

# Birational Equivalences and Generalized Weyl Algebras

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**Abstract.** We calculate suitably localized Hochschild homologies of various quantum groups and Podleś spheres after realizing them as generalized Weyl algebras (GWAs). We use the fact that every GWA is birationally equivalent to a smash product with a 1-torus. We also address and solve the birational equivalence problem, and the birational smoothness problem for GWAs.

*Key words:* noncommutative birational equivalences; generalized Weyl algebras; Hochschild homology; Podleś spheres; quantum groups; quantum enveloping algebras

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## 1 Introduction

A birational equivalence is an algebra morphism that becomes an isomorphism after a suitable localization. In this paper, we show that every generalized Weyl algebra (GWA) is birationally equivalent to a smash product with a rank-1 torus. This fact significantly simplifies their representation theory, and structure problems such as the isomorphism problem [5, 23, 38, 42, 43] and the smoothness problem [4, 23, 30, 31, 41], provided one replaces isomorphisms with suitable noncommutative birational equivalences. We address and solve a relative version of the birational equivalence problem in Section 3.9, and the birational smoothness problem in Section 4.3. We then calculate the Hochschild homology of suitably localized examples of GWAs in Section 5.

Generalized Weyl algebras are defined by Bavula [3, 4], Hodges [23] and Rosenberg [39] independently under different disguises. Their representation theory resembles that of Lie algebras [12, 35] (see Section 3.7), their homologies are extensively studied [13, 30, 31, 41], and they found diverse uses in areas such as noncommutative resolutions of Kleinian singularities [6, 11, 29] and noncommutative geometry of various quantum spheres and lens spaces [9]. Apart from noncommutative resolutions of Kleinian singularities, the class is known to contain the ordinary rank-1 Weyl algebra  $A_1$ , the enveloping algebra  $U(\mathfrak{sl}_2)$  and its primitive quotients, the quantum enveloping algebra  $U_q(\mathfrak{sl}_2)$ , the quantum monoid  $\mathcal{O}_q(M_2)$ , the quantum groups  $\mathcal{O}_q(\mathrm{GL}_2)$ ,  $\mathcal{O}_q(\mathrm{SL}_2)$  and  $\mathcal{O}_q(\mathrm{SU}_2)$ . We reverify that the standard Podleś spheres  $\mathcal{O}_q(S^2)$  [37] are GWAs [30], and then we show that parametric Podleś spheres  $\mathcal{O}_{q,c}(S^2)$  of Hadfield [18] are also GWAs. We finish the paper by calculating localized Hochschild homology of all of these examples.

The Hochschild homology of quantum groups  $\mathcal{O}_q(\mathrm{GL}_n)$  and  $\mathcal{O}_q(\mathrm{SL}_n)$  with coefficients in a 1-dimensional character coming from a modular pair in involution is calculated for every  $n \geq 1$  in [25], and with coefficients in themselves in specific cases in [19, 20, 32, 40]. The Hochschild cohomology of the Podleś sphere was studied by Hadfield [18], and then in the context of van den Bergh duality [44, 45] by Krämer [27]. Both Hadfield and Krämer use twisted Hochschild (co)homology by the Nakayama automorphism with coefficients in themselves. In this

paper, we only calculate the ordinary Hochschild homology of these algebras with coefficients in themselves since one can always move to and from the ordinary Hochschild homology and the twisted homology via isomorphisms coming from suitable cup and cap products [16, 27].

In this paper, we focus on GWAs, i.e., algebras that are birationally equivalent to smash products with rank-1 tori. The higher rank generalized Weyl algebras that (conjecturally) recover enveloping algebras of higher rank Lie algebras and their quantizations are called *twisted generalized Weyl algebras* (TGWAs) [21, 22, 33, 34]. We conjecture that TGWAs are birationally equivalent to smash products with higher rank tori, but we leave this investigation for a future paper.

The celebrated Gelfand–Kirilov conjecture, on the other hand, states that the universal enveloping algebra  $U(\mathfrak{g})$  of a finite dimensional Lie algebra is birationally equivalent to a sufficiently high rank Weyl algebra [15]. One of the equivalent forms of the conjecture is that  $U(\mathfrak{g})$  is birationally equivalent to the smash product of a polynomial algebra with a torus. The conjecture is known to be false in general [2, 10], but is true for a large class of Lie algebras [15, 24, 36]. The quantum analogue of the conjecture (see [7, pp. 19–21 and Section II.10.4] and references therein) is also known to be true many instances [1, 14]. In the light of our conjecture above, we believe that the universal enveloping algebra  $U(\mathfrak{g})$  of a rank- $n$  semi-simple Lie algebra is birationally equivalent to the smash product of a polynomial algebra with an  $n$ -torus. We also believe that the same is true for the quantum enveloping algebras  $U_q(\mathfrak{g})$  and the quantum groups  $\mathcal{O}_q(G)$  where one replaces the  $n$ -torus with a quantum  $n$ -torus.

**Plan of the article.** In Section 2, we recall some basic facts on localizations, relative homology of algebra extensions, smash products and biproducts. In Section 3, we prove two fundamental structure theorems for GWAs in Sections 3.4 and 3.5. Then we state and solve birational equivalence problem for GWAs in Section 3.9. In Section 4, we investigate the interactions between homology, smash biproducts and noncommutative localizations, and in Sections 4.3 and 4.4, we state and solve the birational smoothness problem for GWAs. Finally, we use our machinery to calculate suitably localized Hochschild homologies of various GWAs in Section 5.

**Notations and conventions.** We fix an algebraically closed ground field  $\mathbb{k}$  of characteristic 0, and we set the binomial coefficients  $\binom{n}{m} = 0$  whenever  $m > n$  or  $m < 0$ . All unadorned tensor products  $\otimes$  are taken over  $\mathbb{k}$ . All algebras are assumed to be unital and associative, but not necessarily commutative or finite dimensional. We use the notation  $\mathbb{k}[X]$  for the free unital commutative algebra generated by a set  $X$ , while we use  $\mathbb{k}\{X\}$  for the free unital algebra generated by the same set  $X$ . Throughout the paper, we use  $\mathbb{T}$  to denote the algebra of Laurent polynomials  $\mathbb{k}[x, x^{-1}]$ .

## 2 Preliminaries

### 2.1 Noncommutative localizations

Our main reference for noncommutative localizations is [28, Section 10].

A multiplicative submonoid  $S \subseteq A$  is called a *right Ore set* if for every  $s \in S$  and  $u \in A$

- (i) there are  $s' \in S$  and  $u' \in A$  such that  $su' = us'$ , and
- (ii) if  $su = 0$  then there is  $s' \in S$  such that  $us' = 0$ .

If  $S \subseteq A$  is a right Ore set, then there is a *universal* algebra  $S^{-1}A$  and a morphism of algebras  $\iota_S: A \rightarrow S^{-1}A$  such that  $\varphi(S) \subseteq S^{-1}A^\times$ . The morphism  $\iota_S$  is universal among such  $S$  inverting morphisms, where if  $\varphi: A \rightarrow B$  satisfies  $\varphi(S) \subseteq B^\times$ , then there is a unique morphism of algebras  $\varphi': S^{-1}A \rightarrow B$  with  $\varphi = \varphi' \circ \iota_S$ .

