New Evaluations of Inverse Binomial Series via Cyclotomic Multiple Zeta Values

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Abstract. Through the application of an evaluation technique based on cyclotomic multiple zeta values recently due to Au, we solve open problems on inverse binomial series that were included in a 2010 analysis textbook by Chen.

Key words: binomial coefficients; cyclotomic multiple zeta values; multiple polylogarithms

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1 Introduction

In Hongwei Chen's 2010 textbook on classical analysis [14], the quest for provable closed forms of the following hypergeometric series was highlighted as an open problem [14, p. 215]:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3\binom{2n}{n}}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3\binom{2n}{n}}. \tag{1.2}$$

Series involving inverted binomial coefficients, as in the hypergeometric series in (1.1)–(1.2), are ubiquitous in many areas of analysis, and the problem of evaluating such series is important in computational and experimental mathematics, with a particular regard toward the classic text *Experimentation in Mathematics* [5, Section 1.7]. For inverse binomial series involving higher powers as factors in the denominator, beyond linear or quadratic factors, the evaluation of such series is of a recalcitrant nature, even for the cubic case. This is evidenced by the rich history

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associated with the problem of proving the Chudnovsky brothers' formula [16]

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{3n}{n} 2^n} = \pi G - \frac{33\zeta(3)}{16} + \frac{\log^3 2}{6} - \frac{\pi^2 \log 2}{24}, \tag{1.3}$$

as described in [12], where $G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant and $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$ is Apéry's constant. In addition to the work of the Chudnovsky brothers [16], notable research contributions on inverse binomial series with negative powers as in (1.1)–(1.3) include [1, 2, 7, 15, 17, 18, 24, 25, 31, 36, 37, 38, 39, 41, 42, 43]. These past references further motivate us to investigate the inverse binomial series (1.1) and (1.2). In this work, we solve Hongwei Chen's open problems indicated above [14, p. 215], using a recent evaluation technique due to Au based on *cyclotomic multiple zeta values* (CMZVs) and *multiple polylogarithms* (MPLs).

The rest of this article is organized as follows. In Section 2, we give a gentle introduction to CMZVs and MPLs, which provides background on the algorithms associated with our proofs of Theorems 2.1–2.4. In Section 3, we present computer-assisted proofs for the closed-form evaluations of (1.1) and (1.2), while handling integral representations of these series by Au's `MultipleZetaValues` package [3]. In Section 4, we perform an in-depth analysis of CMZVs and MPLs, accommodating to convergent series in the form of

$$\mathscr{S}_k(z) := \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)^k \binom{2n}{n}} \tag{1.4}$$

for suitable positive integers k and complex numbers z, which generalize (1.1) and (1.2). In addition to Au's software [3], Panzer's `HyperInt` package [34] will also be essential to our manipulations of MPLs related to these generalizations of Chen's series.

References [2] and [17] provide the computational methods applied in this paper. These applications are based on Mellin transform representations of factors appearing within summands, and these factors can be expressed with a single Mellin transform. The desired sum over non-negative integers n is then given by the integral of

$$\sum_{n=0}^{\infty} x^n z^n = \frac{1}{1-xz},$$

for a possibly subsidiary parameter z. This integral is then used to obtain evaluations for the desired sum, by setting $z \to 1$.

2 Cyclotomic multiple zeta values (CMZVs) and multiple polylogarithms (MPLs)

Let $\mathbb{Z}_{>0} := \{1, 2, 3, \dots\}$ be the set of positive integers. A convergent series of the form

$$\mathrm{Li}_{s_1,\dots,s_m}(z_1,\dots,z_m) := \sum_{n_1>\cdots>n_m\geq 1} \frac{z_1^{n_1}\cdots z_m^{n_m}}{n_1^{s_1}\cdots n_m^{s_m}} \tag{2.1}$$

is referred to as a *cyclotomic multiple zeta value* (CMZV) of weight $k \in \mathbb{Z}_{>0}$ and level $N \in \mathbb{Z}_{>0}$ if $s_1,\dots,s_m \in \mathbb{Z}_{>0}$, $s_1+\cdots+s_m = k$, and $z_1^N = \cdots = z_m^N = 1$. At level $N=1$, CMZVs are reduced to the *multiple zeta values* (MZVs)

$$\zeta(s_1,\dots,s_m) := \sum_{n_1>\cdots>n_m\geq 1} \frac{1}{n_1^{s_1}\cdots n_m^{s_m}},$$

which play important roles within experimental mathematics, as highlighted in the classic text by Borwein, Bailey, and Girgensohn [5, Section 3]. CMZVs at levels $N \in \{1,2,3,4,6\}$ feature prominently in the perturbative expansions of Feynman diagrams in quantum field theory [1, 2, 17, 18, 24, 25, 29, 30, 35, 39]. The algorithmic structures of MZVs (together with some generalizations) have been elucidated by Brown [8, 9, 10] and implemented by Panzer in the `HyperInt` package [34].

Collections of CMZVs defined in (2.1) with the same weight $k \in \mathbb{Z}_{>0}$ and level $N \in \mathbb{Z}_{>0}$ span a $\mathbb{Q}$-vector space

$$\mathfrak{Z}_k(N) := \operatorname{span}_{\mathbb{Q}}\left\{ \operatorname{Li}_{s_1,\dots,s_m}(z_1,\dots,z_m) \left| \begin{matrix} s_1,\dots,s_m \in \mathbb{Z}_{>0}, \\ z_1^N = \dots = z_m^N = 1, \\ (s_1,z_1) \neq (1,1), \\ \sum_{j=1}^m s_j = k \end{matrix} \right. \right\}. \tag{2.2}$$

We retroactively set $\mathfrak{Z}_0(N) := \mathbb{Q}$ for all $N \in \mathbb{Z}_{>0}$. The $\mathbb{Q}$-vector spaces enjoy a filtration property $\mathfrak{Z}_j(N)\mathfrak{Z}_k(N) \subseteq \mathfrak{Z}_{j+k}(N)$ for $j,k \in \mathbb{Z}_{\geq 0}$ and any fixed level N [22, Section 1.2], namely, whenever we have two numbers $z_j \in \mathfrak{Z}_j(N)$ and $z_k \in \mathfrak{Z}_k(N)$, their product $z_j z_k$ is in $\mathfrak{Z}_{j+k}(N)$.

