}$, we set $\widetilde{\mathbb{D}}^{\lambda(N)}_{\mathfrak{a}} := \pi^{(N)}_{\mathfrak{a}} \circ \widetilde{\mathbb{D}}^{\lambda} \circ \iota^{(N)}_{\mathfrak{a}} - d\mathfrak{a}\,\mathrm{id}_{\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}}$. Then, for any $c\in\mathbb{R}$, we obtain

$$\widetilde{\mathbb{D}}^{\lambda\,(N)}\big(\mathcal{P}_c\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}\big) \subset \mathcal{P}_c\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}} \otimes \Omega^1_{\widetilde{Y}_P}\big(\log \widetilde{H}_P\big),$$
$$\widetilde{\mathbb{D}}^{\lambda\,(N)} \circ \widetilde{\mathbb{D}}^{\lambda\,(N)}\big(\mathcal{P}_c\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}}\big) \subset \mathcal{P}_{c-N}\widetilde{\mathcal{V}}^{(N)}_{\mathfrak{a}} \otimes \Omega^2_{\widetilde{Y}_P}\big(\log \widetilde{H}_P\big).$$

Then, the claim of the lemma is clear. ■

3.7 Perturbation of good filtered λ-flat bundles

3.7.1 Curve case

Let C be a Riemann surface with a finite subset $D\subset C$. Let $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\big)$ be a good filtered λ-flat bundle on (C,D). We set $e := \mathrm{rank}(\mathcal{V})!$. We choose $\eta>0$ such that $10e\eta < \widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V})$.

For any $0<10\,\mathrm{rank}(\mathcal{V})\epsilon\leq\eta$, and for any $P\in D$, let $\psi_{\epsilon,P}$ be a map $\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},P)\longrightarrow\mathbb{R}$ such that (i) $|\psi_{\epsilon,P}(b)-b|<2\epsilon$, (ii) if $e(b_1-b_2)\in\mathbb{Z}$ then $\psi_{\epsilon,P}(b_1)-b_1=\psi_{\epsilon,P}(b_2)-b_2$. We define $\varphi_{\epsilon,P}\colon \mathbb{Z}\times\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},P)\longrightarrow\mathbb{R}$ by

$$\varphi_{\epsilon,P}(k,b) := \psi_{\epsilon,P}(b)+\epsilon k.$$

We take $\boldsymbol{a}\in\mathbb{R}^D$ as in Lemma 3.19. For each $P\in D$ and $b\in\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},P)$, we obtain the endomorphism $\mathrm{Res}_P\big(\mathbb{D}^{\lambda}\big)$ of $\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)$. Let $W_\bullet\,\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)$ denote the weight filtration associated with the nilpotent part of $\mathrm{Res}_P(\mathbb{D}^{\lambda})$. For any $(k,b)\in\mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},P)$, we obtain the subspace $W_k\big(F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)\big)$ as the pull back of $W_k\,\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)$ by the projection $F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)\longrightarrow\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big)$. We define the filtration $F^{(\epsilon)}$ on $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}$ indexed by $]a(P)-1,a(P)]$ as follows:

$$F^{(\epsilon)}_c\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big) := \sum_{\substack{(k,b)\in\mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},P)\\ \varphi_{\epsilon,P}(k,b)\leq c}} W_kF_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\big).$$

We obtain the corresponding filtered bundle $\mathcal{P}^{(\epsilon)}_*\mathcal{V}$. Note the following lemma.

Lemma 3.22. $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^{\lambda}\big)$ *is a good filtered λ-flat bundle.*

Proof. It is enough to consider the case where C is a neighbourhood of $D=\{P\}=\{0\}$ in $\mathbb{C}$. We obtain $\varphi\colon\widetilde{C}\longrightarrow C$, $\widetilde{D}$, G, and $\big(\mathcal{P}_*\widetilde{\mathcal{V}},\widetilde{\mathbb{D}}^{\lambda}\big)$ as in Section 3.5.1. We set $\widetilde{a} := ea(P)$ and $\widetilde{P} := \varphi^{-1}(P)$. For $(k,b)\in\mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\widetilde{\mathcal{V}},\widetilde{a})$, we construct $W_kF_b\big(\mathcal{P}_{\widetilde{a}}\widetilde{\mathcal{V}}_{|\widetilde{P}}\big)$ as above.

For $b \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{a})$, note that $b/e \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V})$, and we set

$$\widetilde{\varphi}_{\epsilon,P}(k,b) := e\psi_{\epsilon,P}(k,b/e) + ke\epsilon.$$

We set

$$F^{(\epsilon)}_d\big(\mathcal{P}_{\widetilde{a}}\widetilde{\mathcal{V}}_{|\widetilde{P}}\big) := \sum_{\widetilde{\varphi}_{\epsilon,P}(k,b)\le d} W_k F_b\big(\mathcal{P}_{\widetilde{a}}\widetilde{\mathcal{V}}_{|\widetilde{P}}\big).$$

We obtain the corresponding G-equivariant filtered bundle $\mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}}$. We can easily observe that $\varphi^*\mathcal{P}^{(\epsilon)}_*\mathcal{V} \simeq \mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}}$ by using $10^2 e\,\mathrm{rank}(\mathcal{V})\epsilon < \widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V})$.

There exists the decomposition

$$\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big) \otimes \mathbb{C}[\![\zeta]\!] = \bigoplus_{\mathfrak{a}\in\mathcal{I}} \big(\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a}}, \widetilde{\mathbb{D}}^\lambda_{\mathfrak{a}}\big),$$

as in (2.4). We apply the same procedure to each $\mathcal{P}_*\widetilde{\mathcal{V}}_{\mathfrak{a}}$ by using $\widetilde{\varphi}_{\epsilon,P}$, and we obtain filtered bundles $\mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}}_{\mathfrak{a}}$ for which $\widetilde{\mathbb{D}}^\lambda_{\mathfrak{a}} - d\mathfrak{a}\,\mathrm{id}$ are logarithmic. Because $\mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}} \otimes \mathbb{C}[\![\zeta]\!] = \bigoplus \mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}}_{\mathfrak{a}}$, we obtain that $\big(\mathcal{P}^{(\epsilon)}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)$ is unramifiedly good. Hence, we obtain that $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ is good. ■

Suppose that C is compact and connected. We clearly obtain $\lim_{\epsilon\to 0} c_1\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V}\big) = c_1(\mathcal{P}_*\mathcal{V})$. The following is also standard.

Lemma 3.23. *Suppose that $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ is stable. Then, if $\epsilon > 0$ is sufficiently small, $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ is also stable.*

Proof. For any positive integer $s \le \mathrm{rank}(\mathcal{V})$, and for any $P \in D$, let $\mathcal{T}(s,P)$ be the set of real numbers expressed as

$$\sum_{b\in\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},P)} b\, s_b,$$

where s_b are non-negative integers such that $\sum s_b = s$. Let $\mathcal{S}(s)$ denote the set of real numbers expressed as $\frac{1}{s}\big(m + \sum_{P\in D} c_P\big)$, where $m \in \mathbb{Z}$ and $c_P \in \mathcal{T}(s,P)$. Then, $\bigcup_{0<s\le\mathrm{rank}\,\mathcal{V}} \mathcal{S}(s)$ is discrete in $\mathbb{R}$. Hence, there exists $\delta > 0$ such that if $t \in \bigcup_{0<s\le\mathrm{rank}\,\mathcal{V}} \mathcal{S}(s)$ satisfies $t < \mu(\mathcal{P}_*\mathcal{V})$, we obtain $t < \mu(\mathcal{P}_*\mathcal{V}) - \delta$. Then, the claim of the lemma is clear. ■

3.7.2 Surface case

Let X be a complex projective surface with a simple normal crossing hypersurface $H = \bigcup_{i\in\Lambda} H_i$. Let $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$ be a good filtered λ-flat bundle on (X,H). We shall explain a similar perturbation of good filtered λ-flat bundles. Set $e := \mathrm{rank}(\mathcal{V})!$. We choose $\eta > 0$ such that $0 < 10e\eta < \widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V})$.

For any $0 < 10\,\mathrm{rank}(\mathcal{V})\epsilon \le \eta$, let $\psi_{\epsilon,i}$ be a map $\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}, i) \longrightarrow \mathbb{R}$ such that (i) $|\psi_{\epsilon,i}(b) - b| < 2\epsilon$, (ii) if $e(b_1 - b_2) \in \mathbb{Z}$ then $\psi_{\epsilon,i}(b_1) - b_1 = \psi_{\epsilon,i}(b_2) - b_2$. We define $\varphi_{\epsilon,i} \colon \mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V}, i) \longrightarrow \mathbb{R}$ by $\varphi_{\epsilon,i}(k,b) := \psi_{\epsilon,i}(b) + \epsilon k$.

We take $\boldsymbol{a} \in \mathbb{R}^\Lambda$ for $\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}, i)$ $(i \in \Lambda)$ as in Lemma 3.19. The eigenvalues of the endomorphism $\mathrm{Res}_i\big(\mathbb{D}^\lambda\big)$ on $\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)$ are constant on H_i because H_i are compact. We obtain the well defined nilpotent part $N_{i,b}$ of $\mathrm{Res}_i\big(\mathbb{D}^\lambda\big)$. There exists a finite subset $Z_i \subset H_i$ such that the conjugacy classes of the nilpotent part of $N_{i,b|Q}$ $(Q \in H_i \setminus Z_i)$ are constant. We obtain the weight filtration W of $\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i\setminus Z_i}\big)$ by algebraic vector subbundles whose restriction to $Q \in H_i \setminus Z_i$

are the weight filtration of $N_{i,b|Q}$. It uniquely extends to a filtration of $\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)$ by algebraic subbundles, which is also denoted by W.

For any $(k,b)\in\mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},i)$, let $W_kF_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)$ denote the subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}$ obtained as the pull back of $W_k\,\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)$ by the projection $F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)\longrightarrow\mathrm{Gr}^F_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}\big)$. We define the filtration $F^{(\epsilon)}$ on $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}$ indexed by $]a_i-1,a_i]$ as follows:

$$F^{(\epsilon)}_c\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}:=\sum_{\substack{(k,b)\in\mathbb{Z}\times\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},\boldsymbol{a},i)\\ \varphi_{\epsilon,i}(k,b)\le c}}W_kF_b\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|H_i}.$$

We obtain the corresponding filtered bundle $\mathcal{P}^{(\epsilon)}_*\mathcal{V}$ over $\mathcal{V}$. As in the curve case (see Lemma 3.22), we obtain the following.

Lemma 3.24. *$\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^\lambda\big)$ is a good filtered λ-flat bundle.*

We clearly have $\lim_{\epsilon\to 0}c_1\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V}\big)=c_1(\mathcal{P}_*\mathcal{V})$ and $\lim_{\epsilon\to 0}\mathrm{ch}_2\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V}\big)=\mathrm{ch}_2(\mathcal{P}_*\mathcal{V})$. The following is standard, and similar to Lemma 3.23. (See also [46, Proposition 3.28].)

Lemma 3.25. *Let L be an ample line bundle on X. Suppose that $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ is μ_L-stable. Then, if ϵ is sufficiently small, $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^\lambda\big)$ is also μ_L-stable.*

3.8 Some families of auxiliary metrics on a punctured disc

3.8.1 Regular model case

Let V be a finite dimensional vector space over $\mathbb{C}$ with a nilpotent endomorphism f. Let $(a,\alpha)\in\mathbb{R}\times\mathbb{C}$. Let X be a neighbourhood of 0 in $\mathbb{C}$. We set $H:=\{0\}$. We set $\mathcal{V}=\mathcal{O}_X(*H)\otimes V$. From (a,α) and (V,f), we obtain a model filtered λ-flat bundle $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ by applying the construction in Section 3.6.1 in the case $\mathfrak{a}=0$ and $e=1$.

Fix $0<\eta<1$. For any $0\le 10\,\mathrm{rank}(\mathcal{V})\epsilon\le\eta$, we take $a(\epsilon)$ such that $|a-a(\epsilon)|<2\epsilon$, and we obtain the regular filtered λ-flat bundle $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^\lambda\big)$ as in Section 3.7.1. We set $E:=\mathcal{V}_{|X\setminus H}$. We consider the Kähler metric $g_\epsilon:=\big(\eta^2|z|^{2\eta-2}+\epsilon^2|z|^{2\epsilon-2}\big)\,\mathrm{d}z\,\mathrm{d}\overline{z}$ of $X\setminus H$.

Proposition 3.26. *There exist Hermitian metrics $h^{(\epsilon)}$ of E for $0\le 10\,\mathrm{rank}(\mathcal{V})\epsilon\le\eta$ such that the following holds:*

- $\mathcal{P}^{h^{(\epsilon)}}_*E=\mathcal{P}^{(\epsilon)}_*\mathcal{V}$.
- *$h^{(0)}$ is a Hermitian metric of $\big(E,\mathbb{D}^\lambda\big)$, and $h^{(\epsilon)}\to h^{(0)}$ in the C^∞-sense locally on $X\setminus H$.*
- *There exist $C_i>0$ $(i=0,1,2)$ such that the following conditions are satisfied for any ϵ:*

$$\big|G\big(h^{(\epsilon)}\big)\big|_{g_\epsilon,h^{(\epsilon)}}\le C_0,\qquad C_1^{-1}|z|^{2\,\mathrm{rank}(\mathcal{V})\epsilon}h^{(0)}\le h^{(\epsilon)}\le C_1|z|^{-2\,\mathrm{rank}(\mathcal{V})\epsilon}h^{(0)},$$
$$C_2^{-1}\det(h_0)\le\det\big(h^{(\epsilon)}\big)\le C_2\det\big(h^{(0)}\big).$$

- *Let $B^{(\epsilon)}_i$ $(i=1,2)$ be the C^∞-endomorphisms of E determined by the condition $\mathbb{D}^{\lambda\star}_{h^{(\epsilon)}}(v)=B^{(\epsilon)}_1(v)\,\mathrm{d}z/z+B^{(\epsilon)}_2(v)\,\mathrm{d}\overline{z}/\overline{z}$ for $v\in V$. Then, there exists $C_3>0$ such that $\big|B^{(\epsilon)}_i\big|_{h^{(\epsilon)}}\le C_3$ holds for any ϵ.*

Moreover, for any $v_1,v_2\in V$, $h^{(\epsilon)}(v_1,v_2)$ depends only on $|z|$, where v_j are naturally regarded as holomorphic sections of $\mathcal{V}$.

In the case $\lambda\ne 0$, such a family of Hermitian metrics $h^{(\epsilon)}$ is constructed in [49, Sections 4.3 and 4.4.1]. We shall explain the case $\lambda=0$ in Section 3.8.4.

3.8.2 General case

Let S_0 be a finite set. Let S_1 be a finite subset of $]0,1]$. Let S_2 be a finite subset of $\mathbb{C}$. Let $V_{i,a,\alpha}$ ($(i,a,\alpha)\in S_0\times S_1\times S_2$) be finite dimensional $\mathbb{C}$-vector spaces equipped with a nilpotent endomorphism $f_{i,a,\alpha}$. Set $r:=\sum\dim V_{i,a,\alpha}$ and $e:=r!$. Take $\eta>0$ such that

$$10e\eta<\min\bigl\{|a-b|\mid a,b\in S_1\cup\{0,1\},\ a\neq b\bigr\}.$$

As in Section 3.8.1, we obtain the regular filtered λ-flat bundles $\bigl(\mathcal{P}_*\mathcal{V}_{i,a,\alpha},\mathbb{D}^\lambda_{i,a,\alpha}\bigr)$ from $(V_{i,a,\alpha},f_{i,a,\alpha})$. For $0\leq\epsilon<\eta/10r$, we take $a(\epsilon)$ such that $|a-a(\epsilon)|<2\epsilon$, and we obtain $\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}_{i,a,\alpha},\mathbb{D}^\lambda_{i,a,\alpha}\bigr)$. We set $E_{i,a,\alpha}:=\mathcal{V}_{i,a,\alpha|X\setminus H}$. We obtain the metrics $h^{(\epsilon)}_{i,a,\alpha}$ of $E_{i,a,\alpha}$ as in Proposition 3.26. We set $\mathcal{P}^{(\epsilon)}_*\mathcal{V}:=\bigoplus\mathcal{P}^{(\epsilon)}_*\mathcal{V}_{i,a,\alpha}$ and $h^{(\epsilon)}:=\bigoplus h^{(\epsilon)}_{i,a,\alpha}$.

Fix a positive integer m and a positive number C. We consider the following data:

- For each $i\in S_0$, let $\mathfrak{a}(i)$ denote a polynomial $\sum_{j=1}^m\mathfrak{a}(i)_jz^{-j}\in z^{-1}\mathbb{C}\bigl[z^{-1}\bigr]$ such that $|\mathfrak{a}(i)_j|\leq C$.
- Let A be a holomorphic section of $\mathcal{E}nd(\mathcal{V})$ with the decomposition $A=\sum A_{(j,b,\beta),(i,a,\alpha)}$, where

  $$A_{(j,b,\beta),(i,a,\alpha)}\in\mathcal{H}\mathrm{om}(\mathcal{V}_{i,a,\alpha},\mathcal{V}_{j,b,\beta})$$

  If $i\neq j$, we obtain $|A_{(j,b,\beta),(i,a,\alpha)}|_{h^{(0)}}\leq C|z|^{10m}$, and if $i=j$, we obtain $|A_{(i,b,\beta),(i,a,\alpha)}|_{h^{(0)}}\leq C|z|^{4\eta}$.

We define the flat λ-connection $\mathbb{D}^\lambda$ on $\mathcal{V}$ as follows:

$$\mathbb{D}^\lambda=\bigoplus_{i,a,\alpha}\bigl(d\mathfrak{a}(i)\,\mathrm{id}_{\mathcal{V}_{i,a,\alpha}}+\mathbb{D}^\lambda_{i,a,\alpha}\bigr)+A\frac{dz}{z}.$$

Let g_ϵ be the Kähler metrics of $X\setminus H$ as in Section 3.8.1.

Proposition 3.27. *There exists a constant C' depending only on m and C such that*

$$\bigl|G\bigl(h^{(\epsilon)}\bigr)\bigr|_{g_\epsilon,h^{(\epsilon)}}\leq C'.$$

3.8.3 A consequence

Let Y be a neighbourhood of 0 in $\mathbb{C}$. We set $H:=\{0\}$. Let $\bigl(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\bigr)$ be any good filtered λ-flat bundle on (Y,H). Let $\bigl(E,\mathbb{D}^\lambda\bigr)$ be the λ-flat bundle obtained as the restriction of $\bigl(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\bigr)$ to $Y\setminus H$. We set $e:=\mathrm{rank}(\mathcal{V})!$. Let φ be as in Section 3.6, and we set $\bigl(\mathcal{P}_*\widetilde{\mathcal{V}},\widetilde{\mathbb{D}}^\lambda\bigr):=\varphi^*\bigl(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\bigr)$. We take $\eta>0$ such that $10e\eta<\widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V})$. Let g be the Kähler metric $\eta^2|z|^{2\eta-2}\,dz\,d\overline{z}$ of $Y\setminus H$. By using a special case of Proposition 3.27, we obtain the following corollary.

Corollary 3.28. *There exists a Hermitian metric h of $\bigl(E,\mathbb{D}^\lambda\bigr)$ such that (i) $\mathcal{P}^h_*E=\mathcal{P}_*\mathcal{V}$, (ii) $|G(h)|_{h,g}$ is bounded on $Y\setminus H$.*

Proof. Let $\bigl(\mathcal{P}_*\mathcal{V}_0,\mathbb{D}^\lambda_0\bigr)$ be a model filtered λ-flat bundle with an isomorphism $\Phi_m\colon\mathcal{P}_*\mathcal{V}_0\simeq\mathcal{P}_*\mathcal{V}$ as in Lemma 3.21, where m is a sufficiently large integer. We recall that $\bigl(\mathcal{P}_*\mathcal{V}_0,\mathbb{D}^\lambda_0\bigr)$ is obtained as the descent of the G-equivariant model filtered λ-flat bundle $\bigl(\mathcal{P}_*\widetilde{\mathcal{V}}_0,\widetilde{\mathbb{D}}^\lambda_0\bigr)=\bigoplus_{\mathfrak{a},a,\alpha}\bigl(\mathcal{P}_*\mathcal{V}_{\mathfrak{a},a,\alpha},\mathbb{D}^\lambda_{\mathfrak{a},a,\alpha}\bigr)$, and Φ_m is induced by a G-equivariant isomorphism $\mathcal{P}_*\widetilde{\mathcal{V}}_0\simeq\mathcal{P}_*\widetilde{\mathcal{V}}$. Let $\widetilde{h}^{(0)}_0$ be the Hermitian metric of $\widetilde{\mathcal{V}}_{0|\widetilde{Y}\setminus\widetilde{H}}$ as in Proposition 3.26 with $\epsilon=0$. By the isomorphism $\widetilde{\Phi}_m$, it induces a G-equivariant Hermitian metric $\widetilde{h}$ of $\widetilde{\mathcal{V}}_{|\widetilde{Y}\setminus\widetilde{H}}$. Applying Proposition 3.27 to $\bigl(\mathcal{P}_*\widetilde{\mathcal{V}},\widetilde{\mathbb{D}}^\lambda\bigr)$ with $\widetilde{h}$, we obtain the boundedness of $G\bigl(\widetilde{h}\bigr)$ with respect to φ^*g and $\widetilde{h}$. Because $\widetilde{h}$ is G-equivariant, we obtain the Hermitian metric h of E which has the desired property. ■

3.8.4 Proof of Proposition 3.26

Let $V_2 := \mathbb{C}\, v_1 \oplus \mathbb{C} v_2$ with the nilpotent map determined by $f_2(v_1) = v_2$ and $f_2(v_2) = 0$. We obtain $\mathcal{V}_2 = V_2 \otimes \mathcal{O}_X(*H)$ and the Higgs field θ_2 determined by $\theta_2(v) = f_2(v)\, dz/z$. Let $\bigl(E_2, \overline{\partial}_{E_2}, \theta_2\bigr)$ be the Higgs bundle on $X \setminus H$ obtained as the restriction of $(\mathcal{V}_2, \theta_2)$. For any $\epsilon > 0$, we set $L_\epsilon(z) := \epsilon^{-1}(|z|^{-\epsilon} - |z|^{\epsilon})$. We also set $L_0 := -\log |z|^2$.

Lemma 3.29. *We obtain $L_0(z) \leq L_\epsilon(z) \leq |z|^{-\epsilon} L_0(z)$. There exists $C > 0$ such that*

$$\bigl|\partial_z \log L_\epsilon(z)\bigr| \leq C|z|^{-1}$$

on $\{|z| \leq 1/2\}$ for any $0 \leq \epsilon \leq 1/2$.