In the sequel, we are going to drop the requirement that  $S$  is a multiplicative submonoid and consider the conditions above within the submonoid generated by  $S$ . In such cases, we are still going to use the notation  $S^{-1}A$  for the localization.

## 2.2 Birational equivalences

Given two irreducible algebraic varieties  $X$  and  $Y$ , an algebraic map  $f: X \rightarrow Y$  is called a birational equivalence if  $f$  restricted to an open subvariety is an isomorphism [17, Section 4.2]. For the purposes of this article, an algebra  $A$  is an *irreducible noncommutative affine variety* if  $A$  has no primitive idempotents other than 0 and 1. Thus we call a morphism of algebras  $\varphi: A \rightarrow A'$  a *birational equivalence* if there are two Ore sets  $S \subset A$  and  $S' \subset A'$  such that  $\varphi(S) \subseteq S'$  and the extension of  $\varphi$  to the localization  $S^{-1}\varphi: S^{-1}A \rightarrow (S')^{-1}A'$  is an isomorphism of unital associative algebras.

## 2.3 Smash biproducts

Assume  $A$  and  $B$  are two unital associative algebras. A  $\mathbb{k}$ -linear map  $R: B \otimes A \rightarrow A \otimes B$  is called a distributive law if the following diagrams of algebras commute:

$$\begin{array}{ccc} B \otimes B \otimes A & \xrightarrow{B \otimes R} & B \otimes A \otimes B \xrightarrow{R \otimes B} A \otimes B \otimes B \\ \mu_B \otimes A \downarrow & & \downarrow A \otimes \mu_B \\ B \otimes A & \xrightarrow{R} & A \otimes B \\ B \otimes \mu_A \uparrow & & \uparrow \mu_A \otimes B \\ B \otimes A \otimes A & \xrightarrow{R \otimes B} A \otimes B \otimes A \xrightarrow{A \otimes R} A \otimes A \otimes B, \end{array}$$

$$\begin{array}{ccc} & B & \\ B \otimes 1 \swarrow & & \searrow 1 \otimes B \\ B \otimes A & \xrightarrow{R} & A \otimes B \\ 1 \otimes A \swarrow & & \searrow A \otimes 1 \\ & A & \end{array}$$

For notational convenience, we write  $R(b \otimes a) = R_{(1)}(a) \otimes R_{(2)}(b)$  for every  $a \in A$  and  $b \in B$ .

For a distributive law  $R: B \otimes A \rightarrow A \otimes B$ , there is a corresponding smash biproduct algebra  $A \#_R B$  which is  $A \otimes B$  as vector spaces with the multiplication  $(a \otimes b)(a' \otimes b') = aR_{(1)}(a') \otimes R_{(2)}(b)b'$  for every  $a, a' \in A$  and  $b, b' \in B$ . We will use  $A \# B$  instead of  $A \#_R B$  if the distributive law is clear from the context.

## 2.4 Hochschild homology

Let  $A$  be a unital associative algebra, and let  $M$  be an  $A$ -bimodule. Consider the graded  $\mathbb{k}$ -vector space  $\text{CH}_*(A, M) = \bigoplus_{n \geq 0} M \otimes A^{\otimes n}$  together with linear maps  $b_n: \text{CH}_n(A, M) \rightarrow \text{CH}_{n-1}(A, M)$  defined for  $n \geq 1$  via

$$\begin{aligned} b_n(m \otimes a_1 \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

These maps satisfy  $b_n b_{n+1} = 0$  for every  $n \geq 1$ , and we define  $H_*(A, M) = \ker(b_n) / \text{im}(b_{n+1})$ . We use the notation  $HH_*(A)$  for  $H_*(A, A)$ .

## 2.5 Smooth algebras

An algebra  $A$  is said to be *smooth* if it has finite Hochschild homological dimension, i.e., when

$$\text{hh.dim}(A) := \sup\{n \in \mathbb{N} \mid H_n(A, M) \neq 0, M \in A^e\text{-Mod}\}$$

is finite. We call an algebra  $A$  *m-smooth* if  $\text{hh.dim}(A) = m$ , for  $m \in \mathbb{N}$ .

The simplest 0-smooth algebras are groups ring  $\mathbb{k}[G]$  over finite groups where  $|G|$  does not divide the characteristic of  $\mathbb{k}$ , and quotients of polynomial algebras  $\mathbb{k}[x]/\langle f(x) \rangle$  where  $f(x)$  is a separable polynomial. For  $m \geq 1$ , the simplest examples of  $m$ -smooth algebras one can consider are the polynomial algebras  $\mathbb{k}[t_i \mid i = 0, \dots, m]$  and the Laurent polynomial algebras  $\mathbb{k}[t_i, t_i^{-1} \mid i = 0, \dots, m]$  with  $m \geq 0$ , and their smash biproducts with 0-smooth algebras.

## 2.6 Homology of smash biproducts with smooth algebras

Now, let us recall the following result from [25, Proposition 1.5].

**Proposition 2.1.** *Assume  $P$  and  $Q$  are two unital algebras together with a left flat algebra morphism  $\varphi: Q \rightarrow P$ . Let  $M$  be a  $P$ -bimodule. Then there is a spectral sequence whose first page is given by*

$$E_{i,j}^1 = H_j(Q, \underbrace{M \otimes_Q P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}})$$

that converges to the Hochschild homology  $H_*(P, M)$ .

Let us write  $R_{(1)}(b \otimes \mathbf{a}) \otimes R_{(2)}(b \otimes \mathbf{a})$  for  $R(b \otimes \mathbf{a})$ , and defined

$$\mathrm{CH}_n(A, M)_B := \frac{\mathrm{CH}_n(A, M)}{\{\mathbf{a} \otimes m \triangleleft b - R_{(1)}(b \otimes \mathbf{a}) \otimes R_{(2)}(b \otimes \mathbf{a}) \triangleright m \mid b \in B, \mathbf{a} \otimes m \in \mathrm{CH}_n(A, M)\}}.$$

**Corollary 2.2.** *Let  $A$  and  $B$  be two algebras, and let  $R: B \otimes A \rightarrow A \otimes B$  be any distributive law. Then for any  $A \# B$ -bimodule  $M$  and for all  $n \geq 0$  we have  $H_n(A \# B, M) \cong H_n(\mathrm{CH}_*(A, M)_B)$ , when  $B$  is 0-smooth.*

**Proof.** We set  $P = A \# B$  and  $Q = B$  together with  $\varphi(b) = 1 \otimes b$ , and then we use Proposition 2.1. ■

## 3 Generalized Weyl algebras

### 3.1 Generalized Weyl algebras

Assume  $A$  is a unital associative algebra, let  $a \in Z(A)$  and  $\sigma \in \mathrm{Aut}(A)$  be fixed. Define a new algebra  $W_{a,\sigma}$  as the quotient of the free algebra generated by  $A$  and two non-commuting indeterminates  $x$  and  $y$  subject to the following relations

$$yx - a, \quad xy - \sigma(a), \quad xu - \sigma(u)x, \quad y\sigma(u) - uy$$

for every  $u \in A$ . The algebra  $W_{a,\sigma}$  is called *generalized Weyl algebra* [3, 5].