For small weights k and levels $N \in \{1,2,3,4,5,6,7,8,9,10,12\}$, Au's `MultipleZetaValues` package [3] allows us to express every member of $\mathfrak{Z}_k(N)$ as a $\mathbb{Q}$-linear combination of the numbers in a spanning set,[1] extending the support of $N \in \{1,2,4\}$ cases in Panzer's `HyperInt` package [34]. For example, we have

$$\mathfrak{Z}_2(4) = \operatorname{span}_{\mathbb{Q}}\{\mathrm{i}G, \pi^2, \pi\mathrm{i}\log 2, \log^2 2\}$$

involving Catalan's constant $G := \operatorname{Im}\operatorname{Li}_2(\mathrm{i})$.

One may also consider convergent series in the form of (2.1) without imposing the cyclotomic constraint that $z_1^N = \dots = z_m^N = 1$. This defines a *multiple polylogarithm* (MPL) $\operatorname{Li}_{s_1,\dots,s_m}(z_1,\dots,z_m)$ of weight $k = s_1 + \dots + s_m$. As a special case of MPLs, we have the *polylogarithm* function $\operatorname{Li}_s(z) := \sum_{n=1}^\infty \frac{z^n}{n^s}$ of weight $s \in \mathbb{Z}_{>0}$. Panzer's `HyperInt` package [34] allows one to reduce certain expressions involving MPLs, via Brown's algorithm [8, 9, 10].

In particular, the polylogarithm of weight 3 leads us to a Catalan-like constant [11, 13]

$$\mathcal{G} := \operatorname{Im}\operatorname{Li}_3\left(\frac{\mathrm{i}+1}{2}\right) = -\operatorname{Im}\operatorname{Li}_{2,1}(\mathrm{i},1) - \frac{G\log 2}{2} + \frac{\pi\log^2 2}{32} + \frac{3\pi^3}{128}, \tag{2.3}$$

which becomes useful in the construction of a spanning set for $\mathfrak{Z}_3(4)$, namely

$$\mathfrak{Z}_3(4) = \operatorname{span}_{\mathbb{Q}}\big\{\zeta(3), \mathrm{i}\mathcal{G}, \pi G, \mathrm{i}G\log 2, \mathrm{i}\pi^3, \pi^2\log 2, \mathrm{i}\pi\log^2 2, \log^3 2\big\}.$$

Here, $\zeta(3) := \operatorname{Li}_3(1)$ is Apéry's constant.

From now on, we will follow the practices of [36, 41, 42], where special values of natural logarithms are abbreviated as follows:

$$\begin{aligned}
\lambda &:= \log 2 = -\operatorname{Li}_1(-1) \in \mathfrak{Z}_1(2), \\
\mathit{\Lambda} &:= \log 3 = -2\operatorname{Re}\operatorname{Li}_1\big(\mathrm{e}^{2\pi\mathrm{i}/3}\big) \in \mathfrak{Z}_1(3), \\
\pounds &:= \log\frac{1+\sqrt{5}}{2} = \operatorname{Re}\big[\operatorname{Li}_1\big(\mathrm{e}^{2\pi\mathrm{i}/5}\big) - \operatorname{Li}_1\big(\mathrm{e}^{4\pi\mathrm{i}/5}\big)\big] \in \mathfrak{Z}_1(5), \\
\mathscr{L} &:= \log 5 = -2\operatorname{Re}\big[\operatorname{Li}_1\big(\mathrm{e}^{2\pi\mathrm{i}/5}\big) + \operatorname{Li}_1\big(\mathrm{e}^{4\pi\mathrm{i}/5}\big)\big] \in \mathfrak{Z}_1(5),
\end{aligned}$$

[1] Putatively, the spanning set produced by the command `MZBasis[M,k]` in Au's `MultipleZetaValues` package [3] is indeed a $\mathbb{Q}$-vector basis for $\mathfrak{Z}_k(M)$, but there is no definitive evidence for such claims beyond the cases of $\mathfrak{Z}_1(M)$, $\mathfrak{Z}_2(1)$, and $\mathfrak{Z}_2(2)$.

$$\widetilde{\lambda} := \log\left(1+\sqrt{2}\right) = \operatorname{Re}\left[\operatorname{Li}_1\left(e^{\pi i/4}\right) - \operatorname{Li}_1\left(e^{3\pi i/4}\right)\right] \in \mathfrak{Z}_1(8),$$
$$\widetilde{\Lambda} := \log\left(2+\sqrt{3}\right) = 2\operatorname{Re}\operatorname{Li}_1\left(e^{\pi i/6}\right) \in \mathfrak{Z}_1(12). \tag{2.4}$$

We bear in mind that products of these listed logarithms are CMZVs of higher weights, such as

$$\lambda\Lambda \in \mathfrak{Z}_2(6), \qquad \lambda\pounds^2 \in \mathfrak{Z}_3(10),$$

by virtue of the natural embedding $\mathfrak{Z}_k(N) \subseteq \mathfrak{Z}_k(M)$ for $N \mid M$, together with Goncharov's filtration $\mathfrak{Z}_j(M)\mathfrak{Z}_k(M) \subseteq \mathfrak{Z}_{j+k}(M)$ [22, Section 1.2].

With the preparations so far, we can state the next two theorems to be proved in Section 3.

Theorem 2.1. *Recall $\mathfrak{Z}_k(N)$ from (2.2), $\mathcal{G}$ from (2.3), and the abbreviations for special logarithms from (2.4). Chen's series*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3\binom{2n}{n}}$$

admits the evaluation

$$\frac{32\mathcal{G}}{3} - \frac{4\pi\operatorname{Li}_2\left(2-\sqrt{3}\right)}{3} - \frac{\pi^3}{9} - \frac{\pi\left(\lambda - \widetilde{\Lambda}\right)^2}{3}, \tag{2.5}$$

which belongs to the $\mathbb{Q}$-vector space $i\mathfrak{Z}_3(12)$.

Theorem 2.2. *Set ϕ as the golden ratio $\frac{\sqrt{5}+1}{2}$, so that $\pounds = \log\phi$. Chen's series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3\binom{2n}{n}}$$

evaluates to

$$-\frac{4\operatorname{Li}_3\left(\frac{1}{\phi^3}\right)}{3} - 4\operatorname{Li}_2\left(\frac{1}{\phi^3}\right)\pounds + \operatorname{Li}_3\left(\frac{1}{\phi}\right) - \frac{25\pounds^3}{3} + 6\lambda\pounds^2 + \frac{\pi^2\pounds}{10} + \frac{12\zeta(3)}{5} - \frac{\pi^2\lambda}{3}, \tag{2.6}$$

a number that lies in $\mathfrak{Z}_3(10)$.

In Section 4, we will unify Theorems 2.1 and 2.2 into a form given below.