Proof. As proved in [49, Section 4.2], $L_0(z) \leq L_\epsilon(z)$ holds. We set $g_1(\epsilon) := -\epsilon \log |z|^2 - \bigl(1 - |z|^{2\epsilon}\bigr)$ for any $z \in \Delta^*$ and for $\epsilon > 0$. It is easy to check that $\partial_\epsilon g_1(\epsilon) \geq 0$ and $\lim_{\epsilon \to 0} g_1(\epsilon) = 0$. Hence, we obtain $L_\epsilon(z) \leq |z|^{-\epsilon} L_0(z)$. For $0 < a < 1$ and $0 < \epsilon$, we set $g_2(\epsilon, a) := \frac{\epsilon}{2}(a^{-\epsilon} - a^{\epsilon})^{-1}(a^{-\epsilon} + a^{\epsilon})$. Then, $\partial \log L_\epsilon(z) = -g_2(\epsilon, |z|)\frac{dz}{z}$. Then, we can check that $\partial_a g_2(\epsilon, a) \geq 0$ and $\partial_\epsilon g_2(\epsilon, a) \geq 0$. Then, we obtain the second claim of the lemma. ■

Let $h_2^{(\epsilon)}$ be the C^∞-metric of E_2 given by

$$h_2^{(\epsilon)}(v_1, v_1) = L_\epsilon, \qquad h_2^{(\epsilon)}(v_2, v_2) = L_\epsilon^{-1}, \qquad h_2^{(\epsilon)}(v_1, v_2) = 0.$$

Lemma 3.30. *$\bigl(E_2, \overline{\partial}_{E_2}, \theta_2, h_2^{(\epsilon)}\bigr)$ are harmonic bundles. Moreover, the family of metrics $h_2^{(\epsilon)}$ satisfies the condition in Proposition 3.26 for $(\mathcal{P}_*\mathcal{V}_2, \theta_2)$.*

Proof. Let H_ϵ be the matrix valued function on $X \setminus D$ determined by $(H_\epsilon)_{i,j} := h_2^{(\epsilon)}(v_i, v_j)$. Then, the following holds:

$$\overline{\partial}\bigl(H_\epsilon^{-1}\partial H_\epsilon\bigr) = \begin{pmatrix} \overline{\partial}\partial \log L_\epsilon & 0 \\ 0 & -\overline{\partial}\partial \log L_\epsilon \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\epsilon^2 |z|^{-2}\, d\overline{z}\, dz}{\bigl(|z|^{-\epsilon} - |z|^{\epsilon}\bigr)^2}$$
$$= -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L_\epsilon^{-2} \frac{dz\, d\overline{z}}{|z|^2}.$$

Let Θ be the matrix valued function representing θ_2 with respect to the frame (v_1, v_2), i.e., $\theta_2(v_1, v_2) = (v_1, v_2)\Theta$. Let $\theta^\dagger_{2,h^{(\epsilon)}}$ denote the adjoint of θ_2 with respect to $h_2^{(\epsilon)}$. Let $\Theta_\epsilon^\dagger$ denote the matrix valued function representing $\theta^\dagger_{2,h^{(\epsilon)}}$. The following holds:

$$\Theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{dz}{z}, \qquad \Theta_\epsilon^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} L_\epsilon^{-2} \frac{d\overline{z}}{\overline{z}}.$$

Hence, we obtain

$$\bigl[\Theta, \Theta_\epsilon^\dagger\bigr] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot L_\epsilon^{-2} \cdot \frac{dz\, d\overline{z}}{|z|^2}.$$

It implies that

$$\overline{\partial}\bigl(H_\epsilon^{-1}\partial H_\epsilon\bigr) + \bigl[\Theta, \Theta_\epsilon^\dagger\bigr] = 0.$$

It is exactly the Hitchin equation for $\bigl(E_2, \overline{\partial}_{E_2}, \theta_2, h_{2,\epsilon}\bigr)$. The other claim is easy to see. ■

For each $\ell \in \mathbb{Z}_{>0}$, we set $(V_\ell, f_\ell) := \mathrm{Sym}^{\ell-1}(V_2, f_2)$. We set $\mathcal{V}_\ell := V_\ell \otimes \mathcal{O}_X(*H)$ and $\theta_\ell := f_\ell \, \mathrm{d}z/z$. We obtain the regular filtered Higgs bundles $\big(\mathcal{P}^{(\epsilon)}_* \mathcal{V}_\ell, \theta_\ell\big)$. Note that $\big(\mathcal{P}^{(\epsilon)}_* \mathcal{V}_\ell, \theta_\ell\big)$ is naturally isomorphic to the $(\ell-1)$-th symmetric product of $\big(\mathcal{P}^{(\epsilon)}_* \mathcal{V}_2, \theta_2\big)$. Hence, $h^{(\epsilon)}_2$ induce harmonic metrics $h^{(\epsilon)}_\ell$ of $\big(E_\ell, \overline{\partial}_{E_\ell}, \theta_\ell\big) := (\mathcal{V}_\ell, \theta_\ell)_{|X\setminus H}$ satisfying the conditions in Proposition 3.26 for $\big(\mathcal{P}^{(\epsilon)}_* \mathcal{V}_\ell, \theta_\ell\big)$.

Let (V, f), (a, α) and $(\mathcal{V}, \theta)$ be as in Section 3.8.1. There exist integers $\ell_1, \dots, \ell_n$ such that

$$(V, f) \simeq \bigoplus_{j=1}^{n} \big(V_{\ell_j}, f_{\ell_j}\big).$$

We obtain $\mathcal{P}^{(\epsilon)}_* \mathcal{V} \simeq \bigoplus \mathcal{P}^{(\epsilon)}_* \mathcal{V}_{\ell_j}$. We obtain the harmonic metrics $\bigoplus_{j=1}^n |z|^{-2a(\epsilon)} h^{(\epsilon)}_{\ell_j}$ for $\big(E, \overline{\partial}_E, \theta\big)$. We can easily check that they satisfy the conditions in Proposition 3.26 for $(\mathcal{P}_* \mathcal{V}, \theta)$. ∎

3.8.5 Proof of Proposition 3.27

We set $\Phi := \bigoplus_{i,a,\alpha} \mathrm{d}\mathfrak{a}(i) \, \mathrm{id}_{\mathcal{V}_{i,a,\alpha}}$ and $\mathbb{D}^{\lambda\,\mathrm{reg}} := \bigoplus \mathbb{D}^{\lambda}_{i,a,\alpha}$. Let $\Phi^\dagger_{h^{(\epsilon)}}$ and $A^\dagger_{h^{(\epsilon)}}$ denote the adjoint of Φ and A with respect to $h^{(\epsilon)}$, respectively. We obtain the decompositions $\mathbb{D}^\lambda = \mathbb{D}^{\lambda\,\mathrm{reg}} + \Phi + A \, \mathrm{d}z/z$ and $\mathbb{D}^{\lambda\star}_{h^{(\epsilon)}} = \mathbb{D}^{\lambda\,\mathrm{reg}\,\star}_{h^{(\epsilon)}} - \Phi^\dagger_{h^{(\epsilon)}} - A^\dagger_{h^{(\epsilon)}} \, \mathrm{d}\overline{z}/\overline{z}$. Note that $\big[\mathbb{D}^{\lambda\,\mathrm{reg}}, \Phi^\dagger_{h^{(\epsilon)}}\big] = \big[\mathbb{D}^{\lambda\,\mathrm{reg}\,\star}_{h^{(\epsilon)}}, \Phi\big] = \big[\Phi, \Phi^\dagger_{h^{(\epsilon)}}\big] = 0$. By the assumption, we obtain

$$\big|\big[\Phi, A^\dagger_{h^{(\epsilon)}}\big]\big|_{g_\epsilon, h^{(\epsilon)}} = \big|\big[\Phi^\dagger_{h^{(\epsilon)}}, A\big]\big|_{g_\epsilon, h^{(\epsilon)}} \le 2\eta^{-2} C^2 C_1^2 |z|^{5m - 4\,\mathrm{rank}(\mathcal{V})\epsilon - 2\eta}.$$

We also obtain $\big|\big[A, A^\dagger_{h^{(\epsilon)}}\big]\big|_{h^{(\epsilon)}} \le 2C^2 C_1^2 |z|^{4\eta - 4\,\mathrm{rank}(\mathcal{V})\epsilon}$. Because

$$\big[\mathbb{D}^{\lambda\,\mathrm{reg}\,\star}_{h^{(\epsilon)}}, A \, \mathrm{d}z/z\big] = \big[B^{(\epsilon)}_2, A\big] \, |z|^{-2} \mathrm{d}\overline{z} \, \mathrm{d}z,$$

we obtain

$$\big|\big[\mathbb{D}^{\lambda\,\mathrm{reg}\,\star}_{h^{(\epsilon)}}, A \, \mathrm{d}z/z\big]\big|_{g_\epsilon, h^{(\epsilon)}} = \Big|\big[\mathbb{D}^{\lambda\,\mathrm{reg}}, A^\dagger_{h^{(\epsilon)}} \, \mathrm{d}\overline{z}/\overline{z}\big]\big|_{g_\epsilon, h^{(\epsilon)}} \le C_3 C |z|^{2\eta}.$$

Hence, we obtain the desired estimate for $G\big(h^{(\epsilon)}\big) = \big[\mathbb{D}^\lambda, \mathbb{D}^{\lambda\star}_{h^{(\epsilon)}}\big]$. ∎

3.9 Estimate of the curvature for Hermitian–Einstein metrics of a Higgs bundle

Let X be a complex surface. Let $\big(E, \overline{\partial}_E, \theta\big)$ be a Higgs bundle on X. Let g_i be a sequence of Kähler metric on X which is convergent to a Kähler metric g_∞ in the C^∞-sense locally on X. Let h_i $(i = 1, 2, \dots)$ be Hermitian–Einstein metrics of the Higgs bundle. We assume the following:

- $\int_X |G(h_i)|^2_{h_i, g_i} \to 0$.

Let ∇_{h_i} be the Chern connection of $\big(E, \overline{\partial}_E, h_i\big)$. Let $R(h_i)$ denote the curvature of ∇_{h_i}. The following proposition is a refinement contained in the argument in [46, Section 9.1.1].

Proposition 3.31. *For any relatively compact open subset $U \subset X$, and for any $p \ge 1$, the L^p_2-norms of $R(h_i)_{|U}$ with respect to h_i and g_∞ are bounded.*

Proof. Let P be any point of X. Let (X_P, z_1, z_2) be a holomorphic coordinate neighbourhood of X around P. Let us describe θ as $\theta = \sum_{j=1,2} f_j \, \mathrm{d}z_j$. Let $X_{P,0}$ be a relatively compact neighbourhood of P in X_P. According to [46, Lemma 2.13], there exist $C_k > 0$ $(k = 1, 2)$, which are independent of h_i, such that the following inequalities hold on $X_{P,0}$:

$$|f_j|^2_{h_i} \leq C_1 \cdot \exp\left(C_2 \int_X |G(h_i)|^2_{h_i}\right).$$

(Note that $G(h_i)$ is denoted as $F(h_i)$ in [46].) Let ϵ denote a positive number. After rescaling the coordinate system, we may assume the following on $X_{P,0}$:

$$\sum |f_j|^2_{h_i} \leq \epsilon/100.$$

There exists i_0 such that for $i \geq i_0$ we obtain $\int |G(h_i)|^2_{h_i,g_i} \leq \epsilon/100$. Because the L^2-norms are scale invariant, we obtain

$$\int_{X_{P,0}} \left|R(h_i)\right|^2_{h_i,g_{i,P}} \leq \epsilon/10.$$

Let $X_{P,1}$ be a relatively compact neighbourhood of P in $X_{P,0}$. If ϵ is sufficiently small, by the theorem of Uhlenbeck [74, Corollary 2.2], there exists an orthonormal frame $\boldsymbol{v}_i$ of $(E, h_i)_{|X_{P,1}}$ for each i such that the connection form A_i of ∇_{h_i} with respect to $\boldsymbol{v}_i$ satisfies (i) A_i is L^2_1, (ii) $\|A_i\|_{L^2_1} \leq C_3 \|R(h_i)\|_{L^2}$ on $X_{P,1}$ for a positive constant C_3 independently from i, (iii) A_i satisfies $\mathrm{d}^*_{g_{i,P}} A_i = 0$, where $\mathrm{d}^*_{g_{i,P}}$ denotes the adjoint of d with respect to the metric $g_{i,P}$. Let Θ_i and $\Theta_i^\dagger$ represent θ and $\theta^\dagger_{h_i}$ with respect to the frame $\boldsymbol{v}_i$. Then, A_i satisfies

$$\Lambda_{g_{P,i}}(\mathrm{d}A_i + A_i \wedge A_i) + \Lambda_{g_{P,i}}\left[\Theta_i, \Theta_i^\dagger\right] = 0.$$

Let $X_{P,2}$ be a relatively compact neighbourhood of P in $X_{P,1}$. By the argument of Donaldson in the proof of [15, Corollary 23], we obtain that A_i are L^p_1 for any $p \geq 2$ on $X_{P,2}$, and that there exists $C_{4,p} > 0$ such that $\|A_i\|_{L^p_1} \leq C_{4,p}$ on $X_{P,2}$, where $C_{4,p}$ are independent of i. In particular, there exists $C_{5,p} > 0$ independently from i such that $\left\|R(h_i)_{|X_{P,2}}\right\|_{L^p} \leq C_{5,p}$.

Let $A_i = A_i^{0,1} + A_i^{1,0}$ be the decomposition into the $(0,1)$-part and the $(1,0)$-part. Because $\overline{\partial}\theta = 0$, we obtain $\overline{\partial}\Theta_i + \left[A_i^{0,1}, \Theta_i\right] = 0$. Hence, there exist $C_{6,p} > 0$ independently from i such that $\left\|\Theta_{i|X_{P,2}}\right\|_{L^p_2} \leq C_{6,p}$.

Note that $\overline{\partial}_E R(h_i) = 0$ and $\partial_{E,h_i} R(h_i) = 0$. Let $\overline{\partial}^*_{E,h_i,g_{i,P}}$ denote the formal adjoint of $\overline{\partial}_E$ with respect to h_i and $g_{i,P}$. Because $\Lambda_{g_{i,P}} R(h_i) + \Lambda_{g_{i,P}}\left[\theta_i, \theta_i^\dagger\right] = 0$, there exists $C_{7,p} > 0$ such that $\left\|\overline{\partial}^*_{E,h_i,g_{i,P}} R(h_i)_{|X_{P,2}}\right\|_{L^p_1} < C_{7,p}$. Let $X_{P,3}$ be a relatively compact neighbourhood of P in $X_{P,2}$. There exists $C_{8,p} > 0$ independently from i such that $\left\|R(h_i)_{|X_{P,3}}\right\|_{L^p_2} < C_{8,p}$. It implies the claim of the proposition. ■

4 Existence and continuity of harmonic metrics in the curve case

4.1 Existence of Hermitian–Einstein metric

Let X be a compact Riemann surface. Let $D \subset X$ be a finite subset. Let $\left(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\right)$ be a stable good filtered λ-flat bundle on (X, D). Let $\left(E, \mathbb{D}^\lambda\right)$ be the λ-flat bundle on $X \setminus D$ obtained as the restriction of $\left(\mathcal{V}, \mathbb{D}^\lambda\right)$. Let ω be any Kähler form of X. Let $h_{\det(E)}$ be a Hermitian metric of $\det(E)$ such that (i) $\Lambda_\omega R(h_{\det(E)}) = 2\pi \deg(\mathcal{P}_*\mathcal{V})\left(\int_X \omega\right)^{-1}$, (ii) $h_{\det(E)}$ is adapted to $\mathcal{P}_* \det(\mathcal{V})$, i.e., $\mathcal{P}_*^{h_{\det(E)}} \det(E) \simeq \mathcal{P}_* \det(\mathcal{V})$. (See Proposition 3.2.)

Theorem 4.1 (Biquard–Boalch)**.** *There exists a unique Hermitian–Einstein metric h of $\big(E, \mathbb{D}^\lambda\big)$ adapted to $\mathcal{P}_*\mathcal{V}$ such that $\det(h) = h_{\det(E)}$. If $\deg(\mathcal{P}_*\mathcal{V}) = 0$, h is a harmonic metric.*

Proof. It is enough to prove the case $\deg(\mathcal{P}_*\mathcal{V}) = 0$. We explain an outline of the proof based on the fundamental theorem of Simpson [62, Theorem 1] (and its variant [49, Proposition 2.49]) because we obtain a consequence on the Donaldson functional from the proof, which will be useful in the proof of Proposition 4.5 below. Set $e := \operatorname{rank}(\mathcal{V})!$. Take $\eta > 0$ such that $10e\eta < \widetilde{\operatorname{gap}}(\mathcal{P}_*\mathcal{V}, \boldsymbol{a})$. (See Section 3.5.4 for $\widetilde{\operatorname{gap}}$.)

Let $\big(X_P, z_P\big)$ be an admissible coordinate neighbourhood around P. Set $X_P^* := X_P \setminus \{P\}$. We take a Kähler metric $g_{X\setminus D}$ of $X \setminus D$ satisfying the following condition:

- $g_{X\setminus D|X_P^*}$ is mutually bounded with $|z_P|^{-2+\eta} \mathrm{d}z_P\, \mathrm{d}\overline{z}_P$ on X_P^* for each $P \in D$.

Recall that the Kähler manifold $(X \setminus D, g_{X\setminus D})$ satisfies the assumptions given in [62, Section 2], according to [62, Proposition 2.4].

Lemma 4.2. *There exists a Hermitian metric h_0 of E such that the following holds:*

(*a*) $\big(E, \mathrm{d}''_E, h_0\big)$ *is acceptable, and* $\mathcal{P}_*^{h_0} E = \mathcal{P}_*\mathcal{V}$.

(*b*) $G(h_0)$ *is bounded with respect to* h_0 *and* $g_{X\setminus D}$.

(*c*) $\det(h_0) = h_{\det(E)}$.

Proof. By Corollary 3.28, we obtain a Hermitian metric h'_0 of E satisfying (a) and (b). We define the function $\varphi\colon X \setminus D \longrightarrow \mathbb{R}$ by $h_{\det(E)} = \det(h'_0)\mathrm{e}^{\varphi}$. Then, φ induces a C^∞-function on X. We set $h_0 := h'_0 \mathrm{e}^{\varphi/\operatorname{rank}(E)}$. Then, the metric h_0 has the desired property. ■

For any λ-flat subbundle $E' \subset E$, let h'_0 denote the Hermitian metric of E' induced by h_0. Let $\mathbb{D}^\lambda_{E'}$ denote the Higgs field of E' obtained as the restriction of $\mathbb{D}^\lambda$. We obtain the Chern connection $\nabla_{h'_0}$ from the $(0,1)$-part of $\mathbb{D}^\lambda_{E'}$ and h'_0. Let $R(h'_0)$ denote the curvature of $\nabla_{h'_0}$. We set

$$\deg(E', h_0) := \frac{\sqrt{-1}}{2\pi} \frac{1}{1+|\lambda|^2} \int_{X\setminus D} \operatorname{Tr} G\big(E', \mathbb{D}^\lambda_{E'}, h'_0\big) = \frac{\sqrt{-1}}{2\pi} \int_{X\setminus D} \operatorname{Tr} R(h'_0).$$

Let $\Lambda_{g_{X\setminus D,\eta}}$ denote the adjoint of the multiplication by the Kähler form associated with $g_{X\setminus D,\eta}$. Because $G(h_0)$ is bounded with respect to h_0 and $g_{X\setminus D,\eta}$, $\deg(E', h_0)$ is well defined in $\mathbb{R}\cup\{-\infty\}$ by the Chern–Weil formula [62, Lemma 3.2] (see also [49, Lemma 2.34]):

$$\deg(E', h_0) = \frac{\sqrt{-1}}{2\pi} \frac{1}{1+|\lambda|^2} \int \operatorname{Tr}\big(\Lambda_{g_{X\setminus D,\eta}} G(h_0) \pi_{E'}\big) - \frac{1}{2\pi} \frac{1}{1+|\lambda|^2} \int \big|\mathbb{D}^\lambda \pi_{E'}\big|^2.$$

Here, $\pi_{E'}$ denotes the orthogonal projection $E \longrightarrow E'$ with respect to h_0.

Lemma 4.3. $\deg(E', h_0)/\operatorname{rank}(E') < \deg(E, h_0)/\operatorname{rank}(E)$ *holds. Namely,* $(E, \mathbb{D}^\lambda, h_0)$ *is analytically stable in the sense of* [63, Section 6] (*see also* [49, Section 2.3]).

Proof. By [63, Lemma 6.1], we have $\deg(E, h_0) = \deg\big(\mathcal{P}_*^{h_0} E\big) = 0$. Let $0 \neq E' \subsetneq E$ be a λ-flat subbundle on $X \setminus D$. By [63, Lemma 6.2], if $\deg(E', h_0) \neq -\infty$, E' extends to a filtered subbundle $\mathcal{P}_*^{h'_0} E' \subset \mathcal{P}_*^{h_0} E$, and $\deg(E', h_0) = \deg\big(\mathcal{P}_*^{h'_0} E'\big)$ holds. Because $(\mathcal{P}_*^{h_0} E, \mathbb{D}^\lambda)$ is assumed to be stable, we obtain $\deg(E', h_0)/\operatorname{rank} E' < \deg\big(\mathcal{P}_*^{h_0} E\big)/\operatorname{rank} E = 0$. Hence, $\big(E, \overline{\partial}_E, \theta, h_0\big)$ is analytically stable. ■

According to the existence theorem of Simpson [62, Theorem 1] (see also [49, Proposition 2.49]), there exists a Hermitian–Einstein metric h of $\big(E, \mathbb{D}^\lambda\big)$ such that $\det(h) = \det(h_0)$ and that h and h_0 are mutually bounded. We already know the uniqueness as in Proposition 2.22. Thus, we obtain Theorem 4.1. ■

4.1.1 Complement on the Donaldson functional

Let h_0 and $g_{X\setminus D}$ be as in the proof of Theorem 4.1. Let $\mathcal{H}(h_0)$ be the space of C^∞-Hermitian metrics h_1 of E satisfying the following condition:

- Let u_1 be the endomorphism of E such that (i) $h_1 = h_0 e^{u_1}$, (ii) u_1 is self-adjoint with respect to both h_0 and h_1. Then, $\sup_{Q\in X\setminus D} |u_1|_{h_0}(Q) + \|\mathbb{D}^\lambda u_1\|_{L^2} + \left\|\mathbb{D}^\lambda \mathbb{D}^{\lambda\star}_{h_0} u_1\right\|_{L^1} < \infty$. Here, we use the L^p-norms induced by h_0 and $g_{X\setminus D}$.

The Donaldson functional $M(h_0, \bullet)\colon \mathcal{H}(h_0) \longrightarrow \mathbb{R}$ is defined as in [62, Section 5] and [49, Section 2.4].

