Isomonodromy and Painlevé Type Equations, Case Studies

Marius VAN DER PUT and Jaap TOP

Bernoulli Institute, Nijenborgh 9, 9747 AG Groningen, The Netherlands
E-mail: m.van.der.put@rug.nl, j.top@rug.nl
URL: http://www.math.rug.nl/~top/

Abstract. There is an abundance of equations of Painlevé type besides the classical Painlevé equations. Classifications have been computed by the Japanese school. Here we consider Painlevé type equations induced by isomonodromic families of linear ODE's having at most $z = 0$ and $z = \infty$ as singularities. Requiring that the formal data at the singularities produce isomonodromic families parametrized by a single variable t leads to a small list of hierarchies of cases. The study of these cases involves Stokes matrices and moduli for linear ODE's on the projective line. Case studies reveal interesting families of linear ODE's and Painlevé type equations. However, rather often the complexity (especially of the Lax pair) is too high for either the computations or for the output. Apart from classical Painlevé equations one rediscovers work of Harnad, Noumi and Yamada. A hierarchy, probably new, related to the classical $P_3(D_8)$, is discovered. Finally, an amusing "companion" of P_1 is presented.

Key words: moduli space for linear connections; irregular singularities; Stokes matrices; monodromy spaces; isomonodromic deformations; Painlevé equations; Lax pairs; Hamiltonians

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1 Introduction and background

1.1 Introduction

Every classical Painlevé equation can be obtained by an isomonodromy of a family $\mathcal{M}$ of linear differential equations over the differential field $\mathbb{C}(z)$ (or connections on $\mathbb{P}^1$). The monodromy space $\mathcal{R}$, associated to $\mathcal{M}$, consists of all possibilities for the ordinary monodromy and the Stokes data of the solutions for the linear differential equations belonging to $\mathcal{M}$.

An isomonodromic subfamily of $\mathcal{M}$ is defined by "the monodromy is constant" condition, in other words, it is a fibre of the Riemann–Hilbert map RH: $\mathcal{M} \to \mathcal{R}$ which sends a connection in $\mathcal{M}$ to its monodromy data in $\mathcal{R}$. An isomonodromic subfamily turns out to be given by first-order, nonlinear, differential equations (called here the Painlevé vector field) which can be transformed into a single higher-order nonlinear differential equation.

For example (see also [23, Section 4.9]),

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t} = \frac{\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2}{2q} + 4q^2 + 2tq - \frac{\theta_0^2}{2q},$$

which is the Flaschka–Newell form $P_{2,fn}$ of the second Painlevé equation, is obtained from the family

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} p & \frac{p^2 - \theta_0^2 - 2qtz + q^2 z + qz^2}{q} \\ z - q & -p \end{pmatrix}.$$

This 4-dimensional family of operators is defined by the conditions: the trace is zero, the singularities are $z = 0$, which is regular singular, and $z = \infty$ which is irregular singular and has (generalized) eigenvalues $\pm\left(z^{3/2} + t z^{1/2}\right)$.

For this example, the monodromy space $\mathcal{R}$ consists of the ordinary monodromy mon_0 at $z = 0$ and Stokes data at $z = \infty$. It can be shown that it consists of Stokes matrices $\left(\begin{smallmatrix} 1 & 0 \\ a_1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & a_2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ a_3 & 1 \end{smallmatrix}\right)$ for directions $\{d_i\}$ with $1 > d_1 > d_2 > d_3 > 0$. There is one relation, the monodromy identity: mon_0 is conjugated to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ a_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a_2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ a_3 & 1 \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the formal monodromy. Hence $\mathcal{R} = \mathbb{C}^3$. It can be shown that the fibres of the Riemann–Hilbert map RH: $\mathcal{M} \to \mathcal{R}$ are parametrized by t. A fibre has the form

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} p & \frac{p^2-\theta_0^2-2qtz+q^2z+qz^2}{q} \\ z - q & -p \end{pmatrix}$$

with now p, q functions of t and θ_0 a parameter (independent of t).

The functions p, q of t cannot directly be computed by the Riemann–Hilbert map, since the latter is (in general) highly transcendental and not computable. However, the condition "isomonodromic" is equivalent to the above operator commuting with an operator of the form $\frac{\mathrm{d}}{\mathrm{d}t} + B(z, t)$, where the 2×2 matrix $B(z, t)$ depends analytically on t and rationally on z. This is the Lax pair condition and leads to the nonlinear differential equations

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 2p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = 2q^2 + tq + \frac{p^2 - \theta_0^2/4}{q}$$

(the Painlevé vector field) and as consequence to $P_{2,fn}$. Finally, the system of equations for q, p is Hamiltonian with function

$$H = \frac{-p^2 + \theta_0^2/4}{q} + q^2 + tq.$$

The parameter space $\mathcal{P}$ for $\mathcal{R}$ consists of the data for the local formal monodromy at $z = 0$ and $z = \infty$. The parameter space $\mathcal{P}^+$ for $\mathcal{M}$ consists of the data of the residue matrices of the connection at $z = 0$ and $z = \infty$. There is an obvious exponential map $\mathcal{P}^+ \to \mathcal{P}$ commuting with the Riemann–Hilbert map RH: $\mathcal{M} \to \mathcal{R}$.

In the above example, $\mathcal{P}^+$ is given by θ_0 and $\mathcal{P}$ is given by $\mathrm{e}^{2\pi \mathrm{i} \theta_0}$.

Apart from the classical cases $P_1 - P_6$, there is an abundance of families $\mathcal{M}$ of linear differential equations over $\mathbb{C}(z)$ producing Painlevé type equations. The Japanese school has an extensive literature on equations of Painlevé type and also developed classifications.

We apologize for citing only a few items from the extensive literature, [7, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22]. In the literature, other methods than isomonodromy, e.g., middle convolution and constructions with Hamiltonians, are used for producing Painlevé type equations.

