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Sign Convention for A_∞-Operations in Bott–Morse Case

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Abstract. We describe the sign and orientation issue appearing the filtered A_∞-formulae in Lagrangian Floer theory using de Rham model in Bott–Morse setting. After giving the definition of filtered A_∞-operations in a Fukaya category, we verify the filtered A_∞-formulae.

Key words: filtered A_∞-operation; Kuranishi structure; bordered stable map

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1 Introduction

The aim of this note is to describe the sign and orientation issue appearing the filtered A_∞-formulae in Lagrangian Floer theory using de Rham model in Bott–Morse setting. When we work with only one relatively spin Lagrangian submanifold, we constructed the filtered A_∞-algebra in [4, 5] using the singular chain complex model. The sign and orientation are explained in [5, Sections 8.3–8.5]. In the de Rham model version, see [6, Section 22.4] and also [7]. We gave a construction of the filtered A_∞-bimodule using the singular chain model in [4, 5], especially, the sign and orientation are described in [5, Section 8.8]. Sign and orientation in Bott–Morse Hamiltonian Floer complex using the de Rham model version, see [6, Definition 19.3 and Proposition 19.5]. In this note, we discuss the sign and orientation issue appearing in the construction of the filtered A_∞-category for a collection of finitely many (relatively) spin Lagrangian submanifolds. The construction of Kuranishi structures (a version of a tree-like K-system in the sense of [6]) on moduli spaces of stable holomorphic polygons is discussed in other papers [1, 3]. Here, we give a definition of A_∞-operations in Bott–Morse case (see Definition 3.3) using such Kuranishi structures. We verify the sign convention by showing the filtered A_∞-relation (see Theorem 4.4).

2 Preliminaries

We use the convention on orientation on the fiber product (in the sense of Kuranishi structure) as in [5, Section 8.2]. Let $p\colon M \to N$ be a fiber bundle with oriented relative tangent bundle. Restrict the fiber bundle to an open subset U of N, we may assume that U is oriented. Then we give an orientation on $p^{-1}(U) \subset M$ using the isomorphism $TM = p^*TN \oplus T_{\rm fiber}M$, where $T_{\rm fiber}M$ is the relative tangent bundle. Then our convention of the integration along fibers of $p\colon M \to N$ is

$$\int_U \alpha \wedge p_!\beta = \int_{p^{-1}(U)} p^*\alpha \wedge \beta,$$

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where $\alpha \in \Omega^*(U)$ and $\beta \in \Omega^*\big(p^{-1}(U)\big)$, Reversing the orientation of U induces reversing of the orientation of $p^{-1}(U)$, hence the push-forward $p_!\beta$ does not depend on the choice of the orientation of U. Therefore, for a proper submersion $p\colon M \to N$ with the oriented relative tangent bundle, the integration along fibers

$$p_!\colon\ \Omega^k(M) \to \Omega^{k-\dim M+\dim N}(N)$$

is well defined.

We have the following properties.

Proposition 2.1.

(1) $p_!((p^*\theta)\wedge\beta) = \theta\wedge(p_!\beta)$, *where* $\theta \in \Omega^*(N)$ *and* $\beta \in \Omega^*(M)$.

(2) *Let* $p\colon M \to N$ *and* $q\colon N \to B$ *be fiber bundles with oriented relative tangent bundles. For* $\beta \in \Omega^*(M)$, *we have* $(q\circ p)_!\beta = q_!\circ p_!(\beta)$.

Using them, we find the following.

Corollary 2.2. $(q\circ p)_!(p^*\theta\wedge\beta) = q_!(\theta\wedge p_!\beta)$.

Proposition 2.3 (base change)**.** *Let* $f\colon S \to N$ *be a smooth map. Denote by* $\overline{p}\colon f^*M \to S$ *the pullback of the fiber bundle* $p\colon M \to N$ *and* $\widetilde{f}\colon f^*M \to M$ *the bundle map covering* f. *Then we have* $f^*\circ p_! = \overline{p}_!\circ\widetilde{f}^*$.

Proposition 2.4 (Stokes type formula, [6, Theorem 9.28])**.** *Let* $p\colon M \to N$ *be a smooth map* (*or a strongly smooth map from a space with Kuranishi structure to a smooth manifold*)

$$\mathrm{d}p_!\beta = p_!\mathrm{d}\beta + (-1)^{\dim M+\deg\beta}p|_{\partial M}\beta.$$

We introduced the notions of a strongly smooth map and a weakly submersive strongly smooth map from a space equipped with Kuranishi structure to a smooth manifold in [6, Definition 3.40 (4), (5)]. We call a space equipped with a Kuranishi structure a K-space for short. For a proper weakly submersive strongly smooth map p from a K-space X to a manifold N, we define the integration along fibers using a CF-perturbation, see [6, Section 9.2]. In this note, we suppress the notation for Kuranishi structures or good coordinate systems as well as CF perturbations. Refer the indicated places in [6] for detailed statements. For the verification of the sign convention in the filtered A_∞-relations, it is sufficient to treat the integration along fibers of a proper weakly submersive strongly smooth map as if the one for proper submersion between smooth manifolds.

The statements above holds for a proper weakly submersive strongly smooth map p. For Proposition 2.3, f^*M is the fiber product of $f\colon S \to N$ and $p\colon M \to N$. When S and M are K-spaces with a strongly smooth map $f\colon S \to N$ in the sense of [6, Definition 3.40 (4)] and a weakly submersive strongly smooth map $p\colon M \to N$, we have a compatible system of smooth maps from Kuranishi charts of the fiber product $S\times_N M$ to the manifold N and the obstruction bundle on a fiber product Kuranishi chart of f^*M contains the pullback of the obstruction bundle on a Kuranishi chart of M as a subbundle. Using the pullback CF perturbation on f^*M, we obtain Proposition 2.3 in such a situation.

The integration along fibers changes the degree of differential forms by

$$\deg p_!\beta = \deg\beta - \mathrm{reldim}\ p. \tag{2.1}$$

Here $\mathrm{reldim}\ p = \dim X - \dim N$, where $\dim X$ is the dimension of X in the sense of K-space, see [6, p. 52]. A tuple $(X, f_1\colon X \to M_1, f_2\colon X \to M_2)$ is called a smooth correspondence, if X is a K-space, f_1, f_2 are strongly smooth maps and f_1 is weakly submersive. After taking CF-perturbations, we define

$$\mathrm{Corr}_X\colon\ \Omega^*(M_2) \to \Omega^*(M_1)$$

by $(f_1)_! \circ (f_2)^*$. For flat vector bundles $\mathcal{L}_i$ on M_i, $i = 1, 2$, with a given isomorphism

$$f_1^*\mathcal{L}_1 \cong O_{f_1} \otimes f_2^*\mathcal{L}_2,$$

where O_{f_1} is the orientation bundle of the relative tangent bundle of $f_1\colon X \to M_1$, Theorem 27.1 in [5] gives

$$\mathrm{Corr}_X\colon\ \Omega^*(M_2, \mathcal{L}_2) \to \Omega^*(M_1, \mathcal{L}_1).$$

Using Proposition 2.4, we have the following.

Proposition 2.5 ([6, Proposition 27.2])**.**

$$d \circ \mathrm{Corr}_X \xi = \mathrm{Corr}_X \circ d\xi + (-1)^{\dim X + \deg \xi}\mathrm{Corr}_{\partial X}\xi \qquad \text{for } \xi \in \Omega^*(M_2; \mathcal{L}_2).$$

Let $(X_{12}, f_{1,12}\colon X_{12} \to M_1, f_{2,12}\colon X_{12} \to M_2)$ and $(X_{23}, f_{2,23}\colon X_{23} \to M_2, f_{3,23}\colon X_{23} \to M_3)$ be smooth correspondences with given isomorphisms

$$f_{1,12}^*\mathcal{L}_1 \cong O_{f_{1,12}} \otimes f_{2,12}^*\mathcal{L}_2, \qquad f_{2,23}^*\mathcal{L}_2 \cong O_{f_{2,23}} \otimes f_{3,23}^*\mathcal{L}_3. \tag{2.2}$$

Taking the fiber product X_{13} over $f_{2,12}$ and $f_{2,23}$, we obtain a smooth correspondence

$$(X_{13}, f_{1,13}\colon X_{13} \to M_1, f_{3,13}\colon X_{13} \to M_3)$$

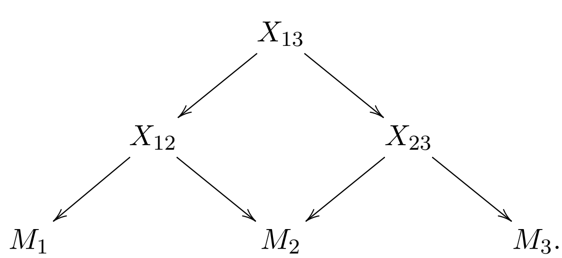
with the isomorphism

$$f_{1,13}^*\mathcal{L}_1 \cong O_{f_{1,13}} \otimes f_{3,13}^*\mathcal{L}_3$$

induced by (2.2) and

$$O_{f_{1,13}} \cong g_1^*O_{f_{1,12}} \otimes g_2^*O_{f_{2,23}}.$$

Here we denote by $g_1\colon X_{13} \to X_{12}$ and $g_2\colon X_{13} \to X_{23}$ the projections of the fiber product of Kuranishi charts,



Then we have the following.

