The Clebsch–Gordan Rule for $U(\mathfrak{sl}_2)$, the Krawtchouk Algebras and the Hamming Graphs

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Abstract. Let $D \geq 1$ and $q \geq 3$ be two integers. Let $H(D) = H(D,q)$ denote the D-dimensional Hamming graph over a q-element set. Let $\mathcal{T}(D)$ denote the Terwilliger algebra of $H(D)$. Let $V(D)$ denote the standard $\mathcal{T}(D)$-module. Let ω denote a complex scalar. We consider a unital associative algebra $\mathfrak{K}_\omega$ defined by generators and relations. The generators are A and B. The relations are $A^2B - 2ABA + BA^2 = B + \omega A$, $B^2A - 2BAB + AB^2 = A + \omega B$. The algebra $\mathfrak{K}_\omega$ is the case of the Askey–Wilson algebras corresponding to the Krawtchouk polynomials. The algebra $\mathfrak{K}_\omega$ is isomorphic to $\mathrm{U}(\mathfrak{sl}_2)$ when $\omega^2 \neq 1$. We view $V(D)$ as a $\mathfrak{K}_{1-\frac{2}{q}}$-module. We apply the Clebsch–Gordan rule for $\mathrm{U}(\mathfrak{sl}_2)$ to decompose $V(D)$ into a direct sum of irreducible $\mathcal{T}(D)$-modules.

Key words: Clebsch–Gordan rule; Hamming graph; Krawtchouk algebra; Terwilliger algebra

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1 Introduction

Throughout this paper, we adopt the following conventions: Fix an integer $q \geq 3$. Let $\mathbb{C}$ denote the complex number field. An algebra is meant to be a unital associative algebra. An algebra homomorphism is meant to be a unital algebra homomorphism. A subalgebra has the same unit as the parent algebra. In an algebra the commutator $[x,y]$ of two elements x and y is defined as $[x,y] = xy - yx$. Note that every algebra has a Lie algebra structure with Lie bracket given by the commutator.

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is a three-dimensional Lie algebra over $\mathbb{C}$ with a basis e, f, h satisfying

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h.$$

Definition 1.1. The *universal enveloping algebra* $\mathrm{U}(\mathfrak{sl}_2)$ *of* $\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ generated by E, F, H subject to the relations

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H.$$

Using Definition 1.1, it is straightforward to verify the following lemma:

Lemma 1.2. *Given any integer* $n \geq 0$ *there exists an* $(n+1)$*-dimensional* $\mathrm{U}(\mathfrak{sl}_2)$*-module* L_n *that has a basis* $\{v_i\}_{i=0}^n$ *such that*

$$\begin{aligned} &Ev_i = (n-i+1)v_{i-1} && \text{for } i = 1,2,\dots,n, && Ev_0 = 0,\\ &Fv_i = (i+1)v_{i+1} && \text{for } i = 0,1,\dots,n-1, && Fv_n = 0,\\ &Hv_i = (n-2i)v_i && \text{for } i = 0,1,\dots,n. && \end{aligned}$$

Note that the $\mathrm{U}(\mathfrak{sl}_2)$-module L_n is irreducible for any integer $n \geq 0$. Furthermore, the finite-dimensional irreducible $\mathrm{U}(\mathfrak{sl}_2)$-modules are classified as follows:

Lemma 1.3. *For any integer $n \geq 0$, each $(n+1)$-dimensional irreducible* $\mathrm{U}(\mathfrak{sl}_2)$*-module is isomorphic to* L_n.

Proof. See [10, Section V.4] for example. ■

It is well known that the universal enveloping algebra of a Lie algebra is a Hopf algebra. For example, see [12, Section 5].

Lemma 1.4. *The algebra* $\mathrm{U}(\mathfrak{sl}_2)$ *is a Hopf algebra on which the counit* $\varepsilon\colon \mathrm{U}(\mathfrak{sl}_2) \to \mathbb{C}$*, the antipode* $S\colon \mathrm{U}(\mathfrak{sl}_2) \to \mathrm{U}(\mathfrak{sl}_2)$ *and the comultiplication* $\Delta\colon \mathrm{U}(\mathfrak{sl}_2) \to \mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{U}(\mathfrak{sl}_2)$ *are given by*

$$\begin{aligned} &\varepsilon(E) = 0, && \varepsilon(F) = 0, && \varepsilon(H) = 0, \\ &S(E) = -E, && S(F) = -F, && S(H) = -H, \\ &\Delta(E) = E \otimes 1 + 1 \otimes E, && \Delta(F) = F \otimes 1 + 1 \otimes F, && \Delta(H) = H \otimes 1 + 1 \otimes H. \end{aligned}$$

Every $\mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{U}(\mathfrak{sl}_2)$-module can be viewed as a $\mathrm{U}(\mathfrak{sl}_2)$-module via the comultiplication of $\mathrm{U}(\mathfrak{sl}_2)$. The Clebsch–Gordan rule for $\mathrm{U}(\mathfrak{sl}_2)$ is as follows:

Theorem 1.5. *For any integers $m, n \geq 0$, the* $\mathrm{U}(\mathfrak{sl}_2)$*-module* $L_m \otimes L_n$ *is isomorphic to*

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.$$

Proof. See [10, Section V.5] for example. ■

For the rest of this paper, let ω denote a scalar taken from $\mathbb{C}$.

Definition 1.6. The *Krawtchouk algebra* $\mathfrak{K}_\omega$ is an algebra over $\mathbb{C}$ generated by A and B subject to the relations

$$A^2B - 2ABA + BA^2 = B + \omega A, \tag{1.1}$$
$$B^2A - 2BAB + AB^2 = A + \omega B. \tag{1.2}$$

The algebra $\mathfrak{K}_\omega$ is the case of the Askey–Wilson algebra corresponding to the Krawtchouk polynomials [22, Lemma 7.2]. Define C to be the following element of $\mathfrak{K}_\omega$:

$$C = [A, B].$$

Lemma 1.7. *The algebra $\mathfrak{K}_\omega$ has a presentation with the generators A, B, C and the relations*

$$[A, B] = C, \tag{1.3}$$
$$[A, C] = B + \omega A, \tag{1.4}$$
$$[C, B] = A + \omega B. \tag{1.5}$$

Proof. The relation (1.3) is immediate from the setting of C. Using (1.3), the relations (1.1) and (1.2) can be written as (1.4) and (1.5), respectively. The lemma follows. ■

Let $\mathcal{K}_\omega$ denote a three-dimensional Lie algebra over $\mathbb{C}$ with a basis a, b, c satisfying

$$[a, b] = c, \qquad [a, c] = b + \omega a, \qquad [c, b] = a + \omega b.$$

By Lemma 1.7, the algebra $\mathfrak{K}_\omega$ is the universal enveloping algebra of $\mathcal{K}_\omega$. There is a connection between $\mathfrak{K}_\omega$ and $\mathrm{U}(\mathfrak{sl}_2)$:

Theorem 1.8. *There exists a unique algebra homomorphism* $\zeta\colon \mathfrak{K}_\omega \to \mathrm{U}(\mathfrak{sl}_2)$ *that sends*

$$A \mapsto \frac{1+\omega}{2}E + \frac{1-\omega}{2}F - \frac{\omega}{2}H, \qquad B \mapsto \frac{1}{2}H, \qquad C \mapsto -\frac{1+\omega}{2}E + \frac{1-\omega}{2}F.$$

Moreover, if $\omega^2 \neq 1$*, then* ζ *is an isomorphism and its inverse sends*

$$E \mapsto \frac{1}{1+\omega}A + \frac{\omega}{1+\omega}B - \frac{1}{1+\omega}C, \qquad F \mapsto \frac{1}{1-\omega}A + \frac{\omega}{1-\omega}B + \frac{1}{1-\omega}C, \qquad H \mapsto 2B.$$

Proof. It is routine to verify the result by using Definition 1.1 and Lemma 1.7. Here we provide another proof by applying [13, Lemmas 2.12 and 2.13].