Here, we *modestly restrict ourselves* to classifying and studying rather special cases of $\mathcal{M}$, namely assuming that at most $z = 0$ and $z = \infty$ are singular and assuming that the fibres of the Riemann–Hilbert map RH: $\mathcal{M} \to \mathcal{R}$ are parametrized by one variable t, called the *time variable*. In general there are more times variables.

One reason for this restriction is that the quantum differential equations (see [2, 8, 9, 10]), associated to algebraic varieties have two singularities $z = 0$ (regular singular) and $z = \infty$ (irregular singular). A further reason is that the theory of Stokes matrices, and the algorithm we describe to make $\mathcal{M}$ explicit in the case of at most two singularities, provide most of the information for $\mathcal{R}$ and $\mathcal{M}$ used here. Furthermore, we are also interested in *hierarchies* of families instead of individual families.

The aim of this paper is to find these families $\mathcal{M}$ and classify them by their formal singularities at $z = 0$ and $z = \infty$ (see the list in Section 2).

For each item in this list, we try to make the family of connections $\mathcal{M}$ explicit by computing a matrix differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$. This can be rather involved. Further we want to make the monodromy space $\mathcal{R}$ explicit, to produce a Lax pair and compute the Painlevé type equation (or vector field), produce a Hamiltonian and find an identification (if this exists) with some known Painlevé type equation.

Due to complexity, we obtained for many items in the list of Section 2, only partial information. However, we highlight here: *A list of the most detailed and interesting explicit cases.* The formulas give representatives for the Galois orbit(s) of the eigenvalues and n is the rank of the connection.

- Section 3: $z^{2/n} + tz^{1/n}$, $n \geq 3$. We rediscover the hierarchy studied by Noumi and Yamada [20]. The spaces $\mathcal{M}$ and $\mathcal{R}$ and the fibres of $\mathcal{M} \to \mathcal{R}$ are made explicit. For any $n \geq 3$, explicit formulas for the Lax pair and the Painlevé vector field are computed. A Hamiltonian is computed for $n = 3, 4, 6$. The Painlevé equations for $n = 3$ and $n = 4$ are identified with P_4 and P_5. An identification of the Hamiltonian for $n = 6$ is missing.

- Section 6.1: $z^{1/2}$, $tz^{1/2}$, $n = 4$. Apart from the Hamiltonian all data are made explicit. The fibres of the map $\mathcal{R} \to \mathcal{P}$ are affine cubic surfaces. From their equation we expected, in view of the list [23, pp. 26–27], that this case is a pull back of P_4. Instead, the computation in [5, Section 1] provides an explicit identification with the sixth Painlevé equation in Okamoto's form.

- Section 8: z, tz, $(-1 - t)z$, $n = 3$. Except for the Hamiltonian, all data (including the Stokes matrices) are made explicit. However the expected identification with P_6 is not verified.

  In fact, we rediscover the family found by Harnad and studied by Mazzocco, Degano and Guzetti [4, 11, 18]. They provide the verification of the equivalence to P_6.

- Section 10: z^2, $-z^2 - tz$, tz, $n = 3$. The results, including an explicit Hamiltonian, are complete. The Painlevé type equation is a second-order explicit equation and therefore related to one of the classical Painlevé equations. From the cubic equation of the fibres of $\mathcal{R} \to \mathcal{P}$, we expected a relation with P_1. Instead, the computation in [5, Section 2] provides an explicit identification with the fourth Painlevé equation in Okamoto's form.

- Section 13: $z^{-1/n}$ and $tz^{1/n}$, $n \geq 2$. This hierarchy is probably new. For $n = 2$, this defines the standard family leading to $P_3(D_8)$. For general n, the spaces of connections and of monodromy are made explicit. The explicit formulas for the Lax pairs and the Painlevé vector field have a structure, similar to the ones for the Noumi–Yamada family in Section 3.

  For $n = 3$, we claim an identification with an item in the classification by Kawakami [14, p. 35].

- Section 14: $z^{5/2} + tz^{1/2}$, $n = 2$. What we like to call "a companion of P_1" is given by the family of connections of rank 2 with a regular singularity at $z = 0$ and an irregular singularity at $z = \infty$ with generalized eigenvalues $\pm(z^{5/2} + tz^{1/2})$.

  We recall that the standard family of connections for P_1 has the same definition except for the assumption "$z = 0$ is regular".

  For this companion of P_1, all data are computed, including the Painlevé equation which is a nonlinear, explicit, fourth-order differential equation.

  The above family is not present in the list of Section 2, since it is a subfamily of the *family with two time variables* t_1, t_2. This family is given by $z = 0$ is regular singular and $z = \infty$ has generalized eigenvalues $\pm(z^{5/2} + \frac{t_1}{2}z^{3/2} + \frac{t_2}{2}z^{1/2})$.

Also in this case an operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$, representing $\mathcal{M}$, is computed. Further, $\mathcal{R}$ and the fibres $\mathcal{R} \to \mathcal{P}$ are made explicit. The Lax pair equations have now the form $\left[z\frac{\mathrm{d}}{\mathrm{d}z} + A, \frac{\mathrm{d}}{\mathrm{d}t_i} + B_i\right] = 0$ for $i = 1, 2$ for certain matrices B_1, B_2 depending on z, t_1, t_2. The Painlevé vector field, solution of the Lax pairs, is computed. It seems that this companion of P_1 and its extension to a "two time variables system" are new.

1.2 Background

For the convenience of the reader, we describe terminology and results concerning the formal classification of differential modules, irregular singularities and Stokes matrices etc. More details can be found in [24].