Proposition 2.6 (composition formula, [6, Theorem 10.21])**.**

$$\mathrm{Corr}_{X_{13}} = \mathrm{Corr}_{X_{12}} \circ \mathrm{Corr}_{X_{23}}.$$

See [6, Chapter 27] in the case with coefficients in local systems, see [6, Theorems 27.1 and 27.2]. In fact, the composition formula is a consequence of the properties mentioned above.

3 Definition of A_∞-operations

Let $\{L_i\}$ be a relatively spin collection of Lagrangian submanifolds, which intersects cleanly in (X,ω). In a later argument, we glue the linearization operator of holomorphic polygons with a Cauchy–Riemann type operator at each boundary marked point, which is sent to the clean intersection of two branches of relatively spin Lagrangian submanifolds, to obtain a Cauchy–Riemann type operator on the unit disk. For the orientation issue, the argument works for clean intersections of distinct relative spin pair of Lagrangian submanifolds and clean self-intersection of a relative spin Lagrangian submanifold. The description of the boundary of holomorphic polygons in Lagrangian immersion case is found in the paper by Akaho and Joyce [2]. For the sign and orientation issue, the argument presented here is also valid for immersed Lagrangian submanifolds. Denote by R_α a connected component of L_i and L_j. (We also consider the case of self clean intersection.)

Let $(\Sigma,\partial\Sigma)$ be a bordered Riemann surface Σ of genus 0 and with connected boundary and $\vec{z}=(z_0,\dots,z_k)$ boundary marked points respecting the cyclic order on $\partial\Sigma$. Let $u\colon(\Sigma,\partial\Sigma)\to(X,\cup L_i)$ be a smooth map such that $u\big(\overset{\frown}{z_j z_{j+1}}\big)\subset L_{i_j}$, $j \mod k+1$, $u(z_j)\in R_{\alpha_j}$, where R_{α_j} is a connected component of $L_{i_{j-1}}\cap L_{i_j}$. (For an immersed Lagrangian with clean self intersection, R_α is a connected component of the clean intersection.) For such u and u', we introduce the equivalence relation $\sim$ so that $u\sim u'$ when $\int_\Sigma\omega=\int_{\Sigma'}\omega$ and the Maslov indices of u and u' are the same. Denote by B the equivalence class. In this note, the dimension of moduli spaces means their virtual dimension.

Consider the moduli space

$$\mathcal{M}_{k+1}\big(B;L_{i_0},\dots,L_{i_k};R_{\alpha_0},\dots R_{\alpha_k}\big)$$

of bordered stable maps of genus 0, with connected boundary and $(k+1)$ boundary marked points, representing the class B.

Set $\mathcal{L}=\big(L_{i_0},\dots,L_{i_k}\big)$ and $\mathcal{R}=\big(R_{\alpha_0},\dots R_{\alpha_k}\big)$ and write

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\mathcal{M}_{k+1}\big(B;L_{i_0},\dots,L_{i_k};R_{\alpha_0},\dots R_{\alpha_k}\big).$$

Denote by $\mathrm{ev}_j^B\colon\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_j}$ the evaluation map at z_j.

For a pair of Lagrangian submanifolds L, L' which intersect cleanly, we constructed the $O(1)$-local system $\Theta^-_{R_\alpha}$ on R_α in [5, Proposition 8.1.1]. Here R_α is a connected component of $L\cap L'$. In this note, we simply write it as Θ_{R_α}.

We recall the construction of Θ_{R_α} briefly. We assume that L, L' are equipped with spin structures. In the case of a relative spin pair, we take $TX\oplus(V\otimes\mathbb{C})$ (on the 3-skeleton of X) instead of TX and $TL\oplus V$ (resp. $TL'\oplus V$) (on the 2-skeleton of L (resp. L') instead of TL, (resp. TL'). Here V is an oriented real vector bundle on the 3-skeleton of X such that the restriction of $w_2(V)$ to the 2-skeleton of L (resp. L') coincides $w_2(TL)$ (resp. $w_2(TL')$). The relative spin structure with the background V is a choice of spin structure of $V\oplus TL$, (resp. $V\oplus TL'$). Then the argument goes in the same way. See the proof of [5, Theorem 8.1.1]. For a point p in the self clean intersection of a Lagrangian immersion $i\colon\widetilde{L}\to X$, there are two local branches of the Lagrangian immersion, i.e., $i_*\big(T_{p'}\widetilde{L}\big)$ and $i_*\big(T_{p''}\widetilde{L}\big)$ where $p',p''\in\widetilde{L}$ with $p=i(p')=i(p'')$. Then we run the argument below by replacing T_pL and T_pL' by $i_*\big(T_{p'}\widetilde{L}\big)$ and $i_*\big(T_{p''}\widetilde{L}\big)$, respectively.

As written in [5, Section 8.8], we consider the space $\mathcal{P}_{R_\alpha}(TL,TL')$ of paths of oriented Lagrangian subspaces in T_pX, $p\in R_\alpha$, of the form $R_\alpha\oplus\lambda(t)$ such that $R_\alpha\oplus\lambda(0)=T_pL$ and $R_\alpha\oplus\lambda(1)=T_pL'$. Here λ is regarded as a path of Lagrangian subspaces in

$$V_{R_\alpha}=\big(T_pL+T_pL'\big)/\big(T_pL+T_pL'\big)^{\perp_\omega}=\big(T_pL+T_pL'\big)/\big(T_pL\cap T_pL'\big),$$

which is a symplectic vector space. Pick a compatible complex structure on it and consider the Dolbeault operator $\overline{\partial}_\lambda$ on $Z_-=\big(D^2\cap\{\operatorname{Re}z\le 0\}\big)\cup([0,\infty)\times[0,1])$.

We set $\mu(R_\alpha;\lambda) = \text{Index}\,\overline{\partial}_\lambda$. The parity of $\mu(R_\alpha;\lambda)$ is independent of the choice of λ above, since $\lambda \oplus T_pR_\alpha$ is a path of oriented totally real subspaces of T_pX with fixed end points, T_pL, T_pL', $p \in R_\alpha$ which are oriented. Denote by $\mu(R_\alpha) = \mu(R_\alpha;\lambda) \mod 2$. Then we have

$$\dim \mathcal{M}_{k+1}(B;\mathcal{L},\mathcal{R}) \equiv \dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \sum_{i=1}^{k} \mu\big(R_{\alpha_i}\big) + k - 2 \mod 2. \tag{3.1}$$

We have the determinant line bundle of $\big\{\text{Index}\,\overline{\partial}_\lambda\big\}_{\lambda \in \mathcal{P}_{R_\alpha}(TL,TL')}$. Pick a hermitian metric on X. Denote by $P_{\text{SO}}(T_pR_\alpha \oplus \lambda)$ the associated oriented orthogonal frame bundle of $T_pR_\alpha \oplus \lambda$. Note that $P_{\text{SO}}(T_pR_\alpha \oplus \lambda)|_{t=0}$ and $P_{\text{SO}}(T_pR_\alpha \oplus \lambda)|_{t=1}$ are canonically identified with $P_{\text{SO}}(L)|_p$ and $P_{\text{SO}}(L')|_p$, respectively. We glue the principal spin bundle $P_{\text{Spin}}(T_pR_\alpha \oplus \lambda)$ at $t = 0, 1$ with $P_{\text{Spin}}(L)|_p$ and $P_{\text{Spin}}(L')|_p$. There are two isomorphic classes of resulting spin structure on the bundle $TL \cup (\lambda \oplus T_pR_\alpha) \cup TL'$ on $L \cup [0,1] \cup L'$, where $p \in L$ and $p \in L'$ are identified with $0, 1 \in [0,1]$, respectively. This gives an $O(1)$-local system O_{Spin} on $\mathcal{P}_{R_\alpha}(TL,TL')$. Proposition 8.1.1 in [5] states that the tensor product $\det \overline{\partial}_\lambda \otimes O_{\text{Spin}}$ descends to an $O(1)$-local system Θ_{R_α} on R_α.

