Virasoro Action on the Q-Functions

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Abstract. A formula for Schur Q-functions is presented which describes the action of the Virasoro operators. For a strict partition, we prove a concise formula for $L_{-k}Q_\lambda$, where L_{-k} ($k \geq 1$) is the Virasoro operator.

Key words: Q-functions; Virasoro operators

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To Minoru Wakimoto on his 80th birthday

1 Introduction

The aim of this paper is to discuss Schur Q-functions in connection with a representation of the Virasoro algebra. Schur Q-functions are labelled by strict partitions and are defined as the Pfaffian of an alternating matrix. Let $A = (a_{ij})_{1 \leq i,j \leq 2m}$ be an alternating $2m \times 2m$ matrix. The Pfaffian of A is

$$\mathrm{Pf}(A) := \sum_{\sigma \in F_{2m}} (\mathrm{sgn}\,\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2m-1)\sigma(2m)},$$

where

$$F_{2m} := \{\sigma \in S_{2m}; \sigma(1) < \sigma(3) < \cdots < \sigma(2m-1), \sigma(i) < \sigma(i+1)\,(i=1,3,\ldots,2m-1)\}.$$

We see that $|F_{2m}| = (2m-1)!!$. The Laplace expansion of $\mathrm{Pf}(A)$ is as follows. For $1 \leq i_1 < \cdots < i_{2\ell} \leq 2m$, let $A_{i_1 i_2 \ldots i_{2\ell}}$ be the $2\ell \times 2\ell$ alternating matrix consisting of i_1th row, i_2th row, $\ldots$, and i_1th column, i_2th column, $\ldots$. Then

$$\mathrm{Pf}(A) = \sum_{i=2}^{2m} (-1)^i \,\mathrm{Pf}(A_{1i}) \,\mathrm{Pf}(A_{2\ldots\widehat{i}\ldots 2m}).$$

Here $\widehat{i}$ means the omission of i. We will utilize this quadratic relation to derive formulas for Q-functions.

Our previous paper [1] gives a formula of $L_k Q_\lambda$ for $k \geq 1$, where L_k denotes the Virasoro operator. As a continuation of [1] we give in the present paper a formula for $L_{-k}Q_\lambda$. Section 2 is a review of Q-functions containing some identities which do not seem to be obviously derived from Pfaffian identities. In Section 3 we first recall the reduced Fock representation of the

Virasoro algebra on the space of the Q-functions. Then the main result is given. Proofs consist of direct, simple calculations.

The Virasoro representations of this paper may be applied to, for example, the Kontsevich matrix models by certain rescaling. However we will not discuss here any relationship. Our motivation is to clarify the representation theoretical nature of the Hirota equations for certain soliton type hierarchies. In the final section we will give a conjecture on the Hirota equations for the KdV hierarchy.

2   Schur's Q-functions

A partition is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$, whose size is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. The number of nonzero parts is the length of λ, denoted by $\ell(\lambda)$. Let $\mathcal{SP}(n)$ be the set of partitions of n into distinct parts. We call a $\lambda \in \mathcal{SP}(n)$ strict partition of n. Let $V = \mathbb{C}[t_j; j \geq 1, \text{odd}]$. This is decomposed as $V = \bigoplus_{n=0}^\infty V(n)$, where $V(n)$ is the space of homogeneous polynomials of degree n, according to the counting $\deg t_j = j$. An inner product of V is defined by $\langle F, G \rangle = F\big(2\widetilde{\partial}\big)\overline{G}(t)|_{t=0}$, where $2\widetilde{\partial} = \big(2\partial_1, \frac{2}{3}\partial_3, \frac{2}{5}\partial_5, \dots\big)$ with $\partial_j = \frac{\partial}{\partial t_j}$.

Schur's Q-functions are defined in our context as follows. Put $\xi(t, u) = \sum_{j \geq 1, \text{odd}} t_j u^j$ and define $q_n(t) \in V(n)$ by

$$e^{\xi(t,u)} = \sum_{n=0}^\infty q_n(t)u^n.$$

For integers a, b with $a > b > 0$, define

$$Q_{ab}(t) := q_a(t)q_b(t) + 2\sum_{i=1}^b (-1)^i q_{a+i}(t)q_{b-i}(t),$$

$$Q_{ba}(t) := -Q_{ab}(t).$$

Finally, the Q-function labelled by the strict partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m})$ $(\lambda_1 > \lambda_2 > \cdots > \lambda_{2m} \geq 0)$ is defined by

$$Q_\lambda(t) = Q_{\lambda_1 \lambda_2 \dots \lambda_{2m}}(t) = \text{Pf}\,(Q_{\lambda_i \lambda_j})_{1 \leq i,j \leq 2m}.$$

The Q-function $Q_\lambda(t)$ is homogeneous of degree $|\lambda|$. It is known that $\{Q_\lambda(t); |\lambda| = n\}$ forms an orthogonal basis for $V(n)$, with respect to the above inner product. As Pfaffians, they satisfy the quadratic relations (cf. [2]):