1.2.1 The formal classification of differential modules

Differential modules M over the field $\mathbb{C}\big(\!\big(z^{-1}\big)\!\big)$ are classified in terms of tuples $(V, \{V_q\}_q, \gamma)$. A tuple consists of a complex vector space V of dimension n with additional structure. The q's denote elements of $\bigcup_{r \geq 1} z^{1/r}\mathbb{C}\big[\!\big[z^{1/r}\big]\!\big]$. For each q there is given a linear subspace $V_q \subset V$ and $V = \oplus V_q$. The *eigenvalues* are the finitely many $q_1, \ldots, q_r$ with $V_q \neq 0$. The multiplicity $m = m(q)$ is the dimension of V_q. The dimension of V is therefore equal to $\sum m(q_j)$. One writes $(q)_m$ to denote an eigenvalue q with multiplicity m.

The *ramification index* e is the smallest positive integer such that $q_j \in z^{1/e}\mathbb{C}\big[\!\big[z^{1/e}\big]\!\big]$ for all j. The degree of q_j is the highest (rational) power of z occurring in q_j. The *Katz invariant* κ is the maximum of the degrees of the q_j's. The Galois group of $\bigcup_{r \geq 1} \mathbb{C}\big(\!\big(z^{-1/r}\big)\!\big)/\mathbb{C}\big(\!\big(z^{-1}\big)\!\big)$ has a topological generator σ which acts by $\sigma\big(z^\lambda\big) = \mathrm{e}^{2\pi\mathrm{i}\lambda}z^\lambda$ for $\lambda \in \mathbb{Q}$.

Further, γ, the *formal monodromy*, is an automorphism of V and has the property $\gamma(V_q) = V_{\sigma(q)}$ for all q.

This classification of differential modules M over $\mathbb{C}\big(\!\big(z^{-1}\big)\!\big)$ by the tuples $(V, \{V_q\}_q, \gamma)$ is based upon the fact that the solutions of M (where M is represented in the form of an ordinary scalar linear differential equation with coefficients in $\mathbb{C}\big(\!\big(z^{-1}\big)\!\big)$) can be written as a sum of expressions $\exp\big(\int q\frac{\mathrm{d}z}{z}\big) \cdot G$ with q as above and G a combination of formal power series in roots of z and $\log(z)$. The space V associated to M is the space of these formal or symbolic expressions. It has a natural decomposition as $\oplus V_q$. Further, the formal monodromy γ, is given by σ applied to the above expressions. The functor $M \mapsto (V, \{V_q\}_q, \gamma)$ is an equivalence of Tannakian categories.

Let a family of data for eigenvalues $q_1, \ldots, q_r$, multiplicities and formal monodromies γ be given. This gives rise to a family of formal differential operators $z\frac{\mathrm{d}}{\mathrm{d}z} + F$.

Example 1.1 (see Section 7). Eigenvalues c_1z, c_2z, c_3z with multiplicities 1 and formal monodromy

$$\gamma = \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix},$$

combine to the formal differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + F$, where

$$F = \begin{pmatrix} c_1z + a_1 & & \\ & c_2z + a_2 & \\ & & c_3z + a_3 \end{pmatrix}$$

and $b_j = \mathrm{e}^{2\pi\mathrm{i}a_j}$ for $j = 1, 2, 3$.

The space of connections $\mathcal{M}$ is the "universal family" (details later), represented by a family of differential operators $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ over $\mathbb{C}(z)$ with singular points $z = 0$ and $z = \infty$. The condition is that this operator is at $z = \infty$ is formally equivalent to $z\frac{\mathrm{d}}{\mathrm{d}z} + F$. Moreover, at $z = 0$ the operator should be regular, or regular singular, or formally equivalent to another given formal operator.

1.2.2   Details on Stokes matrices and construction of the space $\mathcal{R}$

Let M be a differential module over $\mathbb{C}(\{z^{-1}\})$, the field of convergent power series at $z = \infty$. Write $(V, \{V_{q_k}\}, \gamma)$ for its formal classification, i.e., the classification of $\widehat{M} = \mathbb{C}((z^{-1})) \otimes M$. The formal solutions of $\widehat{M}$ lift, by multisummation, to solutions of M on sectors at $z = \infty$. The jumps of these solutions from one sector to another are measured by Stokes matrices $\{\mathrm{St}_d\}$ at the singular directions d. In fact, M is classified by its formal classification and the Stokes matrices. Moreover, the formalism, detailed below, produces all possibilities for the Stokes data and determines therefore the structure of $\mathcal{R}$.

For each difference $q_k - q_l$, $k \neq l$ of eigenvalues, one considers a solution $y \neq 0$ of $z\frac{\mathrm{d}y}{\mathrm{d}z} = q_k - q_l$. The singular directions $d \in \mathbb{R}$ for $q_k - q_l$ are defined by the condition that $y(\mathrm{e}^{2\pi\mathrm{i}d}r)$ tends to zero for $r \to +\infty$ with maximal speed (maximal descent).

The Stokes matrix $\mathrm{St}_d \in \mathrm{GL}(V)$ for direction d reads $\mathbf{1}_V + \sum_{k,l} m_{k,l}$, where the sum is taken over the pairs such that d is singular for $q_k - q_l$ and $m_{k,l}$ denotes a linear map

$$V \overset{\text{projection}}{\to} V_{q_k} \to V_{q_l} \subset V.$$

We note that $\mathrm{St}_d = \mathbf{1}_V$ if d is not a singular direction and that $\mathrm{St}_{d+1} = \gamma^{-1}\mathrm{St}_d\gamma$ holds for $d \in \mathbb{R}$.