We denote by $\overline{\partial}_{R_\alpha}$ is the Dolbeault operator acting on sections of the trivial bundle $Z_- \times (T_pR_\alpha \otimes \mathbb{C})$ on Z_- with totally real boundary condition T_pR_α. Then the operator $\overline{\partial}_{R_{\alpha_i} \oplus \lambda_i} = \overline{\partial}_{R_{\alpha_i}} \oplus \overline{\partial}_{\lambda_i}$ is the Dolbeault operator acting on the trivial bundle $Z_- \times T_pX$ on Z_- with the totally real boundary condition $T_pR_\alpha \oplus \lambda$. After gluing the linearization operator $D\overline{\partial}$ for a holomorphic polygon with $\overline{\partial}_{R_{\alpha_i} \oplus \lambda_i}$, where $R_{\alpha_i} \oplus \lambda_i \in \mathcal{P}_{R_{\alpha_i}}(TL_{i-1}, TL_i)$, $i = 0, \dots, k$, we obtain a Cauchy–Riemann type operator on the unit disk. By [5, Theorem 8.1.1], the relative spin structure for $\{L_i\}$, namely relative spin structures for each L_i with a common oriented vector bundle $V \to X^{[3]}$, determines an isomorphism Φ^B below. For the definition and properties of relative spin structure, see [5, Section 8.1.1].

Proposition 3.1 (cf. [5, Theorem 8.1.1])**.** *A choice of relative spin structure determines the following isomorphisms.*

(1) *Case that $k = 0$ (L is an immersed Lagrangian submanifold with clean self intersection or $R_{\alpha_0} = L$):*

$$\Phi^B\colon\ \text{ev}_0^{B*}\Theta_{R_{\alpha_0}} \to \text{ev}_0^{B*}O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_1(B;L)}.$$

(2) *Case that $k = 1$:*

$$\begin{aligned}\Phi^B\colon\ \text{ev}_0^{B*}\Theta_{R_{\alpha_0}} &\to \text{ev}_0^{B*}O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_2(B;\mathcal{L};\mathcal{R})} \otimes \mathbb{R}_B \otimes \text{ev}_1^{B*}\Theta_{R_{\alpha_1}} \\ &\cong (-1)^{\mu_{\alpha_1}} \text{ev}_0^{B*}O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_2(B;\mathcal{L};\mathcal{R})} \otimes \text{ev}_1^{B*}\Theta_{R_{\alpha_1}} \otimes \mathbb{R}_B.\end{aligned}$$

(3) *Case that $k \geq 2$:*

$$\begin{aligned}\Phi^B\colon\ \text{ev}_0^{B*}\Theta_{R_{\alpha_0}} &\to \text{ev}_0^{B*}O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} \otimes \mathfrak{forget}^* O^*_{\mathcal{M}_{k+1}} \\ &\otimes \text{ev}_1^{B*}\Theta_{R_{\alpha_1}} \otimes \cdots \otimes \text{ev}_k^{B*}\Theta_{R_{\alpha_k}}.\end{aligned}$$

In item (1), we suppress the orientation bundle of the biholomorphic automorphism group $\text{Aut}\big(D^2,1\big)$, since $\text{Aut}\big(D^2,1\big)$ is two-dimensional and does not affect the sign when we exchange $\text{Aut}\big(D^2,1\big)$ with other factors. In item (2), $\mathbb{R}_B$ is the group of translations in the domain $D^2 \setminus \{\pm 1\} \cong \mathbb{R} \times [0,1]$ and $\mathcal{M}_2(B;\mathcal{L};\mathcal{R})$ is the quotient of $\widetilde{\mathcal{M}}_2(B;\mathcal{L};\mathcal{R})$ by the translation action of $\mathbb{R}_B$ on the domain,

$$\widetilde{\mathcal{M}}_2(B;\mathcal{L};\mathcal{R}) = \mathcal{M}_2(B;\mathcal{L};\mathcal{R}) \times \mathbb{R}_B.$$

The sign of the exchange of $\mathbb{R}_B$ and the index of $\overline{\partial}_{\lambda_{R_{\alpha_1}}}$ is $(-1)^{\mu_{\alpha_1}}$. In item (3), $\mathcal{M}_{k+1}$ is the moduli space of bordered Riemann surfaces of genus 0, connected boundary and $(k+1)$ marked points on the boundary and $\mathfrak{forget}\colon \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to \mathcal{M}_{k+1}$ sends $[(\Sigma,\partial\Sigma,\vec{z}),u]$ to $[(\Sigma,\partial\Sigma,\vec{z})]$. Here $O_{R_{\alpha_0}}$, $O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$ and $O_{\mathcal{M}_{k+1}}$ are orientation bundles of R_{α_0}, $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ and $\mathcal{M}_{k+1}$, respectively. We consider $\mathrm{ev}_0^* O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$ the orientation bundle of the relative tangent bundle of $\mathrm{ev}_0\colon \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_{\alpha_0}$. In the notation in [5], we write

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = R_{\alpha_0} \times {}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$$

and

$$\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^{\circ} \times \mathcal{M}_{k+1}.$$

These descriptions are considered as the splitting of tangent spaces in the sense of Kuranishi structures. One may consider ${}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ and $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^{\circ}$ as a fiber of ev_0 and a fiber of $\mathfrak{forget}$, respectively. Using these notations, we have

$$\begin{aligned}&\mathrm{ev}_0^* O^*_{R_{\alpha_0}} \otimes O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} = O_{{}^{\circ}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})},\\ &O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} \otimes \mathfrak{forget}^* O^*_{\mathcal{M}_{k+1}} = O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^{\circ}}.\end{aligned}$$

If we denote by $\mathcal{M}_2$ the quotient stack of a point by $\mathbb{R}_B$, (2) is written in (3) with $k=2$. We give an orientation of $\mathcal{M}_{k+1} = \left(\partial D^2\right)^{k+1}/\mathrm{Aut}\left(D^2,\partial D^2\right)$ as the orientation of the quotient space following [5, convention (8.2.1.2)]. Then the orientation bundle of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ is canonically isomorphic to the one of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^{\circ}$. Hence, for $\mathbf{u} = \left[u\colon (\Sigma,\partial\Sigma,\vec{z}) \to \left(X, \bigcup_{L\in\mathcal{L}} L, \bigcup_{R_\alpha\in\mathcal{R}} R_\alpha\right)\right]$, the relative spin structure of $\mathcal{L}$, local sections σ_{α_i} of $O(1)$-local systems Θ_{α_i} around $u(z_i)$, $i=0,1,\dots,k$, determines a local orientation of the relative tangent bundle of $\mathrm{ev}_0^B\colon \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_\alpha$, at $\mathbf{u}$, i.e., the kernel of $T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to T_{u(z_0)}R_{\alpha_0}$, which is denoted by $o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$.

Remark 3.2. When $k=0$ and $R_{\alpha_0} = L$, the orientation on $\mathcal{M}_1(B;L)$ is given in [5, Section 8.4.1] When $k=1$, the orientation bundle of $\mathcal{M}_2(B;\mathcal{L};\mathcal{R})$ is given in [5, Proposition 8.8.6]. Note that $\Theta^+_{R_\alpha} \otimes O_{R_\alpha} \otimes \Theta^-_{R_\alpha}$ is canonically trivialized. We write $\Theta_{R_\alpha} = \Theta^-_{R_\alpha}$ in this note.

