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Hexagonal Circular 3-Webs with Reducible Polar Curves of Degree Three

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Abstract. The paper reports the progress with the classical problem, posed by Blaschke and Bol in 1938. We present new examples and new classifications of natural classes of hexagonal circular 3-webs. The main results is the classification of hexagonal circular 3-webs with reducible polar curves of degree 3 and description of hexagonal circular 3-webs admitting a one-parameter Möbius symmetry.

Key words: circular hexagonal 3-webs

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1 Introduction

The problem to describe hexagonal 3-webs formed by circles in the plane appeared in the first monograph on the web theory published by Blaschke and Bol in 1938 (see [4, p. 31]). The authors presented an example with 3 elliptic pencils of circles, each pair of pencils sharing a common vertex, and observed that one can construct hexagonal circular 3-webs from hexagonal linear 3-webs, completely described by Graf and Sauer [7] as being formed by tangents to a fixed curves of third class. The construction involves a central projection from a plane to a unit sphere followed by stereographic projection to a plane. The corresponding circular 3-webs were described earlier by Volk [14] and Strubecker [13].

Stereographic projection puts the problem into a natural framework of the Möbius geometry: instead of planar circular webs we study circular webs on the unit sphere, thus treating circles and straight lines on equal footing. Möbius geometry assigns points outside the unit sphere to circles on this sphere: the assigned point is the polar point of the plane that cuts the circle on the sphere. This sphere is called also *Darboux quadric*. Thus any circular 3-web on the unit sphere determines locally 3 curve arcs outside the Darboux quadric, one arc per web foliation. Globally these arcs may belong to one irreducible algebraic curve. In what follows we call this set of polar points a *polar curve* of the web. For example, the polar curve of the hexagonal circular 3-web obtained from a linear 3-web is a planar cubic, possibly reducible. The polar curve of the cited example from [4] splits into 3 non-coplanar lines.

In the same year as the book [4] appeared, Wunderlich published a new remarkable example of hexagonal circular 3-web. Its polar curve splits into 3 conics lying in 3 different planes. Since through a point p on the unit sphere pass the circles whose polar points are intersection of the web polar curve with the plane tangent to the unit sphere at p, the Wunderlich web is actually 6-web, containing 8 hexagonal 3-subwebs.

Wunderlich gave also a construction of hexagonal 3-webs whose polar curve splits into 3 non-coplanar lines, two being dual with respect to the Darboux quadric and the third joining them. These webs were later rediscovered by other authors.

Further, he presented the following way to construct hexagonal 3-webs: for any one-parametric group acting in the plane, choose 2 transversally intersecting curve arcs that are also transversal to the group orbits; acting on the arcs by the group one gets 2 foliations; the third is composed by the group orbits. These 3 foliations compose a (local) hexagonal 3-web. Choosing a one-parameter group either of translations, or of dilatations, or of rotations and taking two intersecting circles (a straight line counts as a circle), we get circular hexagonal 3-webs.

Blaschke was well aware of the difficulty of the posed problem and, in his last book on the web geometry [3], discussed the simpler problem of classifying hexagonal circular 3-webs whose polar curve splits into 3 non-coplanar lines. Note that to a line corresponds a pencil of circles that is *hyperbolic* if the line spears the Darboux quadric, *elliptic* if the line completely misses the Darboux quadric, or *parabolic* if the line touches the Darboux quadric.

By the year 1977, the list of 6 types (one from [4] and five indicated in [15]) of circular hexagonal 3-webs whose polar curve splits into 3 non-coplanar lines was completed by Erdoğan [5] and Lazareva [8]. The first attempt to prove that the list is actually complete was published in 1989 by Erdoğan [6]. Based on direct computational approach, it did not provide the crucial computation: in fact, a modern computer systems for symbolic computations shows that there must be a mistake in the proof presented in [6] (see concluding remarks for further detail).

The Erdoğan's claim was proved only in 2005 by Shelekhov [12]. His insight was to look into the singular set of the webs: defined globally, the webs under study inevitably have singularities. Shelekhov considered the simplest possible singularities where two of the three circular foliations are tangent. It turns out that hexagonality imposes a strong restriction: locally, such singular set is either a circle arc of the 3rd foliation or the common circle arc of the first two. The restriction was rigid enough to obtain all the types on the list.

Five new types of hexagonal circular 3-webs were presented by Nilov in 2014 [10]. Polar curves for four of them split into a line and a conic. The fifth example may be viewed as a 5-web whose polar curve is a union of a line and two conics. Taking the line and two arcs on different conics as the polar curve, one gets a hexagonal 3-subwebs.

One can not help to observe that the polar curves of all the known examples are algebraic. Motivated also by the dual reformulation of the Graf and Sauer theorem, we consider the following natural class of 3-webs: hexagonal circular 3-webs with polar curve of degree three. The main result of the paper is the complete classification of such webs with reducible polar curves.

The case of planar polar curve follows immediately from the Graf and Sauer theorem: 3 points on the polar curve corresponding to 3 circles through a point p on the sphere are the ones where the plane, tangent to the sphere at p meets the polar curve. This plane cuts the polar curve plane along the line. On the polar curve plane we get the configuration dual to the Graf and Sauer theorem.

The case of non-planar set of 3 lines was finally settled by Shelekhov [12].

We obtain a classification of 3-webs whose planar polar curve splits into a line and a smooth conic. Up to Möbius transformation, there are 15 types, most of them depending on one parameter. Four types of five in Nilov's paper [10] are webs of this list, namely, of the types 6, 10, 11 and 15, presented in Section 5. (In fact, Nilov has found only one Möbius orbit from one-parametric family of orbits of our type 6.)

Another natural class that we study in this paper is the set of hexagonal circular 3-webs symmetric by action of one-parameter subgroup of the Möbius group. We also give a complete classification of such webs.

To select candidates for hexagonal webs we exploit further the above mentioned observation of Shelekhov on simplest singularities of hexagonal 3-webs. The proof of the observed property in [12], based on considering the normal form of the web function is not complete: this normal form often does not exists at singular points (see concluding remarks for more detail). We make precise the ideas about the type of singularities and then prove the key singularity property.

For completeness, we also present the classification of hexagonal webs with 3 non-coplanar polar lines. The proof mainly follows the line taken by Shelekhov in [12].

2 Hexagonal 3-webs, Blaschke curvature, singularities

A planar 3-web $\mathcal{W}_3$ in a planar domain is a superposition of 3 foliations $\mathcal{F}_i$, which may be given by integral curves of three ODEs $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, where σ_i are differential one-forms. At non-singular points, where the kernels of these forms are pairwise transverse, we normalize the forms so that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The *connection form* of the web $\mathcal{W}_3$ is a one-form γ determined by the conditions $\mathrm{d}\sigma_i + \gamma \wedge \sigma_i = 0$, $i = 1, 2, 3$. The connection form depends on the normalization of the forms σ_i, the *Blaschke curvature* $d\gamma$ does not.

Definition 2.1. A 3-web is hexagonal if for any non-singular point there are a neighbourhood and a local diffeomorphism sending the web leaves of this neighbourhood in 3 families of parallel line segments.

Topologically, hexagonality means the following incidence property that has given its name to the notion: for any point m, each sufficiently small curvilinear triangle with the vertex m and sides formed by the web leaves, may be completed to the curvilinear hexagon, whose sides are web leaves and whose "large" diagonals are the web leaves meeting at m (see the gallery of pictures illustrating hexagonal webs in the next section). Computationally, hexagonality amounts to vanishing of the Blaschke curvature [4].

Up to a suitable affine transformation, the forms σ_i may be normalized as follows:

$$\sigma_1 = (Q-R)(\mathrm{d}y - P\mathrm{d}x), \qquad \sigma_2 = (R-P)(\mathrm{d}y - Q\mathrm{d}x), \qquad \sigma_3 = (P-Q)(\mathrm{d}y - R\mathrm{d}x),$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$ are the slopes of the tangent lines to the web leaves at (x,y). Vanishing of the curvature writes as

$$\begin{aligned}
&(R-Q)[P_{xx} + (Q+R)P_{xy} + QRP_{yy}] + (P-R)[Q_{xx} + (P+R)Q_{xy} + PRQ_{yy}] \\
&\quad + (Q-P)[R_{xx} + (P+Q)R_{xy} + PQR_{yy}] \\
&\quad + \frac{(Q-R)(2P-Q-R)\big[P_x^2 + (Q+R)P_xP_y + QRP_y^2\big]}{(P-Q)(P-R)} \\
&\quad + \frac{(R-P)(2Q-P-R)\big[Q_x^2 + (P+R)Q_xQ_y + PRQ_y^2\big]}{(Q-R)(Q-P)} \\
&\quad + \frac{(P-Q)(2R-P-Q)\big[R_x^2 + (P+Q)R_xR_y + PQR_y^2\big]}{(R-Q)(R-P)} \\
&\quad + \frac{(2R-P-Q)P_xQ_x}{P-Q} + \frac{(2P-Q-R)Q_xR_x}{Q-R} + \frac{(2Q-R-P)R_xP_x}{R-P} \\
&\quad + \frac{\big(R^2-PQ\big)[P_xQ_y + P_yQ_x]}{(P-Q)} + \frac{\big(P^2-QR\big)[Q_xR_y + Q_yR_x]}{Q-R} \\
&\quad + \frac{\big(Q^2-PR\big)[R_xP_y + R_yP_x]}{R-P} + \frac{\big(2PQR - (P+Q)R^2\big)P_yQ_y}{Q-P} \\
&\quad + \frac{\big(2PQR - (Q+R)P^2\big)Q_yR_y}{R-Q} + \frac{\big(2PQR - (P+R)Q^2\big)R_yP_y}{P-R} = 0.
\end{aligned} \tag{2.1}$$

If only one slope, say R, is given as an explicit functions of x, y and P, Q are roots of a quadratic equation $P^2 + AP + B = 0$, then one finds the first derivatives of P, Q by differentiating the Vieta relations

$$P + Q = -A, \qquad PQ = B, \tag{2.2}$$

as a functions of P, Q and the first derivatives of A, B. Differentiating these expressions, one gets also the second derivatives. Finally, excluding P and Q with the help of (2.2), one can rewrite (2.1) in terms of A, B, R and their derivatives. The result is presented in Appendix A.

The webs considered later will inevitably have singularities: some kernels of the forms σ_i can be not transverse or the forms can vanish at some points. We call a singular 3-web hexagonal if its Blaschke curvature vanishes identically at regular points. The simplest type of singularities of hexagonal 3-webs have the following remarkable property, first observed by Shelekhov [12].

Lemma 2.2. *Suppose that a hexagonal 3-web, defined by three analytic direction fields ξ_1, ξ_2, ξ_3, has a singular point p_0 such that*

(1) *all ξ_i are well defined at p_0,*

(2) *ξ_1 and ξ_2 are transverse at p_0,*

(3) *$\xi_1 = \xi_3$ at p_0,*

then either the leaves of ξ_1 and ξ_3 through p_0 coincide or $\xi_1 = \xi_3$ along the leaf of ξ_2 through p_0.

Proof. The property is a consequence of separation of variables for hexagonal webs. The second condition implies that we can rectify ξ_1 and ξ_2, i.e., choose some local coordinates u, v so that $\xi_1 = \partial_v$ and $\xi_2 = \partial_u$. Then $\xi_3 = -f(u,v)\partial_u + \partial_v$ with $f(u_0, v_0) = 0$, where $p_0 = (u_0, v_0)$. Consider the analytic functions $\varphi(u) := f(u, v_0)$.

If $\varphi(u) \equiv 0$, then $f(u, v_0) = 0$ and the leaves of ξ_1 and ξ_3 are tangent along the line $v = v_0$, which is the leaf of ξ_2.

If $\varphi(u) \not\equiv 0$, then $\varphi(u) = (u-u_0)^n\psi(u)$ with natural n and $\psi(u_0) \neq 0$. Thus for any u_1 close to u_0 with $u_1 \neq u_0$ holds $f(u_1, v_0) = (u_1 - u_0)^n\psi(u_1) \neq 0$. Now the hexagonality amounts to $\partial_v\big(\frac{\partial_u f}{f}\big) = 0$ hence the germ $\tilde{f}$ of f at (u_1, v_0) factors $\tilde{f}(u,v) = a(u)b(v)$ with analytic germs a, b at u_1 and v_0 respectively. One can choose $b(u)$ so that $b(v_0) = 1$. Then $a(u) = \psi(u) = (u-u_0)^n\psi(u)$ is analytic also at u_0, the function of two variables $a(u)b(v)$ is analytic at (u_0, v_0) and coincides with $f(u,v)$ at some neighborhood of (u_1, v_0) included in the domain of f. Then by uniqueness $f(u,v) = a(u)b(v)$. Now observe that $a(u_0) = 0$ which implies $f(u_0, v) \equiv 0$ and the integral curves of ξ_1 and ξ_3 passing through p_0 coincide. ∎

3 Projective model of Möbius geometry

Following Blaschke [2], we call the subgroup PSO(3,1) of projective transformations of $\mathbb{RP}^3$, leaving invariant the quadric $X^2 + Y^2 + Z^2 - U^2 = 0$, the Möbius group. For the reference, we present here infinitesimal generators of Möbius group in homogeneous coordinates $[X : Y : Z : U]$ in $\mathbb{P}^3$, affine coordinates $x = \frac{X}{U}$, $y = \frac{Y}{U}$, $z = \frac{Z}{U}$ in $\mathbb{R}^3$ and cartesian coordinates $(\bar{x}, \bar{y})$ in $\mathbb{R}^2$ related to points (x, y, z) on the unit sphere via stereographic projection $x = \frac{2\bar{x}}{1+\bar{x}^2+\bar{y}^2}$, $y = \frac{2\bar{y}}{1+\bar{x}^2+\bar{y}^2}$, $z = \frac{1-\bar{x}^2-\bar{y}^2}{1+\bar{x}^2+\bar{y}^2}$. There are 3 rotations around the affine axes:

$$R_z = Y\partial_X - X\partial_Y = y\partial_x - x\partial_y = \bar{y}\partial_{\bar{x}} - \bar{x}\partial_{\bar{y}},$$

$$R_x = Z\partial_Y - Y\partial_Z = z\partial_y - y\partial_z = \bar{x}\bar{y}\partial_{\bar{x}} + \frac{1}{2}\big(1 - \bar{x}^2 + \bar{y}^2\big)\partial_{\bar{y}},$$

$$R_y = X\partial_Z - Z\partial_X = x\partial_z - z\partial_x = -\frac{1}{2}\big(1 + \bar{x}^2 - \bar{y}^2\big)\partial_{\bar{x}} - \bar{x}\bar{y}\partial_{\bar{y}},$$

and 3 boosts (or "hyperbolic rotations")

$$B_x = U\partial_X + X\partial_U = \partial_x - x(x\partial_x + y\partial_y + z\partial_z) = \frac{1}{2}\big(1 - \bar{x}^2 + \bar{y}^2\big)\partial_{\bar{x}} - \bar{x}\bar{y}\partial_{\bar{y}},$$

$$B_y = U\partial_Y + Y\partial_U = \partial_y - y(x\partial_x + y\partial_y + z\partial_z) = -\bar{x}\bar{y}\partial_{\bar{x}} + \frac{1}{2}\big(1 + \bar{x}^2 - \bar{y}^2\big)\partial_{\bar{y}},$$

$$B_z = U\partial_Z + Z\partial_U = \partial_z - z(x\partial_x + y\partial_y + z\partial_z) = -\bar{x}\partial_{\bar{x}} - \bar{y}\partial_{\bar{y}}.$$

The identity component of $\mathrm{PSO}(3,1)$ is well known to be isomorphic to the group $PSL_2(\mathbb{C})$, the isomorphism being given by the action $A(V) = AVA^*$ of $A \in SL_2(\mathbb{C})$ on the vector space of matrices

$$V = \begin{pmatrix} X+U & Y+\mathrm{i}Z \\ Y-\mathrm{i}Z & U-X \end{pmatrix}$$

with real X, Y, Z, U. This action preserves determinant of V, which is $U^2 - X^2 - Y^2 - Z^2$. By this isomorphism, the generators are represented by the following matrices:

$$R_x = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \qquad R_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad R_z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$B_x = \mathrm{i}R_x, \qquad B_y = \mathrm{i}R_y, \qquad B_z = \mathrm{i}R_z.$$

Two points $p_1 = [X_1 : Y_1 : Z_1 : U_1]$ and $p_2 = [X_2 : Y_2 : Z_2 : U_2]$ in $\mathbb{RP}^3$ determine a line with the Plücker coordinates

$$a := X_1U_2 - X_2U_1, \qquad b := Y_1U_2 - Y_2U_1, \qquad c := Z_1U_2 - Z_2U_1,$$
$$f := Y_1Z_2 - Y_2Z_1, \qquad g := Z_1X_2 - Z_2X_1, \qquad h := X_1Y_2 - X_2Y_1.$$

By direct computation, one proves the following fact.