Therefore, the Stokes data can be identified with the space of all Stokes matrices St_d with $d \in [0, 1)$ and can be identified with a vector space of dimension

$$N := \sum_{k \neq l} \deg(q_k - q_l) \cdot \dim V_{q_k} \cdot \dim V_{q_l}.$$

The formal data combined with the data of the Stokes matrices classify the analytic singularity at $z = \infty$. In particular, mon_∞, the topological monodromy at $z = \infty$, is equivalent to the product $\gamma \circ \mathrm{St}_{d_s} \circ \cdots \circ \mathrm{St}_{d_1}$, where $d_s > \cdots > d_1$ are the singular directions in $[0, 1)$. This property will be called *the monodromy identity*. For the construction of $\mathcal{R}$, we have to consider various cases.

(i) $z = 0$ *is regular singular.* Given are $V = V_{q_1} \oplus \cdots \oplus V_{q_r}$, an action of σ on $\{q_1, \ldots, q_r\}$, the singular directions $1 > d_s > \cdots > d_1 \geq 0$ with the corresponding differences $q_k - q_l$. We note, in passing, that the highest coefficient of a difference $q_k - q_l$ may depend on t. In such a case the singular directions also depend on t. In the cases that we computed, $\mathcal{R}$ itself is independent of t.

In general, $\mathcal{R}$ is defined as the set of equivalence classes of all possibilities for the Stokes data and the topological monodromies. In the present case, $\mathcal{R}$ consists of the equivalence classes of all possible tuples $(\gamma, \mathrm{St}_{d_s}, \ldots, \mathrm{St}_{d_1}) \in \mathrm{SL}(V)^{s+1}$, where, by assumption, γ is supposed to have distinct eigenvalues. We now make $\mathcal{R}$ explicit.

Let $(V, \{V_q\}, \gamma, \{\mathrm{St}_d\})$ be given. The action of σ on the eigenvalues has orbits (i.e., Galois orbits) $\overline{Q}_1, \ldots, \overline{Q}_r$ and $\overline{Q}_i = \{q_{i,0}, \ldots, q_{i,\ell_i-1}\}$ for all i. Let $d_i = \dim V_{q_{i,0}}$. Put $d = \sum_{i=1}^r d_i$ and as before we write

$$N = \sum \deg(q_{i,j} - q_{k,l}) \cdot \dim(V_{q_{i,j}}) \cdot \dim(V_{q_{k,l}}).$$

Lemma 1.2. *The monodromy space $\mathcal{R}$ is isomorphic to the quotient of the space $(\mathbb{C}^*)^{d-1} \times \mathbb{C}^N$ by the action of a group isomorphic to $(\mathbb{C}^*)^{d-1}$. This quotient has an open, affine, dense subspace isomorphic to $(\mathbb{C}^*)^{d-1} \times \mathbb{C}^{N-d+1}$. In particular, $\dim \mathcal{R} = N$.*

Proof. There is no restriction on the possibilities for the St_{d_j}. This produces the vector space $\mathbb{C}^N$. In order to make the restrictions on γ explicit, we consider a Galois orbit, here written as, $\overline{Q} = \{q_0, \ldots, q_{\ell-1}\}$ with $\dim V_{q_0} = f$. Choose a basis $e_1, \ldots, e_f$ of eigenvectors for the action of γ^ℓ on V_{q_0}. Let $\alpha_1, \ldots, \alpha_f$ denote the distinct eigenvalues. This basis is unique up to permuting and scaling of the basis vectors. Consider, for $i = 1, \ldots, \ell - 1$, the basis of V_{q_i} to be $\gamma^i(e_1), \ldots, \gamma^i(e_f)$. One concludes that the data for the matrix of γ on the space $\oplus V_{q_i}$ is equivalent to the tuple $(\alpha_1, \ldots, \alpha_f)$. Moreover, the set of all ℓth roots of all α_i is the set of eigenvalues of γ.

Thus the total data for γ on V is given by the eigenvalues of γ^{l_i} on the space $V_{q_{i,0}}$ for $i = 1, \ldots, r$. Since, by assumption, γ has determinant 1, the space of possibilities for γ is $(\mathbb{C}^*)^{-1+\sum d_i}$.

The automorphisms of $(V, \{V_q\}, \gamma)$ are the $\tau \in \mathrm{PGL}(V)$ such that $\tau(V_q) = V_q$ for all q and $\tau\gamma = \gamma\tau$. One concludes that τ is determined by its action on all $V_{q_{i,0}}$. Furthermore, τ has on this space the same eigenvectors as γ^{ℓ_i}. This implies that the group of automorphism is isomorphic to $(\mathbb{C}^*)^{d-1}$. It is seen that the group acts faithfully on the Stokes data. Finally, by scaling suitable Stokes data to 1, one obtains this affine, open, dense subspace of $\mathcal{R}$. ∎

(ii) $z = 0$ *is regular.* As above in (i), but now with the additional restriction $\gamma \circ \mathrm{St}_{d_s} \circ \cdots \circ \mathrm{St}_{d_1} = \mathbf{1}_V$. For every candidate, a computation is needed to find out whether $\mathcal{R}$ is not empty and to find its dimension.

(iii) $z = 0$ *is irregular singular.* Let W denote the solution space at $z = 0$. It has similar additional data as V, namely q's, γ, St_d, mon_0. The *link* (in [13] called 'connection') is a linear bijection $L: W \to V$ commuting with the mon_*. The space $\mathcal{R}$ is the space of equivalence classes of the data at V, W and the link L.

The parameter space $\mathcal{P}$ is defined by data of the topological and the formal monodromies, more precisely by their characteristic polynomials. By "fibre" we will mean a fibre of $\mathcal{R} \to \mathcal{P}$, which has the interpretation as space of initial conditions. Each fibre determines a Painlevé vector field (or scalar differential equation) of rank (or order) equal to the dimension of the fibre.

1.2.3 The rules used for composing our list of families of connections

The requirements concern the formal data at $z = \infty$ (and also at $z = 0$ if this point irregular singular).