Theorem 2.3. *If $|w| \le 1$, $\operatorname{Re} w > 0$, $\operatorname{Im} w \ge 0$, and $\left|1-w^2\right| \le 2|w|$, then we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3\binom{2n}{n}}\left(\frac{1-w^2}{w}\right)^{2n+1} = {} & -2\left[\operatorname{Li}_3\left(\frac{1+w}{2}\right) - \operatorname{Li}_3\left(\frac{1-w}{2}\right) - \operatorname{Li}_3\left(\frac{1+\frac{1}{w}}{2}\right)\right.\\ & \left. + \operatorname{Li}_3\left(\frac{1-\frac{1}{w}}{2}\right)\right] + \left[\operatorname{Li}_2\left(\frac{1+w}{2}\right) - \operatorname{Li}_2\left(\frac{1-w}{2}\right)\right.\\ & \left. + \operatorname{Li}_2\left(\frac{1+\frac{1}{w}}{2}\right) - \operatorname{Li}_2\left(\frac{1-\frac{1}{w}}{2}\right)\right]\log w\\ & + \pi i \log\left(\frac{1+w}{2}\right)\log\left(\frac{1+\frac{1}{w}}{2}\right),\end{aligned} \tag{2.7}$$

where the dilogarithm $\operatorname{Li}_2(z)$ and the trilogarithm $\operatorname{Li}_3(z)$ are defined by (analytic continuations of) $\operatorname{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ for $|z| \le 1$ and $s > 1$.

In Section 4, we will also reveal the CMZV structures for further generalizations of Chen's series, as stated in the theorem below.

Theorem 2.4. *Denote the least common multiple of two numbers a and b by* $\operatorname{lcm}(a,b)$*, and recall the definition of* $\mathscr{S}_k(z)$ *from* (1.4).

(a) *For $k-1, N-2 \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$, we have*

$$\mathscr{S}_k\left(4\sin^2\frac{2m\pi}{N}\right)\sin\frac{2m\pi}{N} \in \mathrm{i}\mathfrak{Z}_k(\operatorname{lcm}(2,N)). \tag{2.8}$$

(b) *For $k-1 \in \mathbb{Z}_{>0}$, the following relations hold true:*

$$\mathscr{S}_k\left(-\frac{9}{4}\right) \in \mathfrak{Z}_k(6), \tag{2.9}$$

$$\mathscr{S}_k(-4) \in \mathfrak{Z}_k(8), \tag{2.10}$$

$$\sqrt{2}\mathscr{S}_k\left(-\frac{1}{2}\right) \in \mathfrak{Z}_k(8), \tag{2.11}$$

$$\mathscr{S}_k(-1) \in \mathfrak{Z}_k(10), \tag{2.12}$$

$$\sqrt{5}\mathscr{S}_k\left(-\frac{16}{5}\right) \in \mathfrak{Z}_k(10), \tag{2.13}$$

$$\sqrt{3}\mathscr{S}_k\left(-\frac{4}{3}\right) \in \mathfrak{Z}_k(12). \tag{2.14}$$

Our proof of the last theorem will be both constructive and algorithmic. In particular, Au's `MultipleZetaValues` package [3] will provide us with many concrete CMZV characterizations of the infinite series $\mathscr{S}_k(z)$, such as (cf. (2.3) and (2.4) for notations)

$$\begin{aligned}\sqrt{2}\mathscr{S}_3(2) ={}& -8\operatorname{Im}\operatorname{Li}_3\left(\frac{1-e^{\pi i/4}}{2}\right) - 4\operatorname{Im}\operatorname{Li}_3\big(i(\sqrt{2}-1)\big)\\ &-\frac{\pi\big[48\operatorname{Li}_2\big(\sqrt{2}-1\big)-12\lambda\widetilde{\lambda}+20\widetilde{\lambda}^2+9\lambda^2\big]}{32}+\frac{15\pi^3}{128} \in \mathrm{i}\mathfrak{Z}_3(8),\end{aligned} \tag{2.15}$$

$$\begin{aligned}\sqrt{2}\mathscr{S}_4(2) ={}& -36\operatorname{Im}\operatorname{Li}_4\big(1-e^{\pi i/4}\big) - 12\operatorname{Im}\operatorname{Li}_4\left(\frac{1-e^{\pi i/4}}{2}\right) - 12\operatorname{Im}\operatorname{Li}_4\big(i\big(1-e^{\pi i/4}\big)\big)\\ &-12\operatorname{Im}\operatorname{Li}_4\big(i(\sqrt{2}-1)\big)-\frac{9\beta(4)}{2}-14\sqrt{2}L_{8,4}(4)+\frac{10\pi\sqrt{2}L_{8,2}(3)}{3}\\ &-\frac{9\pi\operatorname{Li}_3\big(\frac{1}{\sqrt{2}}\big)}{2}+\frac{63\pi\zeta(3)}{128}+\frac{\pi\big(78\lambda^2\widetilde{\lambda}-12\lambda\widetilde{\lambda}^2-24\widetilde{\lambda}^3+47\lambda^3\big)}{256}\\ &-\frac{3\pi^3\big(141\lambda-98\widetilde{\lambda}\big)}{1024} \in \mathrm{i}\mathfrak{Z}_4(8),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\sqrt{3}\mathscr{S}_3(3) ={}& -8\operatorname{Im}\operatorname{Li}_3\left(\frac{1-i\sqrt{3}}{4}\right) - 5\operatorname{Im}\operatorname{Li}_3\left(\frac{1+\frac{i}{\sqrt{3}}}{2}\right)+\frac{\pi\operatorname{Li}_2\big(\frac{1}{4}\big)}{3}\\ &+\frac{\pi\Lambda^2}{48}-\frac{7\pi^3}{432} \in \mathrm{i}\mathfrak{Z}_3(6),\end{aligned} \tag{2.17}$$

$$\begin{aligned}\sqrt{3}\mathscr{S}_4(3) ={}& 8\operatorname{Im}\operatorname{Li}_4\left(\frac{3+i\sqrt{3}}{4}\right) - 8\operatorname{Im}\operatorname{Li}_4\left(\frac{1-i\sqrt{3}}{4}\right)\\ &-5\operatorname{Im}\operatorname{Li}_4\left(\frac{1+\frac{i}{\sqrt{3}}}{2}\right)-\frac{45\sqrt{3}L_{3,2}(4)}{16}+\frac{\pi\big[\operatorname{Li}_3\big(\frac{1}{3}\big)+\operatorname{Li}_3\big(\frac{1}{4}\big)\big]}{3}-\frac{19\pi\zeta(3)}{36}\\ &+\frac{\pi\big(64\lambda^3-192\lambda^2\Lambda+144\lambda\Lambda^2-41\Lambda^3\big)}{288}+\frac{\pi^3(144\lambda-41\Lambda)}{864} \in \mathrm{i}\mathfrak{Z}_4(6),\end{aligned} \tag{2.18}$$