Lemma 3.1. *All points of a line with Plücker coordinates $[a : b : c : f : g : h]$ are stable with respect to subgroup with the infinitesimal generator $aR_x + bR_y + cR_z + fB_x + gB_y + hB_z$.*

Observe that the line dual to $[a : b : c : f : g : h]$ is the one with coordinates $[-f : -g : -h : a : b : c]$, which corresponds to multiplication by i of the corresponding matrix representation of the generator.

A line in $\mathbb{RP}^3$ can be hyperbolic, elliptic or parabolic with the respect to the Darboux quadric. Considering simple representatives of these classes one sees that

(1) For hyperbolic line, the action of the corresponding operator on the dual line is Möbius conjugate to the action of R_z on the line $Z = U = 0$ at infinity thus not having extra stable points in $\mathbb{RP}^3$.

(2) For elliptic line, the action of the corresponding operator on the dual line is Möbius conjugate to the action of B_z on the line $X = Y = 0$ (which action leaves invariant two extra points $[0 : 0 : \pm 1 : 1]$).

(3) For parabolic line, the corresponding operator moves all points on the dual line (consider $\partial_y = R_x + B_y$).

In the following proposition we summarize further properties of the above correspondence.

Proposition 3.2. *Let $\xi = aR_x + bR_y + cR_z + fB_x + gB_y + hB_z$ be an infinitesimal operator of the Möbius group.*

(1) *The corresponding action of the one-parameter group is not loxodromic and therefore Möbius equivalent (i.e., conjugated) either to rotation, or dilatation, or translation of the plane $(\bar{x}, \bar{y})$ if and only if*

$$af + bg + ch = 0. \tag{3.1}$$

(2) *This action is Möbius equivalent to rotation if and only if (3.1) and $a^2+b^2+c^2 > f^2+g^2+h^2$ are true, the line with Plücker coordinates $[a : b : c : f : g : h]$ being the set of polar points of the circular orbits.*

(3) *This action is Möbius equivalent to dilatation if and only if* (3.1) *and* $a^2 + b^2 + c^2 < f^2 + g^2 + h^2$ *are true, the line with Plücker coordinates* $[a : b : c : f : g : h]$ *being the set of polar points of the orbits.*

(4) *The action is Möbius equivalent to translation if and only if* (3.1) *and*

$$a^2 + b^2 + c^2 = f^2 + g^2 + h^2 \tag{3.2}$$

are true, the line with Plücker coordinates $[a : b : c : f : g : h]$ *being the set of polar points of the orbits.*

Proof. The type of subgroup transformations is determined by the eigenvalues of the matrix representation for the generating operator ξ. Consider the characteristic polynomial for ξ

$$\Psi(\lambda) = \lambda^2 + \frac{1}{4}\left(a^2 + b^2 + c^2 - f^2 - g^2 - h^2\right) + \frac{\mathrm{i}}{2}(af + bg + ch)$$

and the matrices for non-loxodromic Möbius representatives R_z, B_z and $\partial_y = R_x + B_y$. ∎

4 Polar curve splits into 3 non-coplanar lines

Recall that *limit circles* of a hyperbolic pencil correspond to intersection points of the hyperbolic line with the Darboux quadric under the stereographic projection. *Vertexes* of an elliptic pencil are two points common to all the circles of the pencil, the vertexes correspond to the intersection of the Darboux quadric with the line, dual to the polar elliptic line. Considering a parabolic pencil as a limit case of elliptic, one calls the point corresponding to the point of tangency of the Darboux quadric with the parabolic line also the *vertex*. The vertex of a parabolic pencil is the common point for all circles of this parabolic pencil.

The reader may visualize the pencil fixed by a line L thinking of its circles as cut on the Darboux quadric by the pencil of planes containing the dual line L^*.

Let us list hexagonal 3-webs with non-planar polar lines. There are 9 types of such webs up to Möbius transformation.

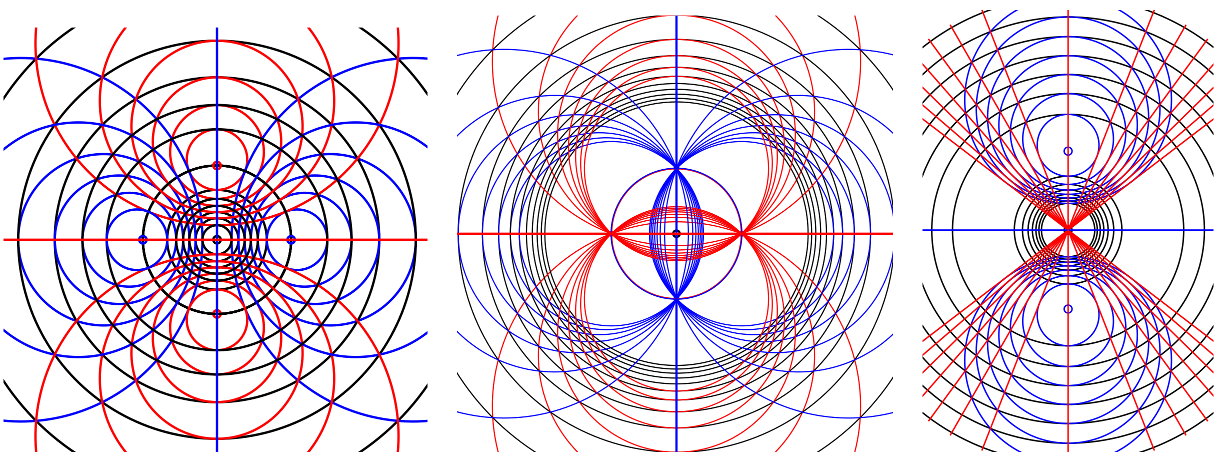

Figure 1. Hexagonal circular 3-webs. 3 hyperbolic pencils (left), 1 hyperbolic and 2 elliptic pencils (center), 1 elliptic and 2 hyperbolic pencils (right).

Take 3 polar lines intersecting inside the Darboux quadric so that each of the three lines contains the point dual to the plane of the other two polar lines, i.e., each pencil has a circle orthogonal to all the circles of the other two pencils. A representative of this web orbit, having one limit circle at infinity, is shown in Figure 1 on the left. This type was described by Lazareva [8].

Replacing two pencils by their orthogonal, we get the web in the center of Figure 1. In the projective model, we replace two polar lines by their dual ones. This web was also described by Lazareva [8].

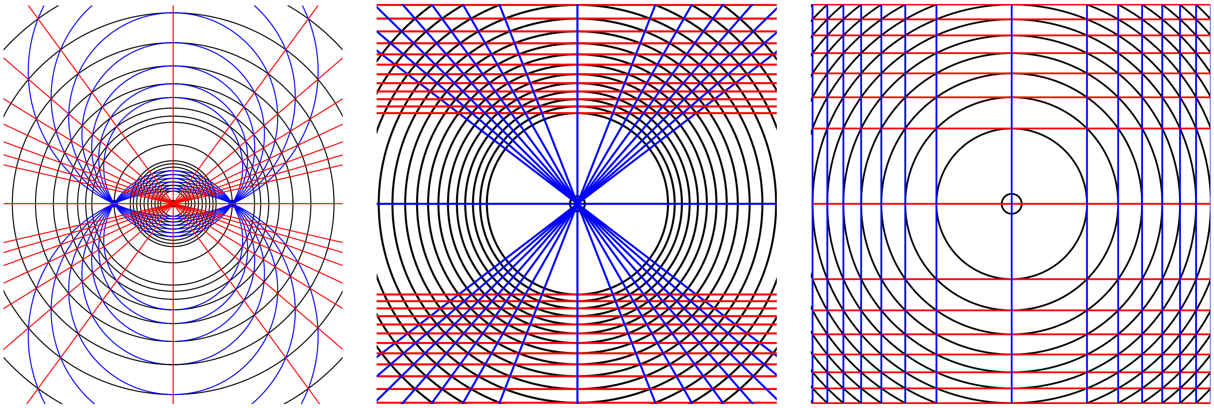

Figure 2. Hexagonal circular 3-webs. 1 hyperbolic and 2 elliptic pencils (left), 1 parabolic, 1 elliptic, and 1 hyperbolic pencil (center), 1 hyperbolic and 2 parabolic pencils (right).

Wunderlich [15] mentioned the following construction, used later also by Balabanova and Erdoğan, to produce hexagonal 3-webs: take two dual polar lines and supplement it by a third intersecting that dual pair. There are four webs in the list, obtained in this way (see also [1]): with two hyperbolic and one elliptic pencils on the right of Figure 1; with one hyperbolic and two elliptic pencils on the left of Figure 2; with one hyperbolic, one elliptic, and one parabolic pencils in the center of Figure 2; and with one hyperbolic and two parabolic pencils on the right of Figure 2.

Another web with two elliptic and one hyperbolic pencils is depicted on the left of Figure 3. Its projective model has two elliptic polar lines, lying in a plane tangent to the Darboux quadric at a point, and a hyperbolic line passing through the points (different from the above tangency point), where duals to elliptic lines intersect the Darboux quadric. Its elliptic pencils share one common vertex at infinity, the other two vertices are also the limit circles of the hyperbolic pencil. This web was described by Erdoğan [5].

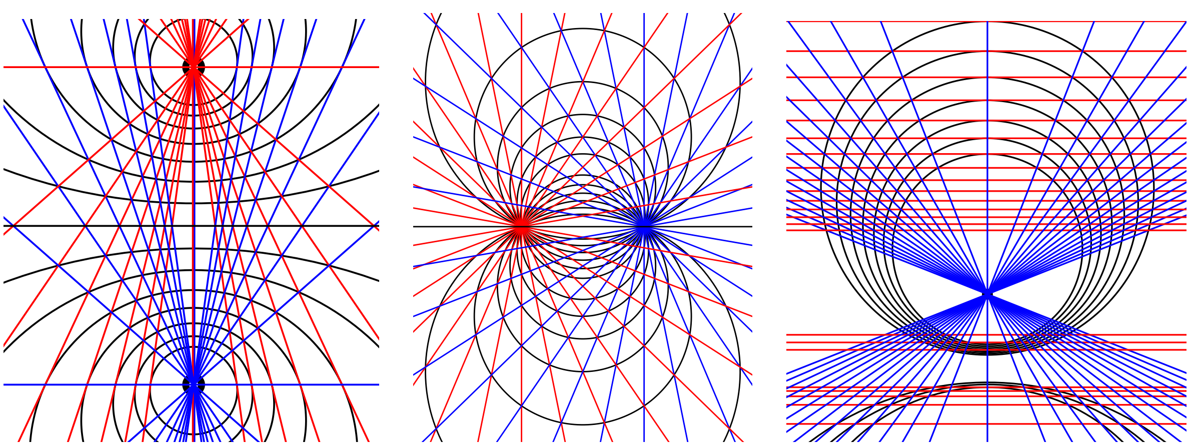

Figure 3. Hexagonal circular 3-webs. 1 hyperbolic and 2 elliptic pencils (left), 3 elliptic pencil (center), 1 hyperbolic, 1 elliptic and 1 parabolic pencil (right).

In the center of Figure 3, there is a web with three elliptic pencils, the vertices of the pencils are two of three fixed points. It is historically the first hexagonal circular 3-web described in the literature [4]. The chosen representative has a vertex at infinity.

Erdoğan [5] found a web with one hyperbolic, one elliptic and one parabolic pencil, arranged so that the vertex P of the parabolic pencil coincides with one vertex of the elliptic pencil, while the other vertex E of the elliptic pencil coincides with one of the limiting circles of the hyperbolic, the common circle of the elliptic and the parabolic pencils being orthogonal to the circle passing through the second limiting circle of the hyperbolic pencil and the points P, E. On the right of Figure 3 is a Möbius representative of this type web with P at infinity. On the projective model, we have 3 pairwise distinct points on the Darboux quadric: E, P and H. The hyperbolic polar line L_h spears the quadric at E and H, the parabolic polar L_p touches the quadric at P and intersects L_h, and the elliptic polar L_e is dual to the line through P and E (and intersects L_p).

Finally, we present a family of hexagonal 3-webs formed by 3 pencils, whose Möbius orbits are parameterized by one parameter. Two pencils are parabolic with distinct vertexes, and the third is elliptic, whose dual to the polar line meets the Darboux quadric at these vertexes. A representative of the family is shown in Figure 4, the vertexes being the origin and the infinite point. In this normalization one can fix the direction of one parabolic line, the direction of the other is arbitrary. Any web in this normalization is symmetric by dilatation $x\partial_x + y\partial_y$.

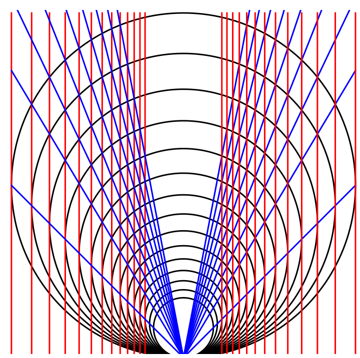

Figure 4. Hexagonal circular 3-web. 1 elliptic and 2 parabolic pencils.