Since the evaluation maps are weakly submersive in the sense of Kuranishi structure, see [6, Definition 3.40 (5)], i.e., after taking sufficiently large obstruction bundles, the evaluation maps on Kuranishi charts are submersive, the push-forward $(\mathrm{ev}_0)_!$ is defined by taking CF-perturbations. Hence, for a smooth correspondence $(\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}), \mathrm{ev}_0, \mathrm{ev}_1\times\dots\times\mathrm{ev}_k)$, Theorem 27.1 in [6] gives

$$\left(\mathrm{ev}_0^B\right)_! \circ \left(\mathrm{ev}_1^{B*}\times\dots\times\mathrm{ev}_k^{B*}\right)\colon\ \Omega^*\left(R_{\alpha_1};\Theta_{R_{\alpha_1}}\right)\otimes\dots\otimes\Omega^*\left(R_{\alpha_k};\Theta_{R_{\alpha_k}}\right) \to \Omega^*\left(R_{\alpha_0};\Theta_{R_{\alpha_0}}\right).$$

Namely, for $\xi_i = \zeta_i\otimes\sigma_{\alpha_i} \in \Omega^*\left(R_{\alpha_i};\Theta_{\alpha_i}\right)$, $i=1,\dots,k$, we define

$$\begin{aligned}&\left(\mathrm{ev}_0^B\right)_! \circ \left(\mathrm{ev}_1^{B*}\times\dots\times\mathrm{ev}_k^{B*}\right)\left(\zeta_1\otimes\sigma_{\alpha_1},\dots,\zeta_k\otimes\sigma_{\alpha_k}\right)\\ &\qquad = \left(\mathrm{ev}_0^B; o\left(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\right)\right)_!\left(\mathrm{ev}_1^{B*}\zeta_1\wedge\dots\wedge\mathrm{ev}_k^{B*}\zeta_k\right)\otimes\sigma_{\sigma_{\alpha_0}}.\end{aligned} \tag{3.2}$$

Here $\left(\mathrm{ev}_0^B; o\left(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\right)\right)_!$ is the integration along fibers with respect to the relative orientation $o\left(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\right)$ of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to R_{\alpha_0}$. Note that the right hand side of (3.2) does not depends on σ_{α_0}, since σ_{α_0} appears twice in the right hand side of (3.2), and gives a differential form on R_{α_0} with coefficients in Θ_{α_0}. For general $\xi_i \in \Omega^*\left(R_{\alpha_i};\Theta_{\alpha_i}\right)$, we use partitions of unity on R_{α_i} and extend the operation $\left(\mathrm{ev}_0^B\right)_! \circ \left(\mathrm{ev}_1^{B*}\times\dots\times\mathrm{ev}_k^{B*}\right)$ multi-linearly.

For $\xi \in \Omega^*(R_\alpha;\Theta_\alpha)$, we define the shifted degree

$$|\xi|' = \deg\xi + \mu(R_\alpha) - 1. \tag{3.3}$$

Definition 3.3. We set $\mathfrak{m}_{0,0} = 0$, $\mathfrak{m}_{(1,0)}\xi = d\xi$ on $\bigoplus \Omega^*(R_\alpha;\Theta_{R_\alpha})$, i.e., the de Rham differential on differential forms with coefficients in the local system Θ_{R_α}. For $(k,B) \neq (1,0)$,

$$\begin{aligned}&\mathfrak{m}_{k,B}(\xi_1,\dots,\xi_k)\\&\quad = (-1)^{\epsilon(\xi_1,\dots,\xi_k)}\big(\mathrm{ev}_0^B\big)_! \circ \big(\mathrm{ev}_1^{B*}\times\cdots\times \mathrm{ev}_k^{B*}\big)(\xi_1\otimes\dots,\otimes\xi_k) \in \Omega^*(R_{\alpha_0};\Theta_{\alpha_0}),\end{aligned}$$

where $\xi_i \in \Omega^*\big(R_{\alpha_i};\Theta_{\alpha_i}\big)$ and

$$\epsilon(\xi_1,\dots,\xi_k) = \left\{\sum_{i=1}^k\left(i+\sum_{p=1}^{i-1}\mu\big(R_{\alpha_p}\big)\right)(\deg\xi_i - 1)\right\} + 1. \tag{3.4}$$

Then we define

$$\begin{aligned}\mathfrak{m}_k = \sum_B \mathfrak{m}_{k,B}T^{\langle\omega,B\rangle}\colon\ &\bigotimes_{i=1}^k \Omega^*\big(R_{\alpha_i};\Theta_{\alpha_i}\otimes\Lambda_0\big)\big[1-\mu\big(R_{\alpha_i}\big)\big]\\&\to \Omega^*(R_{\alpha_0};\Theta_{\alpha_0}\otimes\Lambda_0)[1-\mu(R_{\alpha_0})].\end{aligned}$$

Here

$$\Lambda_0 = \left\{\sum_i a_iT^{\lambda_i} \mid a_i\in\mathbb{R},\ \lambda_i\to\infty \text{ as } i\to\infty\right\}$$

and the symbol $[1-\mu(R_\alpha)]$ is the degree shift by $1-\mu(R_\alpha)$, i.e., the grading of a differential form is given by $|\xi|'$. By (2.1) and (3.1), we find that

$$|\mathfrak{m}_k(\xi_1,\dots,\xi_k)|' \equiv \sum_{i=1}^k|\xi_i|' + 1 \mod 2. \tag{3.5}$$

Remark 3.4. Since the aim of this note is describe the sign and orientation for the filtered A_∞-operations, we use Λ_0 as the coefficient ring. To make $\mathfrak{m}_k$ operations of degree 1, we need to use the universal Novikov ring $\Lambda_{0,\mathrm{nov}}$ introduced in [4].

4 Filtered A_∞-relations

In the rest of this note, we verify the sign convention in the filtered A_∞-relations

$$\sum_{k'+k''=k+1}\mathfrak{m}_{k'}\circ\widehat{\mathfrak{m}}_{k''}(\xi_1,\dots,\xi_k) = 0 \qquad \text{for } k=1,2,\dots$$

under the tree-like K-system and CF-perturbation described in [6]. Here $\hat{\mathfrak{m}}_{k,B}$ is the extension of $\mathfrak{m}_{k,B}$ as a graded coderivation with respect to the shifted degree $|\bullet|'$. This relation is equivalent to the following relations for decompositions of B into B' and B'', $k'+k''=k+1$,

$$\begin{aligned}&\mathfrak{m}_{1,0}\circ\mathfrak{m}_{k,B}(\xi_1,\dots,\xi_k) + \mathfrak{m}_{k,B}\circ\hat{\mathfrak{m}}_{1,0}(\xi_1,\dots,\xi_k)\\&\quad+\sum_{(k',B'),(k'',B'')\neq(1,0)}\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}(\xi_1,\dots,\xi_k) = 0.\end{aligned}$$

We compute $\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}$. For $(k,B)=(1,0)$, $\mathfrak{m}_{1,0}\circ\mathfrak{m}_{1,0}=0$ clearly holds.

From now on, we investigate the case that $(k,B)\neq(1,0)$. Firstly we consider the case that $(k',B')=(1,0)$ or $(k'',B'')=(1,0)$. We find that

$$\mathfrak{m}_{1,0}\circ\mathfrak{m}_{k,B}(\xi_1,\dots,\xi_k) = (-1)^{\epsilon(\xi_1,\dots,\xi_k)}d\big(\mathrm{ev}_0^B\big)_!\big(\mathrm{ev}_1^{B*}\xi_1\wedge\cdots\wedge\mathrm{ev}_k^{B*}\xi_k\big), \tag{4.1}$$

$$
\begin{aligned}
\mathfrak{m}_{k,B} \circ \hat{\mathfrak{m}}_{1,0}(\xi_1,\dots,\xi_k) &= \sum_{j=1}^{k} (-1)^{\sum_{p=1}^{j-1}|\xi_p|'} \mathfrak{m}_{k,B}(\xi_1,\dots,d\xi_j,\dots,\xi_k) \\
&= \sum_{j=1}^{k} (-1)^{\sum_{p=1}^{j-1}|\xi_p|' + \epsilon(\xi_1,\dots,d\xi_j,\dots,\xi_k)} \\
&\quad \times \big(\mathrm{ev}_0^B\big)_! \big(\mathrm{ev}_1^{B*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_j^{B*} d\xi_j \wedge \cdots \wedge \mathrm{ev}_k^{B*}\xi_k\big) \\
&= (-1)^{\epsilon(\xi_1,\dots,\xi_k)+1} \big(\mathrm{ev}_0^B\big)_! \mathrm{d}\big(\mathrm{ev}_1^{B*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_k^{B*}\xi_k\big). \qquad (4.2)
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
\sum_{p=1}^{j-1} |\xi_p|' + \epsilon(\xi_1,\dots,\mathrm{d}\xi_j,\dots,\xi_k) &= \sum_{p=1}^{j-1} \deg\, \xi_p + \sum_{p=1}^{j-1}\big(\mu(R_{\alpha_p}) - 1\big) + \epsilon(\xi_1,\dots,\xi_k) \\
&\quad + \left(j + \sum_{p=1}^{j-1} \mu(R_{\alpha_p}) \right) \\
&\equiv \sum_{p=1}^{j-1} \deg\, \xi_p + \epsilon(\xi_1,\dots,\xi_k) + 1 \mod 2.
\end{aligned}
$$