R1. The (distinct) eigenvalues $q_1, \ldots, q_r$ with multiplicity $m_1, \ldots, m_r$ satisfy: all $q_j \neq 0$, $\sum m_j q_j = 0$ and (in case $e > 1$) invariance under the Galois group of $\overline{\mathbb{C}((1/z))}$ over $\mathbb{C}((1/z))$. Further, one requires that the formal monodromy γ is "generic", meaning that it has n distinct eigenvalues.

R2. If $z = 0$ is regular, then the formal data are normalized by the action of the group $\{z \mapsto az+b\}$. If $z = 0$ is singular, the formal data are normalized using the group $\{z \mapsto az\}$. For the description of all formal data at $z = \infty$ and at $z = 0$ (if this point is also irregular singular) only one variable t is needed. This is the translation of the requirement that the fibres of $\mathcal{M} \to \mathcal{R}$ are locally parametrized by a single t, called *the time variable*.

R3. The data should not define a subfamily of a family with more "time variables". However, in Section 14, we will consider a "companion of P_1", which is a subfamily of an interesting "two time variables family".

R4. Apart from individual families, there is interest in hierarchies. By the latter we mean a sequence of families defined by certain properties of the eigenvalues. For example, $z = 0$ regular singular; $e = 1$, $\kappa = 1$ defines the hierarchy given by the eigenvalues and multiplicities $(z)_{m_1}, (tz)_{m_2}, \left(-\frac{m_1+tm_2}{m_3}z\right)_{m_3}$ for $m_1, m_2, m_3 \geq 1$.

In Section 2, a complete list for the cases with multiplicities 1 is presented. This includes of course the classical Painlevé equations with at most two singular points. This list extends in an obvious way to a complete list of hierarchies by allowing multiplicities.

In the next sections, the cases of the list which are not classical, are studied in more detail.

1.2.4   Details on the definition and construction of the space $\mathcal{M}$

(i) *Case $z = 0$ is regular singular*. We start by assuming that the irregular singularity $z = \infty$ is unramified and is given by data $(V, \{V_q\}, \gamma)$.

We choose a basis of V, consisting of eigenvectors of γ, and consider matrices with respect to this basis. From this, we choose a standard differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + S$ where S is a diagonal matrix with diagonal entries $(Q_1 + a_1, \ldots, Q_n + a_n)$. The $Q_1, \ldots, Q_n$ are the eigenvalues $q_1, \ldots, q_r$, repeated according to their multiplicities. Thus $\sum Q_j = 0$. The $a_1, \ldots, a_n \in \mathbb{C}$ satisfy $\sum a_j = 0$ and are chosen such that the monodromy of the operator $z\frac{\mathrm{d}}{\mathrm{d}z} + \mathrm{diag}(a_1, \ldots, a_n)$ equals γ.

Now we follow [24, Section 12] and consider the fine moduli space defined by the objects (∇, ϕ) on $\mathbb{P}^1$ with the data:

  (a) ∇ is a connection on a trivial vector bundle of rank n on $\mathbb{P}^1$,

  (b) such that $z = 0$ is regular singular, and

  (c) ϕ is a formal isomorphism of ∇ at $z = \infty$ with $z\frac{\mathrm{d}}{\mathrm{d}z} + S$.

The explicit choice ϕ guarantees that the objects have no automorphisms and that a fine moduli space $\mathcal{U}$ (i.e., a universal family) exists.

According to [24, Corollary 12.15 and its proof], the universal family of this fine moduli space is represented by the family of differential operators $\mathrm{Pr}\big(g\big(z\frac{\mathrm{d}}{\mathrm{d}z} + S\big)g^{-1}\big)$ where g runs in the N-dimensional affine space

$$\left\{ \mathbf{1}_V + \sum_{k \neq \ell} \mathrm{Hom}(V_k, V_\ell) \otimes_{\mathbb{C}} \big(\mathbb{C}z^{-1} + \cdots + \mathbb{C}z^{-\deg(q_k - q_\ell)} \big) \right\},$$

seen as subset of the group $\mathrm{SL}_n\big(R[[z^{-1}]]\big)$ with R the polynomial ring $\mathbb{C}\big[\sum_{k \neq \ell} \mathrm{Hom}(V_k, V_\ell)\big]$. The notation Pr denotes "principal part" and is defined here as

$$\mathrm{Pr}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{k \ll \infty} A_k z^k \right) = z\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{0 \leq k \ll \infty} A_k z^k.$$

This ends the construction of the universal family $\mathcal{U}$ for the case $z = 0$ regular singular and $z = \infty$ is unramified. The group of the automorphisms G of the formal operator $z\frac{\mathrm{d}}{\mathrm{d}z} + S$ consists of the diagonal matrices with determinant 1, commuting with S. This group acts on $\mathcal{U}$ and $\mathcal{M}$ is obtained by dividing $\mathcal{U}$ by the action of G. In other words, $\mathcal{M}$ is obtained from $\mathcal{U}$ by "forgetting ϕ".

In general, this categorical quotient has singularities and, moreover need not be the quotient for the set of closed points. In practise, we will consider a dense affine subspace of $\mathcal{M}$, obtained as closed subspace of $\mathcal{U}$ by *normalizing* suitable variables to 1 and so providing representatives for the G-action.

We note in passing that the above describes a (co-adjoint) orbit of a linear algebraic group over $\mathbb{C}$. Therefore, $\mathcal{M}$ has a natural symplectic structure, see also [1].

For the ramified case, one considers the cyclic covering of $\mathbb{P}^1$ of degree e, ramified over 0 and ∞. With respect to the variable $z^{1/e}$, one computes the universal family $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ as above, restricted by the condition of invariance under σ. The next step is a computation of the operator

on a σ-invariant basis (compare [24, Section 12.5])) and, finally, dividing by the action of the group G of automorphisms of $z\frac{\mathrm{d}}{\mathrm{d}z} + S$ (by normalizing suitable variables to 1).