Surprising is not only the fact that the “largest” family was explicitly described only in 1977 by Lazareva [8], even more amazing is that the family falls within the general construction (see Introduction), which seems to have appeared first in the paper of Wunderlich [15] in 1938!

In what follows we refer to the above described webs as *Blaschke–Wunderlich–Balabanova–Erdoğan–Lazareva list*. To prove that there are no other classes, we will strongly use Lemma 2.2. The singularity of the described type occurs when two of the three lines, tangent to Darboux quadric at a point and meeting each its own polar line, coincide and the point is not a vertex or limit circle of any pencil.

Proposition 4.1. *Consider the family of lines such that*

(1) *they meet two fixed lines L_1 and L_2, and*

(2) *they are tangent to the Darboux quadric.*

If the tangent points are on a circle, then either L_1 intersects L_2, or both L_1 and L_2 are tangent to the Darboux quadric, or one meets the dual of the other at some point on the Darboux quadric.

Moreover, in the case of skew L_1, L_2 tangent to the Darboux quadric at two points p_1, p_2, the curve of touching points splits into 2 circles, their planes containing the line p_1p_2 and bissecting the angles between two planes P_1, P_2, where P_i is the plane through L_i and p_1p_2.

Proof. Let $[a : b : c : f : g : h]$ be Plücker coordinates of a line L touching the Darboux quadric. Then, by Proposition 3.2, they satisfy equations (3.1) and (3.2). Therefore, $a^2 + b^2 + c^2 \neq 0$ and we can normalize these coordinates to $a^2 + b^2 + c^2 = f^2 + g^2 + h^2 = 1$. One easily calculates the point $p = (x, y, z) = (cg - bh, ah - cf, bf - ag)$ where the line L touches the quadric. Due to normalization, one can rewrite this as

$$a = hy - gz, \qquad b = fz - hx, \qquad c = gx - fy. \tag{4.1}$$

To simplify calculations, we can bring the Plücker coordinates of L_1 to simple form by Möbius transformation. We suppose that L_1 and L_2 are skew since the case of L_1, L_2 intersecting is obvious.

If L_1 is hyperbolic, we can choose a representative as $L_1 = [0 : 0 : 1 : 0 : 0 : 0]$. Then one has $m \neq 0$ since the lines L_1, $L_2 = [u : v : w : k : l : m]$ do not intersect. (We used the fact that the Plücker coordinates of intersecting lines are orthogonal with respect to the bilinear symmetric form defining the Plücker quadric.) Let us set $m = 1$. Moreover, the coordinates of L_2 satisfy the Plücker equation $ku + lv + w = 0$ thus giving w. Since L intersect L_1 and L_2, we have $h = 0$ and $ka + lb + c + uf + gv + wh = 0$, respectively. The above two equations cut a curve from the three-dimensional variety of lines touching the quadric.

The touching points p trace a curve on the Darboux quadric. This curve can be computed as follows. Equations (3.1) and (4.1) imply $fx + gy + hz = 0$, with $h = 0$ we have $g = -fx/y$. Now normalization $f^2 + g^2 + h^2 = 1$ gives $\left(x^2 + y^2\right)f^2 = y^2$, which means f is not identically zero. Therefore, intersection condition $ka + lb + c + uf + gv + wh = 0$ is equivalent to $kxz + lyz + uy - vx - x^2 - y^2 = 0$. This equation cuts the curve of tangent points on the Darboux quadric (i.e., unit sphere centered at the origin). If this curve is in a plane $z = Ax + By + C$, then by direct calculation one gets $C^2 = 1$. One can choose $C = 1$ and then, by further calculations, we get $A = -k$, $B = -l$, $l = u$, $k = -v$. Thus $L_2 = [u : v : 0 : -v : u : 1]$ lies in the plane tangent to the Darboux quadric at $(0, 0, -1)$, which is the point where the dual to L_2 spears the Darboux quadric. Note that the plane equation is $z = vx - uy + 1$.

If L_1 is hyperbolic, we choose $L_1 = [0 : 0 : 0 : 0 : 0 : 1]$. Then $w \neq 0$ and we can normalize $L_2 = [u : v : 1 : k : l : m]$ and the rest of the reasoning goes in a similar way.

If L_1 is parabolic, we choose $L_1 = [0 : -1 : 0 : 1 : 0 : 0]$. For skew L_1, L_2, we have $-l + u \neq 0$. Normalizing $l - u = 1$ gives $l = u + 1$. Further we rewrite (4.1) as $f = bz - cy$, $g = cx - az$, $h = ay - bx$ and, proceeding as before, obtain by calculation that L_2 satisfy (3.1), which means it is tangent to the Darboux quadric.

To check the last claim, we normalize the configuration so that L_1 and L_2 are tangent to the Darboux quadric at points $(0, 0, \pm 1)$, look for a plane containing a circle of tangent points in the form $y = Ax$, find two solutions for A, and verify the geometry by calculation. ■

Definition 4.2. Motivated by Lemma 2.2, we will refer to the circle of tangent points described in Proposition 4.1 whose polar point is not an intersection of L_1, L_2 as *singular circle.*

Remark 4.3. It is easy to see that the curve of touching points is of degree four. In the hypothesis of the proposition for the case of skew L_1, L_2 not tangent to the Darboux quadric, this curve also splits. One component is a circle and the other has just one real point, where dual to the elliptic line meets the hyperbolic.

Corollary 4.4. *For hexagonal circular 3-webs formed by three pencils with non-planar polar lines, any two L_1, L_2 of polar lines are either dual or obey geometrical restriction described by Proposition 4.1. Moreover, the polar point of a singular circle, defined by two polar lines, belongs to the third one.*

Proof. Let L_1, L_2 be skew but not dual and such that the curve of points, where lines L meeting both L_1, L_2 touch the Darboux quadric, is not planar. Then by Lemma 2.2 any such line meets also the third polar line L_3. Since L_1, L_2 are skew, L_3 cannot intersect neither of L_1, L_2. Thus L_1, L_2, L_3 belong to one ruling of some quadric Q and the touching lines L form the other ruling. But then L_1, L_2, L_3 are also tangent to the Darboux quadric. This contradicts our initial assumption. ■

Let $(p,q),(r,s)\in\mathbb{R}^2$ be vertices of an elliptic pencil, then the pencil circles form the family

$$I(x,y) := \frac{[(p-x)(r-x)+(q-y)(s-y)]^2}{\left[(p-x)^2+(q-y)^2\right]\left[(r-x)^2+(s-y)^2\right]} = \text{const}.$$

The circles are the integral curves of the ODE

$$\omega_e := \mathrm{d}(I) = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y = 0.$$

The circles of the hyperbolic pencil with limit circles at (p,q), (r,s) are orthogonal to the circles of the above elliptic one. They are the integral curves of the ODE

$$\omega_h := g(x,y)\mathrm{d}x - f(x,y)\mathrm{d}y = 0.$$

Now we work out the cases with parabolic pencils. Let $(p,q)\in\mathbb{R}^2$ be the vertex of a parabolic pencil and $[r:1-r]\in\mathbb{P}$ a direction orthogonal to the line tangent to all circles of the pencil. Then the pencil circles form the family

$$\tilde{I}(x,y) := \frac{(x-p)^2+(y-q)^2}{r(p-x)+(r-1)(y-q)} = \text{const}.$$

The circles are the integral curves of the ODE $\omega_p := d\big(\tilde{I}\big) = 0$. For the exceptional direction $(1,-1)$, the pencil circles family is

$$\bar{I}(x,y) := \frac{(x-p)^2+(y-q)^2}{(x-p)-(y-q)} = \text{const}$$

and the corresponding EDO $\omega_{\bar{p}} := \mathrm{d}\big(\bar{I}\big) = 0$.

Observe that differential forms σ_i, describing a 3-web of circles formed by 3 pencils, are algebraic. Thus Lemma 2.2 remains valid also over complex numbers in passing from $\mathbb{RP}^3$ to $\mathbb{CP}^3$. By circles here we understand sections of the complex Darboux quadric by complex planes. The complexification simplifies the proof of the following theorem.

Theorem 4.5 ([12]). *Any hexagonal circular* 3*-webs formed by three pencils with non-coplanar polar lines is Möbius equivalent to one from the Blaschke–Wunderlich–Balabanova–Erdoğan–Lazareva list.*

Proof. We consider all types of non-coplanar polar line triples L_1, L_2, L_3.

• *Three hyperbolic pencils.* By Corollary 4.4, all lines intersect at one point p. This point cannot be outside the Darboux quadric. In fact, applying a suitable Möbius transformation, we send this point to an infinite one, say $p_x = [1:0:0:0]$ and the plane of L_1, L_2 to the plane $Z=0$. Then, by Corollary 4.4, the polar line L_3 joins p_x and $p_z = [0:0:1:0]$, the polar point of the plane $Z=0$. Thus L_3 cannot be hyperbolic as supposed. The point p cannot be on the Darboux quadric either: we can send it to $p=(1,0,0)$ and the plane of L_1, L_2 to the plane $Z=0$. Now Corollary 4.4 implies that L_3 joins p and p_z. Thus L_3 is parabolic and not hyperbolic.

Therefore, p is inside the Darboux quadric and one can send it to the origin $(0,0,0)$. Corollary 4.4 implies that any of the lines L_1, L_2, L_3 contains the point dual to the plane of the other two lines. Thus L_1, L_2, L_3 are orthogonal and we have the web shown on the left of Figure 1. Note that there is only one such web up to Möbius transformation.

• *One elliptic and two hyperbolic pencils.* We can suppose that L_3 is an elliptic polar line and that L_3 is the infinite line in the plane $Z = 0$. Due to Corollary 4.4, the polar lines L_1, L_2, being hyperbolic, must intersect. Then the plane of L_1, L_2 is dual to some point on L_3 and therefore contains the line $X = Y = 0$ dual to L_3 and can be assumed to be the plane $Y = 0$.

Suppose that none of L_1, L_2 is dual to L_3. If both L_1, L_2 intersect L_3, then the intersection point is $p_x = [1 : 0 : 0 : 0]$. Applying Corollary 4.4 to the pair L_1, L_3, we conclude that L_2, joining p_x and the polar point of the plane of L_1, L_3 does not meet the Darboux quadric and is not hyperbolic as supposed. Thus we can suppose that L_1, L_3 are skew and that L_1, by Corollary 4.4, contains $(0, 0, -1)$. Since L_1 is not dual to L_3, we infer by Corollary 4.4 that L_2 contains the dual point of the singular circle for L_1, L_3. This point lies in the tangent plane to the Darboux quadric at $(0, 0, 1)$, therefore L_2, being non-parabolic, cannot meet L_3 and cannot also contain $(0, 0, 1)$. Then it passes through $(0, 0, -1)$, which contradicts the geometry restriction imposed by Corollary 4.4.

So we can assume that L_2 is dual to L_3, i.e., it is the line $X = Y = 0$. Corollary 4.4 prevents L_1 to be skew with L_3. Thus it meets L_3 and therefore intersect L_2 in a point inside the Darboux quadric. We obtain the hexagonal web equivalent to the one on the right of Figure 1.

• *One hyperbolic and two elliptic pencils.* By Corollary 4.4, elliptic lines, say L_2, L_3, intersect. We can assume that the hyperbolic line L_1 is the coordinate axis $X = Y = 0$.

First consider the case when two lines, L_1, L_2, are dual. Then L_2 is the infinite line in the plane $Z = 0$ and we can assume the intersection line of L_2 and L_3 being the point $p_y = [0 : 1 : 0 : 0]$. Then L_3 cannot be skew with L_1 due to Corollary 4.4: the polar of the singular circle determined by L_1, L_3 is finite and cannot lie on the infinite line L_2. Thus L_3, being elliptic, intersect L_1 outside the Darboux quadric and we get the web equivalent to the one shown on the left of Figure 2.

Now suppose that no pair L_1, L_i is skew. Then L_2 and L_3 intersect L_1. Since the triple L_1, L_2, L_3 is not coplanar, all three lines intersect at one point outside the Darboux quadric, which can be taken as $p_z = [0 : 0 : 1 : 0]$. By Corollary 4.4, the line L_3 contains the dual point of the singular circle determined by L_1, L_2 and the line L_2 contains the dual point of the singular circle determined by L_1, L_3. This fixes L_1, L_2, L_3 up to rotation around z-axis and gives the web shown in the center of Figure 1.

Finally, consider the case with skew but not dual L_1, L_2. Applying Möbius transformation, we send the intersection point of L_2 and L_3 to $p_x = [1 : 0 : 0 : 0]$ (preserving the position of L_1). Then L_3 joins p_x and the polar point of the singular circle determined by L_1, L_2. We obtain the web, equivalent to one on the left of Figure 3.

• *Three elliptic pencils.* We treat this case using the complex version of Corollary 4.4. First, we conclude that, being non-coplanar, all three polar lines intersect at one point, which we can send to $p_y = [0 : 1 : 0 : 0]$. None of 3 real planes, containing a pair of polar lines, can miss completely the real Darboux quadric. In fact, if the real plane of L_1, L_2 do not intersect the real Darboux quadric then the polar point of the complex singular circle of L_1, L_2 is inside the real Darboux quadric and L_3, being elliptic, cannot contain this point. None of 3 real planes, containing a pair of polar lines, can intersect the real Darboux quadric. If the real plane of L_1, L_2 cuts the real Darboux quadric, the polar point of the real singular circle of L_1, L_2 is outside the real Darboux quadric and the real plane of L_1, L_3 do not meet the real Darboux quadric. Thus all 3 planes are tangent to the Darboux quadric. A representative of such web is shown in the center of Figure 3.

• *One parabolic and two hyperbolic pencils.* By Corollary 4.4, all 3 polar lines must intersect in one point. The intersection point cannot be inside the Darboux quadric since one line is parabolic. It also cannot be outside: we can send it to $p_x = [1 : 0 : 0 : 0]$ and the parabolic line, joining p_x and the point dual to the plane of hyperbolic lines, will miss the Darboux quadric. Therefore, this point is on the Darboux quadric. One can move the plane of hyperbolic lines to $Z = 0$. Then the parabolic line contains $p_z = [0 : 0 : 1 : 0]$ and none of the hyperbolic lines can pass through the point dual the singular circle of the other two lines. Thus there is no hexagonal web with non-coplanar parabolic and two hyperbolic polar lines.

• *One parabolic, one hyperbolic and one elliptic pencil.* By Corollary 4.4, the parabolic line meets the other two. If the hyperbolic line intersects the elliptic at some point p outside the Darboux quadric, we can move the hyperbolic line to $X = Z = 0$ and p to $p_y = [0 : 1 : 0 : 0]$. Then the point p_s, dual for the singular circle of hyperbolic and elliptic lines, is infinite and the third polar line, joining p_y and p_s cannot be parabolic.