Proposition 4.4. *Let h be the Hermitian–Einstein metric in Theorem 4.1. Then, h is contained in $\mathcal{H}(h_0)$, and $M(h_0, h) \leq 0$ holds.*

Proof. Let b be the automorphism of E which is self-adjoint with respect to both h and h_0, and determined by $h = h_0 \cdot b$. The theorem of Simpson [62, Theorem 1] (see also [49, Proposition 2.49]) implies that b and b^{-1} are bounded, and that $\mathbb{D}^\lambda b$ is L^2 with respect to h_0 and $g_{X\setminus D}$. By [62, Lemma 3.1] (see also [49, Section 2.2.5]), we also obtain $\mathbb{D}^\lambda \mathbb{D}^{\lambda\star}_{h_0} b$ is L^1. Hence, h is contained in $\mathcal{H}(h_0)$. In the proof of [62, Theorem 1] and [49, Proposition 2.39], the metric h is constructed as the limit of a subsequence of the heat flow h_t $(t \geq 0)$ for which $\partial_t M(h_0, h_t) \leq 0$ holds. Because $M(h_0, h_0) = 0$ by the construction, we obtain $M(h_0, h_t) \leq 0$, and hence $M(h_0, h) \leq 0$. ■

4.2 Continuities of some families of Hermitian metrics

4.2.1 Setting

Family of curves. Let Σ be a compact connected oriented real 2-dimensional C^∞-manifold with a finite subset $D \subset \Sigma$. Let J_i $(i = 1, 2, \dots)$ be a sequence of complex structures on Σ such that the sequence J_i is convergent to J in the C^∞-sense. Assume that there exists a neighbourhood $N(D)$ of D in Σ such that $J_{i|N(D)}$ are independent of i. Let X_i denote the compact Riemann surfaces (Σ_i, J_i). Similarly, let X denote the compact Riemann surface (Σ, J). Let $\kappa_i\colon (T\Sigma, J) \simeq (T\Sigma, J_i)$ be isomorphisms of complex vector bundles on Σ such that (i) $\kappa_i \to \mathrm{id}$, (ii) $\kappa_{i|N(D)} = \mathrm{id}$. We regard κ_i as isomorphisms of complex vector bundles $TX \simeq TX_i$.

For $P \in D$, let (X_P, z_P) denote an admissible coordinate neighbourhood of P in X such that $X_P \subset N(D)$. We may regard $\left(X_P, z_P\right)$ as a holomorphic coordinate neighbourhood of P in X_i. Let r be a positive integer, and set $e := r!$. As in Section 3.5.1, let $\varphi_P\colon \widetilde{X}_P \longrightarrow X_P$ be the ramified covering given by $\varphi_P(\zeta_P) = \zeta_P^e$. Let G_P denote the Galois group of the ramified covering φ_P.

Family of good filtered $\boldsymbol{\lambda}$-flat bundles. Let $\left(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\right)$ be a stable good filtered λ-flat bundle of rank r on (X, D) with $\deg(\mathcal{P}_*\mathcal{V}) = 0$. Let $\left(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}^\lambda_i\right)$ be stable good filtered λ-flat bundles of rank r on (X_i, D) with $\deg(\mathcal{P}_*\mathcal{V}_i) = 0$. For each $P \in D$, we set $\left(\mathcal{P}_*\mathcal{V}_P, \mathbb{D}^\lambda_P\right) := \left(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\right)_{|X_P}$ and $\left(\mathcal{P}_*\mathcal{V}_{i,P}, \mathbb{D}^\lambda_{i,P}\right) := \left(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}^\lambda\right)_{|X_P}$. Set $\left(\mathcal{P}_*\widetilde{\mathcal{V}}_P, \widetilde{\mathbb{D}}^\lambda_P\right) := \varphi_P^*\left(\mathcal{P}_*\mathcal{V}_P, \mathbb{D}^\lambda_P\right)$ and $\left(\mathcal{P}_*\widetilde{\mathcal{V}}_{i,P}, \widetilde{\mathbb{D}}^\lambda_{i,P}\right) := \varphi_P^*\left(\mathcal{P}_*\mathcal{V}_{i,P}, \mathbb{D}^\lambda_{i,P}\right)$. There exist G_P-invariant subsets $\mathcal{I}(P), \mathcal{I}(i,P) \subset \zeta_P^{-1}\mathbb{C}[\zeta_P^{-1}]$ and the formal decompositions

$$\left(\mathcal{P}_*\widetilde{\mathcal{V}}_P, \widetilde{\mathbb{D}}^\lambda_P\right) \otimes \mathbb{C}[\![\zeta]\!] = \bigoplus_{\mathfrak{a}\in\mathcal{I}(P)} \left(\mathcal{P}_*\widetilde{\mathcal{V}}_{P,\mathfrak{a}}, \widetilde{\mathbb{D}}^\lambda_{P,\mathfrak{a}}\right),$$

$$\left(\mathcal{P}_*\widetilde{\mathcal{V}}_{i,P}, \widetilde{\mathbb{D}}^\lambda_{i,P}\right) \otimes \mathbb{C}[\![\zeta]\!] = \bigoplus_{\mathfrak{a}\in\mathcal{I}(i,P)} \left(\mathcal{P}_*\widetilde{\mathcal{V}}_{i,P,\mathfrak{a}}, \widetilde{\mathbb{D}}^\lambda_{i,P,\mathfrak{a}}\right),$$

for each $P \in D$. Suppose moreover that there exist G_P-invariant bijections $\rho_{i,P}\colon \mathcal{I}(P) \simeq \mathcal{I}(i,P)$ such that the following holds:

- $\operatorname{rank} \widetilde{\mathcal{V}}_{P,\mathfrak{a}} = \operatorname{rank} \widetilde{\mathcal{V}}_{i,P,\rho_{i,P}(\mathfrak{a})}$.
- $\operatorname{ord} \mathfrak{a} = \operatorname{ord} \rho_{i,P}(\mathfrak{a})$ and $\operatorname{ord}(\mathfrak{a} - \mathfrak{b}) = \operatorname{ord}\bigl(\rho_{i,P}(\mathfrak{a}) - \rho_{i,P}(\mathfrak{b})\bigr)$.
- $\lim_{i\to\infty} \rho_{i,P}(\mathfrak{a}) = \mathfrak{a}$ in $\zeta^{-1}\mathbb{C}\bigl[\zeta^{-1}\bigr]$.

We fix such bijections $\rho_{i,P}$. Let $\pi_{P,\mathfrak{a}}$ denote the projection $\mathcal{P}_*\widetilde{\mathcal{V}}_P \otimes \mathbb{C}[\![\zeta]\!] \longrightarrow \mathcal{P}_*\widetilde{\mathcal{V}}_{P,\mathfrak{a}}$. Similarly, let $\pi_{i,P,\mathfrak{a}}$ denote the projection $\mathcal{P}_*\widetilde{\mathcal{V}}_{i,P} \otimes \mathbb{C}[\![\zeta]\!] \longrightarrow \mathcal{P}_*\widetilde{\mathcal{V}}_{i,P,\mathfrak{a}}$.

***C^∞*-isomorphisms.** We set $\bigl(E, \mathbb{D}^\lambda\bigr) := \bigl(\mathcal{V}, \mathbb{D}^\lambda\bigr)_{|X\setminus D}$ and $\bigl(E_i, \mathbb{D}_i^\lambda\bigr) := \bigl(\mathcal{V}_i, \mathbb{D}_i^\lambda\bigr)_{|X_i\setminus D}$. Let h_0 denote C^∞-metrics of E adapted to $\mathcal{P}_*\mathcal{V}$ such that $R(\det(h_0)) = 0$. Let $h_{0,i}$ denote C^∞-metrics of E_i adapted to $\mathcal{P}_*\mathcal{V}_i$ such that $R(\det(h_{0,i})) = 0$. Let d'' and d_i'' denote the $(0,1)$-parts of $\mathbb{D}^\lambda$ and $\mathbb{D}_i^\lambda$. Suppose that there exist C^∞-isomorphisms $f_i\colon E \simeq E_i$ satisfying the following conditions:

- $f_i^*(h_{0,i}) \to h_0$ in the C^∞-sense locally on $\Sigma \setminus D$.
- On $N(D)\setminus D$, f_i are holomorphic with respect to d'' and d_i'', and f_i extend to isomorphisms of filtered bundles $\mathcal{P}_*\mathcal{V}_{|N(D)} \simeq \mathcal{P}_*\mathcal{V}_{i|N(D)}$.
- For each $P \in D$, we obtain $\operatorname{Gr}^F_c\bigl(f_{i|X_P}\bigr) \circ \operatorname{Res}_P(\mathbb{D}^\lambda) = \operatorname{Res}_P\bigl(\mathbb{D}_i^\lambda\bigr) \circ \operatorname{Gr}^F_c\bigl(f_{i|X_P}\bigr)$ on $\operatorname{Gr}^F_c(\mathcal{V}_P)$ for any $c \in \mathbb{R}$. Moreover, there exists $N(P) \geq 10 \operatorname{rank}(\mathcal{V}) |\operatorname{ord} \mathfrak{a}|$ for any $\mathfrak{a} \in \mathcal{I}(P)$ such that for the induced isomorphisms $\varphi_P^*\bigl(f_{i|X_P}\bigr)\colon \mathcal{P}_*\widetilde{\mathcal{V}}_P \otimes \mathbb{C}[\![\zeta]\!] \simeq \mathcal{P}_*\widetilde{\mathcal{V}}_{i,P} \otimes \mathbb{C}[\![\zeta]\!]$, we obtain
$$\begin{aligned}&\bigl(\pi_{P,\mathfrak{a}} - \varphi_P^*\bigl(f_{i|X_P}\bigr)^{-1} \circ \pi_{i,P,\rho_{i,P}(\mathfrak{a})} \circ \varphi_P^*\bigl(f_{i|X_P}\bigr)\bigr)\mathcal{P}_*\widetilde{\mathcal{V}}_P \otimes \mathbb{C}[\![\zeta]\!] \\ &\qquad \subset \mathcal{P}_{*-N(P)}\widetilde{\mathcal{V}}_{i,P} \otimes \mathbb{C}[\![\zeta]\!],\end{aligned} \tag{4.1}$$
and the sequences (4.1) are convergent to 0 as $i \to \infty$.
- $\mathbb{D}^\lambda - (f_i \otimes \kappa_i)^{-1} \circ \mathbb{D}_i^\lambda \circ f_i \to 0$ in the C^∞-sense with respect to h_0 locally on $\Sigma \setminus D$.

Perturbation. We take η_i $(i = 1,2)$ satisfying $10e\eta_1 < \widetilde{\operatorname{gap}}(\mathcal{P}_*\mathcal{V})$ and $10r\eta_2 < \eta_1$. We take $\boldsymbol{a} \in \mathbb{R}^D$ for $\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}, P)$ $(P \in D)$ as in Lemma 3.19. For any $0 < \epsilon < \eta_2$, by taking $\psi_{P,\epsilon}$ $(P \in D)$ as in Section 3.7.1, we obtain families of good filtered λ-flat bundles $\bigl(\mathcal{P}_*^{(\epsilon)}\mathcal{V}, \mathbb{D}^\lambda\bigr)$ and $\bigl(\mathcal{P}_*^{(\epsilon)}\mathcal{V}_i, \mathbb{D}_i^\lambda\bigr)$. We assume the following for each $P \in D$:
$$\sum_{a(P)-1<c\leq a(P)} \psi_{P,\epsilon}(c) \operatorname{rank} \operatorname{Gr}^F_c\bigl(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\bigr) = \sum_{a(P)-1<c\leq a(P)} c \operatorname{rank} \operatorname{Gr}^F_c\bigl(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_{|P}\bigr).$$
In particular, $\deg(\mathcal{P}_*\mathcal{V}) = \deg\bigl(\mathcal{P}_*^{(\epsilon)}\mathcal{V}\bigr)$ and $\deg(\mathcal{P}_*\mathcal{V}_i) = \deg\bigl(\mathcal{P}_*^{(\epsilon)}\mathcal{V}_i\bigr)$ hold. By making η_2 smaller, we may assume that $\bigl(\mathcal{P}_*^{(\epsilon)}\mathcal{V}, \mathbb{D}^\lambda\bigr)$ are stable for any $0 \leq \epsilon \leq \eta_2$.

4.2.2 Continuity of the family of harmonic metrics

According to Theorem 4.1, there exists a harmonic metric $h_i^{(\epsilon)}$ of $\bigl(E_i, \mathbb{D}^\lambda\bigr)$ adapted to $\mathcal{P}_*^{(\epsilon)}\mathcal{V}_i$ such that $\det h_i^{(\epsilon)} = \det h_{0,i}$. Similarly, there exists a harmonic metric $h^{(0)}$ of $\bigl(E, \mathbb{D}^\lambda\bigr)$ adapted to $\mathcal{P}_*\mathcal{V}$ such that $\det h^{(0)} = \det h_0$. The following proposition is a variant of [49, Propositions 4.1 and 4.2].

Proposition 4.5. *For any sequence $\epsilon_i \to 0$, the sequence $h_i^{(\epsilon_i)}$ is convergent to $h^{(0)}$ locally on $X \setminus D$ in the C^∞-sense.*

Proof. For $0 \le \epsilon \le \eta_2$, let $g_{X\setminus D,\epsilon}$ be the Kähler metric on $X \setminus D$ such that the following holds on X_P^* for any $P \in D$:

$$g_{X\setminus D,\epsilon|X_P^*} = \big(\epsilon^2|z_P|^{2\epsilon} + \eta_1^2|z_P|^{2\eta_1}\big)|z_P|^{-2}\mathrm{d}z_P\,\mathrm{d}\overline{z}_P.$$

Let Λ_ϵ denote the adjoint of the multiplication by the Kähler form $\omega_{X\setminus D,\epsilon}$ associated with $g_{X\setminus D,\epsilon}$.

By the isomorphisms $\kappa_i\colon (T\Sigma, J) \simeq (T\Sigma, J_i)$ and the metrics $g_{X\setminus D,\epsilon}$, we obtain the Kähler metrics $g_{X_i\setminus D,\epsilon}$ of $X_i \setminus D$. Let $\Lambda_{i,\epsilon}$ denote the adjoint of the multiplication by the Kähler form $\omega_{X_i\setminus D,\epsilon}$ associated with $g_{X_i\setminus D,\epsilon}$.

There exists an approximation of $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|X_P}$ by a model filtered λ-flat bundle as in Section 3.6. By using a family of Hermitian metrics for the model λ-flat bundle as in Proposition 3.26, and by using Proposition 3.27, we construct a family of metrics $h^{(\epsilon)}_{\mathrm{in}}$ $(0 \le \epsilon \le \eta_2)$ of E such that the following holds:

- $h^{(\epsilon)}_{\mathrm{in}}$ is adapted to $\mathcal{P}^{(\epsilon)}_*\mathcal{V}$.
- $\det h^{(\epsilon)}_{\mathrm{in}} = \det h_0$.
- $h^{(\epsilon)}_{\mathrm{in}} \longrightarrow h^{(0)}_{\mathrm{in}}$ locally on $X \setminus D$ in the C^∞-sense as $\epsilon \longrightarrow 0$.
- There exists $C_1 > 0$ such that $\big|G\big(h^{(\epsilon)}_{\mathrm{in}}\big)\big|_{g_{X\setminus D,\epsilon},h^{(\epsilon)}_{\mathrm{in}}} < C_1$ for any ϵ.

Let ν_i be the function on $X_i \setminus D$ determined by $\big(f_i^{-1}\big)^*(\det h_0) = \mathrm{e}^{\nu_i}\det h_{i,0}$. We obtain the Hermitian metrics $h^{(\epsilon)}_{i,\mathrm{in}} := \mathrm{e}^{\nu_i/\operatorname{rank}\mathcal{V}}\big(f_i^{-1}\big)^*\big(h^{(\epsilon)}_{\mathrm{in}}\big)$ $(0 \le \epsilon \le \eta_2)$ of E_i. Then, by Proposition 3.27, we obtain the following:

- $h^{(\epsilon)}_{i,\mathrm{in}}$ is adapted to $\mathcal{P}^{(\epsilon)}_*\mathcal{V}_i$.
- $\det h^{(\epsilon)}_{i,\mathrm{in}} = \det h_{i,0}$.
- $h^{(\epsilon)}_{i,\mathrm{in}} \longrightarrow h^{(0)}_{i,\mathrm{in}}$ locally on $X \setminus D$ in the C^∞-sense as $\epsilon \longrightarrow 0$.
- By replacing C_1 with a larger constant, we may assume $\big|G\big(h^{(\epsilon)}_{i,\mathrm{in}}\big)\big|_{g_{X_i\setminus D,\epsilon},h^{(\epsilon)}_{i,\mathrm{in}}} < C_1$ for any ϵ and any i.

Lemma 4.6. *Let $u^{(i)}$ $(\epsilon_i \to 0)$ be automorphisms of E_i which are self-adjoint with respect to $h^{(\epsilon_i)}_{i,\mathrm{in}}$ such that the following holds:*

- $\mathrm{Tr}\,\big(u^{(i)}\big) = 0$.
- $h^{(\epsilon_i)}_{i,\mathrm{in}}\mathrm{e}^{u^{(i)}} \in \mathcal{H}\big(h^{(\epsilon_i)}_{in}\big)$, *i.e.,* $\sup\big\|u^{(i)}\big\|_{h^{(\epsilon_i)}_{i,\mathrm{in}}} + \big\|\mathbb{D}^\lambda_i u^{(i)}\big\|_{L^2} + \big\|\mathbb{D}^\lambda_i \mathbb{D}^{\lambda\star}_{i,h^{(\epsilon_i)}_{i,\mathrm{in}}} u^{(i)}\big\|_{L^1} < \infty$, *where the L^p-norms are taken with respect to $h^{(\epsilon_i)}_{i,\mathrm{in}}$ and $g_{X_i\setminus D,\epsilon_i}$. We do not assume that the estimate is uniform in i.*
- *There exists $C_2 > 0$ such that $\big|\Lambda_{i,\epsilon_i} G\big(h^{(\epsilon_i)}_{i,\mathrm{in}}\mathrm{e}^{u^{(i)}}\big)\big|_{h^{(\epsilon_i)}_{i,\mathrm{in}}} < C_2$ for any i.*

Then, there exists $C_3, C_4 > 0$ such that the following holds for any ϵ_i

$$\sup\big|u^{(i)}\big|_{h^{(\epsilon_i)}_{i,\mathrm{in}}} < C_3 + C_4 M\big(h^{(\epsilon_i)}_{i,\mathrm{in}}, h^{(\epsilon_i)}_{i,\mathrm{in}}\mathrm{e}^{u^{(i)}}\big).$$

Proof. By identifying the vector bundles E_i and E by f_i, we apply the same argument as in the proof of [49, Lemma 2.45]. ∎

Let $b^{(\epsilon)}_{i,1}$ be the automorphism of E_i which is self-adjoint with respect to $h^{(\epsilon)}_{i,\mathrm{in}}$ and $h^{(\epsilon)}_i$ and determined by $h^{(\epsilon)}_i = h^{(\epsilon)}_{i,\mathrm{in}} b^{(\epsilon)}_{i,1}$. Note that $\det(b^{(\epsilon)}_{i,1}) = 1$. Take any sequence $\epsilon_i \to 0$. By Proposition 4.4 and Lemma 4.6, there exists a constant $C_{10} > 0$ such that the following holds for any i:

$$\sup_{Q\in X_i\setminus D} \big|b^{(\epsilon_i)}_{i,1|Q}\big|_{h^{(\epsilon_i)}_{i,\mathrm{in}}} < C_{10}, \qquad \sup_{Q\in X_i\setminus D} \big|\big(b^{(\epsilon_i)}_{i,1|Q}\big)^{-1}\big|_{h^{(\epsilon_i)}_{i,\mathrm{in}}} < C_{10}.$$

Lemma 4.7. *$\int \Lambda_{i,\epsilon_i}\big(\overline{\partial}_{X_i}\partial_{X_i} \operatorname{Tr}\big(b^{(\epsilon_i)}_{i,1}\big)\big)\omega_{X_i\setminus D,\epsilon_i} = 0$ holds.*

Proof. We use Proposition 4.4. Because $\mathbb{D}^{\lambda}_i\mathbb{D}^{\lambda\star}_i b^{(\epsilon_i)}_{i,1}$ is L^1 with respect to $h^{(\epsilon_i)}_{i,\mathrm{in}}$ and $g_{X_i\setminus D,\epsilon_i}$, we obtain that $\Lambda_{i,\epsilon_i}\overline{\partial}_{X_i}\partial_{X_i} \operatorname{Tr}\big(b^{(\epsilon_i)}_{i,1}\big)$ is L^1 with respect to $g_{X_i\setminus D,\epsilon_i}$. Because $\mathbb{D}^{\lambda}_i b^{(\epsilon_i)}_{i,1}$ is L^2 with respect to $g_{X_i\setminus D,\epsilon_i}$ and $h^{(\epsilon_i)}_{i,\mathrm{in}}$, we obtain that $\partial_{X_i} \operatorname{Tr}\big(b^{(\epsilon_i)}_{i,1}\big)$ is L^2 with respect to $g_{X_i\setminus D,\epsilon_i}$. Therefore, we obtain the claim of the lemma by using [62, Lemma 5.2]. ■

By [62, Lemma 3.1], the following holds:

$$\sqrt{-1}\Lambda_{i,\epsilon_i}\overline{\partial}_{X_i}\partial_{X_i} \operatorname{Tr}\big(b^{(\epsilon_i)}_{i,1}\big) = -\operatorname{Tr}\big(b^{(\epsilon_i)}_{i,1}\Lambda_{i,\epsilon_i}G\big(h^{(\epsilon_i)}_{i,\mathrm{in}}\big)\big) - \big|\mathbb{D}^{\lambda}_i\big(b^{(\epsilon_i)}_{i,1}\big)\cdot\big(b^{(\epsilon_i)}_{i,1}\big)^{-1/2}\big|^2_{h^{(\epsilon_i)}_{i,\mathrm{in}},g_{X_i\setminus D,\epsilon_i}}.$$

Therefore, there exists $C_{12} > 0$ such that the following holds for any i:

$$\int \big|\mathbb{D}^{\lambda}_i b^{(\epsilon_i)}_{i,1}\big|^2_{h^{(\epsilon_i)}_{i,\mathrm{in}},g_{X_i\setminus D,\epsilon_i}}\omega_{X_i\setminus D,\epsilon_i} < C_{12}.$$

We also obtain

$$\int \big|\mathbb{D}^{\lambda\star}_i b^{(\epsilon_i)}_{i,1}\big|^2_{h^{(\epsilon_i)}_{i,\mathrm{in}},g_{X_i\setminus D,\epsilon_i}}\omega_{X_i\setminus D,\epsilon_i} < C_{12}. \tag{4.2}$$

Let $\big(E_i,\overline{\partial}^{(\epsilon_i)}_{E_i},\theta^{(\epsilon_i)}_i\big)$ be the Higgs bundles underlying $\big(E_i,\mathbb{D}^{\lambda},h^{(\epsilon_i)}_i\big)$. Then, there exists $C_{13} > 0$ such that the following holds for any i:

$$\int \big|\theta^{(\epsilon_i)}_i\big|^2_{h^{(\epsilon_i)}_{i,\mathrm{in}},g_{X_i\setminus D,\epsilon_i}}\omega_{X_i\setminus D,\epsilon_i} < C_{13}.$$

Then, by applying the argument in [49, Section 4.5.3], we obtain the desired convergence of the sequence $h^{(\epsilon_i)}_i$. ■

4.2.3 Continuity of some families of Hermitian metrics

For $P \in D$, we set $X^*_P := X_P \setminus \{P\}$. We may naturally regard X^*_P as a subset of $X_i \setminus D$. Fix $N > 10$. Let $g_{i,\epsilon}$ be a sequence of Kähler metrics of $X_i \setminus D$, such that $g_{i,\epsilon} \to g_\epsilon$ $(i \to \infty)$ and that

$$g_{i,\epsilon|X^*_P} = \big(\epsilon^{N+2}|z_P|^{2\epsilon} + |z_P|^2\big)\frac{dz_P\, d\overline{z}_P}{|z_P|^2}.$$

Let ϵ_i $(i = 1, 2, \dots)$ be a sequence such that $\epsilon_i \to 0$. The following proposition is a variant and a refinement of [49, Proposition 5.1].