### 3.2 Algebras with automorphisms

One standard source of distributive laws is algebras with a fixed algebra automorphism or endomorphisms. From now on, we will assume  $A$  is a domain with a fixed algebra automorphism  $\sigma \in \mathrm{Aut}(A)$ . Let  $\mathbb{T} = \mathbb{k}[\mathbb{Z}] = \mathbb{k}[x, x^{-1}]$  be the group ring of the free abelian group on a single generator  $\mathbb{Z}$ . Now consider the smash biproduct  $B := A \# \mathbb{T}$  coming from the distributive law  $R: \mathbb{T} \otimes A \rightarrow A \otimes \mathbb{T}$  defined as

$$R(x^n \otimes u) = \sigma^n(u) \otimes x^n \tag{3.1}$$

for every monomial  $x^n \in \mathbb{T}$  with  $n \in \mathbb{Z}$  and  $u \in A$ . Then  $R$  defines an invertible distributive law. In order to simplify the notation, we are going to write  $ux^i$  for every monomial  $u \otimes x^i$  in  $A \# \mathbb{T}$ . If there are more than one automorphisms in the context, we are going to write  $A \#_{R(\sigma)} \mathbb{T}$  instead of  $A \# \mathbb{T}$  to emphasize which automorphism we are using.

### 3.3 The ordinary Weyl algebra

If we consider the subalgebra of operators generated by  $T(f(t)) = tf(t)$  and  $D(f(t)) = f'(t)$  on the space of differentiable functions on  $\mathbb{R}$ , we see that we have the relation  $[D, T] = DT - TD = 1$ . This algebra is the motivating example of the ordinary rank-1 Weyl algebra  $A_1$ , which is defined as the  $\mathbb{k}$ -algebra defined on two non-commuting indeterminates  $x$  and  $y$  subject to the relations  $xy - yx = 1$ . One can define  $A_1$  as a GWA if we let  $A = \mathbb{k}[t]$  where we set the distinguished element  $a = t$ . We define  $\sigma$  to be the algebra automorphism of  $A$  given by  $f(t) = f(t - 1)$  for every  $f(t) \in A$ . Then the GWA  $W_{a,\sigma}$  is the ordinary Weyl algebra  $A_1$ . See [5, Example 2.3].

### 3.4 A structure theorem for GWAs

Given an arbitrary GWA  $W_{a,\sigma}$ , one can also realize it as a unital subalgebra of the smash product  $A \# \mathbb{T}$  where  $\mathbb{T} := \mathbb{k}[x, x^{-1}]$  and  $R: \mathbb{T} \otimes A \rightarrow A \otimes \mathbb{T}$  is defined in equation (3.1). For this, we consider the monomorphism of  $\mathbb{k}$ -algebras  $\varphi: W_{a,\sigma} \rightarrow A \# \mathbb{T}$  given by  $\varphi(u) = u$ ,  $\varphi(x) = x$ ,  $\varphi(y) = ax^{-1}$  for every  $u \in A$ .

**Theorem 3.1.** *For every  $a \in Z(A)$ , the algebra  $W_{a,\sigma}$  is isomorphic to the unital subalgebra of the smash biproduct  $A \# \mathbb{T}$  generated by  $A$ ,  $x$  and  $ax^{-1}$ . Hence  $W_{a,\sigma}$  is isomorphic to  $A \# \mathbb{T}$  for every  $a \in Z(A^\times)$ .*

**Proof.** The result follows from the fact that the image of  $\varphi$  (as  $\mathbb{k}$ -vector spaces) is the direct sum

$$A \otimes \mathbb{k}[x] \oplus \bigoplus_{n=0}^{\infty} \langle a\sigma^{-1}(a) \cdots \sigma^{-n}(a) \rangle \otimes \text{Span}_{\mathbb{k}}(x^{-n-1}),$$

where  $\langle u \rangle$  denotes the two sided ideal in  $A$  generated by an element  $u \in A$ . ■

In specific cases, the fact that GWAs are subalgebras of smash products was already known [6, Lemma 2.3]. However, to the best of our knowledge, the fact that one gets an isomorphism when the distinguished element  $a \in A$  is a unit, even though it implicitly follows from this embedding, is not fully taken advantage of in the literature.

From now on, we identify  $W_{a,\sigma}$  with  $\text{im}(\varphi)$  in  $A \# \mathbb{T}$ .

### 3.5 Localizations of smash products with tori

Let  $A$  be an algebra with a fixed automorphism  $\sigma \in \text{Aut}(A)$ . Assume  $R: \mathbb{T} \otimes A \rightarrow A \otimes \mathbb{T}$  is the distributive law given in equation (3.1). Let  $S \subseteq Z(A)$  be any multiplicative submonoid which stable under the action of  $\sigma$ . The proof of the following lemma is routine verification, and therefore, is omitted.

**Lemma 3.2.** *Any multiplicative monoid  $S$  in  $Z(A)$  which is  $\sigma$ -stable is a right Ore subset in  $A \# \mathbb{T}$ , and  $S^{-1}(A \# \mathbb{T}) = S^{-1}A \# \mathbb{T}$ .*

### 3.6 Localizations of GWAs

As before, assume  $A$  is a unital associative algebra,  $a \in Z(A)$  and  $\sigma \in \text{Aut}(A)$ . Recall that by Theorem 3.1 we identified the GWA  $W_{a,\sigma}$  with the subalgebra of the smash biproduct  $A \# \mathbb{T}$  generated by the algebra  $A$  and the elements  $x$  and  $ax^{-1}$ . Then we have a tower of algebra extensions of the form  $A \# \mathbb{k}[x] \subset W_{a,\sigma} \subseteq A \# \mathbb{T}$ .

**Theorem 3.3.** *Consider the set  $S \subset Z(A)$  of the elements of the form  $\sigma^m(a^n)$ , where  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}$ . Then the embedding of algebras  $W_{a,\sigma} \subseteq A\#\mathbb{T}$  is a birational equivalence with respect to the Ore set generated by  $S$ .*

**Proof.** Now, by Lemma 3.2, we have that  $S^{-1}(A\#\mathbb{T}) = S^{-1}A\#\mathbb{T}$ , and by Theorem 3.1, we see that the algebra  $S^{-1}A\#\mathbb{T}$  is itself generated by  $S^{-1}A$ ,  $x$  and  $ax^{-1}$  since  $a \in S^{-1}A$  is now a unit. ■

### 3.7 Highest weight modules of GWAs

Assume  $A$  is unital associative with a distinguished element  $a \in Z(A)$  and an automorphism  $\sigma \in \text{Aut}(A)$ . Let  $V$  be a representation over the GWA  $W_{a,\sigma}$ . We have an (not necessarily exhaustive) increasing filtration of submodules of the form

$$V^{[\ell]} = \{v \in V \mid v \triangleleft a\sigma^{-1}(a) \cdots \sigma^{-\ell}(a) = 0\}$$

defined for  $\ell \in \mathbb{N}$ . Let us also define

$$V^{[\infty]} = \bigcup_{\ell \geq 0} V^{[\ell]}.$$

We define  $ht_{a,\sigma}(V)$  the height of  $V$  as the smallest integer  $\ell$  such that  $V^{[\ell]} = V^{[\infty]}$ , and if no such integer exists we set  $ht_{a,\sigma}(V) = \infty$ .