Therefore, hyperbolic and elliptic lines are skew and Corollary 4.4 fixes the configuration up to Möbius transformation: if these lines are dual we get the type shown in the center of Figure 2, otherwise the type on the right of Figure 3.

• *Two parabolic and one hyperbolic pencils.* By Corollary 4.4, the hyperbolic line meets both parabolic lines.

If the parabolic lines are skew, then the corresponding two singular circles have their polar points on the line dual to the one joining the points of tangency of parabolic lines with the Darboux quadric. This dual line is elliptic, therefore this configuration is not possible.

If the parabolic lines intersect outside the Darboux quadric, then we can bring the plane of their intersection to $Z = 0$. Now the hyperbolic line contains $p_z = [0 : 0 : 1 : 0]$ by Corollary 4.4 and, intersecting the both elliptic lines, must meet them at their common point. Then it misses the Darboux quadric and is not hyperbolic.

Therefore, the parabolic lines are tangent to the Darboux quadric at the same point. Since the polar lines are not coplanar the hyperbolic line contains this point. We can bring the hyperbolic line to $X = Y = 0$. Then Corollary 4.4 implies that the parabolic lines are dual and we obtain the web type shown on the right of Figure 2.

• *Two parabolic and one elliptic pencils.* By Corollary 4.4, the elliptic line meets both parabolic lines.

If the parabolic lines are skew, then the corresponding two singular circles have their dual points on the line dual to the one joining the points of tangency of parabolic lines with the Darboux quadric. The third polar line must contain these point, it is elliptic and we get the web type presented in Figure 4.

If the parabolic lines intersect outside the Darboux quadric, then we can bring the plane of their intersection to $Z = 0$. By Corollary 4.4, the elliptic line contains $p_z = [0 : 0 : 1 : 0]$ and, intersecting the both elliptic lines, must meet them at their common point. Thus we get the type shown in Figure 4.

If the parabolic lines are tangent to the Darboux quadric at the same point, then the third line, being elliptic, cannot pass through this point. Therefore, it is coplanar with the parabolic lines.

• *Three parabolic pencils.* Suppose that two polar lines L_1 and L_2 intersect. If the intersection point is outside the Darboux quadric, then we bring the plane of L_1, L_2 to $Z = 0$. Since the third polar line L_3 does not lie in this plane it contains $p_z = [0 : 0 : 1 : 0]$ by Corollary 4.4. Therefore, it is skew with at least one of L_1, L_2. Let it be L_1. Then by Corollary 4.4, the line L_2 contains both dual point of two singular circles of L_1, L_3 which is obviously not possible.

If L_1, L_2 touch the Darboux quadric at the same point, then the line L_3 is skew with at least one of L_1, L_2. Again, this is precluded by Corollary 4.4.

Therefore, L_1, L_2, L_3 are pairwise skew. Consider two singular circles C_1, C_2 of L_1, L_2, their polar points p_1, p_2 and the family of lines L touching the Darboux quadric and meeting both L_1, L_2. The third line L_3, being parabolic, cannot contain both points p_1, p_2. If $p_1 \notin L_3$, then by Lemma 2.2 the family of lines L touching the Darboux quadric at points of C_1 must meet also L_3. Therefore, the lines L constitute one ruling of a quadric touching the Darboux quadric along C_1. Therefore, L_1, L_2, L_3 belong to the second ruling. If $p_2 \in L_3$, then, sending the plane of C_1 to $Y = 0$ and the points, where L_1, L_2 touch the Darboux quadric, to $(0, 0, \pm 1)$, we see that $p_2 = [1 : 0 : 0 : 0]$ since C_2 is orthogonal to C_1 and passes through $(0, 0, \pm 1)$. Then L_3 must touch the Darboux quadric also at one of the points $(0, 0, \pm 1)$, which contradicts the initial assumption that all 3 lines are skew. Thus $p_2 \notin L_3$. Then the family of lines L touching the Darboux quadric at points of C_2 must meet also L_3. This is not possible as the lines L, constituting a ruling of a quadric that touch the Darboux quadric along C_1 cannot meet the orthogonal circle C_2. ■

5 Polar curve splits into conic and straight line

First we describe the types then we prove that the list is complete. Some types are one-parametric families and we denote the parameter value by c. The polar conic will be given either by an explicit parametrization of the circle equations and we reserve u for the parameter, or by indicating the conic equations. The former representation gives also a parametrization of the polar conic: to a circle $\epsilon(x^2 + y^2) + \alpha x + \beta y + \gamma = 0$ (where $\epsilon = 0$ or $\epsilon = 1$, the case $\epsilon = 0$ giving a line) corresponds the polar point with the tetracyclic coordinates $[\alpha : \beta : \gamma - \epsilon : -\gamma - \epsilon]$. The parameter for circles in the pencil will be denoted by v.

To check hexagonality, one computes the Blaschke curvature using the formula in Appendix A as follows. The polar conic gives a one-parameter family of circles on the unit sphere. The stereographic projection from the "south" pole

$$X = \frac{2x}{1 + x^2 + y^2}, \qquad Y = \frac{2y}{1 + x^2 + y^2}, \qquad Z = \frac{1 - x^2 - y^2}{1 + x^2 + y^2},$$

transforms this family into a family of circles in the plane parametrized by the points of the conic. The ODE for circles, defined by the conic,

$$P^2 + A(x, y)P + B(x, y) = 0, \qquad P = \frac{\mathrm{d}y}{\mathrm{d}x},$$

is obtained by differentiating and excluding the coordinates of the conic points. The slope R comes from the pencil of circles.

For some types, we add an additional geometric detail in the "title" to separate the types.

1. Polar conic plane does not cut Darboux quadric, hyperbolic pencil. The pencil with limit circles at the origin and in the infinite point gives circles $x^2 + y^2 = v$, the polar conic is the circle $X_0^2 + Y_0^2 + 4cX_0Z_0 = 0$, $U_0 = 0$, $c > 0$, defining the family

$$x^2 + y^2 - \frac{4c}{c^2u^2 + 1}x + \frac{4c^2u}{c^2u^2 + 1}y = 1,$$

the circles of the family enveloping the cyclic

$$\left(x^2 + y^2\right)^2 - \left(x^2 + y^2\right)(4cx + 2) - 4c^2y^2 + 4cx + 1 = 0$$

as shown on the left in Figure 5.

2. Polar conic plane does not cut Darboux quadric, hyperbolic pencil, webs symmetric by rotations. The pencil with limit circles at the origin and at the infinite point gives circles $x^2 + y^2 = v$, the polar conic is the circle $X_0^2 + Y_0^2 = \frac{4Z_0^2}{c^2}$, $U_0 = 0$, defining the family

$$x^2 + y^2 + \frac{2\cos(u)}{c}x + \frac{2\sin(u)}{c}y = 1,$$

the circles of the family enveloping the cyclic

$$c^2\left(x^2 + y^2\right)^2 - \left(2c^2 + 4\right)\left(x^2 + y^2\right) + c^2 = 0,$$

which splits into two concentric circles as shown in the center of Figure 5.

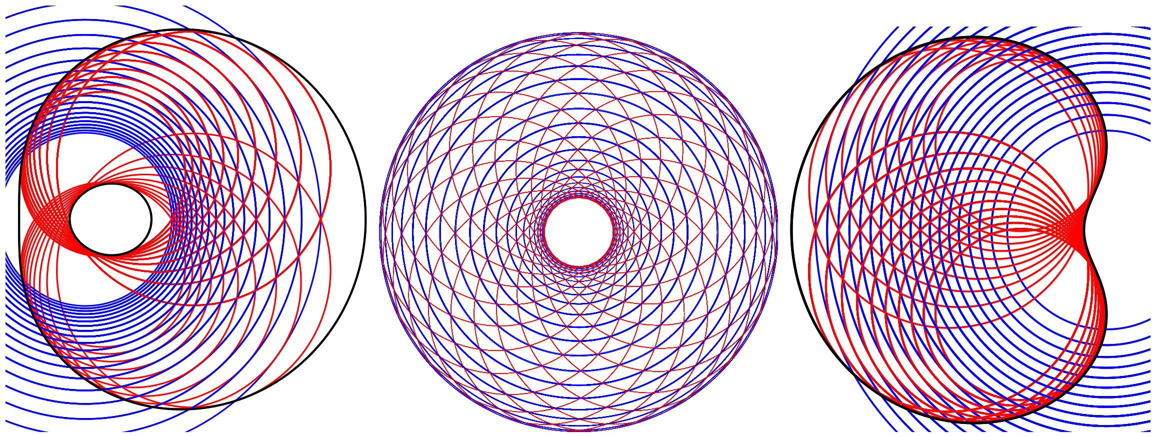

Figure 5. Types 1, 2, 3.

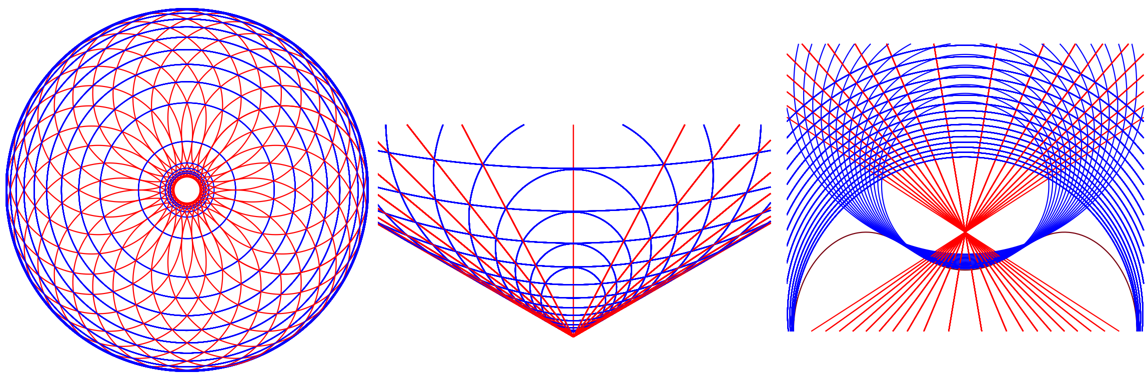

Figure 6. Types 4, 5, 6.

3. Polar conic plane cuts Darboux quadric, hyperbolic pencil. The pencil with limit circles at the origin and at the infinite point gives circles $x^2 + y^2 = v$, the polar conic is the circle $x_0^2 + y_0^2 = 4cx_0$, $z_0 = 0$, $c > 0$, defining the family

$$x^2 + y^2 + \frac{4c}{c^2u^2 + 1}x - \frac{4c^2u}{c^2u^2 + 1}y = -1,$$

the circles of the family enveloping the cyclic

$$\left(x^2 + y^2\right)^2 + \left(x^2 + y^2\right)(4cx + 2) - 4c^2y^2 + 4cx + 1 = 0,$$

as shown on the right of Figure 5.

4. Polar conic plane cuts Darboux quadric, hyperbolic pencil, webs symmetric by rotations. The pencil with limit circles at the origin and at the infinite point gives circles $x^2 + y^2 = v$, the polar conic is the circle $x_0^2 + y_0^2 = \frac{4}{c^2}$, $z_0 = 0$, defining the family

$$x^2 + y^2 - \frac{2\cos(u)}{c}x - \frac{2\sin(u)}{c}y = -1,$$

the circles of the family enveloping the cyclic

$$c^2\left(x^2 + y^2\right)^2 + \left(2c^2 - 4\right)\left(x^2 + y^2\right) + c^2 = 0,$$

which splits into two concentric circles as shown on the left in Figure 6.

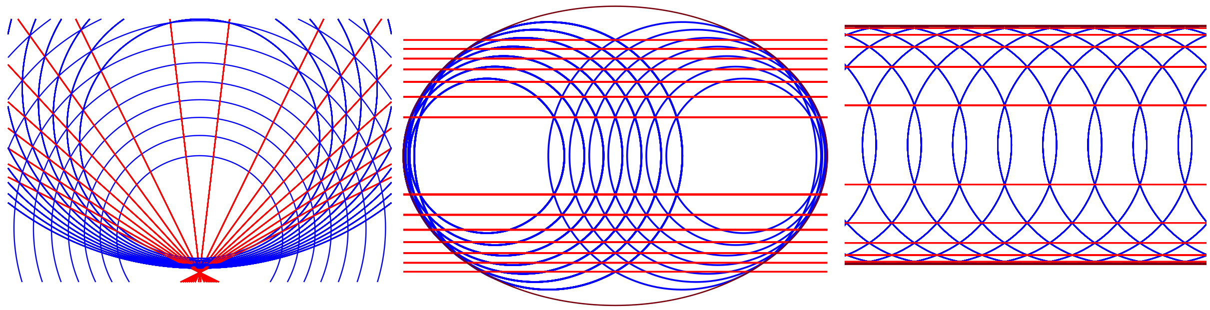

Figure 7. Types 7, 8, 9.

5. Polar conic plane cuts Darboux quadric, elliptic pencil, webs symmetric by homotheties. The pencil with vertexes at the origin and at the infinite point gives lines $y = vx$, the polar conic is $y_0^2 + cz_0^2 = c$, $x_0 = 0$, $c \notin [0, 1]$, defining the family of circles $x^2 + (y + u)^2 = (1 - 1/c)u^2$, the circles enveloping the lines $x^2 = (c - 1)y^2$, real for $c > 1$, as shown in the center in Figure 6. The webs of the family are symmetric by homotheties with the center in the origin.

6. Polar conic plane cuts Darboux quadric, elliptic pencil, polar conic and dual to polar line intersect in 2 points on the Darboux quadric. The pencil with vertexes at the origin and at the infinite point gives lines $y = vx$, the polar conic is $2cy_0 = z_0^2 - 1$, $x_0 = 0$, $c > 0$, defining the family of circles

$$x^2 + y^2 + \frac{(u-1)}{cu}y + \frac{u-1}{u+1} = 0,$$

the circles enveloping the cyclic

$$c^2\left(x^2 + y^2\right)^2 + \left(4cy - 2c^2\right)\left(x^2 + y^2\right) + (2y + c)^2 = 0$$

as shown on the right in Figure 6.

7. Polar conic plane cuts Darboux quadric, elliptic pencil. The pencil with vertexes at the origin and at the infinite point gives lines $y = vx$, the polar conic is $2cy_0z_0 + 1 = z_0^2$, $x_0 = 0$, $c > 0$, defining the family of circles

$$x^2 + y^2 - \frac{c+u}{cu}y + \frac{c+u}{c-u} = 0,$$

as shown on the left in Figure 7.

8. Polar conic plane cuts Darboux quadric, parabolic pencil. The pencil with the vertex at the infinite point gives lines $y = v$, the polar conic $x_0^2 - z_0 = 1$, $y_0 = 0$, gives the family of circles $\left(x - \frac{1}{u}\right)^2 + y^2 = \frac{u^2-1}{u^2}$. The circles envelopes the ellipse $x^2 + 2y^2 = 2$ as shown in the center in Figure 7.