$$\mathscr{S}_3(4) = 4\mathcal{G} - \frac{\pi\lambda^2}{8} - \frac{\pi^3}{32} \in \mathrm{i}\mathfrak{Z}_3(4), \tag{2.19}$$

$$\mathscr{S}_4(4) = 8\,\mathrm{Im}\,\mathrm{Li}_4\left(\frac{1+\mathrm{i}}{2}\right) - 4\beta(4) + \frac{\pi\lambda^3}{24} + \frac{\pi^3\lambda}{32} \in \mathrm{i}\mathfrak{Z}_4(4), \tag{2.20}$$

and

$$\begin{aligned}\mathscr{S}_3\left(-\frac{9}{4}\right) ={}& \frac{4\,\mathrm{Li}_3\left(\frac{1}{3}\right)}{3} + 2\,\mathrm{Li}_3\left(\frac{1}{4}\right) - \frac{5\zeta(3)}{9} + 2\,\mathrm{Li}_2\left(\frac{1}{4}\right)\lambda \\ &+ \frac{2\left(6\lambda^3 - \Lambda^3\right)}{9} - \frac{\pi^2(3\lambda - 2\Lambda)}{9} \in \mathfrak{Z}_3(6),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathscr{S}_4\left(-\frac{9}{4}\right) ={}& \frac{80\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{9} - \frac{40\,\mathrm{Li}_4\left(\frac{1}{3}\right)}{3} + 8\,\mathrm{Li}_4\left(\frac{2}{3}\right) + \frac{7\,\mathrm{Li}_4\left(\frac{1}{4}\right)}{2} \\ &+ \frac{5\,\mathrm{Li}_4\left(\frac{1}{9}\right)}{6} + 4\,\mathrm{Li}_3\left(\frac{1}{3}\right)\lambda + 3\,\mathrm{Li}_3\left(\frac{1}{4}\right)\lambda - \frac{50\zeta(3)\lambda}{9} \\ &- \frac{35\lambda^4 - 54\lambda^2\Lambda^2 + 54\lambda\Lambda^3 - 9\Lambda^4}{27} - \frac{\pi^2\lambda(11\lambda - 36\Lambda)}{54} - \frac{101\pi^4}{1620} \in \mathfrak{Z}_4(6),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mathscr{S}_3(-4) ={}& -2\,\mathrm{Li}_3\left(\sqrt{2}-1\right) + \frac{4\sqrt{2}L_{8,2}(3)}{3} + \frac{25\zeta(3)}{16} \\ &- 2\,\mathrm{Li}_2\left(\sqrt{2}-1\right)\widetilde{\lambda} - \frac{2\widetilde{\lambda}^3}{3} + \frac{\lambda\widetilde{\lambda}^2}{2} - \frac{\pi^2\lambda}{8} \in \mathfrak{Z}_3(8),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\mathscr{S}_4(-4) ={}& \frac{40\,\mathrm{Li}_4\left(1-\frac{1}{\sqrt{2}}\right)}{7} + \frac{4\,\mathrm{Li}_4\left(\sqrt{2}-1\right)}{21} + \frac{4\,\mathrm{Li}_4\left(\frac{1}{\sqrt{2}}\right)}{7} - \frac{27\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{28} - \frac{59\,\mathrm{Li}_4\left(\left(\sqrt{2}-1\right)^2\right)}{14} \\ &+ \frac{19\,\mathrm{Li}_4\left(\left(\sqrt{2}-1\right)^4\right)}{84} - \frac{2\,\mathrm{Li}_4\left(\frac{1-\frac{1}{\sqrt{2}}}{2}\right)}{21} + \frac{8\sqrt{2}L_{8,2}(3)\widetilde{\lambda}}{3} - 4\,\mathrm{Li}_3\left(\frac{1}{\sqrt{2}}\right)\widetilde{\lambda} \\ &+ \frac{7\zeta(3)\widetilde{\lambda}}{16} + \frac{600\lambda^3\widetilde{\lambda} + 1224\lambda^2\widetilde{\lambda}^2 + 96\lambda\widetilde{\lambda}^3 - 752\widetilde{\lambda}^4 - 177\lambda^4}{4032} \\ &- \frac{\pi^2\left(189\lambda\widetilde{\lambda} - 61\widetilde{\lambda}^2 - 30\lambda^2\right)}{504} + \frac{11\pi^4}{7560} \in \mathfrak{Z}_4(8),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\sqrt{2}\mathscr{S}_3\left(-\frac{1}{2}\right) ={}& -80\,\mathrm{Li}_3\left(\frac{1}{\sqrt{2}}\right) + 64\sqrt{2}L_{8,2}(3) + \frac{35\zeta(3)}{4} - 20\,\mathrm{Li}_2\left(\sqrt{2}-1\right)\lambda \\ &+ 10\lambda^2\widetilde{\lambda} - 10\lambda\widetilde{\lambda}^2 + \frac{5\lambda^3}{3} - \frac{15\pi^2\lambda}{4} \in \mathfrak{Z}_3(8),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\sqrt{2}\mathscr{S}_4\left(-\frac{1}{2}\right) ={}& \frac{1669\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{14} - \frac{2112\,\mathrm{Li}_4\left(1-\frac{1}{\sqrt{2}}\right)}{7} - \frac{704\,\mathrm{Li}_4\left(\sqrt{2}-1\right)}{21} + \frac{24\,\mathrm{Li}_4\left(\frac{1}{\sqrt{2}}\right)}{7} \\ &+ \frac{1510\,\mathrm{Li}_4\left(\left(\sqrt{2}-1\right)^2\right)}{7} - \frac{475\,\mathrm{Li}_4\left(\left(\sqrt{2}-1\right)^4\right)}{42} + \frac{352\,\mathrm{Li}_4\left(\frac{1-\frac{1}{\sqrt{2}}}{2}\right)}{21} \\ &- 100\,\mathrm{Li}_3\left(\frac{1}{\sqrt{2}}\right)\lambda + \frac{224\sqrt{2}L_{8,2}(3)\lambda}{3} + \frac{175\zeta(3)\lambda}{16} \\ &+ \frac{2\left(99\lambda^3\widetilde{\lambda} - 297\lambda^2\widetilde{\lambda}^2 - 132\lambda\widetilde{\lambda}^3 + 299\widetilde{\lambda}^4 + 309\lambda^4\right)}{63} \\ &+ \frac{\pi^2\left(1848\lambda\widetilde{\lambda} - 1336\widetilde{\lambda}^2 - 2115\lambda^2\right)}{252} + \frac{397\pi^4}{3780} \in \mathfrak{Z}_4(8),\end{aligned} \tag{2.26}$$