Proposition 4.8. *Let $h^{(\epsilon_i)}_{i,1}$ $(i = 1, 2, \dots)$ be Hermitian metrics of E_i satisfying the following conditions:*

- $\det h^{(\epsilon_i)}_{i,1} = \det h_{i,0}$.

- $\big\|G\big(h_{i,1}^{(\epsilon_i)}\big)\big\|_{L^2,g_{i,\epsilon},h_{i,1}^{(\epsilon_i)}} \to 0$ *as* $i \to \infty$.
- *Let* $s^{(i)}$ *be the automorphism of* E_i *which is self-adjoint with respect to* $h_i^{(\epsilon_i)}$ *and determined by* $h_{i,1}^{(\epsilon_i)} = h_i^{(\epsilon_i)} s^{(i)}$. *Then,* $s^{(i)}$ *and* $\big(s^{(i)}\big)^{-1}$ *are bounded with respect to* $h_i^{(\epsilon_i)}$ *on* $X \setminus D$, *and* $\mathbb{D}_i^\lambda s^{(i)}$ *are* L^2 *with respect to* $h_i^{(\epsilon_i)}$ *and* g_{i,ϵ_i}. *The estimates may depend on* i.

Then, the sequence $\big\{f_i^*\big(s^{(i)}\big)\big\}$ *is weakly convergent to* id_E *in* L_1^2 *locally on* $X \setminus D$. *Moreover, there exists* $A > 0$ *such that* $\big|s^{(i)}\big|_{h_i^{(\epsilon_i)}} < A$ *and* $\big|\big(s^{(i)}\big)^{-1}\big|_{h_i^{(\epsilon_i)}} < A$ *for any* i.

Proof. This is essentially the same as [49, Proposition 5.1]. We explain an outline of the proof. We identify E_i with E by f_i. We set

$$c_i := \sup_{\Sigma \setminus D} \big|s^{(i)}\big|_{h_i^{(\epsilon_i)}}.$$

We set $\widetilde{s}^{(i)} := c_i^{-1} s^{(i)}$. We set $\widetilde{h}_{i,1}^{(\epsilon_i)} := c_i^{-1} h_{i,1}^{(\epsilon_i)} = h_i^{(\epsilon_i)} \widetilde{s}^{(i)}$. The following holds.

$$(1+|\lambda|^2)\Delta_{g_{i,0}} \operatorname{Tr} \widetilde{s}^{(i)} = \operatorname{Tr}\big(\widetilde{s}^{(i)} \sqrt{-1}\Lambda_{g_{i,0}} G\big(\widetilde{h}_{i,1}^{(\epsilon_i)}\big)\big) + \sqrt{-1}\Lambda_{g_{i,0}} \operatorname{Tr}\big(\mathbb{D}_i^\lambda \widetilde{s}^{(i)} \big(\widetilde{s}^{(i)}\big)^{-1} \mathbb{D}^{\lambda\star}_{i,h_i^{(\epsilon_i)}} \widetilde{s}^{(i)}\big).$$

From the boundedness of $\widetilde{s}^{(i)}$ and the L^2-property of $\mathbb{D}^\lambda \widetilde{s}^{(i)}$, we obtain $\int \Delta_{g_{i,0}} \operatorname{Tr}\big(\widetilde{s}^{(i)}\big) \operatorname{dvol}_{g_{i,0}} = 0$ as in Lemma 4.7. We obtain the following for some $A > 0$ and $A' > 0$:

$$\begin{aligned}
\int \big|\mathbb{D}^\lambda\big(\widetilde{s}^{(i)}\big)\big(\widetilde{s}^{(i)}\big)^{-1/2}\big|_{g_{i,0},h_i^{(\epsilon_i)}} \operatorname{dvol}_{g_{i,0}} &\le A \cdot \int \big|\operatorname{Tr} \Lambda_{g_{i,0}} G\big(\widetilde{h}_{i,1}^{(\epsilon_i)}\big)\big| \cdot \operatorname{dvol}_{g_{i,0}} \\
&= A \cdot \int \big|\operatorname{Tr} \Lambda_{g_{i,\epsilon}} G\big(\widetilde{h}_{i,1}^{(\epsilon_i)}\big)\big| \cdot \operatorname{dvol}_{g_{i,\epsilon}} \\
&\le A' \big\|G\big(\widetilde{h}_{i,1}^{(\epsilon_i)}\big)\big\|_{L^2,h_i^{(\epsilon_i)},g_{i,\epsilon}}.
\end{aligned}$$

Hence, the sequence $\widetilde{s}^{(i)}$ is L_1^2-bounded on any compact subset of $X \setminus D$. By taking an appropriate subsequence, it is weakly convergent in L_1^2 locally on $X \setminus D$. Let $\widetilde{s}^{(\infty)}$ denote the weak limit of the sequence. We obtain $\mathbb{D}^\lambda \widetilde{s}^{(\infty)} = 0$. Because $\widetilde{s}^{(i)}$ are self-adjoint and uniformly bounded with respect to $h_i^{(\epsilon_i)}$, $\widetilde{s}^{(\infty)}$ is self-adjoint and bounded with respect to $h^{(0)}$. We can prove that $\widetilde{s}^{(\infty)} \neq 0$ by the same argument as in the proof of [49, Lemma 5.2]. Hence, $\widetilde{s}^{(\infty)}$ is a non-zero endomorphism of $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)$. It implies that $\widetilde{s}^{(\infty)}$ is a multiplication by a positive constant u_∞.

Note that the sequence $\widetilde{s}^{(i)}$ is convergent in L^p for any p locally on $X \setminus D$, and hence $\det\big(\widetilde{s}^{(i)}\big)$ is convergent to $\det\big(\widetilde{s}^{(\infty)}\big)$ in L^p for any p locally on $X \setminus D$. Because $\det\big(s^{(i)}\big) = 1$, we obtain that the sequence $c_i^{-\operatorname{rank}(\mathcal{V})}$ is convergent to $u_\infty^{\operatorname{rank}(\mathcal{V})}$. In particular, it implies that the sequence c_i is bounded. Then, we obtain the claim of the proposition. ■

4.3 Tensor product of stable filtered λ-flat sheaves

Let us state a consequence of Theorem 4.1 on the tensor product of reflexive filtered λ-flat sheaves on arbitrary dimensional projective varieties.

Let X be an n-dimensional non-singular projective variety equipped with a very ample line bundle L. Let H be a simple normal crossing hypersurface of X with the irreducible decomposition $H = \bigcup_{i\in\Lambda} H_i$. Let $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big)$ $(i = 1,2)$ be reflexive filtered λ-flat sheaves on (X, H). We assume the following condition:

Condition 4.9. *There exists a Zariski closed subset* $Z \subset H$ *with* $\dim Z < n-1$ *such that* $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big)_{|X\setminus Z}$ $(i = 1,2)$ *are good filtered* λ*-flat bundles on* $(X \setminus Z, H \setminus Z)$.

For example, if $\mathbb{D}^\lambda$ is logarithmic and if $\lambda \neq 0$, Condition 4.9 is satisfied.

We set $\widetilde{\mathcal{V}} := \mathcal{V}_1 \otimes_{\mathcal{O}_X(*H)} \mathcal{V}_2$ which is equipped with the induced flat λ-connection $\widetilde{\mathbb{D}}^\lambda$. Note that $Z \subset H$ and that $\widetilde{\mathcal{V}}_{|X\setminus Z}$ is a locally free $\mathcal{O}_{X\setminus Z}(*(H\setminus Z))$-module. There exists the natural morphism

$$\varphi\colon\ \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}}^{\vee\vee} := \mathcal{H}om_{\mathcal{O}_X(*H)}\big(\mathcal{H}om_{\mathcal{O}_X(*H)}\big(\widetilde{\mathcal{V}}, \mathcal{O}_X(*H)\big), \mathcal{O}_X(*H)\big).$$

The $\mathcal{O}_X(*H)$-modules $\operatorname{Ker}\varphi$ and $\operatorname{Cok}\varphi$ are coherent, and their supports are contained in $Z \subset H$. Hence, we obtain that $\operatorname{Ker}\varphi = \operatorname{Cok}\varphi = 0$. It implies that $\widetilde{\mathcal{V}} \simeq \widetilde{\mathcal{V}}^{\vee\vee}$, i.e., $\widetilde{\mathcal{V}}$ is a reflexive $\mathcal{O}_X(*H)$-module. For $\boldsymbol{a} \in \mathbb{R}^\Lambda$, we set

$$\mathcal{P}'_{\boldsymbol{a}}\widetilde{\mathcal{V}} := \sum_{\boldsymbol{b}_1+\boldsymbol{b}_2=\boldsymbol{a}} \operatorname{Im}\big(\mathcal{P}_{\boldsymbol{b}_1}\mathcal{V}_1 \otimes_{\mathcal{O}_X} \mathcal{P}_{\boldsymbol{b}_2}\mathcal{V}_2 \longrightarrow \widetilde{\mathcal{V}}\big).$$

Let $\mathcal{P}_{\boldsymbol{a}}\widetilde{\mathcal{V}}$ denote the coherent reflexive subsheaf of $\widetilde{\mathcal{V}}$ generated by $\mathcal{P}'_{\boldsymbol{a}}\widetilde{\mathcal{V}}$. Thus, we obtain a reflexive filtered λ-flat sheaf $\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)$ on (X, H).

Proposition 4.10. *If $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big)$ are μ_L-stable, then $\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)$ is μ_L-polystable.*

Proof. According to Propositions 3.8, 3.9 and Condition 4.9, there exists a positive integer m such that the following holds for any general complete intersection curve Y of $L^{\otimes m}$.

- $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big)_{|Y}$ are stable good filtered λ-flat bundles.
- The natural morphism

$$\operatorname{Hom}\big(\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big), \big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)\big) \longrightarrow \operatorname{Hom}\big(\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)_{|Y}, \big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)_{|Y}\big)$$

  is an isomorphism.

Because $\big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big)_{|Y}$ are stable good filtered λ-flat bundles, each λ-flat bundle $(\mathcal{V}_i, \mathbb{D}_i^\lambda)_{|Y\setminus H}$ are equipped with a Hermitian–Einstein metric h_i adapted to the filtered bundle $\mathcal{P}_*\mathcal{V}_i$ by Theorem 4.1. Because $h_1 \otimes h_2$ is a Hermitian–Einstein metric of the λ-flat bundle $\big(\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)_{|Y\setminus H}$ adapted to the filtered bundle $\mathcal{P}_*\widetilde{\mathcal{V}}$, we obtain that $\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)_{|Y}$ is polystable. By Corollary 3.10, we obtain that $\big(\mathcal{P}_*\widetilde{\mathcal{V}}, \widetilde{\mathbb{D}}^\lambda\big)$ is μ_L-polystable. ∎

5 Preliminary existence theorem for Hermitian–Einstein metrics

5.1 Statements

5.1.1 Kähler metrics

Let X be a smooth projective surface with a simple normal crossing hypersurface $H = \bigcup_{i\in\Lambda} H_i$. Let L be an ample line bundle on X. Let g_X be the Kähler metric of X such that the associated Kähler form ω_X represents $c_1(L)$.

We take Hermitian metrics g_i of $\mathcal{O}(H_i)$. Let $\sigma_i\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(H_i)$ denote the canonical section. Take $N > 10$. There exists $C > 0$ such that the following form defines a Kähler form on $X \setminus H$ for any $0 \leq \epsilon < 1/10$:

$$\omega_\epsilon := \omega_X + \sum_{i\in\Lambda} C\cdot\epsilon^{N+2}\cdot\sqrt{-1}\partial\overline{\partial}|\sigma_i|^{2\epsilon}_{g_i}.$$

It is easy to observe that $\int_X \omega_\epsilon^2 = \int_X \omega_X^2$ and that $\int_X \omega_\epsilon\tau = \int_X \omega_X\tau$ for any closed C^∞-$(1,1)$-form τ on X.

5.1.2 Condition for good filtered λ-flat bundles and initial metrics

Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ be a good filtered λ-flat bundle on (X,H) satisfying the following condition: We set $e:=\mathrm{rank}(\mathcal{V})!$.

Condition 5.1.

- *There exists* $\boldsymbol{c}\in\mathbb{R}^\Lambda$ *and* $m\in e\mathbb{Z}_{>0}$ *such that* $\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},i)=\{c_i+n/m\mid n\in\mathbb{Z}\}$ *for each* $i\in\Lambda$.
- *The nilpotent part of* $\mathrm{Res}_i\big(\mathbb{D}^\lambda\big)$ *on* ${}^i\mathrm{Gr}^F_b(\mathcal{P}_*\mathcal{V})$ *are* 0 *for any* $i\in\Lambda$, $\boldsymbol{a}\in\mathbb{R}^\Lambda$ *and* $b\in\,]a_i-1,a_i[$.

Let $\big(E,\mathbb{D}^\lambda\big)$ denote the λ-flat bundle on $X\setminus H$ obtained as the restriction of $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$.

Let P be any point of $H_i\setminus\bigcup_{j\neq i}H_j$. Let (X_P,z_1,z_2) be an admissible coordinate neighbourhood around P. There exists an open subset X'_P in $\mathbb{C}^2=\{(\zeta_1,\zeta_2)\}$ such that the map $\varphi_P\colon X'_P\longrightarrow X_P$ given by $\varphi_P(\zeta_1,\zeta_2)=(\zeta_1^m,\zeta_2)$ is a ramified covering. We set $H'_P:=\{\zeta_1=0\}\cap X'_P$. We obtain the induced good filtered λ-flat bundle $\big(\mathcal{P}_*\varphi_P^*\mathcal{V},\varphi_P^*\mathbb{D}^\lambda\big)$ on (X'_P,H'_P) such that $\mathcal{P}\mathrm{ar}\big(\mathcal{P}_*\varphi_P^*\mathcal{V}\big)=\{m\cdot c_i\}+\mathbb{Z}$.

Definition 5.2. A Hermitian metric h_P of $E_{|X_P\setminus H}$ is called strongly adapted to $\mathcal{P}_*\mathcal{V}_{|X_P}$ if there exists a C^∞ Hermitian metric h'_P of $\mathcal{P}_{mc_i}\big(\varphi_P^*\mathcal{V}\big)$ on X'_P such that $\varphi^{-1}(h_P)=|\zeta_1|^{-2mc_i}h'_P$.

Let P be any point of $H_i\cap H_j$ $(i\neq j)$. Let (X_P,z_1,z_2) be a admissible coordinate neighbourhood around P such that $X_P\cap H_i=\{z_1=0\}$ and $X_P\cap H_j=\{z_2=0\}$. There exists an open subset X'_P in $\mathbb{C}^2=\{(\zeta_1,\zeta_2)\}$ such that the map $\varphi_P\colon X'_P\longrightarrow X_P$ given by $\varphi_P(\zeta_1,\zeta_2)=(\zeta_1^m,\zeta_2^m)$ is a ramified covering. We set $H'_P:=\{\zeta_1\zeta_2=0\}\cap X'_P$. We obtain the induced good filtered λ-flat bundle $\big(\mathcal{P}_*\varphi_P^*\mathcal{V},\varphi^*\mathbb{D}^\lambda\big)$ on (X'_P,H'_P) such that $\mathcal{P}\mathrm{ar}\big(\mathcal{P}_*\varphi_P^*\mathcal{V},1\big)=\{m\cdot c_i\}+\mathbb{Z}$ and $\mathcal{P}\mathrm{ar}\big(\mathcal{P}_*\varphi_P^*\mathcal{V},2\big)=\{m\cdot c_j\}+\mathbb{Z}$.

Definition 5.3. A Hermitian metric h_P of $E_{|X_P\setminus H}$ is called strongly adapted to $\mathcal{P}_*\mathcal{V}_{|X_P}$ if there exists a C^∞-Hermitian metric h'_P of $\mathcal{P}_{(mc_i,mc_j)}\varphi_P^*(\mathcal{V})$ such that $\varphi^*(h_P)=|\zeta_1|^{-mc_i}|\zeta_2|^{-mc_j}h'_P$.

Definition 5.4. A Hermitian metric h of E is called strongly adapted to $\mathcal{P}_*\mathcal{V}$ if the following holds:

- For any $P\in H$, there exists a neighbourhood X_P of P such that $h_{|X_P\setminus H}$ is strongly adapted to $\mathcal{P}_*\mathcal{V}_{|X_P}$ in the sense of Definitions 5.2 and 5.3.

Lemma 5.5. *Let h be a Hermitian metric of E strongly adapted to $\mathcal{P}_*\mathcal{V}$. Then, the following holds:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\int_{X\setminus H}\mathrm{Tr}\big(R(h)^2\big)=2\int_X\mathrm{ch}_2(\mathcal{P}_*\mathcal{V}).$$

Proof. It is the equality (36) in the proof of [46, Proposition 4.18]. ■

For each $i\in\Lambda$, we choose $b_i\in\mathcal{P}\mathrm{ar}(\mathcal{P}_*\det\mathcal{V},i)$. Set $\boldsymbol{b}=(b_i)\in\mathbb{R}^\Lambda$. We take a Hermitian metric $h_{\det(E)}$ of $\det(E)$ such that $h_{\det(E)}\prod_{i\in\Lambda}|\sigma_i|^{2b_i}_{g_i}$ induces a Hermitian metric of $\mathcal{P}_{\boldsymbol{b}}\det\mathcal{V}$ of C^∞-class.

Proposition 5.6. *There exists a Hermitian metric h_{in} of E such that the following holds:*

- h_{in} *is strongly adapted to* $\mathcal{P}_*\mathcal{V}$.
- $G(h_{\mathrm{in}})$ *is bounded with respect to* h_{in} *and* ω_ϵ, *where* $\epsilon:=m^{-1}$.

- *The following holds:*

$$\int_{X\setminus H} \operatorname{Tr}\bigl(R(h_{\mathrm{in}})^2\bigr) = \frac{1}{(1+|\lambda|^2)^2}\int_{X\setminus H} \operatorname{Tr}\bigl(G(h_{\mathrm{in}})^2\bigr). \tag{5.1}$$

- $\det(h_{\mathrm{in}}) = h_{\det(E)}$.

Such a Hermitian metric h_{in} is called an initial metric of $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr)$.

The case $\lambda = 1$ was studied in [51, Sections 14.1, 14.2 and Lemma 14.4.2]. The case $\lambda \neq 1$ can be argued in the essentially same way. We shall explain the construction in the case $\lambda = 0$ in Section 5.4 after preliminaries in Sections 5.2–5.3.

5.1.3 Preliminary existence theorem for Hermitian–Einstein metrics

Let $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr)$ be a good filtered λ-flat bundle satisfying Condition 5.1. Let h_{in} be an initial metric for $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr)$ as in Proposition 5.6. We shall prove the following theorem in Section 5.5.

Theorem 5.7. *Suppose that $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr)$ is μ_L-stable. Then, there exists a Hermitian–Einstein metric h_{HE} of $\bigl(E, \mathbb{D}^\lambda\bigr)$ with respect to the Kähler form ω_ϵ $\bigl(\epsilon := m^{-1}\bigr)$ satisfying the following conditions:*

(i) *h_{HE} and h_{in} are mutually bounded.*

(ii) *$\mathbb{D}^\lambda\bigl(h_{\mathrm{HE}}\cdot h_{\mathrm{in}}^{-1}\bigr)$ is L^2 with respect to h_{in} and ω_ϵ.*

(iii) *$\det(h_{\mathrm{HE}}) = \det(h_{\mathrm{in}})$ holds. In particular, the following holds:*

$$\frac{1}{1+|\lambda|^2}\operatorname{Tr}\bigl(G(h_{\mathrm{HE}})\bigr) = \frac{1}{1+|\lambda|^2}\operatorname{Tr}\bigl(G(h_{\mathrm{in}})\bigr) = \operatorname{Tr}\bigl(R(h_{\mathrm{in}})\bigr).$$

(iv) *The following equality holds:*

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \frac{1}{(1+|\lambda|^2)^2}\int_{X\setminus H}\operatorname{Tr}\bigl(G(h_{\mathrm{HE}})^2\bigr) = 2\int_X \operatorname{ch}_2(\mathcal{P}_*\mathcal{V}). \tag{5.2}$$

5.2 Around cross points

Let $X_0 := \bigl\{(z_1,z_2)\in\mathbb{C}^2 \mid |z_i|<1\bigr\}$. We set $H_i := X_0\cap\{z_i=0\}$ and $H := H_1\cup H_2$. Let $(\mathcal{P}_*\mathcal{V},\theta)$ be a good filtered Higgs bundle on (X_0,H). We choose $b_i\in\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},i)$ $(i=1,2)$, and set $\boldsymbol{b}=(b_1,b_2)$. We also choose any Hermitian metric $h_{\det(E)}$ of $\det(E)$ such that $h_{\det(E)}|z_1|^{2b_1}|z_2|^{2b_2}$ is a Hermitian metric of $\mathcal{P}_{\boldsymbol{b}}(\det\mathcal{V})$ of C^∞-class.

5.2.1 Unramified case

Suppose that $(\mathcal{P}_*\mathcal{V},\theta)$ satisfies the following condition:

Condition 5.8.

- *There exists $\boldsymbol{c}=(c_1,c_2)\in\mathbb{R}^2$ such that (i) $-1<c_i\le 0$, (ii) $\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V},i)=\{c_i+n\mid n\in\mathbb{Z}\}$.*
- *There exists a decomposition of good filtered Higgs bundles*

$$(\mathcal{P}_*\mathcal{V},\theta) = \bigoplus_{\mathfrak{a}\in\mathcal{I}}\bigoplus_{\boldsymbol{\alpha}\in\mathbb{C}^2}(\mathcal{P}_*\mathcal{V}_{\mathfrak{a},\boldsymbol{\alpha}},\theta_{\mathfrak{a},\boldsymbol{\alpha}}) \tag{5.3}$$

such that $\theta_{\mathfrak{a},\alpha} - \bigl(\mathrm{d}\mathfrak{a} + \sum\alpha_i\mathrm{d}z_i/z_i\bigr)\mathrm{id}_{\mathcal{V}_{\mathfrak{a},\boldsymbol{\alpha}}}$ induce holomorphic Higgs fields of $\mathcal{P}_{\boldsymbol{c}}\mathcal{V}_{\mathfrak{a},\boldsymbol{\alpha}}$.