9. Polar conic plane cuts Darboux quadric, parabolic pencil, webs symmetric by translations. The pencil with the vertex at the infinite point gives lines $y = v$, the polar conic is $x_0^2 + \left(1 - \sqrt{2}\right)z_0^2 - 2\sqrt{2}z_0 = 1 + \sqrt{2}$, $y_0 = 0$, defining the family of circles $(x + u)^2 + y^2 = 1$, the circles touching the lines $y = \pm 1$ as shown on the right in Figure 7.

10. Polar conic plane tangent to Darboux quadric, hyperbolic pencil. The pencil with limit circles at $(1, 0)$ and $(-1, 0)$ gives circles $x^2 + y^2 + 1 = vx$, the polar conic is $x_0^2 + (1 - c)y_0^2 = c$, $z_0 = -1$, $c > 0$, $c \neq 1$, defining the family of lines $x_0x + y_0y = 1$, the lines of the family enveloping the conic $cx^2 + \frac{c}{c-1}y^2 = 1$ with foci $(1, 0)$ and $(-1, 0)$ as shown on the left in Figure 8.

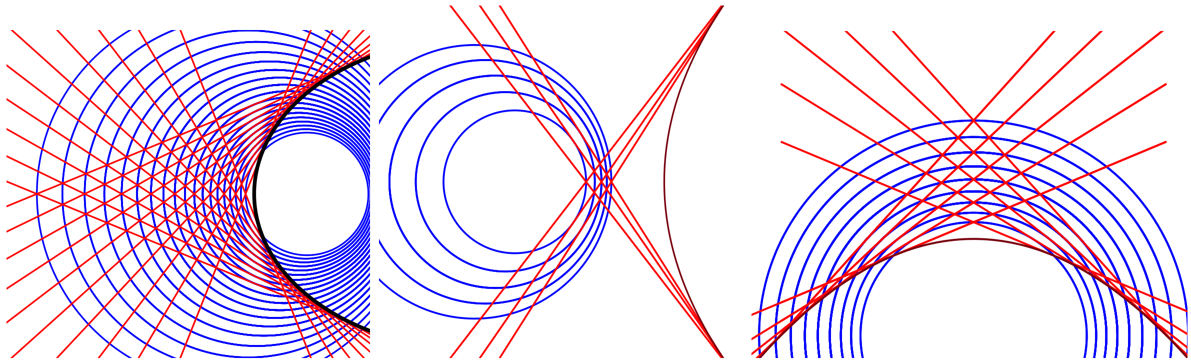

Figure 8. Type 10, 11, 12.

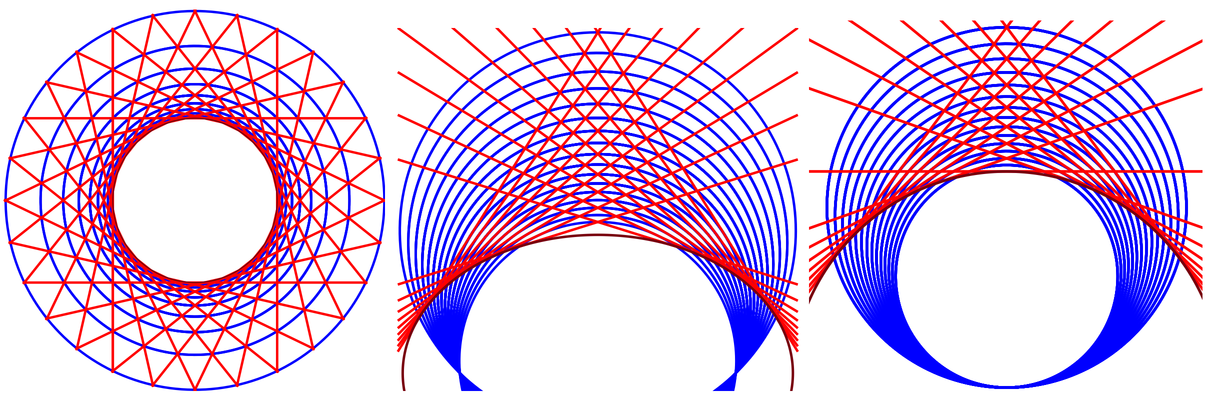

Figure 9. Type 13, 14, 15.

11. Polar conic plane tangent to Darboux quadric, hyperbolic pencil, polar line meets polar conic. The pencil with limit circles at $(1,0)$ and $(-1,0)$ gives circles $x^2+y^2+1 = vx$, the polar conic is $y_0^2 - cx_0y_0 + cy_0 + x_0 = 0$, $z_0 = -1$, $c \geq 0$, defining the family of lines $x_0x + y_0y = 1$, the lines of the family enveloping the parabola

$$(cx - y)^2 - (2c^2 + 4)x - 2cy + c^2 = 0$$

with focus at $(1,0)$ as shown in the center in Figure 8.

12. Polar conic plane tangent to Darboux quadric, hyperbolic pencil, polar line contains the point where polar conic plane touches the Darboux quadric. The pencil with limit circles at the origin and infinity gives circles $x^2 + y^2 = v$, the polar conic is $x_0^2 + y_0^2 - 2y_0 = 0$, $z_0 = -1$ defining the family of lines $x_0x + y_0y = 1$, the lines of the family enveloping the parabola $x^2 + 2y = 1$ with focus at $(0,0)$ as shown on the right in Figure 8.

13. Polar conic plane tangent to Darboux quadric, hyperbolic pencil, web symmetric by rotations. The pencil with limit circles at the origin and infinity gives circles $x^2 + y^2 = v$, the polar conic is $x_0^2 + y_0^2 = 1$, $z_0 = -1$ defining the family of lines $x_0x + y_0y = 1$, the lines of the family enveloping the circle $x^2 + y^2 = 1$ as shown on the left in Figure 9.

14. Polar conic plane tangent to Darboux quadric, elliptic pencil. The pencil with vertexes at $(1,0)$ and $(-1,0)$ gives circles

$$\frac{\left(x^2 + y^2 - 1\right)^2}{\left(x^2 + y^2 - 2x + 1\right)\left(x^2 + y^2 + 2x + 1\right)} = v,$$

the polar conic is $cx_0^2 + (c+1)y_0^2 + 1 = 0$, $z_0 = -1$, defining the family of lines $x_0x + y_0y = 1$, the lines of the family enveloping the conic $\frac{x^2}{c} + \frac{y^2}{c+1} = -1$ with foci $(1,0)$ and $(-1,0)$ as shown in the center in Figure 9.

15. Polar conic plane tangent to Darboux quadric, parabolic pencil. The pencil with vertex at the origin gives circles $x^2 + y^2 = 2vy$, the polar conic is $x_0^2 + y_0^2 = 1$, $z_0 = -1$ defining the family of lines $x_0 x + y_0 y = 1$, the lines of the family enveloping the circle $x^2 + y^2 = 1$ as shown on the right in Figure 9.

Theorem 5.1. *The webs of different types in the above classification list are not Möbius equivalent. The webs of the same type with different normal forms are not Möbius equivalent.*

Proof. The webs from different types are not Möbius equivalent: discrete geometric invariants indicated in the descriptions, such as 1) presence of infinitesimal symmetry and 2) the mutual position of polar line, polar conic and Darboux quadric, effectively separate the types.

To see that the different normal forms within a family are not Möbius equivalent, one computes the subgroup G_p of $\mathrm{PSO}(3,1)$ respecting the positions of the polar conic plane and the polar line in the chosen normalization and checks that the G_p-orbits of the canonical forms are different.

For the first 4 types, G_p is generated by R_z and by reflections in the coordinate planes.

For the 5th, 6th, 7th type, G_p is generated by B_z and by reflections in the coordinate planes.

For the 10th, 11th and 14th types, G_p is discrete and generated by reflections in the planes $x = 0$ and $y = 0$. ■

As in the case of 3 pencils, Lemma 2.2 effectively selects candidates among 3-webs that can be hexagonal. The singularities of the type described by the Lemma arise when either 1) a line, joining 2 different points on the polar conic, touches the Darboux quadric at a point p_t, while the polar conic plane is not tangent the Darboux quadric or 2) a line, joining a point on the polar line with a point on the polar conic, touches the Darboux quadric at a point p_t while the tangent plane to the Darboux quadric at p_t is not tangent to the polar conic.

In the former case, the points p_t trace a circle, which is the intersection of the polar conic plane with the Darboux quadric. Then, by Lemma 2.2, the polar of the polar conic plane lies on the pencil polar line.

In the latter case, consider a point p_r running over the polar line. For non-planar polar set, p_r meets the polar conic plane π_c only at one point. Therefore, the one-parameter family of cones tangent to the Darboux quadric and having their vertexes at p_r cuts the plane π_c in a one-parameter family of conics c_r. Each conic c_r intersect the polar conic c_p at 4 (possibly complex or multiple) points. If the polar conic c_p is not a member of the family $\{c_r\}$, at least one of these 4 intersection points is moving along c_p as p_r runs over the polar line. In fact, if intersection points are stable then $\{c_r\}$ is a pencil of conics containing c_p.

Choose one such moving intersection point p_i. Lines l_r tangent to c_r at p_i form a one-parameter family, or congruence of lines. All the objects in this construction are considered as complex but the polar conic and the polar line must have equations with real coefficients and the real part of the polar conic cannot lie completely inside the Darboux quadric.

Proposition 5.2. *If the polar curve of a hexagonal circular* 3*-web is non-planar and splits into a line and a smooth conic, then for the web complexification hold true*

(1) *the polar of the polar conic plane lies on the polar line, if this plane is not tangent to the Darboux quadric and*

(2) *the congruence of lines l_r is a pencil with the vertex on the polar conic, if the polar conic is not a member of the family $\{c_r\}$.*

Proof. The first claim follows directly from Lemma 2.2: the points, where bisecant lines touch the Darboux quadric, trace a circle whose polar point must lie on the polar line. For conic planes missing the Darboux quadric, the complex version works.

To derive the second claim, observe that the line p_rp_i touches the Darboux quadric at a singular point treated by Lemma 2.2. Thus the tangency point must trace a circle on the Darboux quadric and the polar point p_0 of the circle must lie on the polar conic. Then the plane, tangent to the Darboux quadric and passing through p_rp_i, contains p_0. This plane cuts the plane π_c of the polar conic along a line l_r, passing through the p_i and tangent to the corresponding conic c_r. Hence $\{l_r\}$ is the pencil with vertex at p_0. ∎

Theorem 5.3. *If the polar curve of a hexagonal circular* 3*-web splits into a smooth conic and a straight line, not lying in a plane of the conic, then the web is Möbius equivalent to one from the above presented list of types* 1*–*15*.*

Proof. The polar conic plane either completely misses the Darboux quadric, or cuts it in a circle, or is tangent to it. The polar line is either hyperbolic, or elliptic, or parabolic. Thus we have 9 cases to consider, each case defining a set of webs (possibly empty).

• *Polar conic plane misses the Darboux quadric.* Applying a suitable Möbius transformation, we can send the polar conic plane to infinity. Then by Proposition 5.2 the polar line contains the origin of the affine chart and therefore meets the Darboux quadric at 2 points. Thus the polar line is hyperbolic. Applying a rotation around the origin, we map these two points to $(0,0,\pm 1)$. In the affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ on the polar conic plane $U = 0$, the conics c_r are $x^2 + y^2 = r$, where r is considered as a complex parameter. These conics are real only for real non-negative r. If the polar conic coincides with one of the conics c_r, then we get Type 2, symmetric by R_z.

If the polar conic is not one of c_r, then, by Proposition 5.2, the points, where lines of pencil with the vertex at some point $p_0 = (x_0, y_0) \in \mathbb{C}^2$ touch the circles c_r, run over the polar conic. The line of the pencil $y = y_0 + k(x - x_0)$ corresponding to the parameter $k \in \mathbb{C}$, is tangent to the conic c_r for

$$r = \frac{x_0^2k^2 - 2x_0y_0k + y_0^2}{k^2 + 1}$$

at the point

$$p(k) = (x(k), y(k)) = \left(\frac{k(x_0k - y_0)}{k^2 + 1}, \frac{y_0 - x_0k}{k^2 + 1}\right).$$

The points $p(k)$ run over the complex conic $x^2 + y^2 = x_0x + y_0x$. This conic is real if and only if x_0 and y_0 are real. It is smooth if and only if $p_0 \neq (0,0)$. We got the first web of the list, Type 1. This conic is the circle, passing through the origin $O = (0,0)$ and p_0 and having its center at the midpoint of the segment Op_0. (We obtained a theorem of scholar geometry.) Using R_z, we normalize p_0 to $y_0 = 0$, $x_0 > 0$.

For infinite p_0, different from the cyclic points, the line $l(k)$ of the pencil $y = ax + k$, $a \in \mathbb{C}$ touches a unique conic of the family $\{c_r\}$ at the point $p(k) = \left(-\frac{ak}{a^2+1}, \frac{k}{a^2+1}\right)$. The points $p(k)$ are collinear: $x(k) + ay(k) = 0$ and we cannot obtain a smooth polar conic in this way. Finally, if p_0 is cyclic, the lines from the pencil touch the conics c_r at the very point p_0 and we get no conic at all.

• *Polar conic plane cuts the Darboux quadric, hyperbolic pencil.* We send the conic plane to $Z = 0$. Now the polar line contains the infinite point $[0:0:1:0]$ by Proposition 5.2. The subgroup of the Möbius group, preserving the plane $Z = 0$, is generated by rotations around Z-axis and the boosts B_x and B_y. Using this subgroup, we normalize the polar line to $x = y = 0$. The conics c_r are circles in the plane $z = 0$ with the center at the origin. Repeating the arguments that we used above for conic planes missing the Darboux quadric, we get Types 3 and 4.

• *Polar conic plane cuts the Darboux quadric, elliptic pencil.* We normalize the conic plane to $X = 0$. Then by Proposition 5.2 the point $[1 : 0 : 0 : 0]$ lies on the polar line. The polar line, being elliptic, meets the plane $X = 0$ at a point p outside the unit circle. Möbius transformations, preserving the plane $X = 0$, are generated by rotations around X-axis and the boosts B_y and B_z. Using these transformations, we send the point p to $[0 : 1 : 0 : 0 : 0]$. Now the polar line is $U = Z = 0$ and the conics c_r have equations $z^2 + \frac{y^2}{r} = 1$ in the affine coordinates. If the polar conic coincides with one of these conics, then r is real and we get Type 5.