In order to compute $\mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}$ for $(k',B'), (k'',B'') \neq (1,0)$, we discuss the relation between the orientation bundle of

$$
\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')
$$

and the orientation bundle of the boundary of $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. The codimension 1 boundary of the moduli space $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ is the union of the fiber products of $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and $\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$ with respect to the evaluation maps $\mathrm{ev}_j^{B'} : \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}') \to R_\alpha$ and $\mathrm{ev}_0^{B''} : \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \to R_\alpha$, where

$$
\begin{aligned}
\mathcal{L}' &= \big(L_{i_0},\dots,L_{i_{j-1}}, L_{i_{j+k''-1}},\dots,L_{i_k}\big), & \mathcal{L}'' &= \big(L_{i_{j-1}},\dots,L_{i_{j+k''-1}}\big), \\
\mathcal{R}' &= \big(R_{\alpha_0},\dots,R_{\alpha_{j-1}}, R_\alpha, R_{\alpha_{j+k''}},\dots,R_{\alpha_k}\big), & \mathcal{R}'' &= \big(R_\alpha, R_{\alpha_j}, \dots R_{\alpha_{i_{j+k''-1}}}\big).
\end{aligned}
$$

Here the union is taken over k', k'' such that $k'+k'' = k+1$, all possible decomposition of B into B' and B'', $j = 1,\dots,k'$, and R_α a connected component of $L_{i_{j-1}} \cap L_{j+k''-1}$,.

Proposition 4.1.

$$
(-1)^\kappa \mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big) \subset \partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}),
$$

where

$$
\begin{aligned}
\kappa &\equiv (k''-1)(k'-j) + (k'-1)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1} \mu\big(R_{\alpha_p}\big)\right) \\
&\quad + \left(\sum_{p=1}^{j-1} \mu\big(R_{\alpha_p}\big)\right)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1} \mu\big(R_{\alpha_p}\big)\right) \\
&\quad + \dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \left(\sum_{p=1}^{j-1} \mu\big(R_{\alpha_p}\big) + \mu(R_\alpha) + \sum_{p=j+k''}^{k} \mu\big(R_{\alpha_p}\big)\right) + k'.
\end{aligned}
$$

Proof. Denote by Sw the operation, which exchanges

$$\Theta_{R_{\alpha_1}} \otimes \cdots \otimes \Theta_{R_{\alpha_{j-1}}} \qquad \text{and} \qquad O^*_{R_\alpha} \otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}.$$

Set the weight of $\Theta_{R_{\alpha_i}}$, O_{R_α} and $O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ}$ as $\mu(R_{\alpha_i})$, $\dim R_\alpha$ and

$$\dim \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ = \dim \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) - \dim \mathcal{M}_{k+1},$$

respectively. Then the weighted sign of Sw is $(-1)^{\delta_1}$, where

$$\begin{aligned}\delta_1 &= \left(\sum_{p=1}^{j-1} \mu(R_{\alpha_p})\right)\big(\dim \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') - \dim R_\alpha - \dim \mathcal{M}_{k''+1}\big) \\ &\equiv \left(\sum_{p=1}^{j-1} \mu(R_{\alpha_p})\right)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1} \mu(R_{\alpha_p})\right) \mod 2.\end{aligned}$$

Comparing Φ^B and $Sw \circ \big(\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Phi^{B''} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}\big) \circ \Phi^{B'}$, we find that

$$O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ} \to O_{\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ} \otimes O^*_{R_\alpha} \otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}$$

is $(-1)^{\delta_1}$-orientation preserving.[1] Here $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ$ is the fiber of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) \to \mathcal{M}_{k+1}$, i.e., the moduli space of bordered stable maps with a fixed domain bordered Riemann surface equipped with fixed boundary marked points. The $O(1)$-local system

$$O_{\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ} \otimes O^*_{R_\alpha} \otimes O_{\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ}$$

is the orientation bundle of the fiber product

$$\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)^\circ,$$

which is the moduli space of bordered stable maps with a fixed boundary nodal Riemann surface equipped with fixed boundary marked points.

Now we compare the orientations of

$$\partial \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = \partial\big(\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})^\circ \times \mathcal{M}_{k+1}\big)$$

and

$$\begin{aligned}&\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big) \\ &\quad = \big(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ \times \mathcal{M}_{k'+1}\big)_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \big(\mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)^\circ \times \mathcal{M}_{k''+1}\big).\end{aligned}$$

We note that $O_{\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})} = \mathbb{R}_{\mathrm{out}} \otimes O_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}$. Here $\mathbb{R}_{\mathrm{out}}$ is the normal bundle of the boundary oriented by the outer normal vector. We pick local flat sections $\sigma_{\alpha_0}, \dots, \sigma_{\alpha_k}, \sigma_\alpha$ of $O(1)$-local systems $\Theta_{R_{\alpha_0}}, \dots, \Theta_{R_{\alpha_k}}, \Theta_{R_\alpha}$ and a local orientation $o_{R_{\alpha_0}}$ of R_{α_0} around $u(z_0)$. Then we can equip $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$, $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and the relative tangent bundle of

$$\mathrm{ev}_0^{B''}\colon\ \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \to R_\alpha$$

with local orientations induced by them. Then a local orientation of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}) = R_{\alpha_0} \times^\circ \mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ is given by $o_{R_{\alpha_0}} \times o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$. As the fiber product of spaces with Kuranishi structures equipped with local orientations,

$$\begin{aligned}&\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big) \\ &\quad = \mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big) \times {}^\circ\mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)\end{aligned}$$

[1] (-1)-orientation preserving means orientation reversing.

is locally oriented by

$$o_{R_{\alpha_0}} \times o\big(\sigma_{R_{\alpha_0}}; \sigma_{R_{\alpha_1}}, \dots, \sigma_{R_{\alpha_{j-1}}}, \sigma_{R_\alpha}, \sigma_{R_{\alpha_{j+k''}}}, \dots, \sigma_{R_{\alpha_k}}\big) \times o\big(\sigma_{R_\alpha}; \sigma_{R_{\alpha_j}}, \dots, \sigma_{R_{\alpha_{j+k''-1}}}\big).$$

We fix $z_0 = +1$, $z_j = -1$ and consider the spaces of J-holomorphic maps $\widetilde{\mathcal{M}}_{k+1}(B; \mathcal{L}, \mathcal{R})$, $\widetilde{\mathcal{M}}_{k'+1}(B'; \mathcal{L}', \mathcal{R}')$ and $\widetilde{\mathcal{M}}_{k''+1}(B''; \mathcal{L},'' \mathcal{R}'')$ such that

$$\mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R}) = \widetilde{\mathcal{M}}_{k+1}(B; \mathcal{L}, \mathcal{R})/\mathbb{R}_B,$$
$$\mathcal{M}_{k'+1}\big(B'; \mathcal{L}'; \mathcal{R}'\big) = \widetilde{\mathcal{M}}_{k'+1}\big(B'; \mathcal{L}'; \mathcal{R}'\big)/\mathbb{R}_{B'},$$

and

$$\mathcal{M}_{k''+1}\big(B''; \mathcal{L}''; \mathcal{R}''\big) = \widetilde{\mathcal{M}}_{k''+1}\big(B''; \mathcal{L},'' \mathcal{R}''\big)/\mathbb{R}_{B''}.$$

We may also write

$$\widetilde{\mathcal{M}}_{k+1}(B; \mathcal{L}; \mathcal{R}) = \mathcal{M}_{k+1}(B; \mathcal{L}, \mathcal{R}) \times \mathbb{R}_B, \qquad \text{etc.},$$

as oriented spaces.

The case that $z_0 = +1$, $z_1 = -1$ is discussed in [5, p. 699]. The case that $z_0 = +1$, $z_j = -1$ differs from the case that $z_0 = +1$, $z_1 = -1$ by an additional factor $(-1)^{j-1}$ as below.