(ii) *The case $z = 0$ regular.* From the data $(V, \{V_q\}, \gamma)$ one first computes, as above in (i), a normalized universal family of matrix differential operators $z\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{0 \leq k \ll \infty} A_k z^k$. We propose for $\mathcal{M}$ the subfamily defined by the condition that all the entries of A_0 are zero. An explicit computation is needed to verify whether $\mathcal{M}$ is not empty and to compute its dimension.

(iii) *The case $z = 0$ and $z = \infty$ irregular singular.* One expects a universal family of differential operators

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \sum_{-\infty \ll k \ll \infty} A_k z^k.$$

For the right-hand part $\sum_{0 \leq k \ll \infty} A_k z^k$, the method of (i) produces a proposal. The same holds for the left-hand part $\sum_{-\infty \ll k \leq 0} A_k z^k$. Gluing of the two proposals may result in a suitable family. A priori, it is not clear whether the formal data at $z = 0$ and at $z = \infty$ can be combined to a family $\mathcal{M}$ and a corresponding monodromy space $\mathcal{R}$.

Comments. The explicit computation of $\mathcal{R}$ works quite well. The computation of $\mathcal{M}$ in cases (i) or (ii) may fail or may lead to a result unsuitable for further analysis, due to complexity. For case (iii), one needs a good guess to start the computation.

The number of cases where a complete computation of the Lax pairs can be given is, again due to complexity, rather small. In case (i), the differential equations involve

$$N = \sum_{k,\ell} \deg(q_k - q_\ell) \cdot \dim V_{q_k} \cdot \dim V_{q_\ell}$$

functions of t. This system is mostly too large.

A first step towards simplification is normalization by scaling the basis vectors of V (i.e., the step from $\mathcal{U}$ to $\mathcal{M}$). A next step is to reduce this system by the use of invariants, which are independent of t (i.e., considering fibres of $\mathcal{M} \to \mathcal{P}^+$ and $\mathcal{R} \to \mathcal{P}$). This reduces the number of functions of t involved in the Lax pair equations but can be a source of complexity.

2   List of all cases with one time variable

The computation of this list is straightforward, but somewhat long.

With $z = 0$ regular singular or regular. The condition "one time variable t" implies that there are at most three Galois orbits of eigenvalues. A regular singular case can restrict to a regular case, e.g., z, tz, $(-1-t)z$ and $z = 0$ regular exists and produces trivial Stokes data. In the table representatives for the Galois orbits of the eigenvalues are given; the rightmost column indicates in which section this example is discussed and which classical Painlevé equation P_j it is related to. As in Section 1.1, the Flaschka–Newell equation which is a deformation of P_2, is denoted P_{2fn} (see [23, Section 4.9] for this case).

- One Galois orbit, $z = 0$ regular singular,

$$z^{3/2} + tz^{1/2}, \qquad\qquad (P_{2fn})$$
$$z^{2/e} + tz^{1/e} \quad \text{for} \quad e \geq 3. \qquad (\text{Section } 3, P_4, P_5)$$

- One Galois orbit, $z = 0$ regular,

$$z^{5/2} + tz^{1/2}, \qquad (P_1)$$
$$z^{4/3} + tz^{2/3}. \qquad (\text{Section } 4)$$

- Two Galois orbits, $z = 0$ regular singular,

$$z^2 + tz, \ -(z^2 + tz), \qquad\qquad (P_4)$$
$$z + tz^{1/2}, \ -2z. \qquad\qquad\qquad (\text{Section } 5)$$
$$z^{1/e_1}, tz^{1/e_2} \quad \text{for} \quad e_1 \geq e_2 \geq 2. \qquad (\text{Section } 6, \ P_6)$$

- Two Galois orbits, $z = 0$ regular,

$$z^3 + tz, \ -(z^3 + tz). \qquad (\text{Section } 7, \ P_2)$$

- Three Galois orbits, $z = 0$ regular singular,

$$z, \ tz, \ (-1 - t)z, \qquad\qquad (\text{Section } 8, \ P_6)$$
$$z^{1/e}, \ tz, \ -tz \quad \text{with} \quad e > 1. \qquad (\text{Section } 9)$$

- Three Galois orbits, $z = 0$ regular,

$$z^2, \ -z^2 - tz, \ tz. \qquad (\text{Section } 10, \ P_4)$$

With both $z = 0$ and $z = \infty$ irregular singular.

- $1/z, -1/z$ at $z = 0$ and $tz, -tz$ at $z = \infty$ (Section 11, $P_3(D_6)$).
- $z^{-1/2}$ at $z = 0$ and $tz, -tz$ at $z = \infty$ (Section 12, $P_3(D_7)$).
- $z^{-1/n}$ at $z = 0$ and $tz^{1/n}$ at $z = \infty$ with $n \geq 2$ (Section 13, $P_3(D_8)$)

More general: $n_1, n_2 \geq 2$ and z^{-1/n_1} at $z = 0$ and tz^{1/n_2} at $z = \infty$ (with suitable multiplicities).

We discuss these cases in the indicated sections.

3 $\ z^{2/n} + tz^{1/n}$, $n \geq 3$, hierarchy of Noumi and Yamada

We study the structure of the moduli spaces $\mathcal{M}_n$, $\mathcal{R}_n$ and the Lax pair computations, separately for n odd and n even.

3.1 The moduli spaces $\mathcal{M}_n$ and $\mathcal{R}_n$ for odd n

Computations for $\mathcal{R}_n$. A module $M \in \mathcal{M}_n$ has the eigenvalues $q_j = \sigma^j(q_0) = \omega^{2j} z^{2/n} + t\omega^j z^{1/n}$ for $j = 0, \ldots, n - 1$ at $z = \infty$, where $\omega := \mathrm{e}^{2\pi \mathrm{i}/n}$.