$$\sqrt{5}\mathscr{S}_3\left(-\frac{16}{5}\right) = \frac{5\,\mathrm{Li}_3\left(\frac{1}{5}\right)}{4} + \frac{27\,\mathrm{Li}_3\left(\frac{1}{\phi}\right)}{2} - 10\,\mathrm{Li}_3\left(\frac{1}{\sqrt{5}}\right) - \frac{27\zeta(3)}{20}$$

$$+\frac{5\left[\operatorname{Li}_2\left(\frac{1}{5}\right)-4\operatorname{Li}_2\left(\frac{1}{\sqrt{5}}\right)\right]\mathscr{L}}{8}-\frac{9\pounds^3}{2}+\frac{27\pi^2\pounds}{20}-\frac{5\pi^2\mathscr{L}}{16}\in\mathfrak{Z}_3(10), \tag{2.27}$$

$$\begin{aligned}\sqrt{3}\mathscr{S}_3\left(-\frac{4}{3}\right)={}&-\frac{21\operatorname{Li}_3\left(\frac{1}{3}\right)}{10}-\frac{7\operatorname{Li}_3\left(\frac{1}{4}\right)}{40}-\operatorname{Li}_3\left(\frac{\sqrt{3}-1}{2}\right)+\frac{11\operatorname{Li}_3\left(1-\frac{\sqrt{3}}{2}\right)}{20}+\frac{9\operatorname{Li}_3\left(\frac{2-\sqrt{3}}{3}\right)}{5}\\&-7\operatorname{Li}_3\left(2-\sqrt{3}\right)+\frac{24\operatorname{Li}_3\left(2\sqrt{3}-3\right)}{5}+\frac{11\operatorname{Li}_3\left(3\sqrt{3}-5\right)}{5}+\frac{3\sqrt{3}L_{12,4}(3)}{5}\\&+\frac{39\zeta(3)}{10}+\frac{3\operatorname{Li}_2\left(\frac{1}{4}\right)\varLambda}{8}-3\operatorname{Li}_2\left(\frac{\sqrt{3}-1}{2}\right)\varLambda-3\operatorname{Li}_2\left(2-\sqrt{3}\right)\varLambda-\frac{17\lambda^2\widetilde{\varLambda}}{80}\\&+\frac{71\lambda\widetilde{\varLambda}^2}{80}-\frac{3\lambda\varLambda\widetilde{\varLambda}}{4}-\frac{3\varLambda^2\widetilde{\varLambda}}{20}+\frac{21\varLambda\widetilde{\varLambda}^2}{40}-\frac{209\widetilde{\varLambda}^3}{240}+\frac{13\lambda^3}{80}+\frac{3\lambda^2\varLambda}{8}+\frac{\varLambda^3}{20}\\&+\frac{7\pi^2\widetilde{\varLambda}}{40}-\frac{7\pi^2\lambda}{20}-\frac{7\pi^2\varLambda}{80}\in\mathfrak{Z}_3(12).\end{aligned} \tag{2.28}$$

Here in (2.16) and (2.20), the Dirichlet beta value

$$\beta(4):=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}$$

is yet another generalization of Catalan's constant

$$G:=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2};$$

in (2.16), (2.18), (2.23), (2.24), (2.25), (2.26), and (2.28), we have the Dirichlet L-values

$$\begin{aligned}L_{8,2}(3)&:=\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}}{(2n+1)^3},\\L_{8,4}(4)&:=\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(4n+1)^4}+\frac{1}{(4n+3)^4}\right],\\L_{3,2}(4)&:=\sum_{n=0}^{\infty}\left[\frac{1}{(3n+1)^4}-\frac{1}{(3n+2)^4}\right],\\L_{12,4}(3)&:=\sum_{n=0}^{\infty}\left[\frac{1}{(12n+1)^3}-\frac{1}{(12n+5)^3}-\frac{1}{(12n+7)^3}+\frac{1}{(12n+11)^3}\right].\end{aligned}$$

For each positive integer k greater than 1, it is also worth mentioning that the convergent series $\mathscr{S}_k(z)$ in (1.4) can be written as a generalized hypergeometric function

$$\mathscr{S}_k(z)={}_{k+1}F_k\left(\left.\begin{matrix}\overbrace{\frac{1}{2},\dots,\frac{1}{2}}^{k-1},1,1\\\underbrace{\frac{3}{2},\dots,\frac{3}{2}}_{k}\end{matrix}\right|\frac{z}{4}\right),$$

which does not reduce to elementary expressions. (For $k=2$, using `FunctionExpand` in `Mathematica` v14.0, one can check that

$$\begin{aligned}&\frac{1-w^2}{w}{}_3F_2\left(\left.\begin{matrix}\frac{1}{2},1,1\\\frac{3}{2},\frac{3}{2}\end{matrix}\right|-\frac{1}{4}\left(\frac{1-w^2}{w}\right)^2\right)\\&\qquad=-2[\operatorname{Li}_2(w)-\operatorname{Li}_2(-w)]-2\log(w)\log\left(\frac{1-w}{1+w}\right)+\frac{\pi^2}{2}\end{aligned}$$

holds when $0<w<1$, but reductions for larger integers k are not automated yet.) Our results in Theorems 2.3 and 2.4 also admit natural extensions to analytic continuations of these ${}_{k+1}F_k$ functions, outside the domain of convergence for the infinite series $\mathscr{S}_k(z)$.

3 Evaluations of Chen's series

To evaluate (1.1) and (1.2), we first convert them into integrals over polylogarithmic expressions, and then compute their integral representations by the function `MZIntegrate` in Au's `MultipleZetaValues` package [3].