We take any holomorphic frame $\boldsymbol{v} = (v_j)$ of $\mathcal{P}_{\boldsymbol{c}}\mathcal{V}$ compatible with the decomposition (5.3). For $j = 1, \dots, r$, we obtain $(\mathfrak{a}_j, \boldsymbol{\alpha}_j)$ determined by $v_j \in \mathcal{P}_{\boldsymbol{c}}\mathcal{V}_{\mathfrak{a}_j,\boldsymbol{\alpha}_j}$. Let h_0 be the metric of $\mathcal{V}_{|X_0\setminus H}$ determined by $h_0(v_i, v_i) = |z_1|^{-2c_1}|z_2|^{-2c_2}$ and $h_0(v_i, v_j) = 0$ $(i \neq j)$. Note that $\partial_{h_0}\boldsymbol{v} = \boldsymbol{v}\big(-\sum_{k=1,2} c_k \mathrm{d}z_k/z_k\big)\, I$, where I denotes the identity matrix. Hence, $\big[\partial_{h_0}, \overline{\partial}\big] = 0$ holds. We obtain the description $\theta\boldsymbol{v} = \boldsymbol{v}\big(\Lambda_0 + \Lambda_1\big)$ such that the following holds:

- $(\Lambda_0)_{ii} = (\mathrm{d}\mathfrak{a}_i + \sum_{k=1,2} \alpha_{ik}\mathrm{d}z_k/z_k)$ and $(\Lambda_0)_{ij} = 0$ $(i \neq j)$.
- $(\Lambda_1)_{ij}$ are holomorphic 1-forms for any i and j. Moreover, $(\Lambda_1)_{ij} = 0$ holds unless $(\mathfrak{a}_i, \boldsymbol{\alpha}_i) = (\mathfrak{a}_j, \boldsymbol{\alpha}_j)$.

We obtain $\theta^{\dagger}_{h_0}\boldsymbol{v} = \boldsymbol{v}\big(\overline{\Lambda}_0 + {}^t\overline{\Lambda}_1\big)$ and $\big[\theta, \theta^{\dagger}_{h_0}\big]\boldsymbol{v} = \boldsymbol{v}\big[\Lambda_1, {}^t\overline{\Lambda}_1\big]$, where the entries of $\big[\Lambda_1, {}^t\overline{\Lambda}_1\big]$ are C^{∞} on X_0. We have $(\partial_{h_0}\theta)\boldsymbol{v} = \boldsymbol{v}(\partial\Lambda_1)$, where any entries of $\partial\Lambda_1$ are holomorphic 2-forms, and $(\partial\Lambda_1)_{ij} = 0$ unless $(\mathfrak{a}_i, \boldsymbol{\alpha}_i) = (\mathfrak{a}_j, \boldsymbol{\alpha}_j)$.

Note that there exists a C^{∞}-function u on X_0 such that $\det(h_0) = \mathrm{e}^u h_{\det(E)}$. We set $h_{\mathrm{in}} := h_0 \mathrm{e}^{-u/\operatorname{rank} E}$.

Lemma 5.9. *$\big[\theta, \theta^{\dagger}_{h_{\mathrm{in}}}\big]$, $\partial_{h_{\mathrm{in}}}\theta$ and $\overline{\partial}\theta^{\dagger}_{h_{\mathrm{in}}}$ are bounded with respect to h_{in} and $\sum_{k=1,2} \mathrm{d}z_k\, \mathrm{d}\overline{z}_k$.*

5.2.2 Ramified case

Let $\varphi\colon \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $\varphi(\zeta_1, \zeta_2) = (\zeta_1^m, \zeta_2^m)$. We set $X_0' := \varphi^{-1}(X_0)$, $H_i' := X_0' \cap \varphi^{-1}(H_i)$ and $H' := H_1' \cup H_2'$. We set $\mathrm{Gal}(\varphi) := \{(\kappa_1, \kappa_2) \in \mathbb{C}^2 \mid \kappa_i^m = 1\}$, which acts on X_0' by $(\kappa_1, \kappa_2)(\zeta_1, \zeta_2) = (\kappa_1\zeta_1, \kappa_2\zeta_2)$.

Suppose that $\varphi^*(\mathcal{P}_*\mathcal{V}, \theta)$ satisfies Condition 5.8 on (X', H'). We construct a C^{∞}-metric h_0' of $\varphi^*(E)_{|X_0'\setminus H_0'}$ as in the previous subsection. We may assume that h_0' is $\mathrm{Gal}(\varphi)$-invariant. Note that there exists a $\mathrm{Gal}(\varphi)$-invariant C^{∞}-function u on X_0' such that $\det(h_0') = \mathrm{e}^u \varphi^{-1}(h_{\det(E)})$. We set $h_{\mathrm{in}}' := h_0' \mathrm{e}^{-u/\operatorname{rank}(E)}$. Because it is $\mathrm{Gal}(\varphi)$-invariant, we obtain the induced metric h_{in} of E.

Let $g_{X_0'}$ denote the Kähler metric $\sum_{k=1,2} \mathrm{d}\zeta_k\, \mathrm{d}\overline{\zeta}_k$ on X_0'. Because $\varphi\colon X_0' \setminus H' \longrightarrow X_0 \setminus H$ is a covering map, it induces a Kähler metric $\varphi_*(g_{X_0'})$ of $X_0 \setminus H$.

Lemma 5.10. *$\big[\theta, \theta^{\dagger}_{h_{\mathrm{in}}}\big]$, $\partial_{h_{\mathrm{in}}}\theta$ and $\overline{\partial}\theta^{\dagger}_{h_{\mathrm{in}}}$ are bounded with respect to $\big(h_{\mathrm{in}}, \varphi_* g_{X_0'}\big)$.*

5.2.3 An estimate

We set $Y(\epsilon) := \big\{(z_1, z_2) \in X_0 \mid \min(|z_i|) = \epsilon\big\}$.

Lemma 5.11. *We obtain $\lim\limits_{\epsilon\to 0} \int_{Y(\epsilon)} \operatorname{Tr}\big(\theta\overline{\partial}\theta^{\dagger}\big) = 0$ and $\lim\limits_{\epsilon\to 0} \int_{Y(\epsilon)} \operatorname{Tr}\big(\theta^{\dagger}\partial\theta\big) = 0$.*

Proof. It is enough to consider the case where Condition 5.8 is satisfied for $(\mathcal{P}_*\mathcal{V}, \theta)$. Let f be any anti-holomorphic function on X_0. Let us consider $\int_{Y(\epsilon)} \mathrm{d}\mathfrak{a}\, f\, \mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2$. We set $Y_1(\epsilon) := \{|z_1| = \epsilon, |z_2| \geq \epsilon\}$ and $Y_2(\epsilon) := \{|z_2| = \epsilon, |z_1| \geq \epsilon\}$. We have

$$\int_{Y_1(\epsilon)} \mathrm{d}\mathfrak{a} f\, \mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2 = \int_{Y_1(\epsilon)} \partial_2\mathfrak{a}\mathrm{d}z_2 f \mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2.$$

It is of the form

$$\int_{Y_1(\epsilon)} \frac{\mathfrak{b}(z_1, z_2)}{z_1^{\ell_1} z_2^{\ell_2}} f(\overline{z}_1, \overline{z}_2)\mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2. \tag{5.4}$$

Here, $\mathfrak{b}$ is a holomorphic function. We consider the Taylor expansion of $\mathfrak{b}$ and f. Then, the contributions of the terms

$$\frac{z_1^{k_1}\overline{z}_1^{m_1}}{z_1^{\ell_1}}\,\mathrm{d}\overline{z}_1\,\frac{z_2^{k_2}}{z_2^{\ell_2}}\,\overline{z}_2^{m_2}\,\mathrm{d}\overline{z}_2\,\mathrm{d}z_2$$

to (5.4) is 0 unless $k_1-\ell_1-m_1=1$ and $k_2-\ell_2-m_2=0$. If the equalities hold, we have $k_1-\ell_1+m_1=2m_1+1\geq 1$ and $k_2-\ell_2+m_2=2m_2\geq 0$. Hence, we obtain $\lim_{\epsilon\to 0}\int_{Y_1(\epsilon)}\mathrm{d}\mathfrak{a} f\,\mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2=0$. Similarly, we obtain $\lim_{\epsilon\to 0}\int_{Y_2(\epsilon)}\mathrm{d}\mathfrak{a} f\,\mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2=0$. Similarly and more easily, we obtain $\lim_{\epsilon\to 0}\int_{Y(\epsilon)}(\alpha_i\mathrm{d}z_i/z_i)f\mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2=0$. Then, the claim of the lemma follows. ■

5.3 Around smooth points

We set $X_0:=\{(z_1,z_2)\in\mathbb{C}^2\mid |z_i|<1\}$ and $H:=\{z_1=0\}$. Let $\nu\colon X_0\setminus H\longrightarrow\mathbb{R}_{>0}$ be a C^∞-function such that $\nu|z_1|^{-1}$ induces a nowhere vanishing C^∞-function on X_0. Let $(\mathcal{P}_*\mathcal{V},\theta)$ be a good filtered Higgs bundle on (X_0,H). Let $(E,\overline{\partial}_E,\theta)$ be the Higgs bundle obtained as the restriction of $(\mathcal{P}_*\mathcal{V},\theta)$ to $X_0\setminus H$. We choose $b\in\mathcal{P}\mathrm{ar}(\mathcal{P}_*\det\mathcal{V})$ and a Hermitian metric $h_{\det(E)}$ of $\det(E)$ such that $h_{\det(E)}\nu^{2b}$ induces a C^∞ metric of $\mathcal{P}_b(\det\mathcal{V})$.

5.3.1 Unramified case

Suppose that $(\mathcal{P}_*\mathcal{V},\theta)$ satisfies Condition 5.12.

Condition 5.12.

- *There exists* $-1<c\leq 0$ *such that* $\mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V})=\{c+n\mid n\in\mathbb{Z}\}$.
- *There exists a decomposition of good filtered Higgs bundles*

$$(\mathcal{P}_*\mathcal{V},\theta)=\bigoplus_{\mathfrak{a}\in\mathcal{I}}\bigoplus_{\alpha\in\mathbb{C}}(\mathcal{P}_*\mathcal{V}_{\mathfrak{a},\alpha},\theta_{\mathfrak{a},\alpha}).$$

- $\theta_{\mathfrak{a},\alpha}-(\mathrm{d}\mathfrak{a}+\alpha\mathrm{d}z_1/z_1)\,\mathrm{id}_{\mathcal{V}_{\mathfrak{a},\alpha}}$ *are holomorphic Higgs fields of* $\mathcal{P}_c\mathcal{V}_{\mathfrak{a},\alpha}$.

We take C^∞-metrics $h_{\mathfrak{a},\alpha}$ of $\mathcal{P}_c\mathcal{V}_{\mathfrak{a},\alpha}$, and we set $h_0:=\bigoplus\nu^{-2c}h_{\mathfrak{a},\alpha}$. We may assume that $\det(h_0)=h_{\det(E)}$.

Let $\boldsymbol{v}=(v_1,\ldots,v_r)$ be any holomorphic frame of $\mathcal{P}_c\mathcal{V}$ compatible with the decomposition. For each i, $\mathfrak{a}_i$ and α_i are determined by the condition that v_i is a section of $\mathcal{P}_c\mathcal{V}_{\mathfrak{a}_i,\alpha_i}$. There exist matrix valued C^∞-$(1,0)$-forms $A_{\mathfrak{a},\alpha}$ such that

$$\partial_{h_0}\boldsymbol{v}=\boldsymbol{v}\Big((-c\cdot\partial\log\nu^2)I+\sum A_{\mathfrak{a},\alpha}\Big),$$

where I denotes the identity matrix, and $(A_{\mathfrak{a},\alpha})_{i,j}=0$ unless $(\mathfrak{a}_i,\alpha_i)=(\mathfrak{a}_j,\alpha_j)=(\mathfrak{a},\alpha)$. Let Λ denote the matrix valued holomorphic 1-form determined by $\theta\boldsymbol{v}=\boldsymbol{v}\Lambda$. There exists the decomposition $\Lambda=\Lambda_0+\Lambda_1$ such that the following holds:

- $(\Lambda_0)_{ij}=(\mathrm{d}\mathfrak{a}_i+\alpha_i\mathrm{d}z_1/z_1)$ if $i=j$, and $(\Lambda_0)_{ij}=0$ if $i\neq j$.
- $(\Lambda_1)_{ij}$ are holomorphic 1-forms, and $(\Lambda_1)_{ij}=0$ unless $(\mathfrak{a}_i,\alpha_i)=(\mathfrak{a}_j,\alpha_j)$.

There exists a matrix valued C^∞ $(0,1)$-form Λ_2 such that $\theta^\dagger_{h_0}\boldsymbol{v}=\boldsymbol{v}\big(\overline{\Lambda}_0+\Lambda_2\big)$. Moreover, $(\Lambda_2)_{ij}=0$ holds unless $(\mathfrak{a}_i,\alpha_i)=(\mathfrak{a}_j,\alpha_j)$.

We have $R(h_0) = \big(-c\overline{\partial}\partial\log\sigma^2\big)I + \bigoplus R(h_{\mathfrak{a},\alpha})$, where $R(h_{\mathfrak{a},\alpha})$ are C^∞. Note that $d\Lambda_0 = 0$ and $[\Lambda_0,\Lambda_i] = \big[\Lambda_0,\overline{\Lambda}_i\big] = 0$. Hence, $[\theta,\theta^\dagger_{h_0}]$, $\partial_{h_0}\theta$ and $\overline{\partial}\theta^\dagger_{h_0}$ are C^∞. We also have

$$\big(\partial_{h_0}\theta\big)\boldsymbol{v} = \boldsymbol{v}\Big(\partial\Lambda_1 + \Big[\bigoplus A_{\mathfrak{a},\alpha},\Lambda_1\Big]\Big), \qquad \big(\overline{\partial}\theta^\dagger_{h_0}\big)\boldsymbol{v} = \boldsymbol{v}\big(\overline{\partial}\Lambda_2\big).$$

We set $w_1 = z_1|z_1|^{-1}\nu$ and $w_2 = z_2$. Then, it is easy to check that (w_1,w_2) is a C^∞ complex coordinate system. Clearly, $\mathrm{d}\overline{z}_2 = \mathrm{d}\overline{w}_2$. There exists a C^∞-function γ and a C^∞ $(0,1)$-form κ such that $\mathrm{d}\overline{z}_1 = \gamma\mathrm{d}\overline{w}_1 + \overline{w}_1\kappa_1$. We set $Y(\epsilon) = \{\nu = \epsilon\} = \{|w_1| = \epsilon\}$.

Lemma 5.13. *$\lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\big(\theta\overline{\partial}\theta^\dagger\big) = 0$ and $\lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\big(\theta^\dagger\partial\theta\big) = 0$ hold.*

Proof. It is enough to prove $\lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\big(\theta\overline{\partial}\theta^\dagger\big) = 0$. It is easy to see that

$$\lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\big(\Lambda_1\overline{\partial}\Lambda_2\big) = 0.$$

Let us study $\int_{Y(\epsilon)}\mathrm{Tr}\big(\Lambda_0\overline{\partial}\Lambda_2\big)$. For any C^∞-function g, we consider the following integral:

$$\int_{Y(\epsilon)} g(\mathrm{d}\mathfrak{a}\cdot\mathrm{d}\overline{z}_1\,\mathrm{d}\overline{z}_2) = \int_{Y(\epsilon)}(g\gamma)\cdot\mathrm{d}\mathfrak{a}\,\mathrm{d}\overline{w}_1\,\mathrm{d}\overline{w}_2 + \int_{Y(\epsilon)} g\overline{w}_1\cdot\mathrm{d}\mathfrak{a}\,\kappa\,\mathrm{d}\overline{w}_2. \tag{5.5}$$

We can rewrite the first term in the right hand side of (5.5) as follows, for some non-negative integer ℓ and for a C^∞-function $\mathfrak{b}$:

$$\int_{Y(\epsilon)}(g\gamma)\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\overline{w}_1\,\mathrm{d}\overline{w}_2 = \int_{Y(\epsilon)}(g\gamma\mathfrak{b})w_1^{-\ell}\,\mathrm{d}\overline{w}_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2$$

Take $N > \ell$. We consider the expansion

$$g\gamma\mathfrak{b} = \sum_{\substack{k,m\geq 0\\ k+m\leq N}}(g\gamma\mathfrak{b})_{k,m}(w_2)w_1^k\overline{w}_1^m + O\big(|w_1|^N\big).$$

Here, $(g\gamma\mathfrak{b})_{k,m}(w_2)$ are C^∞-functions of w_2. The contributions

$$\int_{Y(\epsilon)}(g\gamma\mathfrak{b})_{k,m}(w_2)\frac{w_1^k\overline{w}_1^m}{w_1^\ell}\mathrm{d}\overline{w}_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2$$

are 0 unless $k-\ell-m = 1$. If $k-\ell-m = 1$, then $k-\ell+m = 2m+1\geq 1$ holds. Hence, we obtain

$$\lim_{\epsilon\to 0}\int_{Y(\epsilon)}(g\gamma)\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\overline{w}_1\mathrm{d}\overline{w}_2 = 0.$$

We rewrite the second term in the right hand side of (5.5) as follows, for some C^∞-functions f_i $(i=1,2)$ and a non-negative integer ℓ:

$$\int_{Y(\epsilon)} g\overline{w}_1\cdot\mathrm{d}\mathfrak{a}\,\kappa\,\mathrm{d}\overline{w}_2 = \int_{Y(\epsilon)} f_1 w_1^{-\ell-1}\overline{w}_1\,\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2 + \int_{Y(\epsilon)} f_2 w_1^{-\ell}\,\overline{w}_1\,\mathrm{d}\overline{w}_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2.$$

Take $N > \ell+1$. Consider the expansions $f_i = \sum(f_i)_{k,m}(w_2)\,w_1^k\overline{w}_1^m + O\big(|w_1|^N\big)$. The contributions

$$\int_{Y(\epsilon)}(f_1)_{k,m}(w_2)\,\frac{w_1^k\overline{w}_1^{m+1}}{w_1^{\ell+1}}\,\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2$$

are 0 unless $k-(\ell+1)-(m+1)=-1$. If $k-(\ell+1)-(m+1)=-1$ holds, then we have $k-(\ell+1)+(m+1)=2m+1\geq 1$. The contributions

$$\int_{Y(\epsilon)}(f_2)_{k,m}(w_2)\,\frac{w_1^k\overline{w}_1^{m+1}}{w_1^{\ell}}\,\mathrm{d}\overline{w}_1\,\mathrm{d}w_2\,\mathrm{d}\overline{w}_2$$

are 0 unless $k-\ell-(m+1)=1$. If $k-\ell-(m+1)=1$ holds, then we have $k-\ell+(m+1)=2(m+1)+1\geq 3$. Hence, we obtain

$$\lim_{\epsilon\to 0}\int_{Y(\epsilon)} g\overline{w}_1\cdot \mathrm{d}\mathfrak{a}\,\kappa\,\mathrm{d}\overline{w}_2=0.$$

Similarly and more easily, we obtain $\lim_{\epsilon\to 0}\int_{Y(\epsilon)} g(\alpha\mathrm{d}z_1/z_1)\,\mathrm{d}\overline{z}_1\mathrm{d}\overline{z}_2=0$ for any $\alpha\in\mathbb{C}$ and for any C^∞-function g. Thus, we obtain the claim of the lemma. ■

5.3.2 Ramified case

Let $\varphi\colon\mathbb{C}^2\longrightarrow\mathbb{C}^2$ be given by $\varphi(\zeta_1,\zeta_2)=\big(\zeta_1^m,\zeta_2\big)$. We set $X_0':=\varphi^{-1}(X_0)$ and $H':=\varphi^{-1}(H)$. Let $\mathrm{Gal}(\varphi):=\{\mu\in\mathbb{C}\mid\mu^m=1\}$, which acts on X_0' by $\mu\cdot(\zeta_1,\zeta_2)=(\mu\zeta_1,\zeta_2)$.

Suppose that $\varphi^*(\mathcal{P}_*\mathcal{V},\theta)$ satisfies Condition 5.12. We construct a Hermitian metric h_0' for $\varphi^*(\mathcal{P}_*\mathcal{V},\theta)$ as in the previous subsection. We may assume that h_0' is $\mathrm{Gal}(\varphi)$-invariant. There exists a C^∞-function f on X_0' determined by $\det(h_0')=e^f\varphi^{-1}(h_{\det(E)})$. We set $h_{\mathrm{in}}':=h_0'e^{-f/\operatorname{rank}(E)}$. Because $h_{\mathrm{in}'}$ is $\mathrm{Gal}(\varphi)$-invariant, we obtain a Hermitian metric h_{in} of E induced by h_{in}'. Let $\varphi_*(g_{X_0'})$ denote the Kähler metric of $X_0\setminus H_0$ induced by $\sum_{k=1,2}\mathrm{d}\zeta_k\,\mathrm{d}\overline{\zeta}_k$.

Lemma 5.14. *$R(h_{\mathrm{in}})$, $\big[\theta,\theta^\dagger_{h_{\mathrm{in}}}\big]$, $\partial_{h_{\mathrm{in}}}\theta$ and $\overline{\partial}\theta^\dagger_{h_{\mathrm{in}}}$ are bounded with respect to $\varphi_*g_{X_0'}$ and h_0. We also have*

$$\lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\left(\theta\overline{\partial}\theta^\dagger_{h_{\mathrm{in}}}\right)=0,\qquad \lim_{\epsilon\to 0}\int_{Y(\epsilon)}\mathrm{Tr}\left(\theta^\dagger\partial_{h_{\mathrm{in}}}\theta\right)=0.$$

5.4 Proof of Proposition 5.6

Let X, H and L be as in Section 5.1.1. Let $(\mathcal{P}_*\mathcal{V},\theta)$ be a good filtered Higgs bundle on (X,H) satisfying Condition 5.1. Note that $(\mathcal{P}_*\mathcal{V},\theta)$ is as in Section 5.2.2 around any cross point of H, and $(\mathcal{P}_*\mathcal{V},\theta)$ is as in Section 5.3.2 around any smooth points of H. There exists a Hermitian metric h_{in} of E such that (i) $\det(h_{\mathrm{in}})=h_{\det(E)}$, (ii) the restriction of h_{in} around any points of H are as in Section 5.2.2 or Section 5.3.2. By the construction, h_{in} is strongly adapted to $\mathcal{P}_*\mathcal{V}$. By Lemmas 5.10 and 5.14, we obtain that $R(h_{\mathrm{in}})$, $\big[\theta,\theta^\dagger_{h_{\mathrm{in}}}\big]$, $\partial_{h_{\mathrm{in}}}\theta$ and $\overline{\partial}\theta^\dagger_{h_{\mathrm{in}}}$ are bounded with respect to h_{in} and ω_ϵ. As in the proof of [46, Proposition 4.18], we have

$$\mathrm{Tr}\big(G(h_{\mathrm{in}})^2\big)=\mathrm{Tr}\big(R(h_{\mathrm{in}})^2\big)+\mathrm{d}\big(\mathrm{Tr}\left(\theta\overline{\partial}\theta^\dagger_{h_{\mathrm{in}}}\right)+\mathrm{Tr}\left(\theta^\dagger_{h_{\mathrm{in}}}\partial_{h_{\mathrm{in}}}\theta\right)\big).$$

Then, we obtain (5.1) from Lemmas 5.11 and 5.14. Thus, we obtain Proposition 5.6. ■

5.5 Proof of Theorem 5.7

Let $E'\subset E$ be any coherent λ-flat $\mathcal{O}_{X\setminus H}$-subsheaf. We assume that E' is saturated, i.e., E/E' is torsion-free. Let $\big(E',\mathbb{D}^\lambda_{E'}\big)$ be the induced λ-flat sheaf on $X\setminus H$. There exists a discrete subset $Z\subset X\setminus H$ such that $E'_{|X\setminus(H\cup Z)}$ is a subbundle of $E_{|X\setminus(H\cup Z)}$. Let h' denote the metric of $E'_{|X\setminus(H\cup Z)}$ induced by h_{in}. We obtain the Chern connection $\nabla_{h'}$ of $\big(E',d''_{E'},h'\big)$ and the operator $\mathbb{D}^{\lambda\star}_{E',h'}$ from $\mathbb{D}^{\lambda}_{E'}$ and h'. Let $R(E',h')$ denote the curvature of $\nabla_{h'}$. We obtain $G(E',h') := \big[\mathbb{D}^{\lambda}_{E'}, \mathbb{D}^{\lambda\star}_{E',h'}\big]$. Following [62], we define

$$\deg_{\omega_\epsilon}(E', h_{\mathrm{in}}) := \frac{\sqrt{-1}}{2\pi}\frac{1}{1+|\lambda|^2}\int_{X\setminus H} \operatorname{Tr}\big(\Lambda_{\omega_\epsilon} G(E',\theta',h')\big)\,\mathrm{dvol}_{\omega_\epsilon}.$$

It is well defined in $\mathbb{R}\cup\{-\infty\}$ by the Chern–Weil formula [62, Lemma 3.2]:

$$\deg_{\omega_\epsilon}(E', h_{\mathrm{in}}) = \frac{\sqrt{-1}}{2\pi}\frac{1}{1+|\lambda|^2}\int_{X\setminus H} \operatorname{Tr}\big(\pi_{E'}\Lambda_{\omega_\epsilon} G(h_{\mathrm{in}})\big) - \frac{1}{2\pi}\frac{1}{1+|\lambda|^2}\int_{X\setminus H}\big|\mathbb{D}^{\lambda}\pi_{E'}\big|^2_{h_{\mathrm{in}},\omega_\epsilon}.$$

Here, $\pi_{E'}$ denotes the orthogonal projection of $E_{|X\setminus(H\cup Z)}$ onto $E'_{|X\setminus(H\cup Z)}$.