Let $\sigma\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{U}(\mathfrak{sl}_2)$ denote the canonical Lie algebra homomorphism that sends e, f, h to E, F, H, respectively. Let $\tau\colon \mathcal{K}_\omega \to \mathfrak{K}_\omega$ denote the canonical Lie algebra homomorphism that sends a, b, c to A, B, C, respectively. By [13, Lemma 2.12], there exists a unique Lie algebra homomorphism $\phi\colon \mathcal{K}_\omega \to \mathfrak{sl}_2(\mathbb{C})$ that sends

$$a \mapsto \frac{1+\omega}{2}e + \frac{1-\omega}{2}f - \frac{\omega}{2}h, \qquad b \mapsto \frac{1}{2}h, \qquad c \mapsto -\frac{1+\omega}{2}e + \frac{1-\omega}{2}f.$$

Applying the universal property of $\mathfrak{K}_\omega$ to the Lie algebra homomorphism $\sigma \circ \phi$, this gives the algebra homomorphism ζ. Suppose that $\omega^2 \neq 1$. Then $\phi\colon \mathcal{K}_\omega \to \mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra isomorphism by [13, Lemma 2.13]. Applying the universal property of $\mathrm{U}(\mathfrak{sl}_2)$ to the Lie algebra homomorphism $\tau \circ \phi^{-1}$, this gives the inverse of ζ. ■

In this paper, we relate the above algebraic results to the Hamming graphs. We now recall the definition of Hamming graphs. Let X denote a q-element set and let D be a positive integer. The *D-dimensional Hamming graph $H(D) = H(D,q)$ over X* is a simple graph whose vertex set is X^D and $x, y \in X^D$ are adjacent if and only if x, y differ in exactly one coordinate. Let ∂ denote the path-length distance function for $H(D)$. Let $\mathrm{Mat}_{X^D}(\mathbb{C})$ stand for the algebra consisting of the square matrices over $\mathbb{C}$ indexed by X^D.

The adjacency matrix $\mathbf{A}(D) \in \mathrm{Mat}_{X^D}(\mathbb{C})$ of $H(D)$ is the 0-1 matrix such that

$$\mathbf{A}(D)_{xy} = 1 \qquad \text{if and only if} \qquad \partial(x,y) = 1$$

for all $x, y \in X^D$. Fix a vertex $x \in X^D$. The dual adjacency matrix $\mathbf{A}^*(D) \in \mathrm{Mat}_{X^D}(\mathbb{C})$ of $H(D)$ with respect to x is a diagonal matrix given by

$$\mathbf{A}^*(D)_{yy} = D(q-1) - q \cdot \partial(x,y)$$

for all $y \in X^D$. The Terwilliger algebra $\mathcal{T}(D)$ of $H(D)$ with respect to x is the subalgebra of $\mathrm{Mat}_{X^D}(\mathbb{C})$ generated by $\mathbf{A}(D)$ and $\mathbf{A}^*(D)$ [16, 17, 18]. Let $V(D)$ denote the vector space consisting of all column vectors over $\mathbb{C}$ indexed by X^D. The vector space $V(D)$ has a natural $\mathcal{T}(D)$-module structure and it is called the standard $\mathcal{T}(D)$-module.

In [18], Terwilliger employed the endpoints, dual endpoints, diameters and auxiliary parameters to describe the irreducible modules for the known families of thin Q-polynomial distance-regular graphs with unbounded diameter. In [14], Tanabe gave a recursive construction of irreducible modules for the Doob graphs and his method can be adjusted to the case of $H(D)$. In [5], Go gave a decomposition of the standard module for the hypercube. In [4], Gijswijt, Schrijver and Tanaka described a decomposition of $V(D)$ in terms of the block-diagonalization of $\mathcal{T}(D)$. In [11], Levstein, Maldonado and Penazzi applied the representation theory of $\mathrm{GL}_2(\mathbb{C})$ to determine the structure of $\mathcal{T}(D)$. In [20], it was shown that $V(D)$ can be viewed as a $\mathfrak{gl}_2(\mathbb{C})$-module as well as a $\mathfrak{sl}_2(\mathbb{C})$-module. In [2], Bernard, Crampé, and Vinet found a decomposition of $V(D)$ by generalizing the result on the hypercube.

In this paper, we view $V(D)$ as a $\mathfrak{K}_{1-\frac{2}{q}}$-module as well as a $\mathrm{U}(\mathfrak{sl}_2)$-module in light of Theorem 1.8. Subsequently, we apply Theorem 1.5 to prove the following results:

Proposition 1.9. *Let D be a positive integer. For any integers p and k with $0 \leq p \leq D$ and $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$, there exists a $(p-2k+1)$-dimensional irreducible $\mathcal{T}(D)$-module $L_{p,k}(D)$ satisfying the following conditions:*

(i) *There exists a basis for $L_{p,k}(D)$ with respect to which the matrices representing $\mathbf{A}(D)$ and $\mathbf{A}^*(D)$ are*

$$\begin{pmatrix} \alpha_0 & \gamma_1 & & & \mathbf{0} \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_{p-2k} \\ \mathbf{0} & & & \beta_{p-2k-1} & \alpha_{p-2k} \end{pmatrix}, \qquad \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_{p-2k} \end{pmatrix},$$

respectively.

(ii) *There exists a basis for $L_{p,k}(D)$ with respect to which the matrices representing $\mathbf{A}(D)$ and $\mathbf{A}^*(D)$ are*

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_{p-2k} \end{pmatrix}, \qquad \begin{pmatrix} \alpha_0 & \gamma_1 & & & \mathbf{0} \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_{p-2k} \\ \mathbf{0} & & & \beta_{p-2k-1} & \alpha_{p-2k} \end{pmatrix},$$

respectively.

Here the parameters $\{\alpha_i\}_{i=0}^{p-2k}$, $\{\beta_i\}_{i=0}^{p-2k-1}$, $\{\gamma_i\}_{i=1}^{p-2k}$, $\{\theta_i\}_{i=0}^{p-2k}$ are as follows:

$$\begin{aligned} \alpha_i &= (q-2)(i+k)+p-D && \text{for } i=0,1,\ldots,p-2k, \\ \beta_i &= i+1 && \text{for } i=0,1,\ldots,p-2k-1, \\ \gamma_i &= (q-1)(p-i-2k+1) && \text{for } i=1,2,\ldots,p-2k, \\ \theta_i &= q(p-i-k)-D && \text{for } i=0,1,\ldots,p-2k. \end{aligned}$$

Given a vector space W and a positive integer p, we let

$$p \cdot W = \underbrace{W \oplus W \oplus \cdots \oplus W}_{p \text{ copies of } W}.$$

Theorem 1.10. *Let D be a positive integer. Then the standard $\mathcal{T}(D)$-module $V(D)$ is isomorphic to*

$$\bigoplus_{p=0}^{D} \bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{D}{p} \binom{p}{k} (q-2)^{D-p} \cdot L_{p,k}(D).$$

The algebra $\mathcal{T}(D)$ is a finite-dimensional semisimple algebra. Following from [3, Theorem 25.10], Theorem 1.10 implies the following classification of irreducible $\mathcal{T}(D)$-modules:

Theorem 1.11. *Let D be a positive integer. Let $\mathbf{P}(D)$ denote the set consisting of all pairs (p,k) of integers with $0 \leq p \leq D$ and $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$. Let $\mathbf{M}(D)$ denote the set of all isomorphism classes of irreducible $\mathcal{T}(D)$-modules. Then there exists a bijection $\mathcal{E}\colon \mathbf{P}(D) \to \mathbf{M}(D)$ given by*

$$(p,k) \mapsto \text{the isomorphism class of } L_{p,k}(D)$$

for all $(p,k) \in \mathbf{P}(D)$.

The paper is organized as follows: In Section 2, we give the preliminaries on the algebra $\mathfrak{K}_\omega$. In Section 3, we prove Proposition 1.9 and Theorems 1.10, 1.11 by using Theorem 1.5. In Appendix A, we give the equivalent statements of Proposition 1.9 and Theorems 1.10, 1.11.

2 The Krawtchouk algebra

2.1 Finite-dimensional irreducible $\mathfrak{K}_\omega$-modules

Recall the $\mathrm{U}(\mathfrak{sl}_2)$-module L_n from Lemma 1.2. Recall the algebra homomorphism $\zeta\colon \mathfrak{K}_\omega \to \mathrm{U}(\mathfrak{sl}_2)$ form Theorem 1.8. Each $\mathrm{U}(\mathfrak{sl}_2)$-module can be viewed as a $\mathfrak{K}_\omega$-module by pulling back via ζ. We express the $\mathrm{U}(\mathfrak{sl}_2)$-module L_n as a $\mathfrak{K}_\omega$-module as follows:

Lemma 2.1. *For any integer $n \geq 0$, the matrices representing A, B, C with respect to the basis $\{v_i\}_{i=0}^n$ for the $\mathfrak{K}_\omega$-module L_n are*

$$\begin{pmatrix} \alpha_0 & \gamma_1 & & & \mathbf{0} \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_n \\ \mathbf{0} & & & \beta_{n-1} & \alpha_n \end{pmatrix}, \quad \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_n \end{pmatrix}, \quad \begin{pmatrix} 0 & -\gamma_1 & & & \mathbf{0} \\ \beta_0 & 0 & -\gamma_2 & & \\ & \beta_1 & 0 & \ddots & \\ & & \ddots & \ddots & -\gamma_n \\ \mathbf{0} & & & \beta_{n-1} & 0 \end{pmatrix}$$

respectively, where

$$\begin{aligned} \alpha_i &= \frac{(2i-n)\omega}{2} && \text{for } i = 0, 1, \ldots, n, \\ \beta_i &= \frac{(i+1)(1-\omega)}{2} && \text{for } i = 0, 1, \ldots, n-1, \\ \gamma_i &= \frac{(n-i+1)(1+\omega)}{2} && \text{for } i = 1, 2, \ldots, n, \\ \theta_i &= \frac{n}{2} - i && \text{for } i = 0, 1, \ldots, n. \end{aligned}$$

The finite-dimensional irreducible $\mathfrak{K}_\omega$-modules are classified as follows:

Theorem 2.2.