Assume  $V$  is a finite dimensional representation. Then  $h = ht_{a,\sigma}(V)$  is necessarily finite. Furthermore, if the height filtration satisfies  $V^{[h]} = V$ , then we get the analogue of a *highest weight module* for the GWA  $W_{a,\sigma}$ . Approaches for such cases can be seen in [12, 35].

**Proposition 3.4.** *Let  $S \subseteq Z(A)$  be the subset of elements of the form  $\sigma^n(a^m)$  with  $n \in \mathbb{Z}$  and  $m \in \mathbb{N}$ , and let  $S^{-1}(W_{a,\sigma})$  be the localization of  $W_{a,\sigma}$  at  $S$ . Assume  $V$  is an arbitrary  $W_{a,\sigma}$ -module, and let  $h = ht_{a,\sigma}(V)$ . Then  $S^{-1}V := V \otimes_{W_{a,\sigma}} S^{-1}(W_{a,\sigma})$  is isomorphic to  $S^{-1}(V/V^{[h]})$ .*

**Proof.** We consider the following short exact sequence of  $W_{a,\sigma}$ -modules:

$$0 \rightarrow V^{[h]} \rightarrow V \rightarrow V/V^{[h]} \rightarrow 0$$

and use the fact that the functor  $S^{-1}(\cdot)$  is exact. ■

### 3.8 Morphisms of algebra extensions

An algebra  $C$  together with a subalgebra  $A$  is called an algebra extension. Given two extensions  $A \subseteq C$  and  $A \subseteq C'$  of a fixed algebra  $A$ , a morphism  $f: (C, A) \rightarrow (C', A)$  of extensions is a commutative triangle of algebra morphisms of the form

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ & \searrow & \nearrow \\ & A & \end{array}$$

### 3.9 Isomorphisms of smash products with tori

In this subsection, we consider the isomorphism problem for smash products with  $\mathbb{T} = \mathbb{k}[x, x^{-1}]$  since all isomorphism problems for GWAs birationally reduce to isomorphism problems for such smash products.

**Theorem 3.5.** *Assume  $\sigma$  and  $\eta$  are two algebra automorphisms of  $A$ . Then the algebra extensions  $A \subseteq A \#_{R(\sigma)} \mathbb{T}$  and  $A \subseteq A \#_{R(\eta)} \mathbb{T}$  are isomorphic if and only if  $\eta = u\sigma^{\pm 1}u^{-1}$  for some  $u \in A^\times$ .*

**Proof.** Assume for now that  $\sigma = u\eta u^{-1}$  or  $\sigma = u\eta^{-1}u^{-1}$ . Consider an arbitrary  $v \in A$ . In the first case, define  $\delta: A \#_{R(\sigma)} \mathbb{T} \rightarrow A \#_{R(\eta)} \mathbb{T}$  by letting  $\delta(x) = ux$  and we get

$$\delta(xv) = uxv = u\eta(v)x = \sigma(v)ux = \delta(\sigma(v)x),$$

which implies  $\delta$  is an isomorphism of smash biproducts. The proof for the second case is similar, and therefore, is omitted. On the opposite direction, assume  $\delta: A \#_{R(\sigma)} \mathbb{T} \rightarrow A \#_{R(\eta)} \mathbb{T}$  is an isomorphism of algebra extensions. The one easily see that  $\delta$  restricted  $\mathbb{T}$  yields an algebra monomorphism, and therefore,  $\delta(x) = ux^{\pm 1}$  for some  $u \in A^\times$  and  $\delta$  restricted to  $A$  is identity. Thus  $\sigma = u\eta^{\pm 1}u^{-1}$  as expected. ■

Notice that given an automorphism  $\sigma \in \text{Aut}(A)$  and its inverse  $\sigma^{-1}$  extended to  $A \#_{R(\sigma)} \mathbb{T}$  are now an inner automorphisms. From this perspective Theorem 3.5 says that given two automorphism  $\sigma$  and  $\eta$ , they define two different smash products if their outer automorphism classes are different. In particular, we have the following result.

**Corollary 3.6.** *If  $\sigma \in \text{Aut}(A)$  is an inner automorphism, then the smash biproduct  $A \#_{R(\sigma)} \mathbb{T}$  is isomorphic to the direct product  $A \times \mathbb{T}$ .*

## 4 Homology of GWAs

### 4.1 Homology of smash products with tori

Assume  $M$  is a  $\mathbb{T}$ -module  $M$  via an automorphism  $\sigma: M \rightarrow M$ . We define  $M^\mathbb{T} = \{m \in M \mid t \triangleright m = m\}$ . We also consider  $M \otimes \mathbb{T}$  as a  $\mathbb{T}$ -bimodule via  $t \triangleright (m \otimes t^i) = \sigma(m) \otimes t^i$ ,  $(m \otimes t^i) \triangleleft t = m \otimes t^{i+1}$ . We denote this bimodule by  $M \# \mathbb{T}$ . Now, we have the following lemma.

**Lemma 4.1.**

$$H_n(\mathbb{T}, M \# \mathbb{T}) \cong \begin{cases} M_\mathbb{T} \# \mathbb{T} & \text{if } n = 0, \\ M^\mathbb{T} \# \mathbb{T} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Using the fact that  $\mathbb{T}$  has Hochschild dimension 1, we can immediately conclude that  $H_n(\mathbb{T}, M \# \mathbb{T}) = 0$  for  $n \geq 2$ . As for degree 0, we have  $H_0(\mathbb{T}, M \# \mathbb{T}) \cong (M \# \mathbb{T})_\mathbb{T}$  and

$$(M \# \mathbb{T})_\mathbb{T} = (M \# \mathbb{T}) / [\mathbb{T}, M \# \mathbb{T}] = \frac{M \# \mathbb{T}}{\{\sigma^i(m) \otimes t^i - m \otimes t^i \mid m \in M, i \in \mathbb{Z}\}} \cong M_\mathbb{T} \# \mathbb{T}$$

As for degree 1,

$$H_1(\mathbb{T}, M \# \mathbb{T}) = \frac{\{m \otimes t^j \otimes t^n \mid m \otimes t^{n+j} = \sigma^n(m) \otimes t^{n+j}\}}{\{m \otimes t^{i+j} \otimes t^n - m \otimes t^i \otimes t^{n+j} + \sigma^n(m) \otimes t^{n+i} \otimes t^j\}}.$$

Thus a copy of  $M^\mathbb{T} \# \mathbb{T}$  generated by elements of the form  $m \otimes t^n \otimes t$ , where  $\sigma(m) = m$  in the first homology. In the quotient, for all  $m \in M$  with  $\sigma^n(m) = m$ , we have

$$m \otimes 1 \otimes t^n = \sigma^{n-1}(m) \otimes t^{n-1} \otimes t + m \otimes t \otimes t^{n-1},$$



and by inductively repeating we get

$$m \otimes 1 \otimes t^n = \sum_{i=0}^{n-1} \sigma^i(m) \otimes t^{n-1} \otimes t$$

for those pairs  $m \otimes t^n \in M \# \mathbb{T}$  with  $\sigma^n(m) = m$ . This means in the first homology terms of the form  $m \otimes t^i \otimes t^n$  with  $\sigma^n(m) = m$  are replaced by terms of the form

$$\left( \sum_{j=0}^{n-1} \sigma^j(m) \right) \otimes t^{n+i-1} \otimes t.$$

Notice that the sum on the left is  $\sigma$ -invariant. Result follows.  $\blacksquare$

Thanks to Lemma 4.1, we now have the following result by using the spectral sequence given in Proposition 2.1.