If $\ell(\lambda)$ is odd,

$$Q_{\lambda_1 \lambda_2 \dots \lambda_\ell}(t) = \sum_{i=1}^\ell (-1)^{i+1} q_{\lambda_i}(t)Q_{\lambda_1 \dots \widehat{\lambda_i} \dots \lambda_\ell}(t).$$

If $\ell(\lambda)$ is even,

$$Q_{\lambda_1 \lambda_2 \dots \lambda_\ell}(t) = \sum_{i=2}^\ell (-1)^i Q_{\lambda_1 \lambda_i}(t)Q_{\lambda_2 \dots \widehat{\lambda_i} \dots \lambda_\ell}(t).$$

It is convenient to define Q-function $Q_\alpha(t)$ for any non-negative integer sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$. We adopt the following rule for permutations of indices:

1. If $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are all distinct, then $\sigma(\alpha)$ is a strict partition for some permutation $\sigma \in S_\ell$, and

$$Q_\alpha(t) = (\operatorname{sgn} \sigma) Q_{\sigma(\alpha)}(t).$$

2. If $\alpha_i = \alpha_j > 0$ for some $i \neq j$, then $Q_\alpha(t) = 0$.

3. Using permutations, 0's should be moved in the tail of α, keeping 0's order. After such permutation, all 0's should be deleted.

For example, we have $Q_{0,2,3,0,1}(t) = -Q_{3,2,1}(t)$. Detailed arguments are found in [2, Theorem 9.2]. Note that the above quadratic relations hold for $Q_\alpha(t)$ with non-negative integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. We also agree that, for $a > 0$, $Q_{a,-a}(t) = (-1)^{a-1}$.

Lemma 2.1. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ *be a non-negative integer sequence, and* x, y *be non-negative integers.*

(1) *If* $\ell(\alpha)$ *is odd,*

$$Q_{\alpha x} = -q_x Q_\alpha - \sum_{i=1}^\ell (-1)^i q_{\alpha_i} Q_{\alpha_1 \ldots \widehat{\alpha_i} \ldots \alpha_\ell x} - \sum_{i=1}^\ell (-1)^i Q_{\alpha_i x} Q_{\alpha_1 \ldots \widehat{\alpha_i} \ldots \alpha_\ell},$$

(2) *If* $\ell(\alpha)$ *is even,*

$$Q_{\alpha x} = -q_x Q_\alpha + \sum_{i=2}^\ell (-1)^i Q_{\alpha_1 \alpha_i} Q_{\alpha_2 \ldots \widehat{\alpha_i} \ldots \alpha_\ell x} + \sum_{i=2}^\ell (-1)^i Q_{\alpha_1 \alpha_i x} Q_{\alpha_2 \ldots \widehat{\alpha_i} \ldots \alpha_\ell},$$

(3) *If* $\ell(\alpha)$ *is odd,*

$$Q_{\alpha x y} = -Q_{xy} Q_\alpha - \sum_{i=1}^\ell (-1)^i q_{\alpha_i} Q_{\alpha_1 \ldots \widehat{\alpha_i} \ldots \alpha_\ell x y} - \sum_{i=1}^\ell (-1)^i Q_{\alpha_i x y} Q_{\alpha_1 \ldots \widehat{\alpha_i} \ldots \alpha_\ell},$$

(4) *If* $\ell(\alpha)$ *is even,*

$$Q_{\alpha x y} = -Q_{xy} Q_\alpha + \sum_{i=2}^\ell (-1)^i Q_{\alpha_1 \alpha_i} Q_{\alpha_2 \ldots \widehat{\alpha_i} \ldots \alpha_\ell x y} + \sum_{i=2}^\ell (-1)^i Q_{\alpha_1 \alpha_i x y} Q_{\alpha_2 \ldots \widehat{\alpha_i} \ldots \alpha_\ell}.$$

Proof. Let $\ell(\alpha, x, y)$ be an even number. From the Pfaffian identity for $Q_{\alpha x y}$, the case (4) follows easily. The cases (3) and (2) follow from (4) by setting $\alpha_1 = 0$ and $y = 0$, respectively. Finally case (1) is obtained from case (3) by setting $y = 0$. ∎

Next, we recall the boson-fermion correspondence for neutral free fermions ϕ_i ($i \in \mathbb{Z}$) (cf. [3]). The Clifford algebra $\mathfrak{B}$ is generated by free fermions ϕ_i ($i \in \mathbb{Z}$) satisfying the anti-commutation relation:

$$[\phi_i, \phi_j]_+ = (-1)^i \delta_{i,-j}.$$

The vector space F_B has a basis consisting of $\phi_{i_1} \phi_{i_2} \cdots \phi_{i_s} |0\rangle$, $i_1 > i_2 > \cdots > i_s \geq 0$, where $|0\rangle$ is the vacuum vector. The Clifford algebra $\mathfrak{B}$ acts on F_B by $\phi_i |0\rangle = 0$, $i < 0$. For odd number n, we define the Hamiltonian by

$$H_n^B = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^{i-1} \phi_i \phi_{-n-i}.$$