Lemma 5.15. *If $\deg_{\omega_\epsilon}(E',\theta)\neq -\infty$, then E' extends to a filtered subsheaf $\mathcal{P}^{h'}_*E'$ of $\mathcal{P}_*\mathcal{V}$ and*

$$\deg_{\omega_\epsilon}(E', h_{\mathrm{in}}) = \int_X c_1\big(\mathcal{P}^{h'}_*E'\big)\omega_X$$

holds. As a result, $\big(E,\overline{\partial}_E,\theta,h_{\mathrm{in}}\big)$ is analytically stable in the sense of [62] (*see also* [49, Section 2.3]).

Proof. If $\deg_{\omega_\epsilon}(E',h_{\mathrm{in}})\neq-\infty$, we obtain $\int |d''\pi_{E'}|^2<\infty$. As studied in [37, 38] on the basis of [68], we obtain a coherent $\mathcal{O}_X(*H)$-submodule $\mathcal{P}^{h'}(E')\subset\mathcal{V}$ as an extension of E'. Moreover, as proved in [46, Lemma 4.20], we obtain the equality $\deg_{\omega_\epsilon}(E',h_{\mathrm{in}}) = \int_X c_1\big(\mathcal{P}^{h'}_*E'\big)\omega_X$. ∎

According to the fundamental theorem of Simpson [62, Theorem 1] and its variant [49, Proposition 2.49], there exists a Hermitian–Einstein metric h_{HE} of $\big(E,\mathbb{D}^{\lambda}\big)$ satisfying the conditions (*i*), (*ii*) and (*iii*). By [62, Proposition 3.5] and [62, Lemma 7.4] (see also [49, Proposition 2.49]), we obtain

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^2\frac{1}{(1+|\lambda|^2)^2}\int_{X\setminus H}\operatorname{Tr}\big(G(h_{\mathrm{HE}})^2\big) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2\frac{1}{(1+|\lambda|^2)^2}\int_{X\setminus H}\operatorname{Tr}\big(G(h_{\mathrm{in}})^2\big).$$

It is equal to $2\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})$ by Lemma 5.5 and Proposition 5.6. Thus, Theorem 5.7 is proved. ∎

6 Bogomolov–Gieseker inequality

Let X be any dimensional smooth connected projective variety with a simple normal crossing hypersurface $H=\bigcup_{i\in\Lambda}H_i$. Let L be any ample line bundle on X.

Theorem 6.1. *Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^{\lambda}\big)$ be a μ_L-polystable good filtered λ-flat bundle on (X,H). Then, the Bogomolov–Gieseker inequality holds:*

$$\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}\leq\frac{\int_X c_1(\mathcal{P}_*\mathcal{V})^2 c_1(L)^{\dim X-2}}{2\operatorname{rank}\mathcal{V}}.$$

Proof. By the Mehta–Ramanathan type theorem (Proposition 3.8), it is enough to study the case $\dim X=2$, which we shall assume in the rest of the proof. We use the notation in Section 3.7.2. Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^{\lambda}\big)=\bigoplus\big(\mathcal{P}_*\mathcal{V}_j,\mathbb{D}^{\lambda}_j\big)$ be the decomposition into the stable components.

We set $e:=\operatorname{rank}(\mathcal{V})!$. We choose $\eta>0$ such that $0<10e\eta<\widetilde{\mathrm{gap}}(\mathcal{P}_*\mathcal{V})$. We take $\boldsymbol{a}\in\mathbb{R}^{\Lambda}$ for $\widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V},i)$ $(i\in\Lambda)$ as in Lemma 3.19.

Let $m \in e\mathbb{Z}_{>0}$ such that $\epsilon := m^{-1} < \eta/10\,\mathrm{rank}(\mathcal{V})$. For any $b \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}, i)$, we set $b(\epsilon) := \max\bigl\{d \in \epsilon\mathbb{Z} \mid d < b\bigr\}$. We set

$$c_{i,j} := \frac{1}{\mathrm{rank}\,\mathcal{V}_j} \sum_{b \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V}_j, \boldsymbol{a}, i)} (b - b(\epsilon))\,\mathrm{rank}\,{}^i\mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_j).$$

We have $0 \le c_{i,j} \le \epsilon$. For any $b \in \widetilde{\mathcal{P}\mathrm{ar}}(\mathcal{P}_*\mathcal{V}_j, i)$, we set $\psi_{\epsilon,i,j}(b) := b(\epsilon) + c_{i,j}$. Then, we obtain $|\psi_{\epsilon,i,j}(b) - b| < 2\epsilon$ and the following equalities:

$$\sum_{b \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V}_j, \boldsymbol{a}, i)} \psi_{\epsilon,i,j}(b)\,\mathrm{rank}\,{}^i\mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_j) = \sum_{b \in \mathcal{P}\mathrm{ar}(\mathcal{P}_*\mathcal{V}_j, \boldsymbol{a}, i)} b\,\mathrm{rank}\,{}^i\mathrm{Gr}^F_b(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}_j).$$

Moreover, we have $\psi_{\epsilon,i,j}(b) - c_{i,j} \in \epsilon\mathbb{Z}$.

Applying the construction in Section 3.7.2, we obtain good filtered λ-flat bundles $\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}_j, \mathbb{D}^\lambda_j\bigr)$ on (X, H). By the construction, they satisfy Condition 5.1. By Lemma 3.25, there exists m_0 such that $\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}_j, \mathbb{D}^\lambda_j\bigr)$ are μ_L-stable if $m \ge m_0$. Let $\bigl(E_j, \mathbb{D}^\lambda_j\bigr)$ be the λ-flat bundle obtained as the restriction of $\bigl(\mathcal{P}_*\mathcal{V}_j, \mathbb{D}^\lambda_j\bigr)$ to $X \setminus H$. We use the Kähler metric g_ϵ of $X \setminus H$ as in Section 5.1.1. There exist Hermitian–Einstein metrics $h^{(\epsilon)}_{j,\mathrm{HE}}$ of the λ-flat bundles $\bigl(E_j, \mathbb{D}^\lambda_j\bigr)$ as in Theorem 5.7 for the good filtered λ-flat bundles $\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}_j, \mathbb{D}^\lambda_j\bigr)$. Note that $\bigoplus h^{(\epsilon)}_{j,\mathrm{HE}}$ is a Hermitian–Einstein metric of $\bigoplus \bigl(E_j, \mathbb{D}^\lambda_j\bigr)$.

By Proposition 3.1, the equality (5.2), and the equality $\frac{\sqrt{-1}}{2\pi}\frac{1}{1+|\lambda|^2}\,\mathrm{Tr}\,G\bigl(h^{(\epsilon)}_{\mathrm{HE}}\bigr) = \frac{\sqrt{-1}}{2\pi}R(h_{\det E})$, we obtain

$$\int_X \mathrm{ch}_2\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}\bigr) \le \frac{\int_X c_1\bigl(\mathcal{P}^{(\epsilon)}_*\mathcal{V}\bigr)^2}{2\,\mathrm{rank}\,\mathcal{V}}.$$

By taking the limit as $m \to \infty$, i.e., $\epsilon \to 0$, we obtain the desired inequality. ■

Corollary 6.2. *Let $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr)$ be a μ_L-polystable good filtered λ-flat bundle on (X, H). Suppose that*

$$\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1} = 0, \qquad \int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2} = 0.$$

Then, $c_1(\mathcal{P}_*\mathcal{V}) = 0$ holds.

Moreover, for any decomposition $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\bigr) = \bigoplus \bigl(\mathcal{P}_*\mathcal{V}_j, \mathbb{D}^\lambda_j\bigr)$ into μ_L-stable good filtered λ-flat bundles, we obtain $c_1(\mathcal{P}_*\mathcal{V}_j) = 0$ and $\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_j)c_1(L)^{\dim X-2} = 0$.

Proof. On one hand, because of the Hodge index theorem and $\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1} = 0$, we obtain

$$\int_X c_1(\mathcal{P}_*\mathcal{V})^2 c_1(L)^{\dim X-2} \le 0,$$

and the equality holds if and only if $c_1(\mathcal{P}_*\mathcal{V}) = 0$. On the other hand, by the Bogomolov–Gieseker inequality and $\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2} = 0$, we obtain

$$\int_X c_1(\mathcal{P}_*\mathcal{V})^2 c_1(L)^{\dim X-2} \ge 0.$$

Hence, we obtain $c_1(\mathcal{P}_*\mathcal{V}) = 0$.

Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)=\bigoplus\big(\mathcal{P}_*\mathcal{V}_j,\mathbb{D}^\lambda_j\big)$ be a decomposition into μ_L-stable good filtered λ-flat bundles. We have $\int_X c_1(\mathcal{P}_*\mathcal{V}_j)c_1(L)^{\dim X-1}=0$. Hence, by the Hodge index theorem, we obtain

$$\int_X c_1(\mathcal{P}_*\mathcal{V}_j)^2 c_1(L)^{\dim X-2}\leq 0.$$

By the Bogomolov–Gieseker type inequality, we obtain

$$\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_j)c_1(L)^{\dim X-2}\leq 0.$$

Because $\sum_j\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_j)c_1(L)^{\dim X-2}=\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})=0$, we obtain

$$\int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_j)c_1(L)^{\dim X-2}=0.$$

Thus, we obtain the claim of the corollary. ■

Remark 6.3. Although H was assumed to be ample in [51, Section 14.4, Corollary 14.5.1], it is not essential. Indeed, for any simple normal crossing hypersurface H, there exists an ample simple normal crossing hypersurface H' such that $H\subset H'$. Let $\big(\mathcal{P}'_*\mathcal{V},\mathbb{D}^\lambda\big)$ be the filtered λ-flat bundle on (X,H') naturally induced by $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$. The Chern characters of $\mathcal{P}_*\mathcal{V}$ and $\mathcal{P}'_*\mathcal{V}$ are equal, and hence the Bogomolov–Gieseker inequalities for $\mathcal{P}_*\mathcal{V}$ and $\mathcal{P}'_*\mathcal{V}$ are equivalent.

7 Existence theorem of pluri-harmonic metrics

7.1 Statement

Let us prove Theorem 2.23. According to Corollary 6.2, it is enough to study the case where $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ is a μ_L-*stable* good filtered λ-flat bundle on (X,H) such that

$$c_1(\mathcal{P}_*\mathcal{V})=0,\qquad \int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}=0.$$

Let $\big(E,\mathbb{D}^\lambda\big)$ be the λ-flat bundle obtained as the restriction $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)_{|X\setminus H}$. Let $h_{\det(E)}$ denote the pluri-harmonic metric of $\big(\det(E),\mathbb{D}^\lambda_{\det(E)}\big)$ strongly adapted to $\mathcal{P}_*(\det(E))$. For the proof of Theorem 2.23, it is enough to prove the following theorem.

Theorem 7.1. *There exists a unique pluri-harmonic metric h of the λ-flat bundle $\big(E,\mathbb{D}^\lambda\big)$ such that $\mathcal{P}^h_*E=\mathcal{P}_*\mathcal{V}$ and* $\det(h)=h_{\det(E)}$.

The proof is given in the rest of this section.

7.2 Surface case

Let us study the case $\dim X=2$. The following argument is essentially the same as the proof of [49, Theorem 5.5]. Let $\big(\mathcal{P}_*\mathcal{V},\mathbb{D}^\lambda\big)$ be as in Section 7.1. We use the notation in the proof of Theorem 6.1. For large $m\in e\mathbb{Z}_{>0}$, we set $\epsilon:=m^{-1}$. We have the perturbations $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^\lambda\big)$. We use the Kähler metrics g_ϵ of $X\setminus H$ as in Section 5.1.1. There exist the Hermitian–Einstein metrics $h^{(\epsilon)}_{\mathrm{HE}}$ of $\big(E,\mathbb{D}^\lambda\big)$ adapted to $\big(\mathcal{P}^{(\epsilon)}_*\mathcal{V},\mathbb{D}^\lambda\big)$ such that $\det\big(h^{(\epsilon)}_{\mathrm{HE}}\big)=h_{\det(E)}$.

Proposition 7.2. *For any sequence $m_i\to\infty$, we set $\epsilon_i:=m_i^{-1}$. Then, after going to a subsequence, $h^{(\epsilon_i)}_{\mathrm{HE}}$ is convergent almost everywhere on $X\setminus H$, and the limit h is a pluri-harmonic metric of the λ-flat bundle $\big(E,\mathbb{D}^\lambda\big)$ adapted to $\mathcal{P}_*\mathcal{V}$ such that* $\det(h)=h_{\det E}$.

7.2.1 Family of ample hypersurfaces

There exists a 0-dimensional closed subset $Z \subset H$ such that (i) Z contains the singular points of H, (ii) any $P \in H \setminus Z$ has a neighbourhood H_P in H on which the conjugacy classes of $\mathrm{Res}(\mathbb{D}^\lambda)_{|Q}$ $(Q \in H_P)$ are constant.

Take a sufficiently large integer M. We set $\mathfrak{Z}_M := H^0\big(X, L^{\otimes M}\big) \setminus \{0\}$. It is equipped with a natural $\mathbb{C}^*$-action. Let p_i denote the projection of $X \times \mathfrak{Z}_M$ onto the i-th component. There exists the universal section $\mathfrak{s}$ of $p_1^*\big(L^{\otimes M}\big)$. Let $\mathfrak{X}_M$ denote the scheme obtained as $\mathfrak{s}^{-1}(0)$. Let $\mathtt{P}_1 \colon \mathfrak{X}_M \longrightarrow X$ and $\mathtt{P}_2 \colon \mathfrak{X}_M \longrightarrow \mathfrak{Z}_M$ denote the morphism induced by p_i. For each $s \in \mathfrak{Z}_M$, let X_s denote the fiber product of $\mathtt{P}_2$ and the inclusion $\{s\} \longrightarrow \mathfrak{Z}_M$.

There exists the $\mathbb{C}^*$-invariant maximal Zariski open subset $\mathfrak{Z}_M^\circ \subset \mathfrak{Z}_M$ such that (i) the induced morphism $\mathtt{P}_2^\circ \colon \mathfrak{X}_M^\circ := \mathfrak{X}_M \times_{\mathfrak{Z}_M} \mathfrak{Z}_M^\circ \longrightarrow \mathfrak{Z}_M^\circ$ is smooth, (ii) $X_s \cup H$ is normal crossing for any $s \in \mathfrak{Z}_M^\circ$, (iii) $(X_s \cap H) \cap Z = \varnothing$. Let $\mathtt{P}_1^\circ$ denote the restriction of $\mathtt{P}_1$ to $\mathfrak{X}_M^\circ$. For any $Q \in \mathfrak{X}_M^\circ$, we obtain the subspace $T_{\mathtt{P}_1(Q)} X_{\mathtt{P}_2(Q)} \subset T_{\mathtt{P}_1(Q)} X$ of codimension 1. It determines a point in $\mathbb{P}\big(T^*_{\mathtt{P}_1(Q)} X\big)$. Hence, we obtain the natural morphism $\widetilde{\mathtt{P}}_1^\circ \colon \mathfrak{X}_M^\circ \longrightarrow \mathbb{P}(T^*X)$. If M is sufficiently large, $\mathtt{P}_1^\circ$ and $\widetilde{\mathtt{P}}_1^\circ$ are surjective.

By the Mehta–Ramanathan type theorem (Proposition 3.8), there exists a non-empty $\mathbb{C}^*$-invariant Zariski open subset $\mathfrak{Z}_M^\triangle$ of $\mathfrak{Z}_M^\circ$ such that the following holds:

- For each $s \in \mathfrak{Z}_M^\triangle$, $\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|X_s}$ is stable.

We set $\mathfrak{X}_M^\triangle := \mathfrak{X}_M^\circ \times_{\mathfrak{Z}_M^\circ} \mathfrak{Z}_M^\triangle$. Note that $W_M := X \setminus \mathtt{P}_1^\circ\big(\mathfrak{X}_M^\triangle\big)$ is a finite set. For each $P \in X \setminus (H \cup W_M)$, the intersection $\widetilde{\mathtt{P}}_1^\circ\big(\mathfrak{X}_M^\triangle\big) \cap \mathbb{P}(T_P^*X)$ in $\mathbb{P}(T^*X)$ is Zariski dense in $\mathbb{P}(T_P^*X)$.

We set $H_s := X_s \cap H$. Let $(E_s, \mathbb{D}_s^\lambda)$ denote the λ-flat bundle on $X_s \setminus H_s$ obtained as the restriction of $\big(E, \mathbb{D}^\lambda\big)$. For each $s \in \mathfrak{Z}_M^\triangle$, there exists a pluri-harmonic metric h_s of $\big(E_s, \mathbb{D}_s^\lambda\big)$ such that (i) h_s is adapted to $\mathcal{P}_*\mathcal{V}_{|X_s}$, (ii) $\det(h_s) = h_{\det(E)|X_s \setminus H_s}$.

Let $\mathtt{P}_1^\triangle \colon \mathfrak{X}_M^\triangle \longrightarrow X$ be the induced map. Let $\mathfrak{H}_M^\triangle := \big(\mathtt{P}_1^\triangle\big)^{-1}(H)$. We set $\big(E^\triangle, \mathbb{D}^\lambda_{E^\triangle}\big) := \big(\mathtt{P}_1^\triangle\big)^{-1}\big(E, \mathbb{D}^\lambda\big)$ on $\mathfrak{X}_M^\triangle \setminus \mathfrak{H}_M^\triangle$. By Lemma 3.21 and Proposition 4.5, the family of pluri harmonic metrics h_s $(s \in \mathfrak{Z}_M^\triangle)$ induces a continuous Hermitian metric $h^\triangle$ of $E^\triangle$. We also obtain Hermitian metrics $h^{\triangle(\epsilon_i)} := \big(\mathtt{P}_1^\triangle\big)^{-1}\big(h_{\mathrm{HE}}^{(\epsilon_i)}\big)$.

7.2.2 Local holomorphic coordinate systems

Let $P \in X \setminus W_M$. We take $s_\infty \in \mathfrak{Z}_M^\triangle$ such that $P \notin X_{s_\infty}$. The following is clear because $\widetilde{\mathtt{P}}_1\big(\mathfrak{X}_M^\triangle\big) \cap \mathbb{P}(T_P^*X)$ is dense in $\mathbb{P}(T_P^*X)$.

Lemma 7.3. *There exist $s_i \in \mathfrak{Z}_M^\triangle$ $(i = 1, 2)$ and $\delta > 0$ such that the following holds:*

- *$P \in X_{s_i}$ $(i = 1, 2)$.*
- *X_{s_1} and X_{s_2} are transversal at P.*
- *$\{s_1 + a s_\infty \mid |a| < \delta\}$, $\{s_2 + a s_\infty \mid |a| < \delta\}$, $\big\{s_1 + s_2 + a s_\infty \mid |a| < \delta\big\}$ and $\big\{s_1 + \sqrt{-1} s_2 + a s_\infty \mid |a| < \delta\big\}$ are contained in $\mathfrak{Z}_M^\triangle$.*

We set $x_i := s_i / s_\infty$ $(i = 1, 2)$. There exists a neighbourhood U_P of P in $X \setminus H$ such that (x_1, x_2) is a holomorphic coordinate system on U_P. Note that $\big\{\sum b_i x_i = c\big\} \cap U_P$ is equal to $U_P \cap X_{b_1 s_1 + b_2 s_2 - c s_\infty}$.

7.2.3 Proof of Proposition 7.2

Take a sequence $m_i \to \infty$ in $\mathbb{Z}$. We set $\epsilon_i := m_i^{-1}$. By Proposition 3.1, we obtain the following convergence:

$$\lim_{i\to\infty} \int_{X\setminus H} \big|G\big(h_{\mathrm{HE}}^{(\epsilon_i)}\big)\big|^2_{h_{\mathrm{HE}}^{(\epsilon_i)},\omega_{\epsilon_i}} = 0.$$

Let $h_{\mathrm{in}}^{(\epsilon_i)}$ be a Hermitian metric for $\big(\mathcal{P}_*^{(\epsilon_i)}\mathcal{V}, \mathbb{D}^\lambda\big)$ as in Proposition 5.6. Let b_i be the automorphism of E determined by $h_{\mathrm{HE}}^{(\epsilon_i)} = h_{\mathrm{in}}^{(\epsilon_i)} \cdot b_i$. Then, for each i, $\mathbb{D}^\lambda(b_i)$ is L^2 with respect to $h_{\mathrm{HE}}^{(\epsilon_i)}$ and ω_{ϵ_i}.

Let $\omega_{\epsilon_i,s}$ denote the Kähler form of $X_s \setminus H_s$ induced by ω_{ϵ_i}. Let $h_s^{(\epsilon_i)}$ denote the restriction of $h_{\mathrm{HE}}^{(\epsilon_i)}$ to $X_s \setminus H_s$.