**Proposition 4.2.** *Let  $\sigma \in \text{Aut}(A)$  and assume  $\sigma$  acts on  $\text{CH}_*(A)$  diagonally extending the action on  $A$ . Let  $\text{CH}_*(A)_{\mathbb{T}}$  and  $\text{CH}_*(A)^{\mathbb{T}}$  respectively be the complex of coinvariants and invariants of  $\sigma$ . Then*

$$HH_n(A \# \mathbb{T}) \cong H_n(\text{CH}_*(A)_{\mathbb{T}}) \otimes \mathbb{T} \oplus H_{n-1}(\text{CH}_*(A)^{\mathbb{T}}) \otimes \mathbb{T}.$$

## 4.2 Algebraic and separable endomorphisms

We call an algebra endomorphism  $\sigma \in \text{End}(A)$  *algebraic* if there is a polynomial  $f(t) \in \mathbb{k}[t]$  such that  $f(\sigma) = 0$  in  $\text{End}(A)$ . For an algebraic endomorphism  $\sigma$  of  $A$ , the monic polynomial  $f(t)$  with the minimal degree that satisfies  $f(\sigma) = 0$  is called *the minimal polynomial of  $\sigma$* . We call an algebraic endomorphism  $\sigma \in \text{End}(A)$  as *separable* if the minimal polynomial of  $\sigma$  is separable.

Notice that all endomorphisms of a finite dimensional  $\mathbb{k}$ -algebra are algebraic. Regardless of the dimension, all automorphisms of finite order and all nilpotent non-unital endomorphisms are also algebraic. If  $\mathbb{k}$  has characteristic 0, automorphisms of finite order are separable, but nilpotent non-unital endomorphisms are not.

## 4.3 Algebras with separable automorphisms

For a fixed algebraic automorphism  $\sigma \in \text{Aut}(A)$ , let  $\text{Spec}(\sigma)$  be the set of unique eigenvalues of  $\sigma$ , and let  $A^{(\lambda)}$  be the  $\lambda$ -eigenspace of  $\sigma$  corresponding to  $\lambda \in \text{Spec}(\sigma)$ .

**Theorem 4.3.** *Assume  $\sigma \in \text{Aut}(A)$  is separable with minimal polynomial  $f(x)$ , and let  $B$  be the quotient  $\mathbb{k}[x]/\langle f(x) \rangle$ . Then*

$$H_n(A \# \mathbb{T}) = H_n(\text{CH}_*^{(1)}(A)) \otimes \mathbb{T} \oplus H_{n-1}(\text{CH}_*^{(1)}(A)) \otimes \mathbb{T}$$

and  $H_n(A \# B) = H_n(\text{CH}_*^{(1)}(A)) \otimes B$ , where  $\text{CH}_*^{(1)}(A)$  is generated by homogeneous tensors of the form  $a_0 \otimes \cdots \otimes a_n$  with  $a_i \in A^{(\lambda_i)}$  and  $\lambda_1 \cdots \lambda_n = 1$  for every  $n \geq 0$ .

**Proof.** One can extend the distributive law  $R: \mathbb{T} \otimes A \rightarrow A \otimes \mathbb{T}$  given in equation (3.1) to a distributive law of the form  $R: B \otimes A \rightarrow A \otimes B$ . Notice that since  $f(x)$  is separable,  $B$  is a product of a finite number of copies of  $\mathbb{k}$ , and therefore, is 0-smooth. Then the result for  $A \# B$  immediately follows from Corollary 2.2. On the other hand, the result for  $A \# \mathbb{T}$  follows from Proposition 4.2.  $\blacksquare$



Note that Theorem 4.3 solves the smoothness problem for smash products with  $\mathbb{T}$ , and therefore *the birational smoothness* problem for all GWAs, provided that the action is implemented via a separable automorphism. Namely, a smash product with  $\mathbb{Z}$  via a separable automorphism is smooth if and only if the complex subcomplex of invariants  $\mathrm{CH}_*^{(1)}(A)$  has bounded homology. In the next subsection, we solve the birational smoothness problem for all GWAs without requiring automorphism to be separable.

#### 4.4 Localization of GWAs in homology

Consider the set  $S$  of elements of the form  $\sigma^m(a^n)$  in  $Z(A)$  where  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}$ . Let  $\mathbb{k}[S]$  be the (commutative) subalgebra of  $A$  generated by  $S$ , and let  $S^{-1}\mathbb{k}[S]$  be its localization at  $S$ . Then we have that  $S^{-1}A = A \otimes_{\mathbb{k}[S]} S^{-1}\mathbb{k}[S]$ . Now let  $\mathbb{k}[S]_{\mathbb{T}}$  be the algebra of coinvariants of  $\mathbb{k}[S]$  which is given by the following quotient:

$$\mathbb{k}[S]_{\mathbb{T}} := \frac{\mathbb{k}[S]}{\langle \sigma(s) - s \mid s \in S \rangle}.$$

**Corollary 4.4.** *We have*

$$\begin{aligned} HH_n(S^{-1}W_{a,\sigma}) &\cong HH_n(S^{-1}A \# \mathbb{T}) \\ &\cong H_n(\mathrm{CH}_*(A)_{\mathbb{T}} \otimes_{\mathbb{k}[S]_{\mathbb{T}}} S^{-1}(\mathbb{k}[S]_{\mathbb{T}})) \otimes \mathbb{T} \oplus H_{n-1}(\mathrm{CH}_*(S^{-1}A)^{\mathbb{T}}) \otimes \mathbb{T}, \end{aligned}$$

where we view  $\mathrm{CH}_*(A)$  as an  $\mathbb{k}[S]$ -module and  $\mathbb{k}[S]_{\mathbb{T}}$ -module on the coefficient.

**Proof.** By Theorem 3.3, we have  $S^{-1}W_{a,\sigma} \cong S^{-1}A \# \mathbb{T}$ . Now, we consider the algebra extension  $S^{-1}A \subseteq (S^{-1}A) \# \mathbb{T}$  for which by [26] there is a spectral sequence whose first page is

$$E_{p,q}^1 = H_q(S^{-1}A, \mathrm{CH}_p(S^{-1}A \# \mathbb{T} | S^{-1}A)) = H_q(S^{-1}A, \mathrm{CH}_p(\mathbb{T}, S^{-1}A \# \mathbb{T}))$$

that converges to  $HH_*(S^{-1}A \# \mathbb{T})$ . Since  $S \subseteq Z(A)$ , by [8], we know that

$$E_{p,q}^1 \cong H_q(A, S^{-1}\mathrm{CH}_p(\mathbb{T}, S^{-1}A \# \mathbb{T})) \cong H_q(A, \mathrm{CH}_p(\mathbb{T}, S^{-1}A \# \mathbb{T})).$$