The operators H_n^B ($n \in \mathbb{Z}_{\mathrm{odd}}$) generate a Heisenberg algebra $\mathfrak{H}^B$ with $\left[H_n^B, H_m^B\right] = \frac{n}{2}\delta_{n,-m}$. It is known that F_B is isomorphic to V:

$$\sigma_B(|0\rangle) = 1, \qquad \sigma_B\left(H_n^B|0\rangle\right) := \frac{\partial}{\partial p_n}, \qquad \sigma_B\left(H_{-n}^B|0\rangle\right) := np_n, \qquad n \geq 1, \ \mathrm{odd}.$$

The map $\sigma_B \colon \mathfrak{H}^B \longrightarrow V$ is called the boson-fermion correspondence of type B. Moreover, for the basis of F_B

$$\sigma_B\left(\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_\ell}|0\rangle\right) = \begin{cases} 2^{-\frac{\ell}{2}}Q_{\lambda_1\lambda_2\ldots\lambda_\ell} & \text{if } \ell \text{ is even,} \\ 2^{-\frac{\ell+1}{2}}Q_{\lambda_1\lambda_2\ldots\lambda_\ell} & \text{if } \ell \text{ is odd.} \end{cases}$$

In what follows, we denote $\sigma_B\left(\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_\ell}|0\rangle\right)$ by $\phi_{\lambda_1}\phi_{\lambda_2}\cdots\phi_{\lambda_\ell}|0\rangle$.

Proposition 2.2. *Let* $n = 2m$. *Then*

$$\sum_{i=0}^{m-1}(2i+1)t_{2i+1}(n-(2i+1))t_{n-(2i+1)} = 2\sum_{i=0}^{m-1}(-1)^i(m-i)Q_{n-i,i}.$$

Proof. First we rewrite the left-hand side of this equation by using power sum symmetric functions.

$$\sum_{i=0}^{m-1}(2i+1)t_{2i+1}(n-(2i+1))t_{n-(2i+1)} = 4\sum_{i\geq 1,\,\mathrm{odd}}^{n} p_i p_{n-i}.$$

For an odd number i, the operator H_i^B acts on F_B. By the boson-fermion correspondence, the right hand side equals

$$\sum_{i\geq 1,\,\mathrm{odd}}^{n}\sum_{j,k\in\mathbb{Z}}(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle.$$

Since free fermions ϕ_i ($i < 0$) act on vacuum vector $|0\rangle$ as 0, the above summation becomes

$$\sum_{i\geq 1,\,\mathrm{odd}}^{n}\sum_{\substack{-n+i\leq j\leq n \\ 0\leq k\leq n-i}}(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle$$

$$= \sum_{i\geq 1,\,\mathrm{odd}}^{n}\sum_{\substack{-n+i\leq j<0 \\ 0\leq k\leq n-i}}(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle \tag{2.1}$$

$$+ \sum_{i\geq 1,\,\mathrm{odd}}^{n}\sum_{\substack{0\leq j\leq i \\ 0\leq k\leq n-i}}(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle \tag{2.2}$$

$$+ \sum_{i\geq 1,\,\mathrm{odd}}^{n}\sum_{\substack{i<j\leq n \\ 0\leq k\leq n-i}}(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle. \tag{2.3}$$

Here it is verified that the part (2.2) equals 0. Next, we consider the parts (2.1) and (2.3). For the term $(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}$, we only need to consider the cases that j or $-j+i$ belongs to $\{-k, -(n-i)+k\}$. That is, $(-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}$

$$= \begin{cases} \phi_{-k}\phi_{i+k}\phi_k\phi_{-k+(n-i)} & \text{if } j = -k, \\ -\phi_{i+k}\phi_{-k}\phi_k\phi_{-k+(n-i)} & \text{if } j = i+k, \\ -\phi_{-(n-i)+k}\phi_{-j+i}\phi_k\phi_{-k+(n-i)} & \text{if } j = -(n-i)+k, \\ \phi_{n-k}\phi_{-j+i}\phi_k\phi_{-k+(n-i)} & \text{if } j = n-k, \end{cases}$$

$$= \begin{cases} (-1)^{k+1}\phi_{i+k}\phi_{-k+(n-i)} & \text{if } j = -k \text{ or } j = i+k, \\ (-1)^{k+1}\phi_k\phi_{-k+n} & \text{if } j = -(n-i)+k \text{ or } j = n-k. \end{cases}$$