For orientation issue, we consider the top-dimensional strata of the moduli spaces and regard $\widetilde{\mathcal{M}}_{k+1}(B; \mathcal{L}; \mathcal{R})$ as an open subset of

$$\mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R})^\circ \times \prod_{i=1}^{j-1} (\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1} (\partial D)_{z_i} \times \prod_{i=j+k''}^{k} (\partial D)_{z_i}.$$

We simply write

$$\widetilde{\mathcal{M}}_{k+1}(B; \mathcal{L}; \mathcal{R}) = (-1)^{j-1} \mathcal{M}_{k+1}(B; \mathcal{L}; \mathcal{R})^\circ \times \prod_{i=1}^{j-1} (\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1} (\partial D)_{z_i} \times \prod_{i=j+k''}^{k} (\partial D)_{z_i},$$

where $z_0 = +1$, $z_j = -1$,

$$\widetilde{\mathcal{M}}_{k'+1}\big(B'; \mathcal{L}'; \mathcal{R}'\big) = (-1)^{j-1} \mathcal{M}_{k'+1}\big(B'; \mathcal{L}'; \mathcal{R}'\big)^\circ \times \prod_{i=1}^{j-1} (\partial D)_{z_i} \times \prod_{i=j+k''}^{k} (\partial D)_{z_i},$$

where $z'_0 = +1$, $z'_j = -1$, and

$$\widetilde{\mathcal{M}}_{k''+1}\big(B''; \mathcal{L}''; \mathcal{R}''\big) = \mathcal{M}_{k''+1}\big(B''; \mathcal{L};'' \mathcal{R}''\big)^\circ \times \prod_{i=j+1}^{j+k''-1} (\partial D)_{z_i},$$

where $z''_0 = +1$, $z''_1 = -1$.

Note that

$$(-1)^{j-1} \prod_{i=1}^{j-1} (\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1} (\partial D)_{z_i} \times \prod_{i=j+k''}^{k} (\partial D)_{z_i} = \mathcal{M}_{k+1} \times \mathbb{R}_B,$$
$$(-1)^{j-1} \prod_{i=1}^{j-1} (\partial D)_{z_i} \times \prod_{i=j+k''}^{k} (\partial D)_{z_i} = \mathcal{M}_{k'+1} \times \mathbb{R}_{B'}$$

and

$$\prod_{i=j+1}^{j+k''-1} (\partial D)_{z_i} = \mathcal{M}_{k''+1} \times \mathbb{R}_{B''}.$$

Marked points of $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and $\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')$ are related to marked points of $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ in the following way.

$$\begin{aligned}
&\big(z'_0,\dots,z'_{k'}\big) = \big(z_0,\dots,z_{j-1},z'_j,z_{j+k''},\dots,z_k\big),\\
&\big(z''_0,z''_1,,\dots,z''_{k''}\big) = \big(z''_0,z_j,\dots,z_{j+k''-1}\big).
\end{aligned}$$

Here z'_j and z''_0 are identified, i.e., the boundary node of the domain curve of an element in $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. Then we find that

$$\begin{aligned}
&\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R})\\
&\quad= (-1)^{\delta_1}\big(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)^\circ\big)\\
&\qquad\times(-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i}\times\prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\times\prod_{i=j+k''}^{k}(\partial D)_{z_i}\\
&\quad= (-1)^{\delta_1+\delta_2}\left(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ\times(-1)^{j-1}\prod_{i=1}^{j-1}(\partial D)_{z_i}\times\prod_{i=j+k''}^{k}(\partial D)_{z_i}\right)\\
&\qquad {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\left(\mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)^\circ\times\prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\right)\\
&\quad= (-1)^{\delta_1+\delta_2}\big(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)\times\mathbb{R}_{B'}\big) {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\big(\mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)\times\mathbb{R}_{B''}\big)\\
&\quad= (-1)^{\delta_1+\delta_2+\delta_3}\mathbb{R}_{B'-B''}\times\big(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big) {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)\big)\times\mathbb{R}_{B'+B''}\\
&\quad= (-1)^{\delta_1+\delta_2+\delta_3}\mathbb{R}_{\mathrm{out}}\times\big(\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big) {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)\big)\times\mathbb{R}_{B},
\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}
&\delta_2 = \big(k''-1\big)\big(k'-j\big)+\big(k'-1\big)\big(\dim\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)^\circ-\dim R_\alpha\big),\\
&\delta_3 = \dim\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big).
\end{aligned}$$

$\mathbb{R}_{B'-B''}$ and $\mathbb{R}_{B'+B''}$ are the oriented lines spanned by $(1,-1),(1,1)\in\mathbb{R}_{B'}\oplus\mathbb{R}_{B''}$, respectively. Note that the ordered bases $(1,0),(0,1)$ and $(1,-1),(1,1)$ give the same orientation of $\mathbb{R}_{B'}\oplus\mathbb{R}_{B''}$, $\mathbb{R}_{B'-B''}$ and $\mathbb{R}_{B'+B''}$ are identified with $\mathbb{R}_{\mathrm{out}}$ and $\mathbb{R}_B$, respectively.

Here is an explanation of the second equality, i.e., the appearance of $(-1)^{\delta_2}$. By the convention in [5, Section 8.2], we have

$$\begin{aligned}
&\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)^\circ\\
&\qquad= \mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^{\circ\circ}\times R_\alpha\times{}^\circ\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)^\circ\\
&\qquad= \mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^{\circ}\times{}^\circ\mathcal{M}_{k''+1}\big(B'';\mathcal{L};''\mathcal{R}''\big)^\circ,
\end{aligned}$$

where

$$\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^\circ = \mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)^{\circ\circ}\times R_\alpha,$$

and

$$\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ = R_\alpha \times {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ.$$

Using these notations, we have

$$\begin{aligned}
&\left(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \ {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\right)\\
&\qquad \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i}\\
&\qquad = (-1)^{\gamma_1}\left(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \ {}_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}} \times \mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\right)\\
&\qquad \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\\
&\qquad = (-1)^{\gamma_1}\left(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ\right)\\
&\qquad \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i} \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\\
&\qquad = (-1)^{\gamma_1+\gamma_2}\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i}\\
&\qquad \times {}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\\
&\qquad = (-1)^{\gamma_1+\gamma_2}\left(\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')^\circ \times \prod_{i=1}^{j-1}(\partial D)_{z_i} \times \prod_{i=j+k''}^{k}(\partial D)_{z_i}\right)\\
&\qquad \times_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \left(\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')^\circ \times \prod_{i=j+1}^{j+k''-1}(\partial D)_{z_i}\right),
\end{aligned}$$

where $\gamma_1 = (k''-1)(k'-j)$, i.e., $(-1)^{\gamma_1}$ is the sign of switching marked points $(z_{j+k''},\dots,z_k)$ and $(z_{j+1},\dots,z_{j+k''-1})$, and $\gamma_2 = \dim({}^\circ\mathcal{M}_{k''+1}(B'';\mathcal{L};''\mathcal{R}'')^\circ)(\dim \mathcal{M}_{k'+1}+1)$. Then $\delta_2 = \gamma_1+\gamma_2$.

Now we return to the discussion on local orientations of the orientation bundle of

$$\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \qquad \text{and} \qquad \partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}).$$

Recall that

$$\widetilde{\mathcal{M}}_{k+1}(B;\mathcal{L};\mathcal{R}) = \mathcal{M}_{k+1}(B;\mathcal{L},\mathcal{R}) \times \mathbb{R}_B. \tag{4.4}$$

Set $\kappa = \delta_1+\delta_2+\delta_3$, i.e.,

$$\begin{aligned}
\kappa \equiv\ & (k''-1)(k'-j) + (k'-1)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1}\mu(R_{\alpha_p})\right)\\
&+ \left(\sum_{p=1}^{j-1}\mu(R_{\alpha_p})\right)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1}\mu(R_{\alpha_p})\right)\\
&+ \dim R_{\alpha_0} + \mu(R_{\alpha_0}) - \left(\sum_{p=1}^{j-1}\mu(R_{\alpha_p}) + \mu(R_\alpha) + \sum_{p=j+k''}^{k}\mu(R_{\alpha_p})\right) + k'.
\end{aligned}$$

Comparing (4.3) and (4.4), we obtain Proposition 4.1. ■

From Corollary 2.2 in the setting of Kuranishi structures, Propositions 4.1 and 2.3, i.e., the base change formula for integration along fibers, we find the following.