Thus we have an isomorphism of the form  $HH_*(S^{-1}A \# \mathbb{T}) \cong H_*(A \# \mathbb{T}, S^{-1}A \# \mathbb{T})$ . Then by Lemma 4.1 and Proposition 4.2, we get

$$HH_n(S^{-1}W_{a,\sigma}) \cong H_n(\mathrm{CH}_*(A, S^{-1}A)_{\mathbb{T}}) \otimes \mathbb{T} \oplus H_{n-1}(\mathrm{CH}_*(S^{-1}A)^{\mathbb{T}}) \otimes \mathbb{T}.$$

Since  $S \subseteq Z(A)$ , we get that  $\mathrm{CH}_*(A, S^{-1}A) = S^{-1}\mathrm{CH}_*(A)$ . On the other hand, both the coinvariants functor  $(\cdot)_{\mathbb{T}}$  and localization functor  $S^{-1}(\cdot)$  are specific colimits, and colimits commute. Then

$$\mathrm{CH}_n(S^{-1}W_{a,\sigma}) \cong S^{-1}\mathrm{CH}_n(A)_{\mathbb{T}} \otimes \mathbb{T} \oplus \mathrm{CH}_*(S^{-1}A)^{\mathbb{T}} \otimes \mathbb{T}.$$

On the other hand,

$$\begin{aligned} (S^{-1}\mathrm{CH}_*(A))_{\mathbb{T}} &\cong (\mathrm{CH}_*(A) \otimes_{\mathbb{k}[S]} S^{-1}\mathbb{k}[S])_{\mathbb{T}} \cong \mathrm{CH}_*(A) \otimes_{\mathbb{k}[S] \times \mathbb{T}} S^{-1}\mathbb{k}[S] \\ S^{-1}\mathrm{CH}_*(A)_{\mathbb{T}} &\cong \mathrm{CH}_*(A)_{\mathbb{T}} \otimes_{\mathbb{k}[S]_{\mathbb{T}}} S^{-1}(\mathbb{k}[S]_{\mathbb{T}}) \end{aligned}$$

since the  $S$ -localization of an  $A$ -module  $M$  can be written as

$$S^{-1}M = M \otimes_A S^{-1}A \cong M \otimes_{\mathbb{k}[S]} S^{-1}\mathbb{k}[S]. \quad \blacksquare$$

## 5 Homology calculations

### 5.1 The rank-1 Weyl algebra

Consider the ordinary Weyl algebra as a GWA as we did in Section 3.3. Now, let  $S$  be the multiplicative system generated by elements of the form  $(t - m)$  where  $m \in \mathbb{Z}$ . Since  $t$  is not a unit in  $A$  we see that  $W_{t,\sigma}$  is the proper subalgebra of  $\mathbb{k}[t] \# \mathbb{T}$  generated by  $x$  and  $tx^{-1}$  where the distributive law is defined as  $R(x \otimes t) = (t - 1) \otimes x$ . Since there is no non-constant rational function invariant under the action  $\sigma(f(t)) = f(t - 1)$ , we get that  $\text{CH}_*(S^{-1}\mathbb{k}[t])^\mathbb{T} = \text{CH}_*(\mathbb{k})$ . Next, we see that the subalgebra generated by  $S$  is  $A = \mathbb{k}[t]$  itself. Moreover, since  $\sigma(t) - t = 1$  we get that  $\mathbb{k}[S]_\mathbb{T}$  is zero, and therefore, we get

$$HH_n(S^{-1}A_1) = \begin{cases} \mathbb{T} & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

for every  $n \geq 0$ .

### 5.2 The enveloping algebra $U(\mathfrak{sl}_2)$

The universal enveloping algebra of  $\mathfrak{sl}_2$  is given by the presentation

$$\frac{\mathbb{k}\{E, F, H\}}{\langle EH - (H - 2)E, FH - (H + 2)F, EF - FE - H \rangle}.$$

The center of this algebra is generated by the Casimir element

$$\Omega = 4FE + H(H + 2) = 4EF + H(H - 2).$$

In this subsection, we would like to write a generalized Weyl algebra isomorphic to  $U(\mathfrak{sl}_2)$ .

Let  $A = \mathbb{k}[c, t]$  and  $a = c - t(t + 1)$ . Define  $\sigma$  to be the algebra automorphism defined by  $\sigma(f(c, t)) = f(c, t - 1)$  for every  $f(c, t) \in A$ . In this case,  $W_{a,\sigma}$  is generated by  $c$ ,  $t$ ,  $x$  and  $(c - t(t + 1))x^{-1}$  in the smash product algebra  $A \# \mathbb{T}$ . The GWA  $W_{a,\sigma}$  is isomorphic to  $U(\mathfrak{sl}_2)$  via an isomorphism defined as  $H \mapsto 2t$ ,  $E \mapsto x$ ,  $F \mapsto (c - t(t + 1))x^{-1}$ , see [13, Example 2.2].

Let us define  $S$  to be the multiplicative system generated by elements of the form  $c - (t - n)(t - n - 1)$ , for  $n \in \mathbb{Z}$ . Then  $S^{-1}W_{a,\sigma}$  is isomorphic to  $S^{-1}\mathbb{k}[c, t] \# \mathbb{T}$ , and  $\text{CH}_*(S^{-1}A)^\mathbb{T} = \text{CH}_*(\mathbb{k}[c])$ . Moreover, the subalgebra of  $A = \mathbb{k}[c, t]$  generated by  $S$  is  $A$  itself and since  $\sigma(t) - t = 1$ , we again get that  $\mathbb{k}[S]_\mathbb{T} = 0$ . Therefore,

$$HH_n(S^{-1}U(\mathfrak{sl}_2)) = \begin{cases} \mathbb{k}[c] \otimes \mathbb{T} & \text{if } n = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

### 5.3 Primitive quotients of $U(\mathfrak{sl}_2)$

One can also consider  $B_\lambda := W_{a,\sigma} / \langle c - \lambda \rangle$ , where  $W_{a,\sigma}$  is  $U(\mathfrak{sl}_2)$  as we defined above. These algebras are also GWAs since we can realize them using  $A = \mathbb{k}[t]$ ,  $a = \lambda - t(t + 1)$  with the  $\sigma$  given by  $t \mapsto t - 1$ . See [5, Section 3].

In this case, using a similar automorphism we used for  $U(\mathfrak{sl}_2)$ , we can replace  $S$  with the multiplicative system generated by elements of the form  $\mu - (t - n)$  and  $\mu + (t - n)$ , where  $\mu \in \mathbb{k}$  is fixed and  $n$  ranges over  $\mathbb{Z}$ . Then  $\mathbb{k}[S] = \mathbb{k}[t]$  and  $S^{-1}B_\lambda \cong S^{-1}\mathbb{k}[t] \# \mathbb{T}$ . In this case,  $\text{CH}_*(S^{-1}\mathbb{k}[t])^\mathbb{T}$  is  $\text{CH}_*(\mathbb{k})$  and  $\mathbb{k}[S]_\mathbb{T} = 0$  since  $\sigma(t) - t = 1$  as before. Then we get

$$HH_n(S^{-1}B_\lambda) \cong \begin{cases} \mathbb{T} & \text{if } n = 1, 2, \\ 0 & \text{otherwise} \end{cases}$$

for every  $n \geq 0$ .