Hence we have

$$\sum_{\substack{i\geq 1,\,\text{odd}}} \sum_{\substack{-n+i\leq j\leq n \\ 0\leq k\leq n-i}} (-1)^{j+k}\phi_j\phi_{-j+i}\phi_k\phi_{-k+(n-i)}|0\rangle$$

$$= \sum_{\substack{i\geq 1,\,\text{odd}}}^{n} \sum_{0\leq k\leq n-i} (-1)^{k+1}\left(2\phi_{i+k}\phi_{-k+(n-i)} + 2\phi_k\phi_{-k+n}\right)|0\rangle$$

$$= 2\sum_{\substack{i\geq 1,\,\text{odd}}}^{n} \sum_{0\leq k\leq n-i} (-1)^{k+1}\left(\phi_{i+k}\phi_{-k+(n-i)} + \phi_k\phi_{-k+n}\right)|0\rangle$$

$$= 2\sum_{i=0}^{m-1} (-1)^i(n-2i)\phi_{n-i}\phi_i|0\rangle$$

$$= 2\sum_{i=0}^{m-1} (-1)^i(m-i)Q_{n-i,i}. \qquad\qquad\blacksquare$$

3 Reduced Fock representation of the Virasoro algebra

For a positive odd integer j, put $a_j = \sqrt{2}\partial_j$ and $a_{-j} = \frac{j}{\sqrt{2}}t_j$ so that they satisfy the Heisenberg relation as operators on V:

$$[a_j, a_i] = j\delta_{j+i,0}.$$

For an integer k, put

$$L_k = \frac{1}{2}\sum_{j\in\mathbb{Z}_{\text{odd}}} :a_{-j}a_{j+2k}: + \frac{1}{8}\delta_{k,0},$$

where

$$:a_j a_i: = \begin{cases} a_j a_i & \text{if } j \leq i, \\ a_i a_j & \text{if } j > i \end{cases}$$

is the normal ordering. For example,

$$L_2 = 2\partial_1\partial_3 + \sum_{j\geq 1,\,\text{odd}} jt_j\partial_{j+4}, \qquad L_1 = \partial_1^2 + \sum_{j\geq 1,\,\text{odd}} jt_j\partial_{j+2},$$

$$L_0 = \sum_{j\geq 1} jt_j\partial_j + \frac{1}{8}\,\text{id},$$

$$L_{-1} = \frac{1}{4}t_1^2 + \sum_{j\geq 3,\,\text{odd}} jt_j\partial_{j-2}, \qquad\text{and}\qquad L_{-2} = \frac{3}{2}t_1t_3 + \sum_{j\geq 5,\,\text{odd}} jt_j\partial_{j-4}.$$

More generally, it is verified, by Proposition 2.2, that

$$L_{-k} = \frac{1}{4}\sum_{i=0}^{k-1} (2i+1)t_{2i+1}(2k-(2i+1))t_{2k-(2i+1)} + \sum_{i\geq 2k+1,\,\text{odd}} it_i\partial_{i-2k}$$

$$= \frac{1}{2} \sum_{j=0}^{k-1} (-1)^j (k-j) Q_{2k-j,j} + \sum_{j \geq 2k+1,\, \text{odd}} j t_j \partial_{j-2k}.$$

It is known that the operators L_k on V satisfy the Virasoro relation:

$$[L_k, L_\ell] = 2(k-\ell) L_{k+\ell} + \frac{k^3 - k}{3} \delta_{k+\ell,0}, \qquad k, \ell \in \mathbb{Z}.$$

A representation of the Virasoro algebra $\mathcal{L} = \oplus_{k \in \mathbb{Z}} \mathbb{C} \ell_k \oplus \mathbb{C} z$ with central charge 1 is given by $\ell_k \mapsto \frac{1}{2} L_k$, $z \mapsto 1$, which we recall the reduced Fock representation. We have $L_k \cdot v \in V(n-2k)$ for $v \in V(n)$. The inner product $\langle\,,\,\rangle$ defined in Section 2 is contravariant:

$$\langle L_k v, w \rangle = \langle v, L_{-k} w \rangle, \qquad v, w \in V.$$

Therefore the reduced Fock representation is infinitesimally unitary. The singular vectors are discussed in [6]. For the non-reduced Fock representation of the Virasoro algebra, see for example [5].

Proposition 3.1.

$$L_{-1} q_n = (n+1) q_{n+2} + \frac{1}{2} Q_{n2}, \qquad n \geq 0.$$

Proof. It is verified that

$$L_{-1} e^{\xi(t,u)} = \left(\frac{t_1^2}{4} + 3t_3 \partial_1 + 5t_5 \partial_3 + \cdots \right) e^{\xi(t,u)}$$

$$= \left(\frac{t_1^2}{4} + 3t_3 u + 5t_5 u^3 + \cdots \right) e^{\sum_{j \geq 1,\, \text{odd}} t_j u^j}$$

and

$$\frac{\partial}{\partial u} e^{\xi(t,u)} = \frac{\partial}{\partial u} \left(e^{\sum_{j \geq 1,\, \text{odd}} t_j u^j} \right)$$

$$= \left(t_1 + 3t_3 u^2 + 5t_5 u^4 + 7t_7 u^6 + \cdots \right) e^{\sum_{j \geq 1,\, \text{odd}} t_j u^j}.$$