The tuple $(V, \{V_q\}, \gamma, \{\mathrm{St}_d\})$ that classifies M at $z = \infty$ has the form $V = \mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{n-1}$ where $\mathbb{C}e_j = V_{q_j}$ for $j = 0, \ldots, n - 1$. This basis is chosen such that γ satisfies $e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$.

The space of the Stokes matrices at $z = \infty$ is isomorphic to $\mathbb{C}^N$, where $N = n(n - 1) \cdot \frac{2}{n} = 2(n - 1)$. Since the basis vectors $e_0, \ldots, e_{n-1}$ of V are unique up to multiplication by the same constant, one finds $\mathcal{R}_n = \mathbb{C}^{2(n-1)}$ (see Lemma 1.2) and $\dim \mathcal{M}_n = 1 + \dim \mathcal{R}_n = 1 + 2(n - 1)$.

For a module $M \in \mathcal{M}_n$, the data of the topological monodromy mon_0 at $z = 0$ is the conjugacy class of a matrix in SL_n. A conjugacy class is mapped to its characteristic polynomial $T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + (-1)^n$ and the parameter space $\mathcal{P}_n$ is the space of all possible characteristic polynomials and thus isomorphic to $\mathbb{C}^{n-1}$. The "monodromy identity" and a nontrivial, explicit computation, similar to the ones in [3], shows:

$\mathcal{R}_n \to \mathcal{P}_n$ *is surjective and the fibres have dimension* $(n-1)$. The fibre for $n = 3$ is computed in [25] to be the affine surface $xyz + x^2 + p_1 x + p_2 y + p_3 z = p_4$ for certain constants p_i. This is expected because the computation of the Lax pair equation leads to an identification with P_4. A further computation shows that $\mathrm{mon}_0 = \mathbf{1}$ is not possible, although characteristic polynomial $(T-1)^3$ is possible.

For $n = 5$, we present details of the computation the fibre. Write $\omega = \mathrm{e}^{2\pi \mathrm{i}/5}$, $q_0 = z^{2/5} + t z^{1/5}$, $q_1 = \omega^2 z^{2/5} + \omega t z^{1/5}, \ldots, q_4 = \omega^3 z^{2/5} + \omega^4 t z^{1/5}$. The singular directions in $[0,1)$ are $7/8$ for $q_0 - q_2$, $q_3 - q_4$, $5/8$ for $q_1 - q_4$, $q_3 - q_2$, $3/8$ for $q_1 - q_2$, $q_3 - q_0$ and $1/8$ for $q_1 - q_0$, $q_4 - q_2$. The fibres of $\mathcal{R}_5 \to \mathcal{P}_5$ are rational 4-folds. After eliminating 3 of the 8 variables for $\mathcal{R}_5$, the affine fibre is given by a degree 5 polynomial equation in 5 variables and with 4 parameters. This computation also shows that $\mathrm{mon}_0 = \mathbf{1}$ is not possible for $n = 5$.

Construction of $\mathcal{M}_n$ *and the Lax pairs.* We make the method explained in Section 1 explicit. A differential module $M \in \mathcal{M}_n$ over $\mathbb{C}(z)$ is replaced by $N := \mathbb{C}(z^{1/n}) \otimes M$. Let D denote the differential operator $\nabla_{z \frac{\mathrm{d}}{\mathrm{d}z}}$ on M. Now D extends uniquely to a differential operator, also called D, on N. This D commutes with the semi-linear automorphism $\sigma \colon N \to N$, induced by the automorphism σ of $\mathbb{C}(z^{1/n})$, given by $\sigma z^{1/n} = \omega z^{1/n}$. Thus the $M \in \mathcal{M}$ are replaced by pairs (N, σ), as above.

Let $e_0, \ldots, e_{n-1}$ be a basis of N over $\mathbb{C}(z^{1/n})$ such that the map σ satisfies $\sigma \colon e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto e_0$. The operator D is determined by $D(e_0)$. The formula

$$D(e_0) = \left(z^{2/n} + t z^{1/n}\right) e_0 + \sum_{i=1}^{n-1} \left(a_i + b_i z^{1/n}\right) e_i$$

is supported by [24, Sections 12.3–12.5] (compare Section 1). For the operator $E := \frac{\mathrm{d}}{\mathrm{d}t} + B$ such that $\{D, E\}$ forms a Lax pair, one can verify the assumption that E is the σ-invariant operator with $E(e_0) = z^{1/n} e_0 + \sum_{j=1}^{n-1} c_j e_j$.

One deduces from this the matrix of D with respect to the basis $B_0, \ldots, B_{n-1}$ of $M := N^{\langle \sigma \rangle}$, where $B_j := \sum_{k=0}^{n-1} \sigma^k \left(z^{j/n} e_0\right)$ for $0 \le j \le n-1$ and $B_n := z B_0$, $B_{n+1} := z B_1$. The formula is

$$D(B_j) = \frac{j}{n} B_j + \sum_{i=1}^{n-1} a_i \omega^{-ij} B_j + t B_{j+1} + \sum_{i=1}^{n-1} b_i \omega^{-i(j+1)} B_{j+1} + B_{j+2}.$$

The formula for E on this basis is

$$E(B_j) = B_{j+1} + \left(\sum_{k=1}^{n-1} \omega^{-kj} c_k\right) B_j.$$

Put $\epsilon_j = \frac{j}{n} + \sum_{i=1}^{n-1} a_i \omega^{-ij}$ and $f_j = t + \sum_{i=1}^{n-1} b_i \omega^{-ij}$. The operator D is

$$z \frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} \epsilon_0 & 0 & 0 & * & * & z & z f_0 \\ f_1 & \epsilon_1 & 0 & 0 & * & 0 & z \\ 1 & f_2 & \epsilon_2 & 0 & * & * & 0 \\ 0 & 1 & f_3 & \epsilon_3 & 0 & * & * \\ * & * & * & * & * & * & * \\ * & * & * & 1 & f_{n-2} & \epsilon_{n-2} & 0 \\ * & * & * & 0 & 1 & f_{n-1} & \epsilon_{n-1} \end{pmatrix},$$

note that $\sum \epsilon_j = \frac{n-1}{2}$ and $\sum f_j = nt$. The $\epsilon_0, \ldots, \epsilon_{n-1}$ are the parameters of the family. The $\left\{\mathrm{e}^{2\pi \mathrm{i} \epsilon_j}\right\}$ are the eigenvalues of the topological monodromy at $z = 0$. These can be seen as parameters for $\mathcal{R}_n$. For an isomonodromic family the ϵ_j are constant and the $f_0, \ldots, f_{n-1}$ are analytic functions of the parameter t.