(i) *If $\omega = -1$, then any finite-dimensional irreducible $\mathfrak{K}_\omega$-module V is of dimension one and there is a scalar $\mu \in \mathbb{C}$ such that $Av = \mu v$, $Bv = \mu v$ for all $v \in V$.*

(ii) *If $\omega = 1$, then any finite-dimensional irreducible $\mathfrak{K}_\omega$-module V is of dimension one and there is a scalar $\mu \in \mathbb{C}$ such that $Av = \mu v$, $Bv = -\mu v$ for all $v \in V$.*

(iii) *If $\omega^2 \neq 1$, then L_n is the unique $(n+1)$-dimensional irreducible $\mathfrak{K}_\omega$-module up to isomorphism for every integer $n \geq 0$.*

Proof. (i) Let $n \geq 0$ be an integer. Let V denote an $(n+1)$-dimensional irreducible $\mathfrak{K}_{-1}$-module. Since the trace of the left-hand side of (1.1) on V is zero, the elements A and B have the same trace on V. If $n = 0$ then there exists a scalar $\mu \in \mathbb{C}$ such that $Av = Bv = \mu v$ for all $v \in V$.

To see Theorem 2.2(i), it remains to assume that $n \geq 1$ and we seek a contradiction. Applying the method proposed in [6, 7, 8], there exists a basis $\{u_i\}_{i=0}^n$ for V with respect to which the matrices representing A and B are of the forms

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_n \end{pmatrix}, \quad \begin{pmatrix} \theta_0 & \varphi_1 & & & \mathbf{0} \\ & \theta_1 & \varphi_2 & & \\ & & \theta_2 & \ddots & \\ & & & \ddots & \varphi_n \\ \mathbf{0} & & & & \theta_n \end{pmatrix},$$

respectively. Here $\{\theta_i\}_{i=0}^n$ is an arithmetic sequence with common difference -1 and the sequence $\{\varphi_i\}_{i=1}^n$ satisfies $\varphi_{i-1} - 2\varphi_i + \varphi_{i+1} = 0$, $1 \leq i \leq n$, where φ_0 and φ_{n+1} are interpreted as zero. Solving the above recurrence yields that $\varphi_i = 0$ for all $i = 1, 2, \ldots, n$. Thus the subspace of V spanned by $\{u_i\}_{i=1}^n$ is a nonzero $\mathfrak{K}_{-1}$-module, which is a contradiction to the irreducibility of V.

(ii) Using Definition 1.6, it is routine to verify that there exists a unique algebra isomorphism $\mathfrak{K}_{-1} \to \mathfrak{K}_1$ that sends A to A and B to $-B$. Theorem 2.2(ii) follows from Theorem 2.2(i) and the above isomorphism.

(iii) Theorem 2.2(iii) follows immediate from Lemma 1.3 and Theorem 1.8. ■

Lemma 2.3. *There exists a unique algebra automorphism of $\mathfrak{K}_\omega$ that sends $A \mapsto B$, $B \mapsto A$, $C \mapsto -C$.*

Proof. It is routine to verify the lemma by using Lemma 1.7. ■

Lemma 2.4. *Suppose that $\omega^2 \neq 1$. For any integer $n \geq 0$, there exists a basis for the $\mathfrak{K}_\omega$-module L_n with respect to which the matrices representing A, B, C are*

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_n \end{pmatrix}, \quad \begin{pmatrix} \alpha_0 & \gamma_1 & & & \mathbf{0} \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_n \\ \mathbf{0} & & & \beta_{n-1} & \alpha_n \end{pmatrix}, \quad \begin{pmatrix} 0 & \gamma_1 & & & \mathbf{0} \\ -\beta_0 & 0 & \gamma_2 & & \\ & -\beta_1 & 0 & \ddots & \\ & & \ddots & \ddots & \gamma_n \\ \mathbf{0} & & & -\beta_{n-1} & 0 \end{pmatrix}$$

respectively, where

$$\begin{aligned} \alpha_i &= \frac{(2i-n)\omega}{2} && \text{for } i = 0, 1, \ldots, n, \\ \beta_i &= \frac{(i+1)(1-\omega)}{2} && \text{for } i = 0, 1, \ldots, n-1, \\ \gamma_i &= \frac{(n-i+1)(1+\omega)}{2} && \text{for } i = 1, 2, \ldots, n, \\ \theta_i &= \frac{n}{2} - i && \text{for } i = 0, 1, \ldots, n. \end{aligned}$$

Proof. Let L_n' denote the irreducible $\mathfrak{K}_\omega$-module obtained by twisting the $\mathfrak{K}_\omega$-module L_n via the automorphism of $\mathfrak{K}_\omega$ given in Lemma 2.3. Recall the basis $\{v_i\}_{i=0}^n$ for L_n from Lemma 2.1. Observe that the three matrices described in Lemma 2.4 are the matrices representing A, B, C with respect to the basis $\{v_i\}_{i=0}^n$ for the $\mathfrak{K}_\omega$-module L_n'. By Theorem 2.2(iii), the $\mathfrak{K}_\omega$-module L_n' is isomorphic to L_n. The lemma follows. ■

Leonard pairs were introduced in [15, 19, 21] by P. Terwilliger. Suppose that $\omega^2 \neq 1$. By Lemmas 2.1 and 2.4, the elements A and B act on the $\mathfrak{K}_\omega$-module L_n as a Leonard pair. The result was first stated in [13, Theorem 6.3].

2.2 The Krawtchouk algebra as a Hopf algebra

Let $\mathcal{H}$ denote an algebra. Recall that $\mathcal{H}$ is called a *Hopf algebra* if there are two algebra homomorphisms $\varepsilon\colon \mathcal{H} \to \mathbb{C}$, $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and a linear map $S\colon \mathcal{H} \to \mathcal{H}$ that satisfy the following properties:

(H1) $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$,

(H2) $m \circ (1 \otimes (\iota \circ \varepsilon)) \circ \Delta = m \circ ((\iota \circ \varepsilon) \otimes 1) \circ \Delta = 1$,

(H3) $m \circ (1 \otimes S) \circ \Delta = m \circ (S \otimes 1) \circ \Delta = \iota \circ \varepsilon$.

Here $m\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ is the multiplication map and $\iota\colon \mathbb{C}\to\mathcal{H}$ is the unit map defined by $\iota(c)=c1$ for all $c\in\mathbb{C}$. Note that m is a linear map and ι is an algebra homomorphism.

Suppose that (H1)–(H3) hold. Then the maps ε, Δ, S are called the *counit*, *comultiplication* and *antipode* of $\mathcal{H}$, respectively. Let n be a positive integer. The n-fold comultiplication of $\mathcal{H}$ is the algebra homomorphism $\Delta_n\colon \mathcal{H}\to\mathcal{H}^{\otimes(n+1)}$ inductively defined by

$$\Delta_n = \left(1^{\otimes(n-1)}\otimes\Delta\right)\circ\Delta_{n-1}.$$

Here Δ_0 is interpreted as the identity map of $\mathcal{H}$. We may regard every $\mathcal{H}^{\otimes(n+1)}$-module as an $\mathcal{H}$-module by pulling back via Δ_n. Note that

$$\Delta_n = \left(1^{\otimes(n-i)}\otimes\Delta\otimes 1^{\otimes(i-1)}\right)\circ\Delta_{n-1} \qquad \text{for all } i=1,2,\dots,n. \tag{2.1}$$

It follows from (2.1) that

$$\Delta_n = (\Delta_{n-1}\otimes 1)\circ\Delta = (1\otimes\Delta_{n-1})\circ\Delta. \tag{2.2}$$

Recall from Section 1 that $\mathfrak{K}_\omega$ is the universal enveloping algebra of $\mathcal{K}_\omega$. Hence $\mathfrak{K}_\omega$ is a Hopf algebra. For the reader's convenience, we give a detailed verification for the Hopf algebra structure of $\mathfrak{K}_\omega$. By an algebra antihomomorphism, we mean a unital algebra antihomomorphism.

Lemma 2.5.