By the relations $t_1 = q_1$ and $t_1^2 = 2q_2$, we have

$$u L_{-1} e^{\xi(t,u)} = \left(\frac{1}{2} u q_2 - q_1 \right) e^{\xi(t,u)} + \frac{\partial}{\partial u} e^{\xi(t,u)}.$$

Here

$$\left(\frac{1}{2} u q_2 - q_1 \right) \sum_{n=0}^{\infty} q_n u^n = \frac{1}{2} q_2 \sum_{n=0}^{\infty} q_n u^{n+1} - q_1 \sum_{n=0}^{\infty} q_n u^n$$

$$= u \left(\frac{1}{2} q_2 \sum_{n=0}^{\infty} q_n u^n - q_1 \sum_{n=0}^{\infty} q_{n+1} u^n \right) - q_1,$$

and

$$\frac{\partial}{\partial u} \sum_{n=0}^{\infty} q_n u^n = \sum_{n=0}^{\infty} n q_n u^{n-1} = u \sum_{n=0}^{\infty} (n+2) q_{n+2} u^n + q_1.$$

Therefore we have

$$L_{-1} q_n = \frac{1}{2} q_2 q_n - q_1 q_{n+1} + (n+2) q_{n+2}$$

$$= (n+1) q_{n+2} + \frac{1}{2} Q_{n2}. \qquad \blacksquare$$

Similarly, we have

Proposition 3.2.

$$L_{-2}q_n = (n+2)q_{n+4} + Q_{n4} - \frac{1}{2}Q_{n3,1}, \qquad n \geq 0.$$

Proof. It is verified that

$$L_{-2}e^{\xi(t,u)} = \left(\frac{3}{2}t_1t_3 + 5t_5\partial_1 + 7t_7\partial_3 + \cdots\right)e^{\xi(t,u)}$$

$$= \left(\frac{3}{2}t_1t_3 + 5t_5u + 7t_7u^3 + \cdots\right)e^{\sum_{j\geq 1,\,\mathrm{odd}}t_ju^j}$$

and

$$\frac{\partial}{\partial u}e^{\xi(t,u)} = \frac{\partial}{\partial u}\left(e^{\sum_{j\geq 1,\,\mathrm{odd}}t_ju^j}\right)$$

$$= \left(t_1 + 3t_3u^2 + 5t_5u^4 + 7t_7u^6 + \cdots\right)e^{\sum_{j\geq 1,\,\mathrm{odd}}t_ju^j}.$$

Therefore

$$u^3L_{-2}e^{\xi(t,u)} = \left(-t_1 - 3t_3u^2 + \frac{3}{2}t_1t_3u^3\right)e^{\xi(t,u)} + \frac{\partial}{\partial u}e^{\xi(t,u)}.$$

We have $t_3 = \frac{1}{3}(q_3 - Q_{2,1})$ and $t_1t_3 = \frac{1}{3}(2q_4 - Q_{3,1})$. Therefore

$$u^3L_{-2}e^{\xi(t,u)} = \left(-q_1 - u^2(q_3 - Q_{2,1}) + \frac{1}{2}u^3(2q_4 - Q_{3,1})\right)e^{\xi(t,u)} + \frac{\partial}{\partial u}e^{\xi(t,u)}.$$

Here the first term equals

$$-q_1\sum_{n=0}^{\infty}q_nu^n - (q_3 - Q_{2,1})\sum_{n=0}^{\infty}q_nu^{n+2} + \frac{1}{2}(2q_4 - Q_{3,1})\sum_{n=0}^{\infty}q_nu^{n+3}$$

$$= u^3\left(-q_1\sum_{n=0}^{\infty}q_{n+3}u^n - (q_3 - Q_{2,1})\sum_{n=0}^{\infty}q_{n+1}u^n + \frac{1}{2}(2q_4 - Q_{3,1})\sum_{n=0}^{\infty}q_nu^n\right)$$

$$+ u^3\left(-q_1\left(u^{-3} + q_1u^{-2} + q_2u^{-1}\right) - (q_3 - Q_{2,1})u^{-1}\right)$$

and the second term equals

$$\sum_{n=0}^{\infty}nq_nu^{n-1} = u^3\sum_{n=0}^{\infty}(n+4)q_{n+4}u^n + u^3\left(q_1u^{-3} + 2q_2u^{-2} + 3q_3u^{-1}\right).$$