Otherwise, by the second claim of Proposition 5.2, the lines l_r meet at some point $p_0 \in c_p$. If $p_0 = (0, y_0, z_0)$ is finite, a line of the pencil of lines $z = z_0 + k(y - y_0)$ with vertex at p_0 is tangent to the conic c_r with

$$r = \frac{y_0^2 k^2 - 2y_0 z_0 k + z_0^2 - 1}{k^2}.$$

Thus the points, where the lines l_r touch c_r, are parametrized by k via

$$y = \frac{y_0^2 k^2 - 2y_0 z_0 k + z_0^2 - 1}{k(y_0 k - z_0)}, \qquad z = \frac{1}{z_0 - y_0 k}.$$

Excluding k, we get the conic $y_0 z^2 - z_0 yz + y - y_0 = 0$. This conic is real only for real y_0, z_0 and is smooth if and only if $y_0\big(z_0^2 - 1\big) \neq 0$. If $z_0^2 < 1$, we normalize to $z_0 = 0$ applying B_z and get Type 6. If $z_0^2 > 1$, we send p_0 to infinity applying B_z.

For infinite point $p_0 = [0 : Y_0 : Z_0 : 0]$, we conclude that $Y_0 \neq 0$ and $Z_0 \neq 0$, otherwise the tangency points p_i do not trace a conic. Thus we can set $p_0 = [0 : y_0 : 1 : 0]$, where $y_0 \neq 0$. In the affine coordinates $u = \frac{U}{Z}$, $y = \frac{Y}{Z}$ in the plane $X = 0$, the conic c_r is $1 + \frac{y^2}{r} = u^2$. A line from the pencil $y = ku + y_0$ is tangent to the conic c_r if and only if $r = k^2 - y_0^2$. The points, where the lines l_r touch c_r, are parameterized by k via $y = \frac{y_0^2 - k^2}{y_0}$, $u = -\frac{k}{y_0}$. This is a parametrization of the polar conic of Type 7 $y_0 u^2 + y = y_0$, which becomes $y_0 U^2 - y_0 Z^2 + YZ = 0$ in homogeneous coordinates.

• *Polar conic plane cuts the Darboux quadric, parabolic pencil.* We normalize the conic plane to $Y = 0$. Then the point $[0 : 1 : 0 : 0]$ is on the polar line. The plane $Y = 0$ is stable under rotations R_y along the y-axis and under boosts B_x and B_z. Rotating, if necessary, around the y-axis, we bring the polar line to $z = -1$, $x = 0$. In the affine coordinates on the plane $Y = 0$, the conics c_r have equations

$$x^2 + \frac{r-1}{r} z^2 - \frac{2}{r} z - \frac{r+1}{r} = 0.$$

If the polar conic coincides with one of c_r, then r is real. The chosen position of polar conic plane and polar line is stable by action of the group generated by B_z. Applying it, one can normalize to $r = \frac{1}{\sqrt{2}}$ and we get Type 9.

Otherwise, consider first the pencil of lines $z = z_0 + k(x - x_0)$ with finite vertex at $p_0 = (x_0, 0, z_0)$. A line from the pencil is tangent to the conic c_r with

$$r = \frac{x_0^2 k^2 - 2x_0(z_0 + 1)k + (z0 + 1)^2}{\big(x_0^2 - 1\big)k^2 - 2x_0 z_0 k + z_0^2 - 1}.$$

Thus the points, where the lines l_r touch c_r, are parametrized by k via

$$x = \frac{k(x_0 k - z_0 - 1)}{k^2 - x_0 k + z_0 + 1}, \qquad z = \frac{\big(x_0^2 - 1\big)k^2 - x_0(2z_0 + 1)k + z_0(z_0 + 1)}{k^2 - x_0 k + z_0 + 1}.$$

Excluding k, we get the conic

$$(z_0 + 1)x^2 - x_0 xz + z^2 - x_0 x + (1 - z_0)z - z_0 = 0. \tag{5.1}$$

This conic is real only for real x_0, z_0 and is smooth if and only if $z_0 \neq -1$. The chosen position of polar conic plane and polar line is stable by action of the group generated by B_z and $B_x - R_y$. Consider the orbits of points in the plane $Y = 0$. The orbit dimension is two for points outside the union of the line $U + Z = 0$ and the circle $X^2 + Z^2 = U^2$, and is one on this union except for their common point. Thus the finite representatives of the orbits are $[0 : 0 : 0 : 1]$, $[0 : 0 : 1 : 1]$ and $[0 : 0 : -1 : 1]$. For the first two points, the conics (5.1) lie inside the Darboux quadric and there is no real circles. For the point $[0 : 0 : -1 : 1]$, the conic (5.1) is not smooth.

For infinite vertexes p_0, the pencil $z = k$ gives a non-smooth conic. Thus the pencil can be chosen as $x = az + k$. A line from the pencil is tangent to the conic c_r with $r = \frac{(k-a)^2}{k^2-a^2-1}$. The points, where the lines l_r touch c_r, are $x = \frac{k-a}{a^2-ak+1}$, $z = \frac{k^2-ak-1}{a^2-ak+1}$. Excluding k, we get the conic $x^2 - axz - ax - z - 1 = 0$, which is real only for real a. Taking into account the action of the group generated by B_z and $B_x - R_y$, we set $a = 0$ and get Type 8.

• *Polar conic plane tangent to Darboux quadric, hyperbolic pencil.* If the hyperbolic line does not contain the point where the polar conic plane touches the Darboux quadric, then we sent this point to $(0, 0, -1)$ and the polar line to $Y = Z = 0$. In the affine coordinates on the plane $z = -1$, the conics c_r have equations $(x - r)^2 + \left(1 - r^2\right)y^2 + 1 - r^2 = 0$. If the polar conic coincides with one of c_r, then the web curvature

$$K_B = \frac{4\left(r^2-1\right)^2\left(x^2-1\right)\left(rx^2+ry^2+\left(r^2-3\right)x+r\right)\left(x^2+y^2-2rx+1\right)^4}{x^4y^3\left(x^2+1-2rx\right)^6}$$

vanishes only for $r = \pm 1$, the conic c_r being non-smooth for these values.

A line from the pencil $y = y_0 + k(x - x_0)$ with finite vertex at $p_0 = (x_0, y_0, -1)$ is tangent to the conic c_r with

$$r = \frac{\left(x_0^2+1\right)k^2 - 2x_0y_0k + y_0^2 + 1}{2k(x_0k - y_0)}.$$

Thus the points, where the lines l_r touch c_r, are parametrized by k via

$$x = \frac{x_0\left(x_0^2-1\right)k^3 - y_0\left(3x_0^2-1\right)k^2 + x_0\left(3y_0^2+1\right)k - y_0\left(y_0^2+1\right)}{k\left(\left(x_0^2-1\right)k^2 - 2x_0y_0k + \left(y_0^2-1\right)\right)},$$
$$y = \frac{2(x_0k - y_0)}{\left(x_0^2-1\right)k^2 - 2x_0y_0k + \left(y_0^2-1\right)}.$$

Excluding k, we get the cubic

$$\left(x_0^2-1\right)y^3 + \left(y_0^2-1\right)x^2y - 2x_0y_0xy^2 + 2y_0x^2 + 2y_0y^2 - 2x_0y_0x + \left(x_0^2-y_0^2\right)y = 0. \tag{5.2}$$

This cubic splits into a smooth real conic and a line in 3 cases:

(1) for $y_0 = 0$, (5.2) factors as $y\left(x^2 + \left(1 - x_0^2\right)y^2 - x_0^2\right) = 0$ and we get Type 10,

(2) for $x_0 = 1$, $y_0 \neq 0$, (5.2) factors as $(x - 1)\left(\left(y_0^2-1\right)xy - 2y_0y^2 + 2y_0x + \left(y_0^2-1\right)y\right) = 0$,

(3) for $x_0 = -1$, $y_0 \neq 0$, (5.2) factors as $(x + 1)\left(\left(y_0^2-1\right)xy + 2y_0y^2 + 2y_0x - \left(y_0^2-1\right)y\right) = 0$.

The cases 2) and 3) give Type 11, the substitution $x \to -x$ reducing one to the other.

For infinite vertexes p_0, the pencil $x = k$ gives a non-smooth conic. Thus the pencil can be chosen as $y = ax + k$. A line from the pencil is tangent to the conic c_r with $r = \frac{k^2+a^2+1}{2ak}$. The points, where the lines l_r touch c_r, are

$$x = \frac{k\left(k^2-a^2+1\right)}{a\left(k^2-a^2-1\right)}, \qquad y = \frac{2k}{a^2-k^2+1}.$$

Excluding k, we get the cubic $y^3+a^2x^2y+2axy^2+2ax+\left(1-a^2\right)y=0$. The cubic splits into a line and a conic only for $a=0$ or for $a^2\pm 2\mathrm{i}a-1=0$. The former case gives non-real and non-smooth conic $y^2+1=0$, the latter – the non-real conic $xy\pm\mathrm{i}\left(y^2+2\right)=0$.

If the hyperbolic line contains the point where the polar conic plane touches the Darboux quadric, then we sent this point to $(0,0,-1)$ and the polar line to $X=Y=0$. In the affine coordinates on the plane $z=-1$, the conics c_r are concentric circles. The case of concentric circles was considered above. We get Types 12 and 13.

• *Polar conic plane tangent to Darboux quadric, elliptic pencil.* If one vertex of the elliptic pencil coincides with the point where the polar conic plane touches the Darboux quadric, then the polar curve is planar.

Thus we can sent the tangent point to $(0,0,-1)$ and the vertexes to $(\pm 1,0,0)$. The conics c_r have equations $\left(r^2+1\right)x^2+(y+r)^2=r^2+1$. If the polar conic coincides with one of c_r, then the web curvature

$$K_B=-\frac{64\left(r^2+1\right)^2x\left(y^2+1\right)\left(rx^2+ry^2+\left(3+r^2\right)y-r\right)\left(2ry-x^2-y^2+1\right)^4}{\left(x^2-y^2-1\right)^4\left(r^2-x^2+1\right)^6}$$

vanishes only for $r=\pm\mathrm{i}$, the conic c_r being non-smooth for these values.

A line from the pencil $y=y_0+k(x-x_0)$ is tangent to the conic c_r with

$$r=\frac{\left(x_0^2-1\right)k^2-2x_0y_0k+\left(y_0^2-1\right)}{2(x_0k-y_0)}.$$

The points, where the lines l_r touch c_r, are

$$x=\frac{2k(x_0k-y_0)}{\left(x_0^2+1\right)k^2-2x_0y_0k+y_0^2+1},$$
$$y=-\frac{x_0\left(x_0^2-1\right)k^3-y_0\left(3x_0^2-1\right)k^2+x_0\left(3y_0^2+1\right)k-y_0\left(y_0^2+1\right)}{\left(x_0^2+1\right)k^2-2x_0y_0k+y_0^2+1}.$$

Excluding k, we get the cubic

$$\left(y_0^2+1\right)x^3-2x_0y_0x^2y+\left(x_0^2+1\right)xy^2-2x_0x^2-2x_0y^2+\left(x_0^2-y_0^2\right)x+2x_0y_0y=0.$$

This cubic splits into a conic and a line in 4 cases:

(1) for $y_0=\pm\mathrm{i}$, the cubic equation factors as

$$(\pm i-y)\left(\pm 2\mathrm{i}x_0x^2-\left(1+x_0^2\right)xy\mp\mathrm{i}\left(1+x_0^2\right)x+2x_0y\right)=0,$$

(2) for $y_0=\pm\mathrm{i}x_0$, the cubic equation factors as

$$(x\pm\mathrm{i}y)\left(\left(1-x_0^2\right)x^2\mp\mathrm{i}\left(x_0^2+1\right)xy-2x_0x\pm 2\mathrm{i}x_0y+2x_0^2\right)=0,$$

(3) for $x_0=0$, the cubic equation factors as $x\left(\left(y_0^2+1\right)x^2+y^2-y_0^2\right)=0$,

(4) for $x_0=\pm 1$, the cubic equation factors as $(x\mp 1)\left(\left(y_0^2+1\right)x^2+2y^2\mp 2y_0xy\pm\left(y_0^2-1\right)x-2y_0y\right)=0$.

In the cases 1) and 2) the conic is not real, the case 3) gives Type 14, for the case 4) the web curvature does not vanish.

For infinite vertexes p_0, the pencil $x = k$ gives a non-smooth conic. Thus the pencil can be chosen as $y = ax + k$. A line from the pencil is tangent to the conic c_r with $r = \frac{a^2-k^2+1}{2k}$. The points, where the lines l_r touch c_r, are

$$x = -\frac{2ak}{k^2+a^2+1}, \qquad y = \frac{k\left(k^2-a^2+1\right)}{k^2+a^2+1}.$$

Excluding k, we get the cubic

$$a^2x^3 - 2ax^2y + xy^2 + \left(1-a^2\right)x + 2ay = 0.$$

The cubic splits into a line and a conic in 2 cases:

(1) for $a = 0$ the cubic equation factors as $x\left(y^2+1\right) = 0$ and the conic is non-smooth

(2) for $a = \pm\mathrm{i}$ the cubic equation factors as $(x \pm \mathrm{i}y)\left(x^2 \pm \mathrm{i}xy - 2\right) = 0$, and the conic is not real.

• *Polar conic plane tangent to Darboux quadric, parabolic pencil.* For non-planar polar curves, the points, where the polar line and polar conic plane touch the Darboux quadric, are different. Thus we can normalize the polar conic plane to $z = -1$ and the polar line to $x = 0$, $z = 1$. This configuration is preserved by B_z. The conics c_r have equations $\frac{r}{4}x^2 + y = \frac{1}{r}$. If the polar conic coincides with one of c_r, then the web curvature

$$K_B = -\frac{4096r^5xy^2\left(ry+2x^2+2y^2\right)\left(ry-x^2-y^2\right)^4}{\left(x^2-y^2\right)^4\left(r^2-4x^2\right)^6}$$

vanishes only for $r = 0$, the conic c_r being non-smooth for this value.

A line from the pencil $y = y_0 + k(x - x_0)$ is tangent to the conic c_r with $r = \frac{k^2+1}{y_0-x_0k}$. The points, where the lines l_r touch c_r, are

$$x = \frac{2k(x_0k-y_0)}{k^2+1}, \qquad y = \frac{x_0k^3-y_0k^2-x_0k+y_0}{k^2+1}.$$

Excluding k, we get the cubic

$$x^3 + xy^2 - 2x_0x^2 - 2x_0y^2 + \left(x_0^2-y_0^2\right)x + 2x_0y_0y = 0.$$

This cubic splits into a conic and a line in 2 cases:

(1) for $y_0 = \alpha x_0$, where $\alpha^2 \pm 2\mathrm{i}\alpha - 1 = 0$, the cubic equation factors as

$$(x \pm \mathrm{i}y)\left(\pm\mathrm{i}xy - x^2 + 2x_0x \mp 2\mathrm{i}x_0y - 2x_0^2\right) = 0,$$

(2) for $x_0 = 0$, the cubic equation factors as $x\left(x^2+y^2-y_0^2\right) = 0$.