Lemma 4.2.

$$\begin{aligned}&\big(\mathrm{ev}_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})};\partial o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)\big)_!\left(\prod_{i=1}^{j-1}\mathrm{ev}_i^{B*}\times\prod_{i=j+k''}^{k}\mathrm{ev}_i^{B*}\times\prod_{i=j}^{j+k''-1}\mathrm{ev}_i^{B*}\right)\\&=(-1)^{\kappa}\big(\mathrm{ev}_0^{B'};o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_{j-1}},\sigma_\alpha,\sigma_{\alpha_{j+k''-1}},\dots,\sigma_{\alpha_k}\big)\big)_!\\&\quad\circ\left(\prod_{i=1}^{j-1}\mathrm{ev}_i^{B'*}\times\prod_{i=j+1}^{k'}\mathrm{ev}_i^{B'*}\times\left(\mathrm{ev}_j^{B'*}\circ\big(\mathrm{ev}_0^{B''};o\big(\sigma_\alpha;\sigma_{\alpha_j},\dots,\sigma_{\alpha_{j+k''-1}}\big)\big)_!\circ\prod_{i=1}^{k''}\mathrm{ev}_i^{B''*}\right)\right)\end{aligned}$$

as operations applied to

$$\left(\bigotimes_{i=1}^{j-1}\zeta_i\right)\otimes\left(\bigotimes_{i=j+k''}^{k}\zeta_i\right)\otimes\left(\bigotimes_{i=j}^{j+k''-1}\zeta_i\right),$$

where $\xi_i=\zeta_i\otimes\sigma_{\alpha_i}$, $i=1,\dots,k$. *Here* $\partial o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$ *is the local orientation of the relative tangent bundle* $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_0}$ *induced from* $o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$.

Note that $\partial o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$ is not the boundary orientation of $\partial^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ induced from the orientation $o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$ of ${}^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. They differ by $(-1)^{\dim R_{\alpha_0}}$. Namely, for $\mathbf{u}\in\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$, the local orientation $o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$ of ${}^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ and the local orientation $\partial o\big(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}\big)$ of the relative tangent bundle of $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$ $\to R_{\alpha_0}$ are related as follows:

$$\begin{aligned}T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})&=\mathbb{R}_{\mathrm{out}}\times T_{\mathbf{u}}\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}),\\T_{\mathbf{u}}\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})&=T_{u(z_0)}R_{\alpha_0}\times T_{\mathbf{u}}{}^\circ\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}).\end{aligned}$$

Then, under the following identification

$$\mathbb{R}_{\mathrm{out}}\times T_{\mathbf{u}}\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})=\mathbb{R}_{\mathrm{out}}\times T_{u(z_0)}R_{\alpha_0}\times T_{\mathbf{u}}{}^\circ\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R}),$$

we define the local orientation $\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})$ of the relative tangent bundle of $\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})\to R_{\alpha_0}$ so that

$$o_{R_{\alpha_0}}\times o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k})=\mathbb{R}_{\mathrm{out}}\times o_{R_{\alpha_0}}\times\partial o(\sigma_{\alpha_0};\sigma_{\alpha_1},\dots,\sigma_{\alpha_k}).$$

Note that

$$\mathrm{ev}_i^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}=\begin{cases}\mathrm{ev}_i^{B'}\circ\pi_{B'}^B, & i=1,\dots,j-1,\\ \mathrm{ev}_{i-j+1}^{B''}\circ\pi_{B''}^B, & i=j,\dots,j+k''-1,\\ \mathrm{ev}_{i-k''+1}^{B'}\circ\pi_{B'}^B, & i=j+k'',\dots,k,\end{cases}$$

where $\pi_{B'}^B$ and $\pi_{B''}^B$ are projections from the fiber product

$$\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)_{\mathrm{ev}_j^{B'}}\times_{\mathrm{ev}_0^{B''}}\mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big)$$

to $\mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')$ and $\mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'')$, respectively. Note that σ_α appears twice in the right hand side of the equality in Lemma 4.2, hence the right hand side does not depends on the choice of local section σ_α of the $O(1)$-local system Θ_α.

Next, we compute $\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}$ with $(k',B')\neq(1,0)$, $(k'',B'')\neq(1,0)$. Armed with Lemma 4.2, we regard ξ_i, $i=1,\dots,k$, as differential forms on R_{α_i} in the computation below.

Lemma 4.3.

$$\mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}(\xi_1,\dots,\xi_k) = (-1)^{\kappa'}\left(\mathrm{ev}_0^{(B',B'')}\right)_!\left(\mathrm{ev}_1^{(B',B'')*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_k^{(B',B'')*}\xi_k\right), \tag{4.5}$$

where

$$\begin{aligned}\kappa' \equiv{}& \epsilon(\xi_1,\dots,\xi_k) + \sum_{i=1}^{k}\deg\xi_i - k - 1 + j + k'\left(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1}\mu\big(R_{\alpha_i}\big)\right)\\ &+ \left(\sum_{p=1}^{j-1}\mu\big(R_{\alpha_p}\big)\right)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1}\mu\big(R_{\alpha_p}\big)\right) + \big(k'-j\big)k'' \mod 2.\end{aligned}$$

Proof. By the definition of $\mathfrak{m}_{k,B}$ and its extension $\widehat{\mathfrak{m}}_{k,B}$ as a graded coderivation, we have

$$\begin{aligned}&\mathfrak{m}_{k',B'} \circ \hat{\mathfrak{m}}_{k'',B''}(\xi_1,\dots,\xi_k)\\ &\quad= \sum_{j=1}^{k}(-1)^{\sum_{i=1}^{j-1}|\xi_i|'}\mathfrak{m}_{k',B'}\big(\xi_1,\dots,\mathfrak{m}_{k'',B''}\big(\xi_j,\dots,\xi_{j+k''-1}\big),\dots,\xi_k\big)\\ &\quad= \sum_{j=1}^{k}(-1)^{\delta_4}\big(\mathrm{ev}_0^{B'}\big)_!\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1}\\ &\qquad\quad \wedge \mathrm{ev}_j^{B'*}\big(\big(\mathrm{ev}_0^{B''}\big)_!\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big) \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big),\end{aligned} \tag{4.6}$$

where

$$\delta_4 = \sum_{i=1}^{j-1}|\xi_i|' + \epsilon\big(\xi_1,\dots,\mathfrak{m}_{k'',B''}\big(\xi_j,\dots,\xi_{j+k''-1}\big),\dots,\xi_k\big) + \epsilon\big(\xi_j,\dots,\xi_{j+k''-1}\big),$$

and $\mathrm{ev}_j^{(B',B'')}\colon \mathcal{M}_{k'+1}(B';\mathcal{L}';\mathcal{R}')_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}(B'';\mathcal{L}'';\mathcal{R}'') \to R_{\alpha_j}$ is the evaluation map at the j-th marked point on the fiber product

$$\mathcal{M}_{k'+1}\big(B';\mathcal{L}';\mathcal{R}'\big)_{\mathrm{ev}_j^{B'}} \times_{\mathrm{ev}_0^{B''}} \mathcal{M}_{k''+1}\big(B'';\mathcal{L}'';\mathcal{R}''\big).$$