Also it is easy to check that

$$u^3\left(-q_1\left(u^{-3} + q_1u^{-2} + q_2u^{-1}\right) - (q_3 - Q_{2,1})u^{-1}\right) + u^3\left(q_1u^{-3} + 2q_2u^{-2} + 3q_3u^{-1}\right)$$

$$= u^2(2q_3 - q_1q_2 + Q_{2,1}) = 0.$$

Hence

$$L_{-2}q_n = -q_1q_{n+3} - (q_3 - Q_{2,1})q_{n+1} + \frac{1}{2}(2q_4 - Q_{3,1})q_n + (n+4)q_{n+4}$$

$$= -q_1q_{n+3} - (3q_3 - q_2q_1)q_{n+1} + \frac{1}{2}(4q_4 - q_3q_1)q_n + (n+4)q_{n+4}$$

$$= (n+2)q_{n+4} + Q_{n4} - \frac{1}{2}Q_{n3,1}. \qquad\blacksquare$$

From Proposition 2.2, we obtain the following formula.

$$L_{-k}(vw) = (L_{-k}v)w + v(L_{-k}w) - \frac{1}{4}\sum_{j=0}^{k-1}(2j+1)t_{2j+1}(2k-(2j+1))t_{2k-(2j+1)}vw$$

$$= (L_{-k}v)w + v(L_{-k}w) - \frac{1}{2}\sum_{j=0}^{k-1}(-1)^j(k-j)Q_{2k-j,j}vw$$

for $v, w \in V$. In particular $k = 1, 2$, we see

$$L_{-1}(vw) = (L_{-1}v)w + v(L_{-1}w) - \frac{1}{2}q_2vw, \tag{3.1}$$

$$L_{-2}(vw) = (L_{-2}v)w + v(L_{-2}w) - \frac{1}{2}(2q_4 - Q_{3,1})vw. \tag{3.2}$$

Proposition 3.3. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a positive integer sequence. Then*

$$(1) \quad L_{-1}Q_\alpha = \sum_{i=1}^{\ell}(\alpha_i + 1)Q_{\alpha+2\epsilon_i} + \frac{1}{2}Q_{\alpha,2},$$

$$(2) \quad L_{-2}Q_\alpha = \sum_{i=1}^{\ell}(\alpha_i + 2)Q_{\alpha+4\epsilon_i} + Q_{\alpha,4} - \frac{1}{2}Q_{\alpha,3,1}.$$

Proof. If $\alpha_i = \alpha_j$ for some $i \neq j$, then the equations hold as $0 = 0$. Therefore, taking the sign (±1) into account, it suffices to prove the equations for the case $\alpha = \lambda$ is a strict partition. Use induction on the length of λ. First we see (1) for the case $\ell(\lambda)$ is odd. By equation (3.1),

$$L_{-1}\left(\sum_{i=1}^{\ell}(-1)^{i+1}q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right)$$

$$= \sum_{i=1}^{\ell}(-1)^{i+1}\left((L_{-1}q_{\lambda_i})Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell} + q_{\lambda_i}(L_{-1}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}) - \frac{1}{2}q_2q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right). \tag{3.3}$$

By induction hypothesis, and the first term and second term in the right hand side equal, respectively,

$$\sum_{i=1}^{\ell}(-1)^{i+1}\left((\lambda_i + 1)q_{\lambda_i+2} + \frac{1}{2}Q_{\lambda_i2}\right)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}, \qquad \text{and}$$

$$\sum_{i=1}^{\ell}(-1)^{i+1}q_{\lambda_i}\left(\sum_{j=1, j\neq i}^{\ell}(\lambda_j + 1)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell+2\epsilon_j} + \frac{1}{2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell2}\right).$$

Hence the equation (3.3) reads

$$\sum_{i=1}^{\ell}(-1)^{i+1}\left((\lambda_i + 1)q_{\lambda_i+2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell} + q_{\lambda_i}\sum_{j=1, j\neq i}^{\ell}(\lambda_j + 1)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell+2\epsilon_j}\right)$$

$$+ \frac{1}{2}\sum_{i=1}^{\ell}(-1)^{i+1}\left(Q_{\lambda_i2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell} + q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell2} - q_2q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right)$$

$$= \sum_{i=1}^{\ell}(\lambda_i + 1)Q_{\lambda+2\epsilon_i} + \frac{1}{2}\sum_{i=1}^{\ell}(-1)^{i+1}\left(Q_{\lambda_i2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell} + q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell2}\right) - \frac{1}{2}q_2Q_\lambda.$$

By Lemma 2.1(1), the result follows. The case of even $\ell(\lambda)$ is similar. Next we prove (2) in Proposition 3.3. Let $\ell(\lambda)$ be odd. By equation (3.2),

$$L_{-2}\left(\sum_{i=1}^{\ell}(-1)^{i+1}q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right) \tag{3.4}$$

$$=\sum_{i=1}^{\ell}(-1)^{i+1}\left((L_{-2}q_{\lambda_i})Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}+q_{\lambda_i}(L_{-2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell})-\frac{1}{2}(2q_4-Q_{3,1})q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right).$$