By Fubini's theorem, after going to a subsequence, there exists a $\mathbb{C}^*$-invariant subset $\mathfrak{Z}_M^\sharp \subset \mathfrak{Z}_M^\triangle$ with the following property:

- $(a1)$ $\lim_{i\to\infty} \int_{X_s\setminus H_s} \big|G\big(h_s^{(\epsilon_i)}\big)\big|^2_{h_s^{(\epsilon_i)},\omega_{\epsilon_i,s}} = 0$ holds for each $s \in \mathfrak{Z}_M^\sharp$.
- $(a2)$ For each $s \in \mathfrak{Z}_M^\sharp$, $\mathbb{D}_s^\lambda\big(b_{i|X_s\setminus H_s}\big)$ is L^2 with respect to $h_s^{(\epsilon_i)}$ and $\omega_{\epsilon_i,s}$.
- $(a3)$ The Lebesgue measure of $\mathfrak{Z}_M^\triangle \setminus \mathfrak{Z}_M^\sharp$ is 0.

Note that the condition $(a2)$ implies the following.

Lemma 7.4. *Let $s \in \mathfrak{Z}_M^\sharp$. Let $\widetilde{h}_s^{(\epsilon_i)}$ be a harmonic metric of $(E_s, \mathbb{D}_s^\lambda)$ adapted to $\big(\mathcal{P}_*^{(\epsilon_i)}\mathcal{V}, \mathbb{D}^\lambda\big)_{|X_s}$ such that $\det\big(\widetilde{h}_s^{(\epsilon_i)}\big) = h_{\det(E)|X_s\setminus H_s}$. Let $\widetilde{b}_{i,s}$ be the automorphism of $E_{|X_s\setminus H_s}$ determined by $h_s^{(\epsilon_i)} = \widetilde{h}_s^{(\epsilon_i)}\widetilde{b}_{i,s}$. Then, $\widetilde{b}_{i,s}$ and $\widetilde{b}_{i,s}^{-1}$ are bounded with respect to $\widetilde{h}_s^{(\epsilon_i)}$, and $\mathbb{D}^\lambda\big(\widetilde{b}_{i,s}\big)$ is L^2 with respect to $\widetilde{h}_s^{(\epsilon_i)}$ and $\omega_{\epsilon_i,s}$*

Proof. Let $b'_{i,s}$ be the automorphism of E_s determined by $\widetilde{h}_s^{(\epsilon_i)} = h^{(\epsilon_i)}_{\mathrm{in}\,|X_s\setminus H_s}$. Then, $b'_{i,s}$ and $(b'_{i,s})^{-1}$ are bounded with respect to $h^{(\epsilon_i)}_{\mathrm{in}\,|X_s\setminus H_s}$, and $\mathbb{D}^\lambda(b'_{i,s})$ is L^2 with respect to $h^{(\epsilon_i)}_{\mathrm{in}\,|X_s\setminus H_s}$ and $\omega_{\epsilon_i,s}$, according to Proposition 4.4. Then, we obtain the claim of the lemma. ■

Lemma 7.5. *There exists a $\mathbb{C}^*$-invariant subset $\mathfrak{X}_M^\sharp \subset \mathfrak{X}_M^\triangle \times_{\mathfrak{Z}_M^\triangle} \mathfrak{Z}_M^\sharp$ such that the following holds:*

- *The measure of $\mathfrak{X}_M^\triangle \setminus \mathfrak{X}_M^\sharp$ is 0.*
- *A subsequence of $h^{\triangle(\epsilon_i)}_{|\mathfrak{X}_M^\sharp}$ is convergent to $h^\triangle_{|\mathfrak{X}_M^\sharp}$ at any point of $\mathfrak{X}_M^\sharp$.*

Proof. By Proposition 4.8, for any $s \in \mathfrak{Z}_M^\sharp$, the sequence $h_s^{(\epsilon_i)}$ is weakly convergent to h_s in L_1^2 locally on $X_s \setminus H_s$. We set $b_s^{(\epsilon_i)} := h_s^{(\epsilon_i)} h_s^{-1}$. We obtain $\det\big(b_s^{(\epsilon_i)}\big) = 1$, and $b_s^{(\epsilon_i)}$ converges to the identity locally on $X_s \setminus H_s$ in L^p for any $p \geq 1$. We set $g_s^{(\epsilon_i)} := \big|b_s^{(\epsilon_i)}\big|_{h_s}$ on $X_s \setminus H_s$. We obtain the function $g^{(\epsilon_i)}$ on $\mathfrak{X}_M^\triangle \times_{\mathfrak{Z}_M^\triangle} \mathfrak{Z}_M^\sharp$ from $g_s^{(\epsilon_i)}$ $(s \in \mathfrak{Z}_M^\sharp)$. By Lemma 3.21 and Proposition 4.8, for any compact subset $K \subset \mathfrak{X}_M^\triangle \setminus \mathfrak{H}_M^\triangle$, the restriction of $g^{(\epsilon_i)}$ to $K \cap \big(\mathfrak{X}_M^\triangle \times_{\mathfrak{Z}_M^\triangle} \mathfrak{Z}_M^\sharp\big)$ are uniformly bounded. Note that the sequence $\big(h^{\triangle(\epsilon_i)}(h^\triangle)^{-1}\big)_{|K\cap X_s} = b^{(\epsilon_i)}_{s|K\cap X_s}$ is convergent in L^p for any $p \geq 1$. By Fubini's theorem and Lebesgue theorem, we obtain the L^p-convergence of $h^{\triangle(\epsilon_i)}\big(h^\triangle\big)^{-1}$ to the identity for any p on $K \cap \big(\mathfrak{X}_M^\triangle \times_{\mathfrak{Z}_M^\triangle} \mathfrak{Z}_M^\sharp\big)$. Then, after going to a subsequence, we obtain the desired convergence. ■

Remark 7.6. If $\lambda = 0$, the argument can be simplified. Indeed, by Proposition 3.31, the curvature $R\big(h_s^{(\epsilon_i)}\big)$ of $h_s^{(\epsilon_i)}$ are bounded locally on $X_s \setminus H$. Hence, we obtain that $h_s^{(\epsilon_i)}$ is weakly convergent to h_s in L_2^2. In particular, $h_s^{(\epsilon_i)}$ is convergent to h_s in the C^0-sense locally on $X_s \setminus H_s$.

There exists a subset $X^\sharp \subset \mathtt{P}_1^\triangle\big(\mathfrak{X}_M^\sharp\big)$ such that for any $P \in X^\sharp$, the measure of $\big(\mathtt{P}_1^\triangle\big)^{-1}(P) \setminus \big(\mathfrak{X}_M^\sharp\big)$ is 0 in $\big(\mathtt{P}_1^\triangle\big)^{-1}(P)$, and that the measure of $X \setminus X^\sharp$ is 0 in X. We obtain that the sequence $h^{(\epsilon_i)}_{\mathrm{HE}\,|X^\sharp}$ is convergent to a Hermitian metric h_∞ of $E_{|X^\sharp}$.

Lemma 7.7. *For any $P \in X^\sharp$ and $s \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big)$, we obtain $(h_s)_{|P} = h_{\infty|P}$.*

Proof. For any $s \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big) \cap \mathfrak{Z}_M^\sharp$, we obtain $(h_s)_{|P} = h_{\infty|P}$. Then, by using the continuity of h_s on s, we obtain $(h_s)_{|P} = h_{\infty|P}$ for any $s \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big)$. ■

Lemma 7.8. *Let $P \in X \setminus (W_M \cup H)$. Then, for any $s_1, s_2 \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big)$, we obtain $h_{s_1|P} = h_{s_2|P}$.*

Proof. For any $P \in X^\sharp$ and for any $s_1, s_2 \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big)$, we obtain $h_{s_1|P} = h_{\infty|P} = h_{s_2|P}$. By the continuity of h_s on s, we obtain the claim of the lemma. ■

Then, h_∞ extends to a Hermitian metric of $E_{|X\setminus(H\cup W_M)}$ by setting $h_{\infty|P} := (h_s)_{|P}$ for $s \in \mathtt{P}_2\big(\big(\mathtt{P}_1^\triangle\big)^{-1}(P)\big)$.

Lemma 7.9. *h_∞ induces a Hermitian metric of $E_{|X\setminus(H\cup W_M)}$ of C^1-class. The C^1-Hermitian metric is also denoted by h_∞.*

Proof. Let P be any point of $X \setminus W_M$. Let (U_P, x_1, x_2) be a holomorphic coordinate neighbourhood as in Section 7.2.2. By using Proposition 4.5, we define the continuous Hermitian metric $h_P^{(i)}$ of $E_{|U_P}$ by the condition that $h_{P|\{x_i=a\}}$ is equal to the restriction of $h_{s_1+as_\infty}$. By the construction of h_∞, we obtain $h_{\infty|U_P} = h_P^{(i)}$. Hence, we obtain that $h_{\infty|U_P\cap X^\sharp}$ induces a continuous Hermitian metric $h_{P,\infty}$ of $E_{|U_P}$, and $h_P^{(1)} = h_{P,\infty} = h_P^{(2)}$ hold. Moreover, by Proposition 4.5, any derivative of $h_P^{(j)}$ with respect to ∂_{z_i} and $\partial_{\overline{z}_i}$ $(i \neq j)$ are continuous. We obtain that $h_{P,\infty}$ is C^1. Thus, we obtain the claim of the lemma. ■

We obtain the operator $\mathbb{D}^{\lambda\star}_{h_\infty}$ from $\mathbb{D}^\lambda$ and h_∞. We define $G(h_\infty) := \big[\mathbb{D}^\lambda, \mathbb{D}^{\lambda\star}_{h_\infty}\big]$ as a current.

Lemma 7.10. *$G(h_\infty)^{(1,1)} = 0$ on $X \setminus (H \cup W_M)$.*

Proof. Let $P \in X \setminus (H \cup W_M)$. Let (U_P, x_1, x_2) be a holomorphic coordinate neighbourhood as in Section 7.2.2. We have the expression

$$G(h_\infty)^{(1,1)} = G(h_\infty)_{11}\mathrm{d}x_1\,\mathrm{d}\overline{x}_1 + G(h_\infty)_{12}\mathrm{d}x_1\,\mathrm{d}\overline{x}_2 + G(h_\infty)_{21}\mathrm{d}x_2\,\mathrm{d}\overline{x}_1 + G(h_\infty)_{22}\mathrm{d}x_2\,\mathrm{d}\overline{x}_2.$$

Because $h_{\infty|\{x_i=a\}}$ is equal to $h_{s_i+as_\infty}$, we obtain $G(h_\infty)_{ii} = 0$ for $i = 1, 2$.

By considering the holomorphic coordinate system $(w_1, w_2) = (x_1 + x_2, x_1 - x_2)$ and the coefficient of $\mathrm{d}w_1\,\mathrm{d}\overline{w}_1$ in $G(h_\infty)^{(1,1)}$, we obtain $G(h_\infty)_{12} + G(h_\infty)_{21} = 0$. By considering the holomorphic coordinate system $(z_1, z_2) = \big(x_1 + \sqrt{-1}x_2, x_1 - \sqrt{-1}x_2\big)$ and the coefficient of $\mathrm{d}z_1\,\mathrm{d}\overline{z}_1$ in $G(h_\infty)^{(1,1)}$, we obtain $G(h_\infty)_{12} - G(h_\infty)_{21} = 0$. Therefore, we obtain that $G(h_\infty)_{ij} = 0$. ■

Lemma 7.11. *We obtain $\Lambda G(h_\infty) = 0$ on $X \setminus (H \cup W_M)$. As a result, h_∞ is C^∞ on $X \setminus (H \cup W_M)$. If moreover $\lambda \neq 0$, then h_∞ is a pluri-harmonic metric of $\big(E, \mathbb{D}^\lambda\big)_{|X\setminus H\cup W_M}$.*

Proof. The first claim immediately follows from Lemma 7.10. We obtain the second claim by the elliptic regularity and a standard bootstrapping argument. The last claim follows from Corollary 2.16. ∎

Lemma 7.12. *In the case $\lambda = 0$, we obtain $\partial_{h_\infty}\theta = 0$, i.e., h_∞ is a pluri-harmonic metric of the Higgs bundle $\big(E, \overline{\partial}_E, \theta\big)_{|X\setminus(H\cup W_M)}$.*

Proof. Let us observe that the sequence $\partial_{h^{(\epsilon_i)}_{\rm HE}} - \partial_{h_\infty}$ is convergent to 0 almost everywhere on $X \setminus H$. It is enough to prove that $\partial_{h^{(\epsilon_i)}_{{\rm HE}\,|X_s}} - \partial_{h_s}$ is convergent to 0 for $s \in \mathfrak{Z}^{\sharp}_M$. Let $b^{(\epsilon_i)}_s$ be the automorphism of $E_{|X_s\setminus H}$ which is self-adjoint with respect to h_s and $h^{(\epsilon_i)}_{{\rm HE}\,|X_s}$ determined by $h^{(\epsilon_i)}_{{\rm HE}\,|X_s} = h_s b^{(\epsilon_i)}_s$. By Proposition 4.8, the sequence $\big(b^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b^{(\epsilon_i)}_s\big)$ is convergent to 0 weakly in L^2 locally on $X_s \setminus H$. By Proposition 3.31, the sequence $\big(b^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b^{(\epsilon_i)}_s\big)$ is bounded in L^p_2 locally on $X_s \setminus H$ for any $p \geq 1$.

Lemma 7.13. *$\big(b^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b^{(\epsilon_i)}_s\big)$ is convergent to 0 in L^p_1 locally on $X_s \setminus H$.*

Proof. Let $\big(b'^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b'^{(\epsilon_i)}_s\big)$ be any subsequence of $\big(b^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b^{(\epsilon_i)}_s\big)$. Because it is bounded in L^p_2 locally on $X_s \setminus H$, it contains a subsequence $\big(b''^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b''^{(\epsilon_i)}_s\big)$ which is weakly convergent in L^p_2 locally on $X_s \setminus H$ for any $p \geq 2$. By the Sobolev embedding theorem, the sequence $\big(b''^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b''^{(\epsilon_i)}_s\big)$ is convergent in L^p_1 locally on $X_s \setminus H$. Because $\big(b^{(\epsilon_i)}_s\big)^{-1}\partial_{h_s}\big(b^{(\epsilon_i)}_s\big)$ is convergent to 0 weakly in L^2 locally on $X_s \setminus H$, the limit should be 0. Therefore, we obtain the claim of Lemma 7.13. ∎

As a result, $\partial_{h^{(\epsilon_i)}_{\rm HE}}\theta$ is convergent to $\partial_{h_\infty}\theta$ almost everywhere. Note that

$$0 \leq \int_{X\setminus H} \big|\partial_{h^{(\epsilon_i)}_{\rm HE}}\theta\big|^2_{h^{(\epsilon_i)}_{\rm HE},\omega_{\epsilon_i}} \leq \int_{X\setminus H} \big|G\big(h^{(\epsilon_i)}_{\rm HE}\big)\big|^2_{h^{(\epsilon_i)}_{\rm HE},\omega_{\epsilon_i}} = -8\pi^2 \int_X {\rm ch}_2\big(\mathcal{P}^{(\epsilon_i)}_*\mathcal{V}\big).$$

We also have $\lim_{i\to\infty} \int_X {\rm ch}_2\big(\mathcal{P}^{(\epsilon_i)}_*\mathcal{V}\big) = 0$. We have the following convergence almost everywhere on $X \setminus H$:

$$\lim_{i\to\infty} \big|\partial_{h^{(\epsilon_i)}_{\rm HE}}\theta\big|^2_{h^{(\epsilon_i)}_{\rm HE},\omega_{\epsilon_i}} = \big|\partial_{h_\infty}\theta\big|^2_{h_\infty,\omega_X}.$$

Therefore, we obtain $\int \big|\partial_{h_\infty}\theta\big|^2_{h_\infty,\omega_X} = 0$ by Fatou's lemma. ∎

Lemma 7.14. *h_∞ induces a C^∞-metric of E on $X \setminus H$, and hence it is a pluri-harmonic metric of $\big(E, \mathbb{D}^\lambda\big)$.*

Proof. It is enough to prove that h_∞ is a C^∞-metric around any point of $W_M \setminus H$. We have only to apply the argument in [49, Lemma 5.15]. ∎

If $\lambda = 0$, we obtain that $\big(E, \overline{\partial}_E, \theta, h_\infty\big)$ is a good wild harmonic bundle on (X, H), because $(\mathcal{P}_*\mathcal{V}, \theta)$ is a good filtered Higgs bundle. If $\lambda \neq 0$, the associated Higgs bundle $\big(E, \overline{\partial}_E, \theta\big)$ with the pluri-harmonic metric h_∞ is a good wild harmonic bundle by [51, Proposition 13.5.1]. We obtain a good filtered λ-flat bundle $\big(\mathcal{P}^{h_\infty}_* E, \mathbb{D}^\lambda\big)$ on (X, H). We put $H^{[2]} = \bigcup_{i\neq j}(H_i \cap H_j)$. For any $P \in H \setminus \big(W_M \cup H^{[2]}\big)$, there exists $s \in \mathfrak{Z}^{\triangle}_M$ such that $P \in X_s$. By the construction, $h_{\infty|X_s\setminus H_s} = h_s$. Hence, we obtain $\mathcal{P}^{h_\infty}_*(E)_{|X_s} = \mathcal{P}_*(\mathcal{V})_{|X_s}$. Let $Y := (H \cap W_M) \cup H^{[2]}$, which is a finite subset of H. We obtain that $\mathcal{P}^{h_\infty}_*(E)_{|X\setminus Y} \simeq \mathcal{P}_*\mathcal{V}_{|X\setminus Y}$. By Hartogs theorem, we obtain that $\mathcal{P}^{h_\infty}_*(E) \simeq \mathcal{P}_*\mathcal{V}$. Thus, the proof of Proposition 7.2 is completed. ∎

7.3 Higher dimensional case

Let us prove Theorem 7.1 in the case $\dim X \geq 3$ by an induction on $\dim X$. Take a sufficiently large integer M. We set $\mathfrak{Z}_M := H^0\bigl(X, L^{\otimes M}\bigr) \setminus \{0\}$, and let $\mathfrak{X}_M \subset X \times \mathfrak{Z}_M$ be defined as $\mathfrak{s}^{-1}(0)$ as in Section 7.2.1. For any $s \in \mathfrak{Z}_M$, set $X_s := s^{-1}(0)$. Let $\mathbb{P}(T^*X)$ denote the projectivization of the cotangent bundle of X. If M is sufficiently large, there exists a Zariski dense open subset $\mathfrak{Z}_M^{\circ} \subset \mathfrak{Z}_M$ such that the following holds:

- $\mathtt{P}_2^{\circ}\colon \mathfrak{X}_M^{\circ} := \mathfrak{X}_M \times_{\mathfrak{Z}_M} \mathfrak{Z}_M^{\circ} \longrightarrow \mathfrak{Z}_M^{\circ}$ is smooth.
- $\mathfrak{X}_M^{\circ} \cup \bigl(H \times \mathfrak{Z}_M^{\circ}\bigr)$ is simply normal crossing. Moreover the intersections of any tuple of irreducible components are smooth over $\mathfrak{Z}_M^{\circ}$.
- The induced map $\mathtt{P}_1\colon \mathfrak{X}_M^{\circ} \longrightarrow X$ is surjective. Moreover, the induced morphism $\widetilde{\mathtt{P}}_1\colon \mathfrak{X}_M^{\circ} \longrightarrow \mathbb{P}(T^*X)$ is surjective.

Let $p_{i,j}$ denote the projection of $X \times \mathfrak{Z}_M^{\circ} \times \mathfrak{Z}_M^{\circ}$ onto the product of the i-th component and the j-th component. For $j = 2, 3$, let $\bigl(\mathfrak{X}_M^{\circ}\bigr)^{(j)}$ denote the pull back of $\mathfrak{X}_M^{\circ}$ by $p_{1,j}$. There exists a Zariski dense open subset $\mathfrak{U}_M \subset \mathfrak{Z}_M^{\circ} \times \mathfrak{Z}_M^{\circ}$ such that the following holds:

- Let $\bigl(\mathfrak{X}_M^{\circ}\bigr)^{(j)}_{\mathfrak{U}_M}$ denote the fiber product of $\bigl(\mathfrak{X}_M^{\circ}\bigr)^{(j)}$ and $\mathfrak{U}_M$ over $\mathfrak{Z}_M^{\circ} \times \mathfrak{Z}_M^{\circ}$. Then, $\bigl(\mathfrak{X}_M^{\circ}\bigr)^{(2)}_{\mathfrak{U}_M} \cup \bigl(\mathfrak{X}_M^{\circ}\bigr)^{(3)}_{\mathfrak{U}_M} \cup (H \times \mathfrak{U}_M)$ is simply normal crossing. Moreover, the intersection of any tuple of irreducible components are smooth over $\mathfrak{U}$.

By the Mehta–Ramanathan type theorem (Proposition 3.8), there exists a Zariski dense open subset $\mathfrak{U}_M^{\triangle} \subset \mathfrak{U}_M$ such that the following holds:

- For $\boldsymbol{s} = (s_1, s_2) \in \mathfrak{U}_M^{\triangle}$, we set $X_{\boldsymbol{s}} := X_{s_1} \cap X_{s_2}$. Then, the restriction $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\bigr)_{|X_{\boldsymbol{s}}}$ is a μ_L-stable good filtered λ-flat bundle on $(X_{\boldsymbol{s}}, H \cap X_{\boldsymbol{s}})$.

Hence, there exists a Zariski dense open subset $\mathfrak{Z}_M^{\triangle} \subset \mathfrak{Z}_M^{\circ}$ such that the following holds:

- For any $s \in \mathfrak{Z}_M^{\triangle}$, $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\bigr)_{|X_s}$ is a μ_L-stable good filtered λ-flat bundle on $(X_s, H \cap X_s)$.
- For any $s_1, s_2 \in \mathfrak{Z}_M^{\triangle}$, there exists a Zariski open subset $\mathfrak{V}(s_1, s_2) \subset \mathfrak{Z}_M^{\triangle}$ such that the restrictions $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\bigr)_{|X_{(s_i,s_3)}}$ $(i = 1, 2)$ are μ_L-stable for any $s_3 \in \mathfrak{V}(s_1, s_2)$.

We set $\mathfrak{X}_M^{\triangle} := \mathfrak{X}_M \times_{\mathfrak{Z}_M} \mathfrak{Z}_M^{\triangle}$. Let $\mathtt{P}_2^{\triangle}\colon \mathfrak{X}_M^{\triangle} \longrightarrow X$ denote the naturally induced morphism. Then, $W_M := X \setminus \mathtt{P}_2^{\triangle}\bigl(\mathfrak{X}_M^{\triangle}\bigr)$ is a finite subset.

For any $P \in X \setminus (H \cup W_M)$, there exists $s \in \mathfrak{Z}_M^{\triangle}$ such that $P \in X_s$. Then, $\bigl(\mathcal{P}_*\mathcal{V}_s, \mathbb{D}^{\lambda}_s\bigr) := \bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\bigr)_{|X_s}$ is μ_L-stable, and the following holds:

$$\int_{X_s} c_1(\mathcal{P}_*\mathcal{V}_s) c_1\bigl(L_{|X_s}\bigr)^{\dim X_s - 1} = 0, \qquad \int_{X_s} \mathrm{ch}_2(\mathcal{P}_*\mathcal{V}_s) c_1\bigl(L_{|X_s}\bigr)^{\dim X_s - 2} = 0.$$

There exists a pluri-harmonic metric h_s of $\bigl(E_s, \overline{\partial}_{E_s}, \mathbb{D}^{\lambda}_s\bigr) := \bigl(E, \overline{\partial}_E, \mathbb{D}^{\lambda}\bigr)_{|X_s \setminus H}$ adapted to $\mathcal{P}_*\mathcal{V}_s$ such that $\det(h_s) = h_{\det(E)|X_s \setminus H}$. Take another $s' \in \mathfrak{Z}_M^{\triangle}$ such that $P \in X_{s'}$. There exists a pluri-harmonic metric $h_{s'}$ of $\bigl(E_{s'}, \overline{\partial}_{E_{s'}}, \mathbb{D}^{\lambda}_{s'}\bigr)$ adapted to $\mathcal{P}_*\mathcal{V}_{s'}$ such that $\det(h_{s'}) = h_{\det(E)|X_{s'} \setminus H}$.