(i) There exists a unique algebra homomorphism $\varepsilon\colon \mathfrak{K}_\omega\to\mathbb{C}$ *given by*

$$\varepsilon(A)=0, \qquad \varepsilon(B)=0, \qquad \varepsilon(C)=0.$$

(ii) There exists a unique algebra homomorphism $\Delta\colon \mathfrak{K}_\omega\to\mathfrak{K}_\omega\otimes\mathfrak{K}_\omega$ *given by*

$$\Delta(A)=A\otimes 1+1\otimes A, \qquad \Delta(B)=B\otimes 1+1\otimes B, \qquad \Delta(C)=C\otimes 1+1\otimes C.$$

(iii) There exists a unique algebra antihomomorphism $S\colon \mathfrak{K}_\omega\to\mathfrak{K}_\omega$ *given by*

$$S(A)=-A, \qquad S(B)=-B, \qquad S(C)=-C.$$

(iv) The algebra $\mathfrak{K}_\omega$ *is a Hopf algebra on which the counit, comultiplication and antipode are the above maps* ε, Δ, S, *respectively.*

Proof. (i)–(iii) It is routine to verify Lemma 2.5(i)–(iii) by using Definition 1.6.

(iv) Using Lemma 2.5(ii), it yields that $(1\otimes\Delta)\circ\Delta$ and $(\Delta\otimes 1)\circ\Delta$ agree at the generators A, B, C of $\mathfrak{K}_\omega$. Since Δ is an algebra homomorphism, the maps $(1\otimes\Delta)\circ\Delta$ and $(\Delta\otimes 1)\circ\Delta$ are algebra homomorphisms. Hence **(H1)** holds for $\mathfrak{K}_\omega$.

Let $k=m\circ(1\otimes(\iota\circ\varepsilon))\circ\Delta$ and $k'=m\circ((\iota\circ\varepsilon)\otimes 1)\circ\Delta$. Evidently, k and k' are linear maps. Using Lemma 2.5(i), (ii) yields that

$$k(1)=k'(1)=1, \qquad k(A)=k'(A)=A, \qquad k(B)=k'(B)=B, \qquad k(C)=k'(C)=C.$$

Let x, y be any two elements of $\mathfrak{K}_\omega$. To see that $k=1$ it remains to check that $k(xy)=k(x)k(y)$. We can write

$$\Delta(x)=\sum_{i=1}^{n} x_i^{(1)}\otimes x_i^{(2)}, \tag{2.3}$$

$$\Delta(y)=\sum_{i=1}^{n} y_i^{(1)}\otimes y_i^{(2)}, \tag{2.4}$$

where $n \geq 1$ is an integer and $x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)} \in \mathfrak{K}_\omega$ for $1 \leq i \leq n$. Then

$$k(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} \cdot y_j^{(1)} \cdot (\iota \circ \varepsilon)\big(x_i^{(2)}\big) \cdot (\iota \circ \varepsilon)\big(y_j^{(2)}\big).$$

Since each of $(\iota \circ \varepsilon)\big(x_i^{(2)}\big)$ and $(\iota \circ \varepsilon)\big(y_j^{(2)}\big)$ is a scalar multiple of 1, it follows that

$$k(xy) = \left(\sum_{i=1}^{n} x_i^{(1)} \cdot (\iota \circ \varepsilon)\big(x_i^{(2)}\big) \right) \left(\sum_{j=1}^{n} y_j^{(1)} \cdot (\iota \circ \varepsilon)\big(y_j^{(2)}\big) \right) = k(x)k(y).$$

By a similar argument, one may show that $k' = 1$. Hence (H2) holds for $\mathfrak{K}_\omega$.

Let $h = m \circ (1 \otimes S) \circ \Delta$ and $h' = m \circ (S \otimes 1) \circ \Delta$. Evidently, h and h' are linear maps. Using Lemma 2.5(ii), (iii) yields that

$$\begin{aligned} &h(1) = h'(1) = (\iota \circ \varepsilon)(1) = 1, && h(A) = h'(A) = (\iota \circ \varepsilon)(A) = 0, \\ &h(B) = h'(B) = (\iota \circ \varepsilon)(B) = 0, && h(C) = h'(C) = (\iota \circ \varepsilon)(C) = 0. \end{aligned}$$

Let x, y be any two elements of $\mathfrak{K}_\omega$ and suppose that $h(x) = (\iota \circ \varepsilon)(x)$ and $h(y) = (\iota \circ \varepsilon)(y)$. To see that $h = \iota \circ \varepsilon$, it suffices to check that $h(xy) = h(x)h(y)$. Applying (2.3) and (2.4), one finds that

$$h(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} y_j^{(1)} S\big(x_i^{(2)} y_j^{(2)}\big).$$

Using the antihomomorphism property of S, we obtain

$$\begin{aligned} h(xy) &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^{(1)} y_j^{(1)} S\big(y_j^{(2)}\big) S\big(x_i^{(2)}\big) = \sum_{i=1}^{n} x_i^{(1)} \left(\sum_{j=1}^{n} y_j^{(1)} S\big(y_j^{(2)}\big) \right) S\big(x_i^{(2)}\big) \\ &= \sum_{i=1}^{n} x_i^{(1)} h(y) S\big(x_i^{(2)}\big). \end{aligned}$$

Since $h(y) = (\iota \circ \varepsilon)(y)$ is a scalar multiple of 1, it follows that

$$h(xy) = \sum_{i=1}^{n} x_i^{(1)} S\big(x_i^{(2)}\big) h(y) = h(x)h(y).$$

By a similar argument, one can show that $h' = \iota \circ \varepsilon$. Hence (H3) holds for $\mathfrak{K}_\omega$. The result follows. ■

Theorem 2.6. *For any integers $m, n \geq 0$, the $\mathfrak{K}_\omega$-module $L_m \otimes L_n$ is isomorphic to*

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.$$

Proof. By Lemmas 1.4 and 2.5 along with Theorem 1.8 the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{K}_\omega & \xrightarrow{\ \zeta\ } & \mathrm{U}(\mathfrak{sl}_2) \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \Delta} \\ \mathfrak{K}_\omega \otimes \mathfrak{K}_\omega & \xrightarrow[\ \zeta \otimes \zeta\ ]{} & \mathrm{U}(\mathfrak{sl}_2) \otimes \mathrm{U}(\mathfrak{sl}_2) \end{array}$$

Here $\Delta\colon \mathrm{U}(\mathfrak{sl}_2) \to \mathrm{U}(\mathfrak{sl}_2)\otimes\mathrm{U}(\mathfrak{sl}_2)$ is the comultiplication of $\mathrm{U}(\mathfrak{sl}_2)$ from Lemma 1.4 and $\Delta\colon \mathfrak{K}_\omega \to \mathfrak{K}_\omega \otimes \mathfrak{K}_\omega$ is the comultiplication of $\mathfrak{K}_\omega$ from Lemma 2.5(ii). Combined with Theorem 1.5, the result follows. ■

For the rest of this paper, the notation Δ will refer to the map from Lemma 2.5(ii) and Δ_n will stand for the corresponding n-fold comultiplication of $\mathfrak{K}_\omega$ for every positive integer n.

3 The Clebsch–Gordan rule for $\mathrm{U}(\mathfrak{sl}_2)$ and the Hamming graph $H(D,q)$

3.1 Preliminaries on distance-regular graphs

Let Γ denote a finite simple connected graph with vertex set $X \neq \varnothing$. Let ∂ denote the path-length distance function for Γ. Recall that the *diameter* D of Γ is defined by

$$D = \max_{x,y\in X} \partial(x,y).$$

Given any $x \in X$ let

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\} \qquad \text{for } i = 0,1,\dots,D.$$

For short, we abbreviate $\Gamma(x) = \Gamma_1(x)$. We call Γ *distance-regular* whenever for all $h,i,j \in \{0,1,\dots,D\}$ and all $x,y \in X$ with $\partial(x,y) = h$ the number $|\Gamma_i(x)\cap\Gamma_j(y)|$ is independent of x and y. If Γ is distance-regular, the numbers a_i, b_i, c_i for all $i = 0,1,\dots,D$ defined by

$$a_i = |\Gamma_i(x)\cap\Gamma(y)|, \qquad b_i = |\Gamma_{i+1}(x)\cap\Gamma(y)|, \qquad c_i = |\Gamma_{i-1}(x)\cap\Gamma(y)|$$

for any $x,y \in X$ with $\partial(x,y) = i$ are called the *intersection numbers* of Γ. Here $\Gamma_{-1}(x)$ and $\Gamma_{D+1}(x)$ are interpreted as the empty set.