By the induction hypothesis, the first and second terms in the right hand side are, respectively,

$$\sum_{i=1}^{\ell}(-1)^{i+1}\left((\lambda_i+2)q_{\lambda_i+4}+Q_{\lambda_i4}-\frac{1}{2}Q_{\lambda_i3,1}\right)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}, \qquad \text{and}$$

$$\sum_{i=1}^{\ell}(-1)^{i+1}q_{\lambda_i}\left(\sum_{j=1,j\neq i}^{\ell}(\lambda_j+2)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell+4\epsilon_j}+Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell4}-\frac{1}{2}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell3,1}\right).$$

Hence the equation (3.4) reads

$$\sum_{i=1}^{\ell}(-1)^{i+1}\left((\lambda_i+2)q_{\lambda_i+4}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}+q_{\lambda_i}\sum_{j=1,j\neq i}^{\ell}(\lambda_j+2)Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell+4\epsilon_j}\right)$$

$$+\sum_{i=1}^{\ell}(-1)^{i+1}\left(Q_{\lambda_i4}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}+q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell4}-q_4q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right)$$

$$+\frac{1}{2}\sum_{i=1}^{\ell}(-1)^{i+1}\left(-Q_{\lambda_i3,1}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}-q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell3,1}+Q_{3,1}q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}\right).$$

Hence

$$\sum_{i=1}^{\ell}(\lambda_i+2)Q_{\lambda+4\epsilon_i}+\sum_{i=1}^{\ell}(-1)^{i+1}\left(Q_{\lambda_i4}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}+q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell4}\right)-q_4Q_\lambda$$

$$+\frac{1}{2}\sum_{i=1}^{\ell}(-1)^{i+1}\left(-Q_{\lambda_i3,1}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell}-q_{\lambda_i}Q_{\lambda_1\ldots\widehat{\lambda_i}\ldots\lambda_\ell3,1}\right)+\frac{1}{2}Q_{3,1}Q_\lambda.$$

The result follows immediately from Lemma 2.1(1) and (3). The case of even $\ell(\lambda)$ is similar. ∎

Theorem 3.4. *Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ be a positive integer sequence. Then*

$$L_{-k}Q_\alpha=\sum_{i=1}^{\ell}(\alpha_i+k)Q_{\alpha+2k\epsilon_i}+\frac{1}{2}\sum_{i=0}^{k-1}(-1)^i(k-i)Q_{\alpha,2k-i,i}, \qquad k\geq 1.$$

Proof. Use induction on k. The cases $k=1,2$ are already shown in Proposition 3.3. Thanks to the Virasoro relations, it suffices to show

$$[L_{-k},L_{-1}]Q_\alpha$$

$$=2(-k+1)\left(\sum_{i=1}^{\ell}(\alpha_i+(k+1))Q_{\alpha+2(k+1)\epsilon_i}+\frac{1}{2}\sum_{i=0}^{k}(-1)^i((k+1)-i)Q_{\alpha,2(k+1)-i,i}\right).$$

Since

$$L_{-k}L_{-1}Q_\alpha = \sum_{i=1}^{\ell}(\alpha_i+1)L_{-k}Q_{\alpha+2\epsilon_i} + \frac{1}{2}L_{-k}Q_{\alpha 2}, \qquad \text{and}$$

$$L_{-1}L_{-k}Q_\alpha = \sum_{i=1}^{\ell}(\alpha_i+k)L_{-1}Q_{\alpha+2k\epsilon_i} + \frac{1}{2}\sum_{i=0}^{k-1}(-1)^i(k-i)L_{-1}Q_{\alpha,2k-i,i},$$

we have

$$[L_{-k}, L_{-1}]Q_\alpha = \sum_{i=1}^{\ell}\left((\alpha_i+1)L_{-k}Q_{\alpha+2\epsilon_i} - (\alpha_i+k)L_{-1}Q_{\alpha+2k\epsilon_i}\right)$$

$$+ \frac{1}{2}\left(L_{-k}Q_{\alpha 2} - \sum_{i=0}^{k-1}(-1)^i(k-i)L_{-1}Q_{\alpha,2k-i,i}\right).$$

We write down terms in the right hand side:

$$\sum_{i=1}^{\ell}(\alpha_i+1)L_{-k}Q_{\alpha+2\epsilon_i} = \sum_{i,j=1,i\neq j}^{\ell}(\alpha_i+1)(\alpha_j+k)Q_{\alpha+2\epsilon_i+2k\epsilon_j}$$

$$+ \sum_{i=1}^{\ell}(\alpha_i+1)(\alpha_i+k+2)Q_{\alpha+2(k+1)\epsilon_i} + \sum_{i=1}^{\ell}(\alpha_i+1)\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j(k-j)Q_{\alpha+2\epsilon_i,2k-j,j},$$

$$\sum_{i=1}^{\ell}(\alpha_i+k)L_{-1}Q_{\alpha+2k\epsilon_i}$$

$$= \sum_{i=1}^{\ell}(\alpha_i+k)\left(\sum_{j=1,j\neq i}^{\ell}(\alpha_j+1)Q_{\alpha+2k\epsilon_i+2\epsilon_j} + (\alpha_i+2k+1)Q_{\alpha+2(k+1)\epsilon_i} + \frac{1}{2}Q_{\alpha+2k\epsilon_i,2}\right),$$