Proof of Theorem 2.1. In view of the beta integral identity

$$\int_0^1\left(\frac{x}{1+x^2}\right)^{2n+1}\frac{\mathrm{d}x}{1+x^2}=\frac{1}{4(2n+1)\binom{2n}{n}}$$

together with the series bisection yielding

$$\sum_{n=0}^\infty\frac{z^{2n+1}}{(2n+1)^2}=\frac{\mathrm{Li}_2(z)-\mathrm{Li}_2(-z)}{2},$$

we may recast Chen's series (1.1) into

$$2\int_0^1\frac{\mathrm{Li}_2\big(\frac{x}{1+x^2}\big)-\mathrm{Li}_2\big(-\frac{x}{1+x^2}\big)}{1+x^2}\,\mathrm{d}x, \tag{3.1}$$

by reversing the order of integration and infinite summation (which is justified by the dominated convergence theorem). Feeding

```
MZIntegrate[(2*(PolyLog[2, x/(1 + x^2)]
 - PolyLog[2, -x/(1 + x^2)]))/(1 + x^2), {x, 0, 1}]
```

to `Mathematica` after loading Au's `MultipleZetaValues` package [3], we receive an output that is equivalent to the desired evaluation in (2.5). ∎

Proof of Theorem 2.2. As a variation on (3.1), we may represent (1.2) by

$$\frac{2}{\mathrm{i}}\int_0^1\frac{\mathrm{Li}_2\big(\frac{\mathrm{i}x}{1+x^2}\big)-\mathrm{Li}_2\big(-\frac{\mathrm{i}x}{1+x^2}\big)}{1+x^2}\,\mathrm{d}x. \tag{3.2}$$

Unfortunately, the function `MZIntegrate` in the current version (v1.2.0) of Au's `MultipleZeta-Values` package [3] does not automatically evaluate (3.2) in terms of CMZVs.

Nevertheless, we may reincarnate (3.2) into a form that is amenable to `MZIntegrate`, by considering symmetries and deforming contours. Concretely speaking, the inversion $x\mapsto\frac{1}{x}$ and reflection $x\mapsto -x$ symmetries allow us to identify (3.2) with

$$\frac{1}{\mathrm{i}}\int_0^\infty\frac{\mathrm{Li}_2\big(\frac{\mathrm{i}x}{1+x^2}\big)-\mathrm{Li}_2\big(-\frac{\mathrm{i}x}{1+x^2}\big)}{1+x^2}\,\mathrm{d}x=\frac{1}{\pi}\int_{-\infty}^\infty\frac{\mathrm{Li}_2\big(\frac{\mathrm{i}x}{1+x^2}\big)-\mathrm{Li}_2\big(-\frac{\mathrm{i}x}{1+x^2}\big)}{1+x^2}\log\frac{x}{\mathrm{i}}\,\mathrm{d}x. \tag{3.2$'$}$$

Subsequently, we may close the contour upwards while exploiting a jump relation

$$\mathrm{Li}_2(\xi+\mathrm{i}0^+)-\mathrm{Li}_2(\xi-\mathrm{i}0^+)=2\pi\mathrm{i}\log\xi$$

for $\xi\in(1,\infty)$, which yields

$$2\mathrm{i}\int_{\mathrm{i}/\phi}^{\mathrm{i}}\frac{\log\big(-\frac{\mathrm{i}z}{1+z^2}\big)}{1+z^2}\log\frac{z}{\mathrm{i}}\mathrm{d}z+2\mathrm{i}\int_{\mathrm{i}}^{\mathrm{i}\phi}\frac{\log\big(\frac{\mathrm{i}z}{1+z^2}\big)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z \tag{3.2$''$}$$

for $\phi:=\frac{\sqrt{5}+1}{2}$. Now, throwing the last two integrals as

```
FullSimplify[MZIntegrate[2*I*(Log[(-I)*(z/(1 +
z^2))]/(1 + z^2))*Log[z/I]*D[I*((Sqrt[5] - 1)/2)*(1 - t) +
I*t, t] //.z -> I*((Sqrt[5] - 1)/2)*(1 - t) +
I*t, {t, 0, 1}] + MZIntegrate[2*I*(Log[I*(z/(1 + z^2))]/(1 +
z^2))*Log[z/I]*D[I*((Sqrt[5] + 1)/2)*t + I*(1 -
t), t] //. z -> I*((Sqrt[5] + 1)/2)*t + I*(1 -
t), {t, 0, 1}]]
```

into `Mathematica`, we may confirm (2.6). ■

4 Further generalizations of Chen's series

Proof of Theorem 2.3. By a natural extension of (3.2″), we may equate the infinite series in (2.7) with

$$2\mathrm{i}\int_{\mathrm{i}w}^{\mathrm{i}}\frac{\log\left(\frac{1-w^2}{\mathrm{i}w}\frac{z}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z+2\mathrm{i}\int_{\mathrm{i}}^{\mathrm{i}/w}\frac{\log\left(-\frac{1-w^2}{\mathrm{i}w}\frac{z}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z. \tag{4.1}$$

Integrals of this type can be handled by Panzer's `HyperInt` package [34] for `Maple`. For instance, one may type

```
f := (w, z) -> 2*I*log(-I*(1 - w^2)*z/(w*(1 + z^2)))
*log(-I*z)/(1 + z^2);
g := (w, z) -> 2*I*log((1 - w^2)*z*I/(w*(1 + z^2)))
*log(-I*z)/(1 + z^2);
fibrationBasis(fibrationBasis(fibrationBasis
(hyperInt(f(w, z), [z = 0 .. I]), [w])
- fibrationBasis(hyperInt(f(w, z), [z = 0 .. I*w]), [w])
- fibrationBasis(hyperInt(g(w, z), [z = 0 .. I]), [w])
+ fibrationBasis(hyperInt(g(w, z), [z = 0 .. I/w]), [w])
+ (2*polylog(3, (1 + w)/2) - 2*polylog(3, (1 - w)/2)
- 2*polylog(3, 1/2*(1 + 1/w)) + 2*polylog(3, 1/2*(1 - 1/w))
- (polylog(2, (1 + w)/2) - polylog(2, (1 - w)/2)
+ polylog(2, (1 + 1/w)/2) - polylog(2, (1 - 1/w)/2))*log(w)
- Pi*delta[w]*log((1 + w)/2)*log((1 + 1/w)/2)*I), [w]));
```

and check that the output is zero. This verifies (2.7) under the condition that $\operatorname{Im} w > 0$, since the symbol `delta[w]` in the last line of our code represents $\operatorname{Im} w/|\operatorname{Im} w|$ when $\operatorname{Im} w \neq 0$ [34, formula (3.6)]. After this, the case where $\operatorname{Im} w = 0$ can be determined by a limit procedure and a scrupulous analysis for the values of polylogarithms on their branch cuts. ■

Remark 4.1. One may recover the evaluations in Theorems 2.1 and 2.2 by setting $w = e^{\pi\mathrm{i}/6}$ and $w = \frac{\sqrt{5}-1}{2}$ in (2.7) before invoking the function `MZExpand` in Au's `MultipleZetaValues` package [3].