We now assume that Γ is distance-regular. Let $\mathrm{Mat}_X(\mathbb{C})$ be the algebra consisting of the complex square matrices indexed by X. For all $i = 0,1,\dots,D$ the i^{th} *distance matrix* $\mathbf{A}_i \in \mathrm{Mat}_X(\mathbb{C})$ is defined by

$$(\mathbf{A}_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases}$$

for all $x,y \in X$. The *Bose–Mesner algebra* $\mathcal{M}$ of Γ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $\mathbf{A}_i$ for all $i = 0,1,\dots,D$. Note that the adjacency matrix $\mathbf{A} = \mathbf{A}_1$ of Γ generates $\mathcal{M}$ and the matrices $\{\mathbf{A}_i\}_{i=0}^D$ form a basis for $\mathcal{M}$.

Since $\mathbf{A}$ is real symmetric and $\dim\mathcal{M} = D+1$, it follows that $\mathbf{A}$ has $D+1$ mutually distinct real eigenvalues $\theta_0,\theta_1,\dots,\theta_D$. Set $\theta_0 = b_0$ which is the valency of Γ. There exist unique $\mathbf{E}_0,\mathbf{E}_1,\dots,\mathbf{E}_D \in \mathcal{M}$ such that

$$\sum_{i=0}^{D} \mathbf{E}_i = \mathbf{I} \qquad \text{(the identity matrix)}, \qquad \mathbf{A}\mathbf{E}_i = \theta_i\mathbf{E}_i \qquad \text{for all } i = 0,1,\dots,D.$$

The matrices $\{\mathbf{E}_i\}_{i=0}^D$ form another basis for $\mathcal{M}$, and $\mathbf{E}_i$ is called the *primitive idempotent* of Γ associated with θ_i for $i = 0,1,\dots,D$.

Observe that $\mathcal{M}$ is closed under the Hadamard product $\odot$. The distance-regular graph Γ is said to be *Q-polynomial* with respect to the ordering $\{\mathbf{E}_i\}_{i=0}^D$ if there are scalars a_i^*, b_i^*, c_i^* for all $i = 0,1,\dots,D$ such that $b_D^* = c_0^* = 0$, $b_{i-1}^*c_i^* \neq 0$ for all $i = 1,2,\dots,D$ and

$$\mathbf{E}_1 \odot \mathbf{E}_i = \frac{1}{|X|}\left(b_{i-1}^*\mathbf{E}_{i-1} + a_i^*\mathbf{E}_i + c_{i+1}^*\mathbf{E}_{i+1}\right) \qquad \text{for all } i = 0,1,\dots,D,$$

where we interpret b_{-1}^*, c_{D+1}^* as any scalars in $\mathbb{C}$ and $\mathbf{E}_{-1}, \mathbf{E}_{D+1}$ as the zero matrix in $\mathrm{Mat}_X(\mathbb{C})$.

We now assume that Γ is Q-polynomial with respect to $\{\mathbf{E}_i\}_{i=0}^D$ and fix $x \in X$. For all $i = 0, 1, \ldots, D$ let $\mathbf{E}_i^* = \mathbf{E}_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ defined by

$$(\mathbf{E}_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \tag{3.1}$$

for all $y \in X$. The matrix $\mathbf{E}_i^*$ is called the i^{th} *dual primitive idempotent* of Γ with respect to x. The *dual Bose–Mesner algebra* $\mathcal{M}^* = \mathcal{M}^*(x)$ of Γ with respect to x is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $\mathbf{E}_i^*$ for all $i = 0, 1, \ldots, D$. Since $\mathbf{E}_i^*\mathbf{E}_j^* = \delta_{ij}\mathbf{E}_i^*$ the matrices $\{\mathbf{E}_i^*\}_{i=0}^D$ form a basis for $\mathcal{M}^*$. For all $i = 0, 1, \ldots, D$ the i^{th} *dual distance matrix* $\mathbf{A}_i^* = \mathbf{A}_i^*(x)$ is the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ defined by

$$(\mathbf{A}_i^*)_{yy} = |X|(\mathbf{E}_i)_{xy} \qquad \text{for all } y \in X. \tag{3.2}$$

The matrices $\{\mathbf{A}_i^*\}_{i=0}^D$ form another basis for $\mathcal{M}^*$. Note that $\mathbf{A}^* = \mathbf{A}_1^*$ is called the *dual adjacency matrix* of Γ with respect to x and $\mathbf{A}^*$ generates $\mathcal{M}^*$ [16, Lemma 3.11].

The *Terwilliger algebra* $\mathcal{T}$ of Γ with respect to x is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$ [16, Definition 3.3]. The vector space consisting of all complex column vectors indexed by X is a natural $\mathcal{T}$-module and it is called the *standard $\mathcal{T}$-module* [16, p. 368]. Since the algebra $\mathcal{T}$ is finite-dimensional, the irreducible $\mathcal{T}$-modules are finite-dimensional. Since the algebra $\mathcal{T}$ is closed under the conjugate-transpose map, it follows that $\mathcal{T}$ is semisimple. Hence the algebra $\mathcal{T}$ is isomorphic to

$$\bigoplus_{\text{irreducible } \mathcal{T}\text{-modules } W} \mathrm{End}(W),$$

where the direct sum is over all non-isomorphic irreducible $\mathcal{T}$-modules W. Since the standard $\mathcal{T}$-module is faithful, all irreducible $\mathcal{T}$-modules are contained in the standard $\mathcal{T}$-module up to isomorphism.

Let W denote an irreducible $\mathcal{T}$-module. The number $\min_{0 \leq i \leq D}\{i \mid \mathbf{E}_i^* W \neq \{0\}\}$ is called the *endpoint* of W. The number $\min_{0 \leq i \leq D}\{i \mid \mathbf{E}_i W \neq \{0\}\}$ is called the *dual endpoint* of W. The *support* of W is defined as the set $\{i \mid 0 \leq i \leq D, \mathbf{E}_i^* W \neq \{0\}\}$. The *dual support* of W is defined as the set $\{i \mid 0 \leq i \leq D, \mathbf{E}_i W \neq \{0\}\}$. The number $|\{i \mid 0 \leq i \leq D, \mathbf{E}_i^* W \neq \{0\}\}| - 1$ is called the *diameter* of W. The number $|\{i \mid 0 \leq i \leq D, \mathbf{E}_i W \neq \{0\}\}| - 1$ is called the *dual diameter* of W.

3.2 The adjacency matrix and the dual adjacency matrix of a Hamming graph

Let X be a nonempty set and let n be a positive integer. The notation

$$X^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in X\}$$

stands for the n-ary Cartesian product of X. For any $x \in X^n$, let x_i denote the i^{th} coordinate of x for all $i = 1, 2, \ldots, n$.

Recall that q stands for an integer greater than or equal to 3. For the rest of this paper, we set

$$X = \{0, 1, \ldots, q-1\}$$

and let D be a positive integer.

Definition 3.1. The *D-dimensional Hamming graph $H(D) = H(D,q)$ over X* has the vertex set X^D and $x, y \in X^D$ are adjacent if and only if x and y differ in exactly one coordinate.

Let ∂ denote the path-length distance function for $H(D)$. Observe that $\partial(x,y) = |\{i \mid 1 \leq i \leq D,\, x_i \neq y_i\}|$ for any $x, y \in X^D$. It is routine to verify that $H(D)$ is a distance-regular graph with diameter D and its intersection numbers are

$$a_i = i(q-2), \qquad b_i = (D-i)(q-1), \qquad c_i = i$$

for all $i = 0, 1, \ldots, D$.

Let $V(D)$ denote the vector space consisting of the complex column vectors indexed by X^D. For convenience we write $V = V(1)$. For any $x \in X^D$, let $\hat{x}$ denote the vector of $V(D)$ with 1 in the x-coordinate and 0 elsewhere. We view any $L \in \mathrm{Mat}_{X^D}(\mathbb{C})$ as the linear map $V(D) \to V(D)$ that sends $\hat{x}$ to $L\hat{x}$ for all $x \in X^D$. We identify the vector space $V(D)$ with $V^{\otimes D}$ via the linear isomorphism $V(D) \to V^{\otimes D}$ given by

$$\hat{x} \to \hat{x}_1 \otimes \hat{x}_2 \otimes \cdots \otimes \hat{x}_D \qquad \text{for all } x \in X^D.$$

Let $\mathbf{I}(D)$ denote the identity matrix in $\mathrm{Mat}_{X^D}(\mathbb{C})$ and let $\mathbf{A}(D)$ denote the adjacency matrix of $H(D)$. We simply write $\mathbf{I} = \mathbf{I}(1)$ and $\mathbf{A} = \mathbf{A}(1)$.