$$L_{-k}Q_{\alpha,2} = \sum_{i=1}^{\ell}(\alpha_i+k)Q_{\alpha+2k\epsilon_i,2} + (2+k)Q_{\alpha,2+2k} + \frac{1}{2}\sum_{i=0}^{k-1}(-1)^i(k-i)Q_{\alpha,2,2k-i,i},$$

$$\sum_{i=0}^{k-1}(-1)^i(k-i)L_{-1}Q_{\alpha,2k-i,i} = \sum_{i=0}^{k-1}(-1)^i(k-i)\sum_{j=1}^{\ell}(\alpha_j+1)Q_{\alpha+2\epsilon_j,2k-i,i}$$

$$+ \sum_{i=0}^{k-1}(-1)^i(k-i)(2k-i+1)Q_{\alpha,2k-i+2,i} + \sum_{i=0}^{k-1}(-1)^i(k-i)(i+1)Q_{\alpha,2k-i,i+2}$$

$$- \frac{1}{2}kQ_{\alpha,2k,2} + \frac{1}{2}\sum_{i=1}^{k-1}(-1)^i(k-i)Q_{\alpha,2k-i,i,2}.$$

Summing up, we have

$$\left(\sum_{i=1}^{\ell}(\alpha_i+1)(\alpha_i+k+2) - \sum_{i=1}^{\ell}(\alpha_i+k)(\alpha_i+2k+1)\right)Q_{\alpha+2(k+1)\epsilon_i}$$

$$+ \frac{1}{2}\left((2+k)Q_{\alpha,2+2k} - \sum_{i=0}^{k-1}(-1)^i(k-i)(2k-i+1)Q_{\alpha,2k-i+2,i}\right.$$

$$+ \frac{1}{2}\sum_{i=0}^{k-1}(-1)^i(k-i)Q_{\alpha,2,2k-i,i} + \frac{1}{2}kQ_{\alpha,2k,2} - \frac{1}{2}\sum_{i=1}^{k-1}(-1)^i(k-i)Q_{\alpha,2k-i,i,2}$$

$$-\sum_{i=0}^{k-1}(-1)^i(k-i)(i+1)Q_{\alpha,2k-i,i+2}\Bigg)$$

$$= 2(-k+1)\sum_{i=1}^{\ell}(\alpha_i+(k+1))Q_{\alpha+2(k+1)\epsilon_i}+(-k+1)(k+1)Q_{\alpha,2+2k}$$

$$-\frac{1}{2}\Bigg(\sum_{i=1}^{k-1}(-1)^i(k-i)(2k-i+1)Q_{\alpha,2k-i+2,i}+kQ_{\alpha,2k,2}$$

$$+\sum_{i=1}^{k-1}(-1)^i(k-i)(i+1)Q_{\alpha,2k-i,i+2}\Bigg)$$

$$= 2(-k+1)\sum_{i=1}^{\ell}(\alpha_i+(k+1))Q_{\alpha+2(k+1)\epsilon_i}+(-k+1)(k+1)Q_{\alpha,2+2k}$$

$$+(-k+1)\sum_{i=1}^{k}(-1)^i((k+1)-i)Q_{\alpha,2(k+1)-i,i}. \qquad\blacksquare$$

Together with the previously proved formula

$$L_kQ_\lambda = \sum_{i=1}^{2m}(\lambda_i-k)Q_{\lambda-2k\epsilon_i}$$

for $\lambda = (\lambda_1,\ldots,\lambda_{2m})$, $k \geq 1$ [1, Theorem 2], Theorem 3.4 completely describes the reduced Fock representation of the Virasoro algebra. Consider the Lie subalgebra $\mathfrak{g} = \sum_{|k|\leq 1}\mathbb{C}l_k$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Let $\mathcal{ESP}$ be the set of the strict partitions whose parts are all even numbers, and let V^{even} be the subspace of V spanned by the Q_λ for $\lambda \in \mathcal{ESP}$.

Corollary 3.5. *The space V^{even} is invariant under the action of $\mathfrak{g}$.*

We are interested in the space V^{even} because of the following conjecture: the set of Hirota bilinear equations

$$Q_\lambda(\widetilde{D})\tau\cdot\tau = 0, \qquad \lambda \in \mathcal{SP}\backslash\mathcal{ESP}$$

coincides with those of the KdV hierarchy, where $\widetilde{D} = \left(D_1,\frac{1}{3}D_3,\frac{1}{5}D_5,\ldots\right)$ is the Hirota differential operator. Namely $\left(V^{\mathrm{even}}\right)^\perp$ is conjecturally the space of Hirota equations for the KdV hierarchy. For example,

$$Q_{3,1}(\widetilde{D}) = \frac{1}{12}D_1^4 - \frac{1}{3}D_1D_3$$

corresponds to the original KdV equation.

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