In the case 1), the conic is not real, the case 2) gives Type 15 after rescaling by B_z.

For infinite vertexes p_0, the pencil $y = k$ gives a non-smooth conic. Thus the pencil can be chosen as $x = ay + k$. A line from the pencil is tangent to the conic c_r with $r = -\frac{a^2+1}{ak}$. The points, where the lines l_r touch c_r, are

$$x = \frac{2k}{a^2+1}, \qquad y = \frac{k\left(1-a^2\right)}{a\left(1+a^2\right)}.$$

Excluding k, we get the line $\left(a^2-1\right)x + 2y = 0$.

Thus we have considered all the cases with non-planar polar curve, the theorem is proved. ■

Corollary 5.4. *Suppose that the polar curve of a hexagonal circular* 3*-web is reducible algebraic of degree three. Then the polar curve is either planar or the web is Möbius equivalent to one described by Theorems* 4.5 *and* 5.3*.*

6 Hexagonal circular 3-webs with Möbius symmetries

A Möbius transformation is a symmetry of a circular 3-web if it maps any web circle to a web circle. The Möbius group in $\mathbb{RP}^3$ can be realized as $PGL_2(\mathbb{C})$, or equivalently, as the group of fractional-linear transformations of $z = x + \mathrm{i}y$, where (x, y) are cartesian coordinates in the plane. A generator of any 1-dimensional subalgebra can be brought to the Jordan normal form by adjoint action. The generator can be chosen either as $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$, where $\lambda = \alpha + \mathrm{i}\beta$ is some complex number with $\mathrm{Re}(\lambda) = \alpha$, $\mathrm{Im}(\lambda) = \beta$.

One way to obtain symmetric hexagonal 3-webs is provided by the Wunderlich construction (see [15] and Introduction).

Another easy way to produce hexagonal circular 3-webs is to choose two orbits of polar points such that one is a conic and the other is a coplanar straight line, these two orbits forming a polar curve. This construction may degenerate if there are 3 orbits which are coplanar straight lines.

For translations $(x, y) \mapsto (x, y + u)$, the orbit of a polar point for a circle $(x - a)^2 + y^2 = r$, parametrized by u as follows $\left[-2a : -2u : a^2 + u^2 - r - 1 : -a^2 - u^2 + r - 1\right]$, is a conic in the plane $-X + a(Z + U) = 0$. For a nonvertical line $y = ax$, the orbit is a line $Z + U = X + aY = 0$, parametrized by u via $[a : -1 : -u : u]$. Thus we immediately get the following hexagonal 3-webs symmetric by translations.

T1. 3 families of parallel lines. Möbius orbits of such webs form a two-parametric family. The polar curve splits into 3 coplanar lines. Observe that any web of this family has a 3-dimensional symmetry group.

T2. Polar curve splits into conic and coplanar line. There is only one Möbius class of such webs, any representative is formed by horizontal lines $y = \mathrm{const}$ and by the orbit of a circle which can be chosen as the unitary one centered at the origin.

T3. Wunderlich's type. There are several types, depending on the position of generating curves.

(1) Nondegenerate type. Webs are formed by vertical lines and by two different orbits of circles. The family of orbits is two-parametric: by translation and rescaling we can fix one orbit.

(2) Coinciding circle orbits. There is only one Möbius class of this type. It was already obtained as Type 9 in the classification of Theorem 5.3.

(3) Generated by a circle and a line. The family of orbits is one-parametric: by translation and rescaling we can fix the generating circle.

For dilatations $(x, y) \mapsto (ux, uy)$, the orbit of a polar point for a circle $(x-a)^2 + (y-b)^2 = r$, parametrized by u via $\left[-2au : -2bu : a^2 + b^2 - r - u^2 : -\left(a^2 + b^2 - r\right) - u^2\right]$, is a conic in the plane $bX - aY = 0$ if $(a, b) \neq (0, 0)$ and $a^2 + b^2 \neq r$, the hyperbolic line $X = Y = 0$ if $(a, b) = (0, 0)$, and the parabolic line $bX - aY = U - Z = 0$ if $a^2 + b^2 = r$. For a line $ax + by + c = 0$ with $c \neq 0$, the orbit is the parabolic line $bX - aU = Z + U = 0$, parametrized by u via $[au : bu : c : -c]$. The lines with $c = 0$ are invariant. Invoking the classification of the webs with 3 pencils, we list the following hexagonal 3-webs symmetric by dilatations.

D1=T1. 3 families of parallel lines.

D2. 2 coplanar parabolic lines and hyperbolic line intersecting them. Family of parallel lines, the parabolic pencil with circles tangent to a line of the family through the origin, and the family of concentric circles with the center at the origin. There is only one Möbius class of such webs.

D3. 2 dual parabolic lines and hyperbolic line through their common point. 2 orthogonal families of parallel lines and the family of concentric circles with the center at the origin. There is only one Möbius class of such webs, we have already presented it on the right of Figure 2.

D4. Polar curve splits into conic and coplanar hyperbolic line. Family of circles obtained by the dilatations from one not passing through the origin and not having its center at the origin and the family of concentric circles with the center at the origin. Möbius orbits of such webs form a one-parameter family.

D5. Polar curve splits into conic and tangent parabolic line. Family of circles obtained by dilatations from one not passing through the origin and not having its center at the origin and the family of parallel lines orthogonal to the orbit of centers of the circles. Möbius orbits of such webs form a one-parameter family.

D6. Wunderlich's type.

(1) Nondegenerate type, polar curve with 2 conics and elliptic line. Webs are formed by pencil of lines centered at the origin and by two different orbits of circles at general position. The family of Möbius orbits is 3-parametric: one can choose the centers of circles on a fixed circle centered at the origin and normalize by rotations.

(2) Polar curve with conic and elliptic line. Webs are formed by pencil of lines centered at the origin and orbit of a circle. The family of Möbius orbits is 1-parametric, it is Type 5 in the classification of Theorem 5.3.

(3) Polar curve with conic, hyperbolic and elliptic line. Webs are formed by pencil of lines centered at the origin, orbit of a circle, and the family of concentric circles with the center at the origin. The family of Möbius orbits is 1-parametric.

(4) Polar curve with conic, parabolic and elliptic line. Webs are formed by pencil of lines centered at the origin, orbit of a circle, and a family of parallel lines. The family of Möbius orbits is 2-parametric.

(5) Polar curve with hyperbolic line, elliptic line dual to hyperbolic, and parabolic line intersecting them. Webs are formed by the pencil of lines centered at the origin, the family of concentric circles with the center at the origin, family of parallel lines. There is only one Möbius type. This web is shown in the center of Figure 2.

(6) Polar curve with 2 parabolic lines touching Darboux quadric at the same point and coplanar elliptic line. Webs are formed by the pencil of lines centered at the origin and 2 families of parallel lines. The family of Möbius orbits is 1-parametric.

(7) Polar curve with 2 parabolic lines and elliptic line whose dual joins the touching points of parabolic lines with Darboux quadric. Webs are formed by the pencil of lines centered at the origin, a family of parallel lines and a parabolic pencil with the vertex at the origin. The family of Möbius orbits is 1-parametric. A representative of this web is shown in Figure 4.

For rotations $(x, y) \mapsto (x\cos(t) + y\sin(t), -x\sin(t) + y\cos(t))$, the orbit of a polar point for a circle $(x-a)^2+y^2 = r$, parameterized by t via $\big[-2a\cos(t) : -2a\sin(t) : a^2-r-1 : -a^2+r-1\big]$, is a circle in the plane $\big(a^2 - r + 1\big)Z + \big(a^2 - r - 1\big)U = 0$ if $a \neq 0$. For a line $ax + by + c = 0$ with $c \neq 0$, the orbit of its polar is also a circle $[a\cos(t) - b\sin(t) : a\sin(t) + b\cos(t) : c : -c]$ in the plane $Z + U = 0$. The circles with $a = 0$ are invariant, the orbit of a polar point of a line with $c = 0$ is an elliptic line. Thus one can easily list the following hexagonal 3-webs symmetric by rotations.

R1. Polar curve is a circle symmetric by rotations around z-axis and coplanar elliptic line. The family of Möbius orbits is 1-parametric.

R2. Wunderlich's type. One foliation is formed by orbits of points by rotations and the other two are images of two circles by rotation, these circles can coincide or "degenerate" into straight lines. Examples of such webs with coinciding generators are shown in Figure 5 (center), Figure 6 (left), and Figure 9 (left). The reader easily describes different types and computes corresponding Möbius orbit dimension of the webs.

Theorem 6.1. *Hexagonal circular* 3*-web with* 1*-dimensional Möbius symmetry is Möbius equivalent to one of the above described T-, D-, or R-types.*

Proof. A circular 3-web, symmetric by a given 1-parametric subgroup of the Möbius group and not obtained by the Wunderlich construction, is fixed by a choice of three curves, each being either a circle or a straight line, one from each foliation. Two circles can coincide: an orbit of circle still gives a (singular) 2-web. Moreover, one can move around these curves by the stabilizer of the infinitesimal generator of the subgroup.

• *Webs symmetric by translations* ∂_y. Consider a point such that none of the leaf tangents is parallel to the field ∂_y. If a generating curve C_1 of some foliation is a circle, then there are 2 circles from the orbit of C_1 passing through this point. Thus there are two locally defined direction fields $\partial_x + P_\pm\partial_y$ tangent to these 2 circles. Globally they are not separable: one direction swaps for the other upon running along C_1.

Consider one of the foliations and the corresponding direction field $\partial_x + P\partial_y$. Since the foliation is symmetric, the slope P does not depend on y. Since all the leaves are circles of the same curvature, the foliation has a first integral $\frac{(P')^2}{(P^2+1)^3}$, where $P' = \frac{\mathrm{d}P}{\mathrm{d}x}$. The differential equation $\frac{(P')^2}{(P^2+1)^3} = A^2 = \mathrm{const}$ has general solution

$$P(x) = \frac{A(x-x_0)}{\sqrt{1-A^2(x-x_0)^2}},$$

the zero value of A corresponding to the foliation by straight parallel lines. Let the slopes of the other two web foliations be Q and R, and the connection form be $\gamma = \alpha(x)\mathrm{d}x + \beta(x)\mathrm{d}y$. Hexagonality condition $\mathrm{d}\gamma = 0$ implies $\beta(x) = \mathrm{const}$, which amounts to

$$k = \frac{P'}{(P-Q)(P-R)} + \frac{Q'}{(Q-P)(Q-R)} + \frac{R'}{(R-P)(R-Q)} = \mathrm{const}. \tag{6.1}$$

If all foliations are formed by circles and not by straight lines, then the above identity is not possible. In fact, each of the slopes P, Q, R, considered as function of complex x, has two ramification points, for example, P has singularities at $x_\pm = x_0 \pm 1/A$. If at least one of the 6 ramification points is not coinciding with one of the others, then, supposing it be x_+ of P and expanding the expression (6.1) for $x = x_+ + t^2$ by t at $t = 0$, one sees that it has a simple pole and therefore cannot be constant. Therefore, each ramification point of any slope coincides with a ramification point of another slope.

The ramification points correspond to the group orbits (lines $x = \mathrm{const}$) tangent to the foliation circles. If only two circles are tangent along some orbit $x = \mathrm{const}$, then, applying Lemma 2.2, we conclude that either this orbit belongs to the third foliation and the web is of Wunderlich's type or the two generating circles coincide. In the latter case, the ramification points of the third foliation again must coincide with the common ramification points of the first two. This means that all orbits of generating circles coincide and we do not have a 3-web.

Therefore, either all leaves of all foliations are straight lines or one of the foliation, say the one corresponding to R, is formed by straight lines with slope $R = \mathrm{const}$ and the other two are formed by orbits of the same circle and $Q = -P$. Then it is immediate that $k = 0$ and $R = 0$. We get the type $T2$. The webs of Wunderlich's type, which are hexagonal, are excluded from consideration by the coordinate choice.

• *Webs symmetric by dilatations* $x\partial_x + y\partial_y$. Stabilizer of the 1-dimensional algebra spanned by $x\partial_x + y\partial_y$ is generated by rotation $y\partial_x - x\partial_y$ and dilatation $x\partial_x + y\partial_y$. The orbit of the polar of the generating curve is either a conic, or a hyperbolic line, or a parabolic line. Parabolic lines touch the Darboux quadric at one of the stationary point of the dilatation.

We use modified polar coordinates rectifying the symmetry

$$u = \arctan\left(\frac{y}{x}\right), \qquad v = \frac{1}{2}\ln\left(x^2+y^2\right). \tag{6.2}$$

A curve $x \mapsto (x, y(x))$ is a circle if and only if $y''' = \frac{3y'(y'')^2}{1+(y')^2}$, therefore integral curves of a symmetric vector field $\partial_u + P(u)\partial_v$ are circles if and only if

$$P'' = \frac{3P}{P^2+1}(P')^2 - P\big(P^2+1\big). \tag{6.3}$$

The integral curves are straight lines if and only if $y'' = 0$, which is equivalent to $P' = P^2+1$. Hence $\big($apply $z \mapsto \frac{1}{z}\big)$ the integral curves are circles of a parabolic pencil with the vertex at the origin if and only if $P' = -\big(P^2+1\big)$ thus giving $P(u) = \tan(u-u_0)$ and $P(u) = -\tan(u-u_0)$ respectively. The second-order equation (6.3) has a first integral

$$\frac{(P')^2}{\big(P^2+1\big)^3} - \frac{1}{P^2+1} = A = \mathrm{const},$$

allowing also to integrate it

$$P(u) = \frac{\sqrt{A+1}\tan(u-u_0)}{\sqrt{1-A\tan^2(u-u_0)}}.$$

To the hyperbolic pencil with the circles centered at the origin corresponds the solution $P \equiv 0$.

For hexagonal webs, the connection form is $\gamma = \alpha(u)\mathrm{d}u + \beta(u)\mathrm{d}v$. Hexagonality condition $\mathrm{d}\gamma = 0$ implies $\beta(u) = \mathrm{const}$, giving again (6.1) (where the slopes of the other foliations are Q and R). Observe that the choice of local coordinates u, v excludes webs of Wunderlich's type and we have to show that hexagonal are only the types D1, D2, D3, D4, D5. This can be done as follows. The types D1, D2, D3 are webs with three pencils, the case being settled earlier. Therefore, we study the webs whose polar curve includes at least one conic.

The approach used for translation works also in this case if we pass to complex webs: the singular points of solutions P, Q, R become complex if the corresponding generating circle on the Darboux quadric separates the stable points $(0,0,\pm1)$ of the dilatation. Considering behavior of the expression k at singular points of P, Q, R, we see that necessarily a singular point of slope for one foliation must coincide with a singular point for another. Singular points are the points (possibly complex) where the group orbit touches the generating circle, or lies either on a line of one foliation or on the common tangent to circles of parabolic pencil with the vertex at $(0,0)$.