Lemma 3.2. *Let $D \geq 2$ be an integer. Then*

$$\mathbf{A}(D) = \mathbf{A}(D-1) \otimes \mathbf{I} + \mathbf{I}(D-1) \otimes \mathbf{A}. \tag{3.3}$$

Proof. Let $x \in X^D$ be given. Applying $\hat{x}$ to the right-hand side of (3.3) a straightforward calculation yields that it is equal to

$$\sum_{i=1}^{D} \sum_{y_i \in X \setminus \{x_i\}} \hat{x}_1 \otimes \cdots \otimes \hat{x}_{i-1} \otimes \hat{y}_i \otimes \hat{x}_{i+1} \otimes \cdots \otimes \hat{x}_D = \mathbf{A}(D)\hat{x}.$$

The lemma follows. ■

Using Lemma 3.2, a routine induction yields that $\mathbf{A}(D)$ has the eigenvalues

$$\theta_i(D) = D(q-1) - qi \qquad \text{for all } i = 0, 1, \ldots, D.$$

Let $\mathbf{E}_i(D)$ denote the primitive idempotent of $H(D)$ associated with $\theta_i(D)$ for all $i = 0, 1, \ldots, D$. We simply write $\mathbf{E}_0 = \mathbf{E}_0(1)$ and $\mathbf{E}_1 = \mathbf{E}_1(1)$. For convenience, we interpret $\mathbf{E}_{-1}(D)$ and $\mathbf{E}_{D+1}(D)$ as the zero matrix in $\mathrm{Mat}_{X^D}(\mathbb{C})$.

Lemma 3.3. *Let $D \geq 2$ be an integer. Then*

$$\mathbf{E}_i(D) = \mathbf{E}_i(D-1) \otimes \mathbf{E}_0 + \mathbf{E}_{i-1}(D-1) \otimes \mathbf{E}_1 \qquad \textit{for all } i = 0, 1, \ldots, D. \tag{3.4}$$

Proof. We proceed by induction on D. Let $\mathbf{E}_i(D)'$ denote the right-hand side of (3.4) for $i = 0, 1, \ldots, D$. Applying Lemma 3.2 along with the induction hypothesis, it follows that

$$\sum_{i=0}^{D} \mathbf{E}_i(D)' = \mathbf{I}(D), \qquad \mathbf{A}(D)\mathbf{E}_i(D)' = \theta_i(D)\mathbf{E}_i(D)' \qquad \text{for all } i = 0, 1, \ldots, D.$$

Hence $\mathbf{E}_i(D) = \mathbf{E}_i(D)'$ for all $i = 0, 1, \ldots, D$. The lemma follows. ■

Applying Lemma 3.3 yields that

$$\mathbf{E}_1(D) \odot \mathbf{E}_i(D) = q^{-D}(b^*_{i-1}\mathbf{E}_{i-1}(D) + a^*_i\mathbf{E}_i(D) + c^*_{i+1}\mathbf{E}_{i+1}(D))$$

for all $i = 0, 1, \ldots, D$, where

$$a_i^* = i(q-2), \qquad b_i^* = (D-i)(q-1), \qquad c_i^* = i$$

for all $i = 0, 1, \ldots, D$. Here b_{-1}^*, c_{D+1}^* are interpreted as any scalars in $\mathbb{C}$. Hence $H(D)$ is Q-polynomial with respect to the ordering $\{\mathbf{E}_i(D)\}_{i=0}^D$.

Observe that the graph $H(D)$ is vertex-transitive. Without loss of generality, we can consider the dual adjacency matrix $\mathbf{A}^*(D)$ of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$. We simply write $\mathbf{A}^* = \mathbf{A}^*(1)$.

Lemma 3.4. *Let $D \geq 2$ be an integer. Then*

$$\mathbf{A}^*(D) = \mathbf{A}^*(D-1) \otimes \mathbf{I} + \mathbf{I}(D-1) \otimes \mathbf{A}^*.$$

Proof. Given $y \in X^D$ let c_y denote the coefficient of $\hat{y}$ in $\mathbf{E}_1(D) \cdot \hat{0}^{\otimes D}$ with respect to the basis $\{\hat{x}\}_{x \in X^D}$ for $V(D)$. By (3.2), we have

$$\mathbf{A}^*(D)\hat{y} = q^D c_y \hat{y} \qquad \text{for all } y \in X^D.$$

Suppose that $D \geq 2$. Using Lemma 3.3 yields that $c_y = q^{-1} c_{(y_1, \ldots, y_{D-1})} + q^{1-D} c_{y_D}$ for all $y \in X^D$. Hence

$$\begin{aligned}
\mathbf{A}^*(D)\hat{y} &= \left(q^{D-1} c_{(y_1, \ldots, y_{D-1})} + q c_{y_D}\right)\hat{y} \\
&= \mathbf{A}^*(D-1)(\hat{y}_1 \otimes \cdots \otimes \hat{y}_{D-1}) \otimes \hat{y}_D + \hat{y}_1 \otimes \cdots \otimes \hat{y}_{D-1} \otimes \mathbf{A}^* \hat{y}_D \\
&= (\mathbf{A}^*(D-1) \otimes \mathbf{I} + \mathbf{I}(D-1) \otimes \mathbf{A}^*)\hat{y}
\end{aligned}$$

for all $y \in X^D$. The lemma follows. ■

Let $\mathbf{E}_i^*(D)$ denote the i^{th} dual primitive idempotent of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$ for all $i = 0, 1, \ldots, D$. We simply write $\mathbf{E}_0^* = \mathbf{E}_0^*(1)$ and $\mathbf{E}_1^* = \mathbf{E}_1^*(1)$. For convenience, we interpret $\mathbf{E}_{-1}^*(D)$ and $\mathbf{E}_{D+1}^*(D)$ as the zero matrix in $\mathrm{Mat}_{X^D}(\mathbb{C})$.

Lemma 3.5. *Let $D \geq 2$ be an integer. Then*

$$\mathbf{E}_i^*(D) = \mathbf{E}_i^*(D-1) \otimes \mathbf{E}_0^* + \mathbf{E}_{i-1}^*(D-1) \otimes \mathbf{E}_1^* \qquad \textit{for all } i = 0, 1, \ldots, D.$$

Proof. It is straightforward to verify the lemma by using (3.1). ■

Using Lemmas 3.4 and 3.5, a routine induction yields that $\mathbf{A}^*(D)\mathbf{E}_i^*(D) = \theta_i^*(D)\mathbf{E}_i^*(D)$ for all $i = 0, 1, \ldots, D$ where $\theta_i^*(D) = D(q-1) - qi$.

3.3 Proofs of Proposition 1.9 and Theorems 1.10, 1.11

In this subsection, we set

$$\omega = 1 - \frac{2}{q}.$$

Let $\mathcal{T}(D)$ denote the Terwilliger algebra of $H(D)$ with respect to $(0, 0, \ldots, 0) \in X^D$.

Definition 3.6. Let V_0 denote the subspace of V consisting of all vectors $\sum_{i=1}^{q-1} c_i \hat{i}$, where $c_1, c_2, \ldots, c_{q-1} \in \mathbb{C}$ with $\sum_{i=1}^{q-1} c_i = 0$. Let V_1 denote the subspace of V spanned by $\hat{0}$ and $\sum_{i=1}^{q-1} \hat{i}$.

Definition 3.7. For any $s \in \{0,1\}^D$, we define the subspace $V_s(D)$ of $V(D)$ by

$$V_s(D) = V_{s_1} \otimes V_{s_2} \otimes \cdots \otimes V_{s_D}.$$

Note that $V_0(1) = V_0$ and $V_1(1) = V_1$.

Lemma 3.8. *The vector space $V(D)$ is equal to*

$$\bigoplus_{s \in \{0,1\}^D} V_s(D).$$

Proof. By Definition 3.6, we have $V = V_0 \oplus V_1$. It follows that

$$V(D) = V^{\otimes D} = (V_0 \oplus V_1)^{\otimes D}.$$

The lemma follows by applying the distributive law of $\otimes$ over $\oplus$ to the right-hand side of the above equation. ∎

Lemma 3.9.

(i) *There exists a unique representation $r_0 \colon \mathfrak{K}_\omega \to \mathrm{End}(V_0)$ that sends*

$$A \mapsto \frac{1}{q}\mathbf{A}|_{V_0} + \frac{1}{q}, \qquad B \mapsto \frac{1}{q}\mathbf{A}^*|_{V_0} + \frac{1}{q}.$$

Moreover, the $\mathfrak{K}_\omega$-module V_0 is isomorphic to $(q-2) \cdot L_0$.

(ii) *There exists a unique representation $r_1 \colon \mathfrak{K}_\omega \to \mathrm{End}(V_1)$ that sends*

$$A \mapsto \frac{1}{q}\mathbf{A}|_{V_1} + \frac{1}{q} - \frac{1}{2}, \qquad B \mapsto \frac{1}{q}\mathbf{A}^*|_{V_1} + \frac{1}{q} - \frac{1}{2}.$$

Moreover, the $\mathfrak{K}_\omega$-module V_1 is isomorphic to L_1.