Now, so long as $\operatorname{Im}(\mathrm{i}w) > 0$ and $\operatorname{Im}(\mathrm{i}/w) > 0$, we may upgrade our derivations for (4.1) into

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^k\binom{2n}{n}}\left(\frac{1-w^2}{w}\right)^{2n+1}&=2\mathrm{i}\int_0^1\frac{\mathrm{Li}_{k-1}\left(\frac{1-w^2}{\mathrm{i}w}\frac{x}{1+x^2}\right)-\mathrm{Li}_{k-1}\left(-\frac{1-w^2}{\mathrm{i}w}\frac{x}{1+x^2}\right)}{1+x^2}\,\mathrm{d}x\\&=\frac{2\mathrm{i}}{(k-2)!}\int_{\mathrm{i}w}^{\mathrm{i}}\frac{\log^{k-2}\left(\frac{1-w^2}{\mathrm{i}w}\frac{z}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z\end{aligned}$$

$$+\frac{2\mathrm{i}}{(k-2)!}\int_{\mathrm{i}}^{\mathrm{i}/w}\frac{\log^{k-2}\left(-\frac{1-w^2}{\mathrm{i}w}\frac{z}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z, \tag{4.2}$$

after recalling a jump relation $\mathrm{Li}_s(\xi+\mathrm{i}0^+)-\mathrm{Li}_s(\xi-\mathrm{i}0^+)=\frac{2\pi\mathrm{i}}{(s-1)!}\log^{s-1}\xi$ for $\xi\in(1,\infty)$.

To evaluate the last two contour integrals in (4.2), we need generalized polylogarithms (GPLs), which are defined recursively by an integral along a straight line segment

$$G(\alpha_1,\dots,\alpha_n;z):=\int_0^z\frac{G(\alpha_2,\dots,\alpha_n;x)\,\mathrm{d}x}{x-\alpha_1} \tag{4.3}$$

for $|\alpha_1|+\dots+|\alpha_n|\neq0$, with the boundary conditions that

$$G(\underbrace{0,\dots,0}_{m};z):=\frac{\log^m z}{m!},\qquad G(—;z):=1. \tag{4.4}$$

One can convert GPLs to (analytic continuations of) MPLs [cf. (2.1)] via the following equation:

$$G(\underbrace{0,\dots,0}_{a_1-1},\widetilde{\alpha}_1,\underbrace{0,\dots,0}_{a_2-1},\widetilde{\alpha}_2,\dots,\underbrace{0,\dots,0}_{a_n-1},\widetilde{\alpha}_n;z)=(-1)^n\,\mathrm{Li}_{a_1,\dots,a_n}\left(\frac{z}{\widetilde{\alpha}_1},\frac{\widetilde{\alpha}_1}{\widetilde{\alpha}_2},\dots,\frac{\widetilde{\alpha}_{n-1}}{\widetilde{\alpha}_n}\right), \tag{4.5}$$

where $\prod_{j=1}^n\widetilde{\alpha}_j\neq0$. The corresponding integral relations date back to the work of E.E. Kummer [26, 27, 28], while the GPL-MPL conversion (4.5) can be found in [6, Section 4.2]. This allows us to reformulate (2.2) into

$$\mathfrak{Z}_k(N):=\operatorname{span}_{\mathbb{Q}}\left\{G(z_1,\dots,z_k;z)\,\middle|\,\begin{matrix}z_1^N,\dots,z_k^N\in\{0,1\},\\ z_1\neq1,\ z_k\neq0,\ z^N=1\end{matrix}\right\}. \tag{2.2′}$$

In view of (4.3) and (4.4), we have $G(\pm1;z/\mathrm{i})=\log(1\pm\mathrm{i}z)$ and $G(0;z/\mathrm{i})=\log\frac{z}{\mathrm{i}}$. The function $\log^{k-2}\left(\pm\frac{z/\mathrm{i}}{1+z^2}\right)\log\frac{z}{\mathrm{i}}$ is a $\mathbb{Q}$-linear combination of GPLs in the form of $G(\alpha_1,\dots,\alpha_{k-1};z/\mathrm{i})$, where $\alpha_1,\dots,\alpha_j\in\{-1,0,1\}$, as can be seen by repeated invocations of a shuffle product [4, 23]

$$\begin{aligned}G(\alpha;t)G(\beta_1,\dots,\beta_r;t)={}&G(\alpha,\beta_1,\dots,\beta_r;t)+\sum_{j=1}^{r-1}G(\beta_1,\dots,\beta_j,\alpha,\beta_{j+1},\dots,\beta_r;t)\\&+G(\beta_1,\dots,\beta_r,\alpha;t).\end{aligned}$$

Therefore, the GPL recursion in (4.3) tells us that

$$\mathrm{i}\int_{\mathrm{i}w}^{\mathrm{i}}\frac{\log^{k-2}\left(\frac{z/\mathrm{i}}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z\in\mathfrak{Z}_k(2)+\mathfrak{H}_k^{(w)}(2), \tag{4.6}$$

$$\mathrm{i}\int_{\mathrm{i}}^{\mathrm{i}/w}\frac{\log^{k-2}\left(-\frac{z/\mathrm{i}}{1+z^2}\right)}{1+z^2}\log\frac{z}{\mathrm{i}}\,\mathrm{d}z\in\mathfrak{Z}_k(2)+\mathfrak{H}_k^{(1/w)}(2), \tag{4.7}$$

where

$$\begin{aligned}\mathfrak{H}_k^{(z)}(2)&:=\operatorname{span}_{\mathbb{Q}}\left\{(\pi\mathrm{i})^{r-\ell}G(\alpha_1,\dots,\alpha_\ell;z)\,\middle|\,\begin{matrix}\alpha_1^2,\dots,\alpha_\ell^2\in\{0,1\},\\ 0\le\ell\le r,\ \alpha_1\neq z\end{matrix}\right\}\\&\overset{[19,\text{ p. }24]}{=\!=\!=}\operatorname{span}_{\mathbb{Q}}\left\{(\pi\mathrm{i})^{r-\ell}G(\alpha_1,\dots,\alpha_m;z)\log^{\ell-m}z\,\middle|\,\begin{matrix}\alpha_1^2,\dots,\alpha_m^2\in\{0,1\},\\ 0\le m\le\ell\le r,\\ \alpha_1\neq z,\ \alpha_m\neq0\end{matrix}\right\}\end{aligned} \tag{4.8}$$

is a $\mathbb{Q}$-vector space of *hyperlogarithms* [32, 33].