Applying Lemma 2.2 as in the case of translation, we infer that, for webs of non-Wunderlich's type, having two foliations with conic polar curves and coinciding singular points, these polar curves must coincide and the third polar must be a line.

Consider the points where the circles from the common orbit are tangent. Lemma 2.2 implies that for non-Wunderlich's type the leaves of the third foliations must be also tangent to the circles and we get either type D4 or D5.

Finally, if two components of the polar curve are lines and the third is a conic then the lines must be parabolic for non-Wunderlich's type. In fact, considering the points where circles of the hyperbolic pencil are tangent to circles corresponding to conic, we conclude by Lemma 2.2 that the third foliation lines are also tangent to the circles at these points but then the singular points of the "conic" solution to (6.3) can not be compensated.

Thus the polar lines are parabolic. The expression k cannot be constant if the poles of Q, R, corresponding to these lines, do not coincide with singular points of P, which represent a conic in the polar curve. Therefore, the generating circle of this "conic" solution does not separate the stationary point of the dilatation and the singular points are real. Then by Lemma 2.2 the web cannot be hexagonal.

• *Webs symmetric by rotations* $y\partial_x - x\partial_y$. Stabilizer of the 1-dimensional subalgebra spanned by $y\partial_x - x\partial_y$ is generated by rotations $y\partial_x - x\partial_y$ and dilatations $x\partial_x + y\partial_y$. The polar orbit for a generating curve is either a circle or an elliptic line.

We again use polar coordinates (6.2) to describe symmetric vector fields $\partial_v + P(v)\partial_u$. The integral curves of such vector field are circles (in coordinates x, y, of course) if and only if

$$P'' = \frac{3P}{P^2+1}(P')^2 + P\big(P^2+1\big). \tag{6.4}$$

The integral curves are straight lines if and only if $P' = -P\big(P^2+1\big)$. The integral curves are circles passing through the origin if and only if $P' = P\big(P^2+1\big)$. Solutions of these two equations are $P(v) = \frac{1}{\sqrt{e^{2(v-v_0)}-1}}$ and $P(v) = \frac{1}{\sqrt{e^{-2(v-v_0)}-1}}$, respectively. The second-order equation (6.4) has a first integral

$$\frac{(P')^2}{\big(P^2+1\big)^3} + \frac{1}{P^2+1} = A^2 = \text{const},$$

allowing also to integrate it

$$P(v) = \frac{\sqrt{A^2-1}\tanh(v-v_0)}{\sqrt{1-A^2\tanh^2(v-v_0)}}, \qquad A^2 \neq 1.$$

The value $A^2 = 1$ gives the special solutions with generating circles passing through the stationary points of the symmetry, i.e., elliptic pencil with the lines passing through the origin with $P \equiv 0$. Analysis of the behavior of k at singular points and use of Lemma 2.2, similar to the ones performed above, show that only the type R1 is hexagonal, the Wunderlich types being excluded by the choice of variables. (In fact, multiplying by i the independent variable of the differential equation (6.3) reduces it to (6.4).)

- *Loxodromic symmetry.* Finally, we show that there is no hexagonal 3-webs symmetric by loxodromic vector field $y\partial_x - x\partial_y + \kappa(x\partial_x + y\partial_y)$ for any $\kappa \neq 0$. Note that the Wunderlich construction does not give circular webs as the symmetry orbits are spirals. We use the following coordinates:

$$s = \frac{\kappa}{2}\ln\big(x^2+y^2\big) - \arctan\left(\frac{y}{x}\right), \qquad t = \kappa\arctan\left(\frac{y}{x}\right) + \frac{1}{2}\ln\big(x^2+y^2\big),$$

the variable t being invariant by the symmetry. Integral curves of a symmetric vector field $\partial_t + P(t)\partial_s$ are circles (in coordinates x, y) if and only if

$$P'' = \frac{3P}{P^2+1}(P')^2 + \frac{\big(P^2+1\big)(\kappa P+1)(P-\kappa)}{\big(\kappa^2+1\big)^2}. \tag{6.5}$$

This equation has a first integral

$$\frac{(P')^2}{\big(P^2+1\big)^3} + \frac{2\kappa P - \kappa^2 + 1}{\big(\kappa^2+1\big)\big(P^2+1\big)} = A = \text{const}.$$

This integral does not allow to integrate (6.5) in elementary functions but allows to study the behavior of solutions at singular points. A singular point emerges when a symmetry orbit is tangent to a circle (or a line) of corresponding foliation.

Consider possible singularity types. If we exclude from consideration the stationary points of symmetry, then the symmetry vector field touches a generic circle at two points and the tangency is simple. The corresponding solution P has two singularities if the tangency points belong to different orbits and only one if the points lie on the same orbit. There are circles, for which two tangency points merge to give only one singularity. If the generating curve is a line or a circle through the origin, then the solution has only one singularity. Finally, there are two constant solutions, namely $P = -1/\kappa$ corresponding to the invariant hyperbolic line $X = Y = 0$ and $P = k$ corresponding to the invariant elliptic line $Z = U = 0$.

If a solution P has two singularities at t_1, t_2, then $A \neq 0$ and the singularity is of the same type as for the non-loxodromic cases

$$P(t) = \frac{1}{\sqrt[4]{4A}\sqrt{t - t_i}} + \left\{\text{analytic function of } \sqrt{t - t_i}\right\}.$$

If a solution P is generated by a circle tangent to the symmetry trajectory $t = t_0$ and the tangency is of second order, then there is only one singularity of the following type at $t = t_0$ $P(t) = c(t - t_0)^{-\frac{2}{3}} + \cdots$, where $c \neq 0$ and the omitted terms are not essential for our analysis. The condition of double tangency is equivalent to $A = 0$. The corresponding generating circle $(x - a)^2 + y^2 = r^2$ verifies the relation $r^2 = \frac{(k^2+1)a^2}{k^2}$. For an orbit $t = t_0$, there is at most one such circle.

Let P, Q, R be solutions giving a hexagonal 3-web. These solutions, as well as the coordinates s, t, are defined only locally but we can prolong them along a symmetry orbit, along a leaf of some of the 3 foliations, or along any curve, as long as we do not meet a singular point of one of P, Q, R. The condition of hexagonality (6.1) remains satisfied along any such prolongations. Therefore, we cannot meet a singularity of only one of P, Q, R, they emerge necessarily at least in pairs.

Suppose there is a symmetric hexagonal 3-web. Consider a non-singular point. There are 3 leaves passing through it. Each leaf can be considered as the generating curve of the respective foliation. At least one of the corresponding solutions P, Q, R has a singular point. Let us run along the respective leaf until we meet a singularity t_1 of P, Q or R. Then at least two of P, Q, R are singular at t_1 and the orbit $t = t_1$ is tangent to at least two web leaves at a singular point p_s. Since the symmetry orbit is not a circle, Lemma 2.2 implies that either exactly two leaves at p_s are tangent and therefore coincide or all three leaves are tangent at p_s.

In the former case, suppose that the coinciding leaves are C_Q and C_R. Then the leaf C_P at p_s is different from $C_Q = C_R$ at p_s. Let us go along C_P keeping track of Q, R until one of the two leaves corresponding to Q and R touches C_P at some $\bar{p}_s$. Such point exists until all the foliations are formed by straight lines. Then by Lemma 2.2 this leaf coincide with C_P at $\bar{p}_s$, thus all three generating curves of the web coincide and there is no 3-web. If all 3 generating curves are straight lines, then C_P is necessary the line through the origin and $P = \kappa$. Applying the map $z \mapsto 1/z$, we transform the lines C_Q and C_R into circles and the above argument applies.

If all three leaves are tangent at p_s, then at least one leaf, say C_P, is different from any of the other two. Let us go again along C_P keeping track of Q, R until one of the two leaves corresponding to Q and R touches C_P. Let it be C_Q. Repeating the above used argument we conclude that such point $\bar{p}_s$ exists and C_P coincides at $\bar{p}_s$ with C_Q. Then C_P coincides with C_Q also at p_s. Now either all 3 generating curves C_P, C_Q, C_R coincide at p_s and we do not have 3-web, or C_R is different from $C_P = C_Q$ at p_s. Now we repeat the trick with prolongation, this time along C_R, and conclude that $C_P = C_Q = C_R$ at p_s. Thus all three generating curves coincide and we can get at most 2-web. ■

7 Concluding remarks

7.1 Circular hexagonal 3-webs on surfaces

Pottmann, Shi, and Skopenkov [11] classified circular hexagonal 3-webs on nontrivial Darboux cyclides: such surfaces carry up to 6 one-parameter families of circles, 3 families can be picked up in 5 different ways to form a hexagonal web. In fact, as was proven by Lubbes [9], if through a general point of a surface in $\mathbb{R}^3$ pass at least 3 circles then the surface is either a plane, or a sphere, or a Darboux cyclid.

7.2 Erdoğan's approach to Theorem 4.5

The first attempt to prove Theorem 4.5 appeared in [6]. The idea was to choose a Möbius normalization sending one of the vertexes to infinity, to set $y = 0$, and to obtain "sufficient number of equations" to fix the pencil configurations. The author claimed that the curvature equation for $y = 0$ is a polynomial one of degree 6 in x, though the calculation itself was not present. Nowadays, armed with a powerful computer (32GB of RAM is enough) and a symbolic computation system like Maple, one can perform this computations and check that the degree may be much higher (in fact, up to 18) for some choices of polar line types.

Anyway, brute computer force does work: with the above mentioned equipment the author of this paper managed to derive the classification results. The treating has the following steps:

(1) choosing an initial Möbius normalization,

(2) computing the curvature,

(3) isolating and factoring the highest homogeneous part of the curvature equation,

(4) Möbius renormalization adjusted to the geometric information obtained in the previous step and repeating from the step 2 until one makes the curvature vanish.

7.3 Boundaries of regular domain for hexagonal 3-webs

To avoid heavy computation of the curvature in proving Theorem 4.5, the author of [12] suggested to use the structure of web singular set (see Lemma 2.2). The presented proof was not correct. The author argued that the web equation $u_3 = F(u_1, u_2)$, relating first integrals u_i of the web foliations $\mathcal{F}_i$, may be rewritten as $u_3 = f(\alpha(u_1) + \beta(u_2))$ and used this form on the curve of singular points Γ_1. This argument is definitely wrong as the functions α, β typically have singularities on Γ_1: a simple counterexample is the web equation $u_3 = u_1 u_2$.

7.4 Hexagonal 3-subwebs

Consider the autodual tetrahedron with vertexes at $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. Lines joining the vertexes of this tetrahedron give a 6-web A_6 with 6 pencils of circles. Any 3-subweb of this 6-web is hexagonal: any 3 lines are either coplanar or give a polar curve of a hexagonal 3-web. By direct computation (better use computer!) one checks the following claim.

Proposition 7.1. *The rank of the* 6*-web* A_6 *is maximal, i.e., is equal to* 10*.*

Another remarkable feature of this autodual 6-web is that the infinitesimal operators, corresponding to the pencils in the sense of Proposition 3.2, form the basis R_x, R_y, R_z, B_x, B_y, B_z of the Lie algebra of the Möbius group so that the commutator of any two of them is either zero or an operator of the basis.

There is also autodual 4-web A_4 whose polar curve is the union of 4 pencils corresponding to R_z, B_z, $B_x - R_y$, $R_x + B_y$. This web has similar properties: any its 3-subweb is hexagonal, the commutator of any two of the four operators is either zero or an operator of the set, its rank is maximal. One finds more examples with hexagonal subwebs among symmetric webs of Wunderlich's type (see Section 6).

7.5 Conjecture

The polar curve of a hexagonal circular 3-web is an algebraic curve such that each its irreducible component is either a twisted cubic or a planar curve of degree at most 3. The author thanks an anonymous referee for providing an example of hexagonal circular 3-web with polar twisted cubic.

A Appendix: Curvature equation

$$
\begin{aligned}
&\big(A^2-4B\big)\big(R^2+AR+B\big)\big[(AR+2B)A_{xx}+A\big(R^2-B\big)A_{xy}-BR(A+2R)A_{yy}\\
&\quad-(A+2R)B_{xx}+\big(A^2-2R^2-2B\big)B_{xy}+R\big(A^2-2B+AR\big)B_{yy}\\
&\quad+\big(4B-A^2\big)R_{xx}+A\big(A^2-4B\big)R_{xy}+B\big(4B-A^2\big)R_{yy}\big]\\
&\quad+\big(A^2-4B\big)^2(A+2R)R_x^2+(A+2R)\big(A^2-4AR-4R^2-8B\big)B_x^2+\\
&\quad-\big(2A^3R^2+A^2R^3+7A^2BR+4ABR^2+4BR^3+4AB^2-4B^2R\big)A_x^2\\
&\quad+\big(B-R^2\big)\big(2A^3R+A^2R^2+A^2B+4BR^2+4B^2\big)A_xA_y\\
&\quad-A\big(A^2-4B\big)^2(A+2R)R_xR_y\\
&\quad+\big(2A^3R-A^4+4A^2R^2-8AR^3-8R^4+8A^2B-16BR^2-8B^2\big)B_xB_y\\
&\quad+BR\big(2A^3R+7A^2R^2+4AR^3+A^2B+4ABR-4BR^2+4B^2\big)A_y^2+\\
&\quad-R\big(A^4-A^3R-6A^2R^2-4AR^3-8A^2B-8ABR+8B^2\big)B_y^2\\
&\quad+B\big(A^2-4B\big)^2(A+2R)R_y^2+\big(A^2-4B\big)\big(A^2R-AR^2-AB-8BR\big)A_xR_x\\
&\quad+\big(3A^3R+13A^2R^2+8AR^3+A^2B+12ABR-4BR^2+12B^2\big)A_xB_x\\
&\quad+2\big(A^2-4B\big)\big(A^2+AR+R^2-3B\big)B_xR_x+\big(5A^2R^3-A^4R-A^3R^2+4AR^4\\
&\quad+A^2BR+4ABR^2-4BR^3-4AB^2-4B^2R\big)[A_xB_y+A_yB_x]\\
&\quad+\big(A^2-4B\big)\big(A^2R^2+A^2B+2ABR-2BR^2-2B^2\big)[A_xR_y+A_yR_x]+\\
&\quad-A\big(A^2-4B\big)\big(A^2+AR+R^2-3B\big)[B_xR_y+B_yR_x]\\
&\quad+B\big(A^2-4B\big)\big(A^2R-AR^2-AB-8BR\big)A_yR_y\\
&\quad+\big(4B-A^2\big)\big(A^3R+A^2R^2-A^2B-6ABR-6BR^2+2B^2\big)B_yR_y+\\
&\quad-R\big(2A^4R+5A^3R^2+3A^2R^3-A^2BR+4ABR^2+4BR^3+8AB^2+20B^2R\big)A_yB_y\\
&\quad=0.
\end{aligned}
$$

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