Proof. (i) The subspace V_0 of V is invariant under $\mathbf{A}$ and $\mathbf{A}^*$ acting as scalar multiplication by -1. By Lemma 2.1, the statement (i) follows.

(ii) The subspace V_1 of V is invariant under $\mathbf{A}$ and $\mathbf{A}^*$ and the matrices representing $\mathbf{A}$ and $\mathbf{A}^*$ with respect to the basis $\hat{0}$, $\sum_{i=1}^{q-1} \hat{\imath}$ for V_1 are

$$\begin{pmatrix} 0 & q-1 \\ 1 & q-2 \end{pmatrix}, \qquad \begin{pmatrix} q-1 & 0 \\ 0 & -1 \end{pmatrix},$$

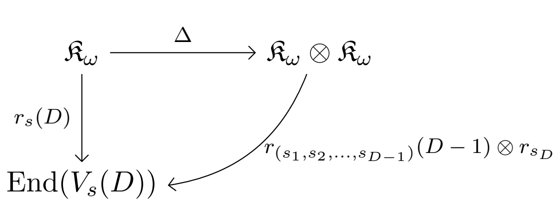
respectively. By Lemma 2.1, the statement (ii) follows. ∎

Definition 3.10. For any $s \in \{0,1\}^D$, we define the representation $r_s(D) \colon \mathfrak{K}_\omega \to \mathrm{End}(V_s(D))$ by

$$r_s(D) = (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_D}) \circ \Delta_{D-1}.$$

Note that $r_0(1) = r_0$ and $r_1(1) = r_1$.

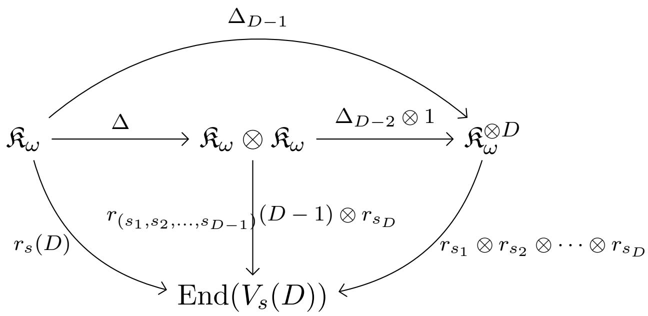
Proposition 3.11. *For any integer $D \geq 2$ and any $s \in \{0,1\}^D$, the following diagram commutes:*



Proof. By Definition 3.10 the map $r_{(s_1,s_2,\ldots,s_{D-1})}(D-1) = (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_{D-1}}) \circ \Delta_{D-2}$. Hence

$$\begin{aligned} r_{(s_1,s_2,\ldots,s_{D-1})}(D-1) \otimes r_{s_D} &= \big((r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_{D-1}}) \circ \Delta_{D-2}\big) \otimes r_{s_D} \\ &= (r_{s_1} \otimes r_{s_2} \otimes \cdots \otimes r_{s_D}) \circ (\Delta_{D-2} \otimes 1). \end{aligned}$$

By (2.2), the map $\Delta_{D-1} = (\Delta_{D-2} \otimes 1) \circ \Delta$. Combined with Definition 3.10, the following diagram commutes:



The proposition follows. ■

Proposition 3.12. *For any $s \in \{0,1\}^D$, the representation $r_s(D)\colon \mathfrak{K}_\omega \to \mathrm{End}(V_s(D))$ maps*

$$A \mapsto \frac{1}{q}\mathbf{A}(D)|_{V_s(D)} + \frac{D}{q} - \frac{1}{2}\sum_{i=1}^{D} s_i, \tag{3.5}$$

$$B \mapsto \frac{1}{q}\mathbf{A}^*(D)|_{V_s(D)} + \frac{D}{q} - \frac{1}{2}\sum_{i=1}^{D} s_i. \tag{3.6}$$

Proof. We proceed by induction on D. By Lemma 3.9, the statement is true when $D = 1$. Suppose that $D \geq 2$. For convenience let $s' = (s_1, s_2, \ldots, s_{D-1}) \in \{0,1\}^{D-1}$. By Lemma 2.5 and Proposition 3.11, the map $r_s(D)$ sends A to

$$r_{s'}(D-1)(A) \otimes 1 + 1 \otimes r_{s_D}(A).$$

Applying the induction hypothesis the above element is equal to

$$\begin{aligned} &\left(\frac{1}{q}\mathbf{A}(D-1)|_{V_{s'}(D-1)} + \frac{D-1}{q} - \frac{1}{2}\sum_{i=1}^{D-1} s_i\right) \otimes 1 + 1 \otimes \left(\frac{1}{q}\mathbf{A}|_{V_{s_D}} + \frac{1}{q} - \frac{s_D}{2}\right) \\ &\quad = \frac{\mathbf{A}(D-1)|_{V_{s'}(D-1)} \otimes 1 + 1 \otimes \mathbf{A}|_{V_{s_D}}}{q} + \frac{D}{q} - \frac{1}{2}\sum_{i=1}^{D} s_i. \end{aligned}$$

By Lemma 3.2, the first term in the right-hand side of the above equation equals $\frac{1}{q}\mathbf{A}(D)|_{V_s(D)}$. Hence (3.5) holds. By a similar argument, (3.6) holds. The proposition follows. ■

In light of Proposition 3.12, the $\mathcal{T}(D)$-module $V_s(D)$ is a $\mathfrak{K}_\omega$-module for all $s \in \{0,1\}^D$. Combined with Lemma 3.8, the standard $\mathcal{T}(D)$-module $V(D)$ is a $\mathfrak{K}_\omega$-module.

Lemma 3.13. *Let p be a positive integer. Then the $\mathfrak{K}_\omega$-module $L_1^{\otimes p}$ is isomorphic to*

$$\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1}\binom{p}{k} \cdot L_{p-2k}.$$

Proof. We proceed by induction on p. If $p = 1$, then there is nothing to prove. Suppose that $p \geq 2$. Applying the induction hypothesis yields that the $\mathfrak{K}_\omega$-module $L_1^{\otimes p}$ is isomorphic to

$$\left(\bigoplus_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{p-2k}{p-k} \binom{p-1}{k} \cdot L_{p-2k-1} \right) \otimes L_1.$$

Applying the distributive law of $\otimes$ over $\oplus$ the above $\mathfrak{K}_\omega$-module is isomorphic to

$$\bigoplus_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{p-2k}{p-k} \binom{p-1}{k} \cdot (L_{p-2k-1} \otimes L_1).$$

By Theorem 2.6, the $\mathfrak{K}_\omega$-module $L_{p-2k-1} \otimes L_1$ is isomorphic to

$$\begin{cases} L_{p-2k} \oplus L_{p-2k-2} & \text{if } 0 \leq k \leq \lfloor \frac{p}{2} \rfloor - 1, \\ L_1 & \text{else} \end{cases}$$

for all $k = 0, 1, \ldots, \lfloor \frac{p-1}{2} \rfloor$. Hence the multiplicity of L_{p-2k} in $L_1^{\otimes p}$ is equal to

$$\frac{p-2k}{p-k} \binom{p-1}{k} + \frac{p-2k+2}{p-k+1} \binom{p-1}{k-1} = \frac{p-2k+1}{p-k+1} \binom{p}{k}$$

for all $k = 0, 1, \ldots, \lfloor \frac{p}{2} \rfloor$. Here $\binom{p-1}{k-1}$ is interpreted as 0 when $k = 0$. The lemma follows. ■

Lemma 3.14. *Let p be an integer with $0 \leq p \leq D$. Suppose that $s \in \{0,1\}^D$ with $p = \sum_{i=1}^D s_i$. Then the $\mathfrak{K}_\omega$-module $V_s(D)$ is isomorphic to*

$$\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{p}{k} (q-2)^{D-p} \cdot L_{p-2k}.$$

Proof. By Definition 3.7, the $\mathfrak{K}_\omega$-module $V_s(D)$ is isomorphic to $V_1^{\otimes p} \otimes V_0^{\otimes (D-p)}$. Applying Lemma 3.9 the above $\mathfrak{K}_\omega$-module is isomorphic to $(q-2)^{D-p} \cdot L_1^{\otimes p}$. Combined with Lemma 3.13, the lemma follows. ■