Lemma 7.15. $h_{s|P} = h_{s'|P}$.

Proof. Suppose that $X_s \cup X_{s'} \cup H$ is simply normal crossing. We set $X_{s,s'} := X_s \cap X_{s'}$. It is smooth and connected. We obtain a good filtered λ-flat bundle $\bigl(\mathcal{P}_*\mathcal{V}, \mathbb{D}^{\lambda}\bigr)_{|X_{s,s'}}$, and $h_{s|X_{s,s'}}$

and $h_{s'|X_{s,s'}}$ are adapted to $\mathcal{P}_*\mathcal{V}_{|X_{s,s'}}$. Let $b_{s,s'}$ be the automorphism of $E_{|X_{s,s'}}$ which is self-adjoint with respect to both $h_{s|X_{s,s'}}$ and $h_{s'|X_{s,s'}}$, and determined by $h_{s'|X_{s,s'}} = h_{s|X_{s,s'}} \cdot b_{s,s'}$. There exists a decomposition

$$\big(\mathcal{P}_*\mathcal{V}, \mathbb{D}^\lambda\big)_{|X_{s,s'}} = \bigoplus \big(\mathcal{P}_*\mathcal{V}_i, \mathbb{D}_i^\lambda\big),$$

which is orthogonal with respect to both $h_{s|X_{s,s'}}$ and $h_{s'|X_{s,s'}}$, and $b_{s,s'} = \bigoplus a_i \,\mathrm{id}_{\mathcal{V}_i}$ for some positive constants a_i.

There exists $s_1 \in \mathfrak{V}(s,s')$. Then, $(\mathcal{P}_*\mathcal{V}, \theta)_{|X_{s_1,s}}$ and $(\mathcal{P}_*\mathcal{V}, \theta)_{|X_{s_1,s'}}$ are μ_L-stable. Therefore, we have $h_{s|X_{s_1,s}} = h_{s_1|X_{s_1,s}}$ and $h_{s'|X_{s_1,s'}} = h_{s_1|X_{s_1,s'}}$. We obtain that $h_{s|X_{s_1}\cap X_s\cap X_{s'}} = h_{s'|X_{s_1}\cap X_s\cap X_{s'}}$. It implies that a_i are 1, and hence $h_{s|P} = h_{s'|P}$.

In general, there exists $s_2 \in \mathfrak{Z}_M^\triangle$ such that (i) $P \in X_{s_2}$, (ii) $X_s \cup X_{s_2} \cup H$ and $X_{s'} \cup X_{s_2} \cup H$ are simply normal crossing. By the above consideration, we obtain $h_{s|P} = h_{s_2|P} = h_{s'|P}$. ■

Therefore, we obtain Hermitian metrics h_P of $E_{|P}$ $(P \in X \setminus (H \cup W_M))$. By using the argument in Lemma 7.9, we can prove that they induce a Hermitian metric h of $E_{|X\setminus(H\cup W_M)}$ of C^1-class. We obtain $G(h)$ from $\mathbb{D}^\lambda$ and h as a current. Because $h_{|X_s}$ $\big(s \in \mathfrak{U}_M^\triangle\big)$ are pluri-harmonic metrics of $\big(E, \mathbb{D}^\lambda\big)_{|X_s\setminus H}$, we obtain that $G(h) = 0$. It also implies that h is C^∞ on $X \setminus (H \cup W_M)$. By using the argument in [49, Lemma 5.15], we obtain that h induces a pluri-harmonic metric of $\big(E, \mathbb{D}^\lambda\big)$ on $X \setminus H$. Then, as in the proof of Proposition 7.2, we can conclude that $\big(E, \mathbb{D}^\lambda, h\big)$ is a good wild harmonic bundle, and that $\mathcal{P}_*^h(E) = \mathcal{P}_*\mathcal{V}$. Thus, we obtain Theorem 7.1. ■

8 Homogeneity with respect to group actions

8.1 Preliminary

8.1.1 Homogeneous harmonic bundles

Let Y be a complex manifold. Let K be a compact Lie group. Let $\rho\colon K \times Y \longrightarrow Y$ be a K-action on Y such that $\rho_k\colon Y \longrightarrow Y$ is holomorphic for any $k \in K$. Let $\kappa\colon K \longrightarrow S^1$ be a homomorphism of Lie groups.

Let $\big(E, \overline{\partial}_E, \theta, h\big)$ be a harmonic bundle on Y. It is called (K, ρ, κ)-homogeneous if $\big(E, \overline{\partial}_E, h\big)$ is K-equivariant and $k^*\theta = \kappa(k)\theta$.

Remark 8.1. According to s3, harmonic bundles are equivalent to polarized variation of pure twistor structure of weight w, for any given integer w. If κ is non-trivial, as studied in [53, Section 3], by choosing a vector $\mathfrak{v}$ in the Lie algebra of K such that $d\kappa(\mathfrak{v}) \neq 0$, we obtain the integrability of the variation of pure twistor structure from the homogeneity of harmonic bundles.

8.1.2 Homogeneous filtered Higgs sheaves and the stability condition with respect to the action

Let X be a connected complex projective manifold with a simple normal crossing hypersurface H. Let G be a complex reductive algebraic group. Let $\rho\colon G \times Y \longrightarrow Y$ be an algebraic G-action on Y which preserves H. Let $\kappa\colon G \longrightarrow \mathbb{C}^*$ be a homomorphism of complex algebraic groups.

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a filtered Higgs sheaf on (Y, H). It is called (G, ρ, κ)-homogeneous if $\mathcal{P}_*\mathcal{V}$ is G-equivariant and $g^*\theta = \kappa(g)\theta$ for any $g \in G$.

Let L be a G-equivariant ample line bundle on X. A (G,ρ,κ)-homogeneous filtered Higgs sheaf $(\mathcal{P}_*\mathcal{V},\theta)$ on (X,H) is called μ_L-stable (resp. μ_L-semistable) with respect to the G-action if the following holds:

- Let $\mathcal{V}'$ be a G-invariant saturated Higgs subsheaf of $\mathcal{V}$ such that $0<\operatorname{rank}\mathcal{V}'<\operatorname{rank}\mathcal{V}$. Then, $\mu_L(\mathcal{P}_*\mathcal{V}')<\mu_L(\mathcal{P}_*\mathcal{V})$ (resp. $\mu_L(\mathcal{P}_*\mathcal{V}')\leq\mu_L(\mathcal{P}_*\mathcal{V})$) holds.

A (G,ρ,κ)-homogeneous filtered Higgs sheaf $(\mathcal{P}_*\mathcal{V},\theta)$ on (X,H) is called μ_L-polystable with respect to the G-action if it is μ_L-semistable with respect to the G-action and isomorphic to a direct sum of (G,ρ,κ)-homogeneous filtered sheaves $\bigoplus(\mathcal{P}_*\mathcal{V}_i,\theta_i)$, where each $(\mathcal{P}_*\mathcal{V}_i,\theta_i)$ is μ_L-stable with respect to the G-action.

Lemma 8.2. *$(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-semistable if and only if $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-semistable with respect to the G-action.*

Proof. The "only if" part is clear. Let us prove that the "if" part. Let $\mathcal{V}_0\subset\mathcal{V}$ be the β-subobject as in Proposition 3.4. Because $g^*\mathcal{V}_0$ also has the same property, we obtain that $\mathcal{V}_0$ is G-invariant. Then, the claim of the proposition is clear. ■

The following lemma is clear.

Lemma 8.3. *If $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-stable, then $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-stable with respect to the G-action.*

Lemma 8.4. *If $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-stable with respect to the G-action, then $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-polystable.*

Proof. According to Lemma 8.2, $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-semistable. Let $\mathcal{V}_1$ be the socle of $(\mathcal{P}_*\mathcal{V},\theta)$ as in Proposition 3.5. Because $g^*\mathcal{V}_1$ also has the same property, $\mathcal{V}_1$ is G-invariant. Moreover, $\mu_L(\mathcal{P}_*\mathcal{V}_1)=\mu_L(\mathcal{P}_*\mathcal{V})$ holds. Hence, we obtain $\mathcal{V}_1=\mathcal{V}$. According to Proposition 3.5, $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-polystable. ■

Remark 8.5. In general, even if $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-stable with respect to the G-action, $(\mathcal{P}_*\mathcal{V},\theta)$ is not necessarily μ_L-stable.

8.1.3 Actions of a complex reductive group and its compact real form

Let X be a complex projective manifold equipped with an algebraic action of a complex reductive group G. Let L be a G-equivariant ample line bundle on X. Let K be a compact real form of G.

Let $(E,\overline{\partial}_E)$ be a G-equivariant holomorphic vector bundle on X. Then, as the restriction, we may naturally regard $(E,\overline{\partial}_E)$ as a K-equivariant holomorphic vector bundle on X.

Lemma 8.6. *The above procedure induces an equivalence between G-equivariant holomorphic vector bundles and K-equivariant holomorphic vector bundles on X.*

Proof. Let $(E,\overline{\partial}_E)$ be a K-equivariant holomorphic vector bundle on X. There exists $m_0>0$ such that $E\otimes L^{\otimes m_0}$ is globally generating. We set $\mathcal{G}_0:=H^0\big(X,E\otimes L^{\otimes m_0}\big)\otimes\big(L^{\otimes m_0}\big)^{-1}$. There exists a naturally induced epimorphism of $\mathcal{O}_X$-modules $\mathcal{G}_0\longrightarrow E$. Let $\mathcal{K}$ denote the kernel. There exists $m_1>0$ such that $\mathcal{K}\otimes L^{\otimes m_1}$ is globally generating. We set $\mathcal{G}_1:=H^0\big(X,\mathcal{K}\otimes L^{\otimes m_1}\big)\otimes\big(L^{\otimes m_1}\big)^{-1}$. There exists a naturally induced epimorphism $\mathcal{G}_1\longrightarrow\mathcal{K}$. Thus, we obtain a resolution $\mathcal{G}_1\longrightarrow\mathcal{G}_0$ of E. Because E is K-equivariant, $H^0\big(X,E\otimes L^{\otimes m_0}\big)$ is naturally a K-representation, $\mathcal{G}_0$ is a K-equivariant holomorphic vector bundle on X, and $\mathcal{G}_0\longrightarrow E$ is K-equivariant. Hence, $\mathcal{K}$ is a K-equivariant holomorphic vector bundle. Similarly $H^0\big(X,\mathcal{K}\otimes L^{\otimes m_2}\big)$ is a K-representation, and $\mathcal{G}_1$ is K-equivariant holomorphic vector bundle, and $\mathcal{G}_1\longrightarrow\mathcal{K}_2$ is K-equivariant.

The K-representations on $H^0\big(X,E\otimes L^{\otimes m_1}\big)$ and $H^0\big(X,\mathcal{K}\otimes L^{\otimes m_2}\big)$ naturally induce G-representations on $H^0\big(X,E\otimes L^{\otimes m_1}\big)$ and $H^0\big(X,\mathcal{K}\otimes L^{\otimes m_2}\big)$. Hence, $\mathcal{G}_i$ are naturally algebraic G-equivariant vector bundles on X. Moreover, the morphism $\mathcal{G}_1\longrightarrow\mathcal{G}_0$ is G-equivariant and algebraic. Hence, E is a G-equivariant algebraic vector bundle on X. ■

8.2 An equivalence

8.2.1 Good filtered Higgs bundles associated with homogeneous good wild Higgs bundles

Let X be a connected complex projective manifold with a simple normal crossing hypersurface H. Let G be a complex reductive group acting on (X,H). Let K be a compact real form of G. The actions of G and K on X are denoted by ρ. Let $\kappa\colon G\longrightarrow \mathbb{C}^*$ be a character. The induced homomorphism $K\longrightarrow S^1$ is also denoted by κ.

Let $\big(E,\overline{\partial}_E,\theta,h\big)$ be a (K,ρ,κ)-homogeneous harmonic bundle on $X\setminus H$ which is good wild on (X,H). We obtain a good filtered Higgs bundle $\big(\mathcal{P}^h_*E,\theta\big)$ on (X,H). Because each $\mathcal{P}^h_{\boldsymbol{a}}E$ is naturally a K-equivariant holomorphic vector bundle on X, $\mathcal{P}^h_*E$ is naturally G-equivariant by Lemma 8.6. Because $k^*\theta=\kappa(k)\theta$ for any $k\in K$, we obtain $g^*\theta=\kappa(g)\theta$ for any $g\in G$. Therefore, $\big(\mathcal{P}^h_*E,\theta\big)$ is a (G,ρ,κ)-homogeneous good filtered Higgs bundle on (X,H).

Let L be a G-equivariant ample line bundle on X.

Proposition 8.7. *$\big(\mathcal{P}^h_*E,\theta\big)$ is μ_L-polystable with respect to the G-action, i.e., there exists a decomposition $\big(E,\overline{\partial}_E,\theta,h\big)=\bigoplus\big(E_i,\overline{\partial}_{E_i},\theta_i,h_i\big)$ of (G,ρ,κ)-homogeneous harmonic bundles such that each $\big(\mathcal{P}^{h_i}_*E_i,\theta_i\big)$ is μ_L-stable with respect to the G-action.*

Proof. Because $\big(\mathcal{P}^h_*E,\theta\big)$ is μ_L-polystable, we obtain that $\big(\mathcal{P}^h_*E,\theta\big)$ is μ_L-semistable with respect to the G-action. Let $\mathcal{V}_1\subset\mathcal{P}^hE$ be a G-invariant saturated Higgs $\mathcal{O}_X(*H)$-submodule such that $\mu_L(\mathcal{P}_*\mathcal{V}_1)=\mu_L\big(\mathcal{P}^h_*E\big)=0$. Let E_1 be the Higgs subsheaf of E obtained as the restriction of $\mathcal{V}_1$ to $X\setminus H$. Then, by the argument in the proof of [51, Proposition 13.6.1], we obtain that E_1 is a subbundle, and the orthogonal complement $E_2:=E_1^{\perp}$ is also a holomorphic subbundle. Moreover, $\theta(E_2)\subset E_2\otimes\Omega^1_{X\setminus H}$, and E_2 is K-equivariant. Hence, we obtain a decomposition $\big(E,\overline{\partial}_E,\theta,h\big)=\big(E_1,\overline{\partial}_{E_1},\theta_1,h_1\big)\oplus\big(E_2\overline{\partial}_{E_2},\theta_2,h_2\big)$ of (K,ρ,κ)-homogeneous harmonic bundles. Then, the claim of the proposition is clear. ∎

8.2.2 Uniqueness

Let $\big(E,\overline{\partial}_E,\theta,h\big)$ be a (K,ρ,κ)-homogeneous harmonic bundle on $X\setminus H$ which is good wild on (X,H). Let h' be another pluri-harmonic metric of $\big(E,\overline{\partial}_E,\theta\big)$ such that (i) h' is K-invariant, (ii) $\mathcal{P}^{h'}_*E=\mathcal{P}^h_*E$. The following is clear from Proposition 2.22.

Proposition 8.8. *There exists a decomposition $\big(E,\overline{\partial}_E,\theta\big)=\bigoplus_{i=1}^m\big(E_i,\overline{\partial}_{E_i},\theta_i\big)$ such that (i) the decomposition is orthogonal with respect to both h and h', (ii) there exist $a_i>0$ $(i=1,\dots,m)$ such that $h'_{|E_i}=a_ih_{E_i}$, (iii) the decomposition $E=\bigoplus E_i$ is preserved by the K-action.*

8.2.3 Existence theorem

Let $(\mathcal{P}_*\mathcal{V},\theta)$ be a (G,ρ,κ)-homogeneous good filtered Higgs bundle on (X,H) such that

$$\int_X c_1(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-1}=0,\qquad \int_X \mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}=0.$$

Let $\big(E,\overline{\partial}_E,\theta\big)$ be the Higgs bundle on $X\setminus H$ obtained as the restriction of $(\mathcal{P}_*\mathcal{V},\theta)$.

Theorem 8.9. *Suppose that $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-stable with respect to the G-action. Then, there exists a K-invariant pluri-harmonic metric h of $\big(E,\overline{\partial}_E,\theta\big)$ such that $\mathcal{P}^h_*E=\mathcal{P}_*\mathcal{V}$. If h' is another K-invariant pluri-harmonic metric of $\big(E,\overline{\partial}_E,\theta\big)$, there exists a positive constant a such that $h'=ah$.*

Proof. By Lemma 8.4, $(\mathcal{P}_*\mathcal{V},\theta)$ is μ_L-polystable. There exists the canonical decomposition

$$(\mathcal{P}_*\mathcal{V},\theta)=\bigoplus_{i=1}^{m}(\mathcal{P}_*\mathcal{V}_i,\theta_i)\otimes U_i,$$

where $(\mathcal{P}_*\mathcal{V}_i,\theta_i)$ are μ_L-stable good filtered Higgs bundles such that $(\mathcal{P}_*\mathcal{V}_i,\theta_i)\not\simeq(\mathcal{P}_*\mathcal{V}_j,\theta_j)$ $(i\neq j)$, and U_i are finite dimensional complex vector spaces. Let $\big(E_i,\overline{\partial}_{E_i},\theta_i\big)$ denote the Higgs bundle obtained as the restriction of $(\mathcal{V}_i,\theta_i)$ to $X\setminus H$. There exist pluri-harmonic metrics h_i of $\big(E_i,\overline{\partial}_{E_i},\theta_i\big)$ adapted to the filtered bundles $\mathcal{P}_*\mathcal{V}_i$. Let $h^{(0)}_{U_i}$ be Hermitian metrics of U_i. We obtain a pluri-harmonic metric $h^{(0)}=\bigoplus\big(h_i\otimes h^{(0)}_{U_i}\big)$ of $\big(E,\overline{\partial}_E,\theta\big)$ adapted to $\mathcal{P}_*\mathcal{V}$. By Proposition 2.22 and the uniqueness of the canonical decomposition, we obtain the following lemma.

Lemma 8.10. *For any pluri-harmonic metric $h^{(1)}$ of $\big(E,\overline{\partial}_E,\theta\big)$ adapted to $\mathcal{P}_*\mathcal{V}$, there uniquely exist Hermitian metrics $h^{(1)}_{U_i}$ of U_i such that $h^{(1)}=\bigoplus\big(h_i\otimes h^{(1)}_{U_i}\big)$.*

For any $k\in K$, we obtain a pluri-harmonic metric $k^*h^{(0)}$ of $\big(E,\overline{\partial}_E,\kappa(k)\theta\big)$ adapted to $\mathcal{P}_*\mathcal{V}$. Because $|\kappa(k)|=1$, $k^*\big(h^{(0)}\big)$ is also a pluri-harmonic metric of $\big(E,\overline{\partial}_E,\theta\big)$ adapted to $\mathcal{P}_*\mathcal{V}$. Hence, there uniquely exist Hermitian metrics $h_{U_i}(k)$ of U_i such that $k^*(h)=\bigoplus_{i=1}^m(h_i\otimes h_{U_i}(k))$. By using the Haar measure $\mathrm{d}k$ on K with $\int_K\mathrm{d}k=1$, we define the Hermitian metric h of E as follows:

$$h:=\int_K k^*\big(h^{(0)}\big)\mathrm{d}k=\bigoplus_{i=1}^{m}\Big(h_i\otimes\int_K h_{U_i}(k)\mathrm{d}k\Big).$$

Then, h is also a pluri-harmonic metric. By the construction, h is K-invariant.

Let h' be another K-invariant pluri-harmonic metric of $\big(E,\overline{\partial}_E,\theta\big)$ adapted to $\mathcal{P}_*\mathcal{V}$. We obtain the decomposition $\big(E,\overline{\partial}_E,\theta\big)=\bigoplus\big(E_i,\overline{\partial}_{E_i},\theta_i\big)$ as in Proposition 2.22, which induces a decomposition of good filtered Higgs bundles $(\mathcal{P}_*\mathcal{V})=\bigoplus(\mathcal{P}_*\mathcal{V}_i,\theta_i)$. Because both h and h' are K-invariant, the decompositions are also K-invariant. Hence, the decomposition $(\mathcal{P}_*\mathcal{V})=\bigoplus(\mathcal{P}_*\mathcal{V}_i,\theta_i)$ is G-invariant. By the μ_L-stability of $(\mathcal{P}_*\mathcal{V})$, we obtain $m=1$, i.e., $h'=ah$ for $a>0$. ■

Corollary 8.11. *We obtain the equivalence between the isomorphism classes of the following objects:*

- *(K,ρ,κ)-homogeneous good wild harmonic bundles on (X,H).*
- *(G,ρ,κ)-homogeneous good filtered Higgs bundles $(\mathcal{P}_*\mathcal{V},\theta)$ such that (i) it is μ_L-polystable with respect to the G-action, (ii) $\mu_L(\mathcal{P}_*\mathcal{V})=0$, $\int_X\mathrm{ch}_2(\mathcal{P}_*\mathcal{V})c_1(L)^{\dim X-2}=0$.*

Acknowledgement

It is my great pleasure to dedicate this paper to Professor Kyoji Saito on the occasion of his 77th birthday. Not to mention his pioneering and profound achievements, I have always been impressed with his kindness and generosity towards the younger generation of mathematicians, as well as his insatiable quest for mathematics. I appreciate the editors of this special issue for the invitation.

I thank Carlos Simpson for his fundamental works on harmonic bundles which are most important in this study. I am grateful to the referees for their careful readings and helpful comments to improve this manuscript. I thank Philip Boalch and Andy Neitzke for their kind comments to a preliminary note on the proof in the one dimensional case. I thank Claude

Sabbah for discussions on many occasions and for his kindness. I thank François Labourie for his comment on the definition of wild harmonic bundles. I thank Pengfei Huang for discussions. I am grateful to Ya Deng for asking questions related to Mehta–Ramanathan type theorems in Section 3.4 and tensor products in Section 4.3. I thank Akira Ishii and Yoshifumi Tsuchimoto for their constant encouragement. A part of this manuscript was prepared for lectures in the Oka symposium and ICTS program "Quantum Fields, Geometry and Representation Theory". I thank the organizers for the opportunities.

I am partially supported by the Grant-in-Aid for Scientific Research (S) (No. 17H06127), the Grant-in-Aid for Scientific Research (S) (No. 16H06335), the Grant-in-Aid for Scientific Research (A) (No. 21H04429), the Grant-in-Aid for Scientific Research (C) (No. 15K04843), and the Grant-in-Aid for Scientific Research (C) (No. 20K03609), Japan Society for the Promotion of Science.

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