Here the numbering of the marked points is the same as that on $\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})$. We also have

$$\begin{aligned}&\big(\mathrm{ev}_0^{B'}\big)_!\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1} \wedge \mathrm{ev}_j^{B'*}\big(\big(\mathrm{ev}_0^{B''}\big)_!\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big)\\ &\qquad \wedge \mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big)\\ &\quad= (-1)^{\eta_1}\big(\mathrm{ev}_0^{B'}\big)_!\big(\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1} \wedge \mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big)\\ &\qquad\qquad \wedge \mathrm{ev}_j^{B'*} \circ \big(\mathrm{ev}_0^{B''}\big)_!\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big)\\ &\quad= (-1)^{\eta_1}\big(\mathrm{ev}_0^{B'}\big)_!\big(\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1} \wedge \mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big)\\ &\qquad\qquad \wedge (\pi_{B'})_! \circ \pi_{B''}^*\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big)\\ &\quad= (-1)^{\eta_1}\big(\mathrm{ev}_0^{B'}\big)_! \circ (\pi_{B'})_!\big(\pi_{B'}^*\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1} \wedge \mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big)\\ &\qquad\qquad \wedge \pi_{B''}^*\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big)\\ &\quad= (-1)^{\eta_1+\eta_2}\big(\mathrm{ev}_0^{B'} \circ \pi_{B'}\big)_!\big(\pi_{B'}^*\big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1}\big)\\ &\qquad\qquad \wedge \pi_{B''}^*\big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\\ &\qquad\qquad \wedge \pi_{B'}^*\big(\mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big)\big)\\ &\quad= (-1)^{\eta_1+\eta_2}\big(\mathrm{ev}_0^{(B',B'')}\big)_!\big(\mathrm{ev}_1^{(B',B'')*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_k^{(B',B'')*}\xi_k\big),\end{aligned}$$

where

$$\eta_1 = \left(\left(\sum_{i=j}^{j+k''-1} \deg \xi_i\right) + \left(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1} \mu\big(R_{\alpha_i}\big) + k'' - 2\right)\right)\left(\sum_{i=j+k''}^{k} \deg \xi_i\right),$$
$$\eta_2 = \left(\sum_{i=j}^{j+k''-1} \deg \xi_i\right)\left(\sum_{i=j+k''}^{k} \deg \xi_i\right).$$

The second equality is a consequence of Proposition 2.3 (base change formula) for integration along fibers, i.e., $\mathrm{ev}_j^{B'*} \circ \big(\mathrm{ev}_0^{B''}\big)_! = (\pi_{B'})_! \circ \pi_{B''}^*$. The third equality follows from Corollary 2.2. Note that

$$\mathrm{ev}_i^{(B',B'')} = \begin{cases} \mathrm{ev}_i^{B'} \circ \pi_{B'}^B & i = 0, 1, \dots, j-1, \\ \mathrm{ev}_{i-j+1}^{B''} \circ \pi_{B''}^B, & i = j, \dots, j+k''-1, \\ \mathrm{ev}_{i-k''+1} B' \circ \pi_{B'}^B, & i = j+k'', \dots, k. \end{cases}$$

We set

$$\delta_5 = \eta_1 + \eta_2 = \left(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1} \mu\big(R_{\alpha_i}\big) + k'' - 2\right)\left(\sum_{i=j+k''}^{k} \deg \xi_i\right).$$

Then we have

$$\begin{aligned} &\big(\mathrm{ev}_0^{B'}\big)_! \big(\mathrm{ev}_1^{B'*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_{j-1}^{B'*}\xi_{j-1} \wedge \mathrm{ev}_j^{B'*}\big(\big(\mathrm{ev}_0^{B''}\big)_! \big(\mathrm{ev}_1^{B''*}\xi_j \wedge \cdots \wedge \mathrm{ev}_{k''}^{B''*}\xi_{j+k''-1}\big)\big) \\ &\qquad \wedge \mathrm{ev}_{j+1}^{B'*}\xi_{j+k''} \wedge \cdots \wedge \mathrm{ev}_{k'}^{B'*}\xi_k\big) \\ &\quad = (-1)^{\delta_5}\big(\mathrm{ev}_0^{(B',B'')}\big)_! \big(\mathrm{ev}_1^{(B',B'')*}\xi_1 \wedge \cdots \wedge \mathrm{ev}_k^{(B',B'')*}\xi_k\big). \end{aligned} \tag{4.7}$$

Set $\kappa' = \delta_4 + \delta_5$, i.e.,

$$\begin{aligned} \kappa' = &\sum_{i=1}^{j-1} |\xi_i|' + \epsilon\big(\xi_1, \dots, \mathfrak{m}_{k'',B''}\big(\xi_j, \dots, \xi_{j+k''-1}\big), \dots, \xi_k\big) + \epsilon\big(\xi_j, \dots, \xi_{j+k''-1}\big) \\ &+ \left(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1} \mu\big(R_{\alpha_i}\big) + k'' - 2\right)\left(\sum_{i=j+k''}^{k} \deg \xi_i\right). \end{aligned}$$

Recall the definitions of the shifted degree in (3.3) and the $\epsilon(\xi_1, \dots, \xi_k)$ in (3.4) and the fact on the degree of $\mathfrak{m}_k$ (3.5), we find that

$$\begin{aligned} \kappa' \equiv\ &\epsilon(\xi_1, \dots, \xi_k) + \sum_{i=1}^{k} \deg \xi_i - k - 1 + j + k'\left(\mu(R_\alpha) - \sum_{i=j}^{j+k''-1} \mu\big(R_{\alpha_i}\big)\right) \\ &+ \left(\sum_{p=1}^{j-1} \mu\big(R_{\alpha_p}\big)\right)\left(\mu(R_\alpha) - \sum_{p=j}^{j+k''-1} \mu\big(R_{\alpha_p}\big)\right) + \big(k' - j\big)k'' \mod 2. \end{aligned}$$

Combining (4.6) and (4.7), we obtain Lemma 4.3. ∎

Now we show the following.

Theorem 4.4. *The operations $\mathfrak{m}_k$, $k = 0, 1, \dots,$ that is the Bott–Morse A_∞-operation in the de Rham model, in Definition 3.3 satisfy the filtered A_∞-relation*

$$\sum_{k'+k''=k+1} \mathfrak{m}_{k'} \circ \widehat{\mathfrak{m}}_{k''}(\xi_1, \dots, \xi_k) = 0 \qquad \text{for } k = 1, 2, \dots.$$

Proof. By Proposition 4.1, we find the following.

Claim 4.5. *The summation of the right hand side of* (4.5) *over* k', k'', B', B'' *such that* $k'+k''=k+1$, $B=B'+B''$, $(k',B'),(k'',B'')\neq(1,0)$ *is equal to*

$$(-1)^{\kappa+\kappa'}\big(\mathrm{ev}_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}\big)_!\big(\mathrm{ev}_1^{B*}\xi_1\wedge\cdots\wedge\mathrm{ev}_k^{B*}\xi_k\big).$$

Note that

$$\begin{aligned}\kappa+\kappa' &\equiv \epsilon(\xi_1,\dots,\xi_k)+1+k+\sum_{i=1}^k\deg\xi_i+\dim R_{\alpha_0}+\mu(R_{\alpha_0})-\sum_{p=1}^k\mu\big(R_{\alpha_p}\big)\\ &\equiv \epsilon(\xi_1,\dots,\xi_k)+1+\dim\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})+\sum_{i=1}^k\deg\xi_i \mod 2.\end{aligned}$$

Using Proposition 2.5, we have

$$\begin{aligned}\mathrm{d}\big(\mathrm{ev}_0^B\big)_!\big(\mathrm{ev}_1^{B*}\xi_1\wedge\cdots\wedge\mathrm{ev}_k^{B*}\xi_k\big) &= \big(\mathrm{ev}_0^B\big)_!\mathrm{d}\big(\mathrm{ev}_1^{B*}\xi_1\wedge\cdots\wedge\mathrm{ev}_k^{B*}\xi_k\big)\\ &\quad+(-1)^{\nu}\big(\mathrm{ev}_0^B|_{\partial\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})}\big)_!\big(\mathrm{ev}_1^{B*}\xi_1\wedge\cdots\wedge\mathrm{ev}_k^{B*}\xi_k\big),\end{aligned} \tag{4.8}$$

where $\nu=\dim\mathcal{M}_{k+1}(B;\mathcal{L};\mathcal{R})+\sum_{i=1}^k\deg\xi_i$.

Combining (4.1), (4.2), Claim 4.5 and (4.8), we have

$$\begin{aligned}&\mathfrak{m}_{1,0}\circ\mathfrak{m}_{k,B}(\xi_1,\dots,\xi_k)+\mathfrak{m}_{k,B}\circ\hat{\mathfrak{m}}_{1,0}(\xi_1,\dots,\xi_k)\\ &\quad+\sum_{(k',B'),(k'',B'')\neq(1,0)}\mathfrak{m}_{k',B'}\circ\hat{\mathfrak{m}}_{k'',B''}(\xi_1,\dots,\xi_k)=0\end{aligned}$$

for all $(k,B)\neq(1,0)$. Recall that, in the case that $(k,B)=(1,0)$, $\mathfrak{m}_{1,0}=\mathrm{d}$ clearly satisfies $\mathfrak{m}_{1,0}\circ\mathfrak{m}_{1,0}=0$. Hence, we obtain Theorem 4.4. ∎

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