Proof of Proposition 1.9. Let p and k be two integers with $0 \leq p \leq D$ and $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$. Pick any $s \in \{0,1\}^D$ with $p = \sum_{i=1}^D s_i$. By Lemma 3.14, the $\mathfrak{K}_\omega$-module $V_s(D)$ contains the irreducible $\mathfrak{K}_\omega$-module L_{p-2k}. Let $\{v_i\}_{i=0}^{p-2k}$ and $\{w_i\}_{i=0}^{p-2k}$ denote the two bases for L_{p-2k} described in Lemmas 2.1 and 2.4 with $n = p - 2k$, respectively. In light of Proposition 3.12, we may view the $\mathfrak{K}_\omega$-submodule L_{p-2k} of $V_s(D)$ as an irreducible $\mathcal{T}(D)$-module and denoted by $L_{p,k}(D)$. To see (i) and (ii), one may evaluate the matrices representing $\mathbf{A}(D)$ and $\mathbf{A}^*(D)$ with respect to the bases $\{v_i\}_{i=0}^{p-2k}$ and $\{w_i\}_{i=0}^{p-2k}$ for $L_{p,k}(D)$, respectively. The proposition follows. ■

Proof of Theorem 1.10. Let p be any integer with $0 \leq p \leq D$. By Lemma 3.14, for any $s \in \{0,1\}^D$ with $p = \sum_{i=1}^D s_i$ the $\mathcal{T}(D)$-submodule $V_s(D)$ of $V(D)$ is isomorphic to

$$\bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{p}{k} (q-2)^{D-p} \cdot L_{p,k}(D).$$

Combined with Lemma 3.8, the result follows. ■

Proof of Theorem 1.11. Since the standard $\mathcal{T}(D)$-module $V(D)$ contains all irreducible $\mathcal{T}(D)$-modules up to isomorphism, the map $\mathcal{E}$ is onto. Suppose that there are two pairs (p,k) and (p',k') in $\mathbf{P}(D)$ such that the irreducible $\mathcal{T}(D)$-module $L_{p,k}(D)$ is isomorphic to $L_{p',k'}(D)$. Since they have the same dimension, it follows that

$$p-2k=p'-2k'. \tag{3.7}$$

Since $\mathbf{A}^*(D)$ has the same eigenvalues in $L_{p,k}(D)$ and $L_{p',k'}(D)$, it follows from Proposition 1.9 that $p-k=p'-k'$. Combined with (3.7), this yields that $(p,k)=(p',k')$. Therefore, $\mathcal{E}$ is one-to-one. ■

Corollary 3.15 ([11, Corollary 3.7])**.** *The algebra $\mathcal{T}(D)$ is isomorphic to*

$$\bigoplus_{p=0}^{D}\bigoplus_{k=0}^{\lfloor \frac{p}{2}\rfloor}\mathrm{Mat}_{p-2k+1}(\mathbb{C}).$$

Moreover, $\dim\mathcal{T}(D)=\binom{D+4}{4}$.

Proof. By Theorem 1.11, the algebra $\mathcal{T}(D)$ is isomorphic to $\bigoplus_{p=0}^{D}\bigoplus_{k=0}^{\lfloor \frac{p}{2}\rfloor}\mathrm{End}(L_{p,k}(D))$. Hence $\dim\mathcal{T}(D)$ is equal to

$$\sum_{p=0}^{D}\sum_{k=0}^{\lfloor \frac{p}{2}\rfloor}(p-2k+1)^2=\sum_{p=0}^{D}\binom{p+3}{3}=\binom{D+4}{4}.$$

The corollary follows. ■

A Restatements of Proposition 1.9 and Theorems 1.10, 1.11

Recall the irreducible $\mathcal{T}(D)$-module $L_{p,k}(D)$ from Proposition 1.9. Let r, r^*, d, d^* denote the endpoint, dual endpoint, diameter, dual diameter of $L_{p,k}(D)$ respectively. It is known from [18, p. 197] that $\lceil\frac{D-d}{2}\rceil\leq r,r^*\leq D-d$. From the results of Section 3.2, we see that

$$r=r^*=D+k-p, \qquad d=d^*=p-2k.$$

In terms of the parameters r and d, the parameters p and k read as

$$p=2D-d-2r, \qquad k=D-d-r.$$

Thus we can restate Proposition 1.9 and Theorems 1.10, 1.11 as follows:

Proposition A.1. *Let D be a positive integer. For any integers d and r with $0\leq d\leq D$ and $\lceil\frac{D-d}{2}\rceil\leq r\leq D-d$, there exists a $(d+1)$-dimensional irreducible $\mathcal{T}(D)$-module $M_{d,r}(D)$ satisfying the following conditions:*

(i) *There exists a basis for $M_{d,r}(D)$ with respect to which the matrices representing $\mathbf{A}(D)$ and $\mathbf{A}^*(D)$ are*

$$\begin{pmatrix}\alpha_0 & \gamma_1 & & & \mathbf{0}\\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_d\\ \mathbf{0} & & & \beta_{d-1} & \alpha_d\end{pmatrix}, \qquad \begin{pmatrix}\theta_0 & & & & \mathbf{0}\\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_d\end{pmatrix},$$

respectively.

(*ii*) *There exists a basis for* $M_{d,r}(D)$ *with respect to which the matrices representing* $\mathbf{A}(D)$ *and* $\mathbf{A}^*(D)$ *are*

$$\begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \theta_2 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \alpha_0 & \gamma_1 & & & \mathbf{0} \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \gamma_d \\ \mathbf{0} & & & \beta_{d-1} & \alpha_d \end{pmatrix},$$

respectively.

Here the parameters $\{\alpha_i\}_{i=0}^d$, $\{\beta_i\}_{i=0}^{d-1}$, $\{\gamma_i\}_{i=1}^d$, $\{\theta_i\}_{i=0}^d$ *are as follows:*

$$\begin{aligned} \alpha_i &= (D-d+i-r)(q-1)-i-r && \text{for} \quad i=0,1,\ldots,d, \\ \beta_i &= i+1 && \text{for} \quad i=0,1,\ldots,d-1, \\ \gamma_i &= (q-1)(d-i+1) && \text{for} \quad i=1,2,\ldots,d, \\ \theta_i &= D(q-1)-q(i+r) && \text{for} \quad i=0,1,\ldots,d. \end{aligned}$$

Theorem A.2. *Let* D *be a positive integer. Then the standard* $\mathcal{T}(D)$*-module* $V(D)$ *is isomorphic to*

$$\bigoplus_{d=0}^{D} \bigoplus_{r=\lceil \frac{D-d}{2} \rceil}^{D-d} \frac{d+1}{D-r+1} \binom{D}{2D-d-2r} \binom{2D-d-2r}{D-d-r} (q-2)^{d-D+2r} \cdot M_{d,r}(D).$$

We illustrate Theorem A.2 for $D=3$ and $D=4$:

D	d	r	The support of $M_{d,r}(D)$	The multiplicity of $M_{d,r}(D)$ in $V(D)$
3	3	0	$\{0,1,2,3\}$	1
	2	1	$\{1,2,3\}$	$3(q-2)$
	1	1	$\{1,2\}$	2
		2	$\{2,3\}$	$3(q-2)^2$
	0	2	$\{2\}$	$3(q-2)$
		3	$\{3\}$	$(q-2)^3$
4	4	0	$\{0,1,2,3,4\}$	1
	3	1	$\{1,2,3,4\}$	$4(q-2)$
	2	1	$\{1,2,3\}$	3
		2	$\{2,3,4\}$	$6(q-2)^2$
	1	2	$\{2,3\}$	$8(q-2)$
		3	$\{3,4\}$	$4(q-2)^3$
	0	2	$\{2\}$	2
		3	$\{3\}$	$6(q-2)^2$
		4	$\{4\}$	$(q-2)^4$

Theorem A.3. *Let* D *be a positive integer. Let* $\mathbf{P}(D)$ *denote the set consisting of all pairs* (d,r) *of integers with* $0 \le d \le D$ *and* $\lceil \frac{D-d}{2} \rceil \le r \le D-d$. *Let* $\mathbf{M}(D)$ *denote the set of all* *isomorphism classes of irreducible* $\mathcal{T}(D)$*-modules. Then there exists a bijection* $\mathbf{P}(D) \to \mathbf{M}(D)$ *given by*

$$(d, r) \mapsto \textit{the isomorphism class of } M_{d,r}(D)$$

for all $(d, r) \in \mathbf{P}(D)$*.*

By Theorem A.3, the structure of an irreducible $\mathcal{T}(D)$-module is determined by its endpoint and its diameter. Also we can restate Corollary 3.15 as follows:

Corollary A.4. *The algebra* $\mathcal{T}(D)$ *is isomorphic to*

$$\bigoplus_{d=0}^{D} \left(\left\lfloor \frac{D-d}{2} \right\rfloor + 1 \right) \cdot \mathrm{Mat}_{d+1}(\mathbb{C}).$$

Moreover, $\dim \mathcal{T}(D) = \binom{D+4}{4}$*.*

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