## 5.4 Quantum 2-torus

Fix an element  $q \in \mathbb{k}^\times$  which is not a root of unity. Let  $A = \mathbb{k}[t, t^{-1}]$  and let  $a = t$  as in the case of the ordinary Weyl algebra. But this time, let us define  $\sigma \in \text{Aut}(A)$  to be the algebra automorphism given by  $\sigma(f(t)) = f(qt)$  for every  $f(t) \in A$ . The smash biproduct algebra  $A \# \mathbb{T}$  is the algebraic quantum 2-torus  $\mathbb{T}_q^2$  and the GWA  $W_{a,\sigma}$  is the quantum torus itself since  $a = t$  is a unit.

Note that for every  $u \in A$  and  $m \in \mathbb{Z}$  we have  $\sigma^m(u) \neq u$  unless  $m = 0$  since  $q$  is not a root of unity. Thus  $\text{CH}_*(A)_{\mathbb{T}} = \text{CH}_*(A)^{\mathbb{T}} = \text{CH}_*^{(0)}(A)$  where

$$\text{CH}_m^{(0)}(A) = \text{Span}_{\mathbb{k}} \left( t^{n_0} \otimes \cdots \otimes t^{n_m} \mid n_1, \dots, n_m \in \mathbb{Z} \text{ with } 0 = \sum_i n_i \right),$$

which gives us just the group homology of  $\mathbb{Z}$ . Then by Proposition 4.2, we get

$$HH_n(\mathbb{T}_q^2) \cong \mathbb{k}^{\binom{2}{n}} \otimes \mathbb{T}$$

for every  $n \geq 0$  as expected.

## 5.5 The quantum enveloping algebra $U_q(\mathfrak{sl}_2)$

For a fixed  $q \in \mathbb{k}^\times$ , the quantum enveloping algebra of the lie algebra  $\mathfrak{sl}_2$  is given by the presentation

$$\frac{\mathbb{k}\{K, K^{-1}, E, F\}}{\langle KE - q^2 EK, KF - q^{-2} FK, EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \rangle}.$$

As before, we assume  $q$  is not a root of unity. There is an element  $\Omega$  in the center of  $U_q(\mathfrak{sl}_2)$  called the *quantum Casimir element* defined as

$$\Omega = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}.$$

See [7, Section I.3]. Our first objective is to give a GWA that is isomorphic to  $U_q(\mathfrak{sl}_2)$ .

We start by setting  $A = \mathbb{k}[c, t, t^{-1}]$  together with  $a = c - (q^{-1}t + qt^{-1})$  and  $\sigma \in \text{Aut}(A)$  given by  $\sigma(f(c, t)) = f(c, q^2t)$  for every  $f(c, t) \in \mathbb{k}[c, t, t^{-1}]$ . Define an algebra map  $\gamma: W_{a,\sigma} \rightarrow U_q(\mathfrak{sl}_2)$  given on the generators by

$$t \mapsto K, \quad c \mapsto (q - q^{-1})^2 \Omega, \quad x \mapsto (q - q^{-1})F, \quad ax^{-1} \mapsto (q - q^{-1})E.$$

Notice that the inverse of  $\gamma$  is defined easily as

$$K \mapsto t, \quad E \mapsto \frac{ax^{-1}}{q - q^{-1}}, \quad F \mapsto \frac{x}{q - q^{-1}}.$$

One can show that both  $\gamma$  and its inverse are well-defined by showing the relations are preserved.

Now, let  $S$  be the multiplicative system in  $A$  generated by the elements of the form

$$c - (q^{-2n+1}t + q^{2n-1}t^{-1}) \quad \text{for } n \in \mathbb{Z}.$$

In this case too, the subalgebra of  $A$  generated by  $S$  is  $A$  itself. Then we have

$$\text{CH}_*(S^{-1}A)^{\mathbb{T}} \cong \text{CH}_*^{(0)}(\mathbb{k}[c, t, t^{-1}]) \cong \text{CH}_*(A)_{\mathbb{T}}.$$

On the other hand, since  $\sigma(t) - t = (q^2 - 1)t$  and  $t$  is a unit, we get that  $\mathbb{k}\langle S \rangle_{\mathbb{T}} = 0$ . Thus, as in the case of  $U(\mathfrak{sl}_2)$  we get

$$HH_n(S^{-1}W_{a,\sigma}) \cong HH_n(S^{-1}U_q(\mathfrak{sl}_2)) \cong \begin{cases} \mathbb{k}[c] \otimes \mathbb{T} & \text{if } n = 1, 2, \\ 0 & \text{otherwise} \end{cases}$$

for every  $n \geq 0$ .

### 5.6 The quantum matrix algebra $\mathcal{O}_q(M_2)$

For a fixed  $q \in \mathbb{k}^\times$  the algebra  $\mathcal{O}_q(M_2)$  of quantum  $2 \times 2$  matrices is given by the presentation

$$\begin{aligned} bc &= cb, & ab &= q^{-1}ba, & ac &= q^{-1}ca, & db &= qbd, \\ dc &= qcd, & ad - da &= (q^{-1} - q)bc. \end{aligned}$$

The quantum determinant  $\Omega = ad - q^{-1}bc = da - qbc$  generates the center of this algebra, see [7, pp. 4–8].

Now, let  $A = \mathbb{k}[u, v, w]$  with the distinguished element  $u + qvw \in A$  where we set  $\sigma(f(u, v, w)) = f(u, q^{-1}v, q^{-1}w)$  for every  $f(u, v, w) \in A$ . Then the GWA  $W_{a,\sigma}$  is the subalgebra of  $A \# \mathbb{T}$  generated by  $A$ ,  $x$  and  $(u + qvw)x^{-1}$ , and it is isomorphic to  $\mathcal{O}_q(M_2)$  via

$$u \mapsto \Omega, \quad v \mapsto b, \quad w \mapsto c, \quad x \mapsto a, \quad (u + qvw)x^{-1} \mapsto d,$$

and its inverse is

$$a \mapsto x, \quad b \mapsto v, \quad c \mapsto w, \quad d \mapsto (u + qvw)x^{-1}.$$

Since  $\mathcal{O}_q(\mathrm{GL}_2)$  is obtained by localizing  $\mathcal{O}_q(M_2)$  at the quantum determinant, we see that  $\mathcal{O}_q(\mathrm{GL}_2)$  is isomorphic to  $u^{-1}W_{a,\sigma}$  which itself is a GWA with  $A$  replaced by  $\mathbb{k}[u, u^{-1}, v, w]$  with the remaining datum unchanged.

On the other hand,  $\mathcal{O}_q(\mathrm{SL}_2)$  is the quotient of  $\mathcal{O}_q(M_2)$  by the two sided ideal generated by  $u - 1$ , and therefore, is again a GWA with the same datum where this time we replace  $A$  by  $\mathbb{k}[u, v, w]/\langle u - 1 \rangle$ . We also know that  $\mathcal{O}_q(\mathrm{GL}_2)$  is isomorphic (as algebras only) to  $\mathcal{O}_q(\mathrm{SL}_2) \times \mathbb{k}[\Omega]$ .

For the remaining of the section, we are going to concentrate on  $\mathcal{O}_q(\mathrm{SL}_2)$  only given as the subalgebra of  $\mathbb{k}[v, w] \# \mathbb{T}$  generated by  $v$ ,  $w$ ,  $x$  and  $(1 + qvw)x^{-1}$ .