Proof of Theorem 2.4. (a) For the left-hand side of (2.8), there are four possible choices of w, namely, $\pm e^{2m\pi i/N}$ and $\pm e^{-2m\pi i/N}$. Without loss of generality, we may assume that $\operatorname{Im}\big(ie^{2m\pi i/N}\big) > 0$ and $\operatorname{Im}\big(ie^{-2m\pi i/N}\big) > 0$, so that $w = e^{2m\pi i/N}$ is applicable to (4.2). Consequently, the right-hand sides of both (4.6) and (4.7) are subsets of $\mathfrak{Z}_k(\operatorname{lcm}(2,N))$, as one can check by comparing (2.2′) and (4.8) against the fact that $\pi i \in \mathfrak{Z}_1(N)$ for $N-2 \in \mathbb{Z}_{>0}$ [3, Lemma 4.1]. In addition, the relation

$$\log^k\left(\frac{1-w^2}{w}\right) = [G(-1;w) - G(0;w) + G(1;w)]^k \in \mathfrak{H}_k^{(w)}(2) \subseteq \mathfrak{Z}_k(\operatorname{lcm}(2,N))$$

also holds for $w = e^{2m\pi i/N}$. Therefore, the right-hand side of (2.8) is a result of Goncharov's filtration $\mathfrak{Z}_j(M)\mathfrak{Z}_k(M) \subseteq \mathfrak{Z}_{j+k}(M)$ [22, Section 1.2] for $M = \operatorname{lcm}(2,N)$.

(b) For $N-2 \in \mathbb{Z}_{>0}$ and $w \neq 0$, we have an embedding $\mathfrak{H}_k^{(w)}(2) \subseteq \mathfrak{Z}_k(N)$ if $\log w \in \mathfrak{Z}_k(N)$ and

```
IterIntDoableQ[{0, 1/w, -1/w}]
```

returns a positive divisor of N in Au's `MultipleZetaValues` package [3]. Particular cases of such embeddings include

$$\begin{aligned}
&\mathfrak{H}_k^{(2)}(2) \subseteq \mathfrak{Z}_k(6), && \mathfrak{H}_k^{(1/2)}(2) \subseteq \mathfrak{Z}_k(6),\\
&\mathfrak{H}_k^{(\sqrt{2}+1)}(2) \subseteq \mathfrak{Z}_k(8), && \mathfrak{H}_k^{(\sqrt{2}-1)}(2) \subseteq \mathfrak{Z}_k(8),\\
&\mathfrak{H}_k^{(\sqrt{2})}(2) \subseteq \mathfrak{Z}_k(8), && \mathfrak{H}_k^{(1/\sqrt{2})}(2) \subseteq \mathfrak{Z}_k(8),\\
&\mathfrak{H}_k^{(\phi)}(2) \subseteq \mathfrak{Z}_k(10), && \mathfrak{H}_k^{(1/\phi)}(2) \subseteq \mathfrak{Z}_k(10),\\
&\mathfrak{H}_k^{(\sqrt{5})}(2) \subseteq \mathfrak{Z}_k(10), && \mathfrak{H}_k^{(1/\sqrt{5})}(2) \subseteq \mathfrak{Z}_k(10),\\
&\mathfrak{H}_k^{(\sqrt{3})}(2) \subseteq \mathfrak{Z}_k(12), && \mathfrak{H}_k^{(1/\sqrt{3})}(2) \subseteq \mathfrak{Z}_k(12).
\end{aligned}$$

Meanwhile, we also have

$$\log\left(\frac{1-w^2}{w}\right) = \begin{cases} \log\frac{3}{2} \in \mathfrak{Z}_1(6) & \text{if } w = \frac{1}{2},\\ \log 2 \in \mathfrak{Z}_1(2) \subset \mathfrak{Z}_1(8) & \text{if } w = \sqrt{2}-1,\\ \log\frac{1}{\sqrt{2}} \in \mathfrak{Z}_1(2) \subset \mathfrak{Z}_1(8) & \text{if } w = \frac{1}{\sqrt{2}},\\ 0 & \text{if } w = \frac{1}{\phi},\\ \log\frac{4}{\sqrt{5}} \in \mathfrak{Z}_1(10) & \text{if } w = \frac{1}{\sqrt{5}},\\ \log\frac{2}{\sqrt{3}} \in \mathfrak{Z}_1(6) \subset \mathfrak{Z}_1(12) & \text{if } w = \frac{1}{\sqrt{3}}. \end{cases}$$

Therefore, the right-hand sides of (2.9)–(2.14) all result from Goncharov's filtrations [22, Section 1.2] $\mathfrak{Z}_j(N)\mathfrak{Z}_k(N) \subseteq \mathfrak{Z}_{j+k}(N)$ for $N \in \{6, 8, 10, 12\}$. ∎

Remark 4.2. Evaluating the integral representations in part (a) explicitly for $w = e^{\pi i/4}$, $w = e^{\pi i/3}$, and $w = e^{\pi i/2}$ in Au's `MultipleZetaValues` package [3], we get (2.15)–(2.20). A similar service on part (b) brings us (2.21)–(2.28).

Remark 4.3. To symbolically check any individual case among (2.15)–(2.28), one simply implements (4.2) as

```
((2*I)/(k - 2)!)*
(MZIntegrate[(Log[((1 - w^2)/(I*w))*(z/(1 + z^2))]^(k - 2)
/(1 + z^2))*Log[z/I]*D[I*w*(1 - t) + I*t, t]
```

```
//. z -> I*w*(1 - t) + I*t, {t, 0, 1}] +
MZIntegrate[(Log[(-((1 - w^2)/(I*w)))*(z/(1 + z^2))]^(k - 2)
/(1 + z^2))*Log[z/I]*D[I*(1 - t) + (I/w)*t, t]
//. z -> I*(1 - t) + (I/w)*t, {t, 0, 1}])
```

in `Mathematica`, after choosing appropriate values for k and w. For example, we need $k = 3$ and $w = \mathrm{e}^{\pi\mathrm{i}/4}$ for the analytic expression of $\frac{1}{\sqrt{2}}\frac{1-w^2}{w} = -\mathrm{i}$ times (2.15).

So far, we have limited the scope of this section to $\mathscr{S}_k(z)$ where $k-1 \in \mathbb{Z}_{>0}$. Here, we point out that both

$$\mathscr{S}_1(z) = \frac{4}{\sqrt{z(4-z)}}\arcsin\frac{\sqrt{z}}{2}, \qquad \mathscr{S}_0(z) = \frac{4}{\sqrt{4-z}(4-z)}\left(\sqrt{4-z} + \sqrt{z}\arcsin\frac{\sqrt{z}}{2}\right)$$

are classical results, while the closed forms of $\mathscr{S}_k(z)$ for negative integers k can be deduced from the well-studied series of Apéry–Lehmer type [20, 21, 40]

$$S_\ell(z) := \sum_{n=1}^{\infty} \frac{n^\ell z^n}{\binom{2n}{n}},$$

where $\ell \in \{1, 2, \dots, |k|\}$.

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