Now, let  $S$  be the Ore set generated by elements of the form  $1 + q^{2n+1}vw$  for  $n \in \mathbb{Z}$ . Then  $S^{-1}\mathcal{O}_q(\mathrm{SL}_2)$  is isomorphic to  $S^{-1}\mathbb{k}[v, w] \# \mathbb{T}$ . In this case, since  $q$  is not a root of unity, we get that  $\mathrm{CH}_*(A, S^{-1}A)^\mathbb{T} = \mathrm{CH}_*(\mathbb{k}) = \mathrm{CH}_*(A)_\mathbb{T}$ . The subalgebra of  $\mathbb{k}[v, w]$  generated by  $S$  is the polynomial algebra  $\mathbb{k}[vw]$  over the indeterminate  $vw$ . Since  $\sigma(vw) - vw = (q^{-2} - 1)vw$ , we get that  $\mathbb{k}[vw]_\mathbb{T} = \mathbb{k}$ . Hence

$$HH_n(S^{-1}\mathcal{O}_q(\mathrm{SL}_2)) \cong \mathbb{k}^{\binom{2}{n}} \otimes \mathbb{T}$$

for every  $n \geq 0$ .

### 5.7 Quantum group $\mathcal{O}_q(\mathrm{SU}_2)$

Let us fix  $q \in \mathbb{k}^\times$ . The algebraic quantum group  $\mathcal{O}_q(\mathrm{SU}_2)$  is the noncommutative  $*$ -algebra generated by two non-commuting indeterminates  $s$  and  $x$  subject to the following relations:

$$x^*x = 1 - s^*s, \quad xx^* = 1 - q^2s^*s, \quad s^*s = ss^*, \quad xs = qsx, \quad xs^* = qs^*x.$$

See [18, p. 4]. One can write  $\mathcal{O}_q(\mathrm{SU}_2)$  as a GWA  $W_{a,\sigma}$  by letting  $A = \mathbb{k}[s, s^*]$  with the distinguished element  $a \in A$  is defined as  $1 - s^*s$  and  $\sigma(f(s, s^*)) = f(qs, qs^*)$  for every  $f(s, s^*) \in \mathbb{k}[s, s^*]$ .

Let  $S$  be the multiplicative system in  $A$  generated by elements of the form  $q^{2n}s^*s - 1$  for  $n \in \mathbb{Z}$ . Then  $S^{-1}\mathcal{O}_q(\mathrm{SU}_2)$  is isomorphic to  $S^{-1}A \# \mathbb{T}$  by Theorem 3.3. If we assume that  $q \in \mathbb{k}^\times$  is not a root of unity, we get that  $\mathrm{CH}_*(S^{-1}A)^\mathbb{T} = \mathrm{CH}_*(\mathbb{k}) = \mathrm{CH}_*(A)_\mathbb{T}$ . We also see that the subalgebra of  $\mathbb{k}[s, s^*]$  generated by  $S$  is the polynomial algebra  $\mathbb{k}[ss^*]$ , and since  $\sigma(ss^*) - ss^* = (q^2 - 1)ss^*$  we get that  $\mathbb{k}\langle S \rangle_\mathbb{T} = \mathbb{k}$ . Then

$$HH_n(S^{-1}\mathcal{O}_q(\mathrm{SU}_2)) \cong \mathbb{k}^{\binom{2}{n}} \otimes \mathbb{T}$$

for every  $n \geq 0$ .

## 5.8 Podleś spheres

For a fixed  $q \in \mathbb{k}^\times$ , the algebra of functions  $\mathcal{O}_q(S^2)$  on standard Podleś quantum spheres [18, 37] is the subalgebra of  $\mathcal{O}_q(\mathrm{SU}_2)$  generated by elements  $s^*s$ ,  $xs$  and  $s^*x^*$ . This means  $\mathcal{O}_q(S^2)$  is the subalgebra of the smash product  $\mathbb{k}[s, s^*] \# \mathbb{T}$  generated by the elements  $s^*s$ ,  $xs$  and  $s^*(1-s^*s)x^{-1}$ . One can give a presentation for the Podleś sphere as

$$xt = q^2tx, \quad yt = q^{-2}ty, \quad yx = -t(t-1), \quad xy = -q^2t(q^2t-1),$$

then we get a GWA structure if we let  $A = \mathbb{k}[t]$  and where we set  $t = s^*s$  with  $a = -t(t-1)$  and  $\sigma(f(t)) = f(q^2t)$  for every  $f(t) \in A$ .

Let  $S$  be the multiplicative system in  $A$  generated by the set  $\{t(t-q^{2n}) \mid n \in \mathbb{Z}\}$ , then  $S^{-1}\mathcal{O}_q(S^2) \cong S^{-1}A \# \mathbb{T}$ . Instead of this generating set, one can use  $\{t\} \cup \{t-q^{2n} \mid n \in \mathbb{Z}\}$  to get the same localization. Then we get that  $\mathbb{k}[S]$  is  $A$  itself. If we assume that  $q \in \mathbb{k}^\times$  is not a root of unity, we get that  $\mathrm{CH}_*(S^{-1}A)^\mathbb{T} = \mathrm{CH}_*^{(0)}(\mathbb{k}[t, t^{-1}])$ , and  $\mathrm{CH}_*(A)_\mathbb{T} = \mathrm{CH}_*(\mathbb{k})$ . In this case,  $\mathbb{k}[S]_\mathbb{T} = \mathbb{k}$  since  $\sigma(t) - t = (q^2 - 1)t$ . Thus

$$HH_n(S^{-1}\mathcal{O}_q(S^2)) \cong \mathbb{k}^{\binom{2}{n}} \otimes \mathbb{T}$$

for every  $n \geq 0$ .

## 5.9 Parametric Podleś spheres

In [18], Hadfield defines another family of Podleś spheres  $\mathcal{O}_{q,c}(S^2)$  given by a presentation is equivalent to the following:

$$xt = q^2tx, \quad x^*t = q^{-2}tx^*, \quad x^*x = c - t(t-1), \quad xx^* = c - q^2t(q^2t-1).$$

If we set  $A = \mathbb{k}[c, t]$ , and let the distinguish element  $a \in A$  be  $c - t(t-1)$  together with  $\sigma(f(c, t)) = f(c, q^2t)$  for every  $f(c, t) \in A$ , we get a GWA structure on  $\mathcal{O}_{q,c}(S^2)$  similar to the GWA structure on  $U(\mathfrak{sl}_2)$  where we changed only the algebra automorphism from  $\sigma(f(c, t)) = f(c, t-1)$  to  $\sigma(f(c, t)) = f(c, q^2t)$ .

Let  $S$  be the multiplicative system in  $A$  generated by the elements of the form  $c - q^{2n}t(q^{2n}t-1)$ . If we assume that  $q \in \mathbb{k}^\times$  is not a root of unity, we conclude that

$$\mathrm{CH}_*(S^{-1}A)^\mathbb{T} = \mathrm{CH}_*(\mathbb{k}[c]) = \mathrm{CH}_*(A)_\mathbb{T}$$

which allows us to conclude

$$HH_n(S^{-1}\mathcal{O}_{q,c}(S^2)) \cong \mathbb{k}^{\binom{2}{n}} \otimes \mathbb{k}[c] \otimes \mathbb{T}$$

for every  $n \geq 0$ .

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