The operator E reads on the above basis

$$\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} g_0 & 0 & 0 & 0 & * & * & z \\ 1 & g_1 & 0 & 0 & * & * & 0 \\ 0 & 1 & g_2 & 0 & * & * & 0 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & g_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & g_{n-1} \end{pmatrix}$$

with $g_j = \left(\sum_{k=1}^{n-1} \omega^{-kj} c_k\right)$ and $\sum g_j = 0$. For an isomonodromic family, the $\{g_j\}$ are functions of t and are in fact eliminated by the Lax pair condition $DE = ED$.

For $n = 5$, the Painlevé type differential system for this Lax pair is

$$f_1' = f_1(-f_1 - 2f_2 - 2f_4 + t) + 2\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4,$$
$$f_2' = f_2(-2f_1 + f_2 - 2f_4 - t) - \epsilon_1 + \epsilon_2, \qquad f_3' = f_3(-2f_1 - f_3 - 2f_4 + t) - \epsilon_2 + \epsilon_3,$$
$$f_4' = f_4(2f_1 + 2f_3 + f_4 - t) - \epsilon_3 + \epsilon_4.$$

For $n = 7$, one has $\sum f_j = 7t$, $\sum \epsilon_j = 3$ and

$$f_0' = f_0(-f_1 + f_2 - f_3 + f_4 - f_5 + f_6) + \epsilon_0 - \epsilon_6 + 1,$$
$$f_1' = f_1(f_0 - f_2 + f_3 - f_4 + f_5 - f_6) - \epsilon_0 + \epsilon_1,$$
$$f_2' = f_2(-f_0 + f_1 - f_3 + f_4 - f_5 + f_6) - \epsilon_1 + \epsilon_2,$$
$$f_3' = f_3(f_0 - f_1 + f_2 - f_4 + f_5 - f_6) - \epsilon_2 + \epsilon_3,$$
$$f_4' = f_4(-f_0 + f_1 - f_2 + f_3 - f_5 + f_6) - \epsilon_3 + \epsilon_4,$$
$$f_5' = f_5(f_0 - f_1 + f_2 - f_4 + f_4 - f_6) - \epsilon_4 + \epsilon_5,$$
$$f_6' = f_6(-f_0 + f_1 - f_2 + f_3 - f_4 + f_5) - \epsilon_5 + \epsilon_6.$$

The general case for odd n is similar.

Observation. Apart from small changes the above is the symmetric Lax pair introduced by Noumi, Yamada et al. (see [20, 26]). The changes are

(a) ϵ_j is changed into $\epsilon_j - \frac{n-1}{2n}$ in order to obtain a matrix with trace zero. This corresponds to a small change in the definition of D, namely

$$D(e_0) = \left(z^{2/n} + tz^{1/n} - \frac{n-1}{2n}\right)e_0 + \sum_{i=1}^{n-1}(a_i + b_i z^{1/n})e_i.$$

(b) A notational change of t into $\frac{t}{n}$.

(c) Transposing the matrix. This is due to the relation between a covariant solution space and a contravariant solution space.

An alternative method for $n = 3$, i.e., the Noumi–Yamada form for P_4. The Lax pair equations are equivalent to $ED(e_0) = DE(e_0)$. The normalized operator D given as

$$De_0 = \left(z^{2/3} + tz^{1/3} + \frac{2}{3}\right)e_0 + (a_1 + b_1 z^{1/3})e_1 + \left(a_2 + b_2 z^{1/3}\right)e_2,$$

where a_1, a_2 are constants and b_1, b_2 are functions of t, commutes with the operator E such that $E(e_0) = z^{1/3}e_0 + c_1 e_1 + c_2 e_2$ (for suitable functions c_1, c_2 of t) if and only b_1, b_2 satisfy the differential equations

$$b_1' = a_1(1 - \omega) + b_1 t(2\omega + 1) + b_2^2(-2\omega - 1),$$
$$b_2' = a_2(\omega + 2) + b_1^2(2\omega + 1) + b_2 t(-2\omega - 1).$$

This is in fact a Hamiltonian system $b_1' = \frac{\partial H}{\partial b_2}$, $b_2' = -\frac{\partial H}{\partial b_1}$ with $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ and

$$H = -\left(\frac{b_1^3}{3} + \frac{b_2^3}{3}\right)(2\omega + 1) + b_1 b_2 t(2\omega + 1) - b_1 a_2(\omega + 2) - b_2 a_1(\omega - 1).$$

After a linear change of variables this Hamiltonian coincides with Okamoto's standard Hamiltonian for P_4 (see [22, p. 265]).

3.2 The moduli spaces $\mathcal{M}_n$ and $\mathcal{R}_n$ for even n

We proceed as in Section 3.1. Write $n = 2m$ and $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$. The eigenvalues at $z = \infty$ are $q_j = \omega^{j/m} z^{1/m} + \omega^{j/2m} t z^{1/2m}$ for $j = 0, \ldots, 2m - 1$.

Now $N := \sum_{i \neq j} \deg(q_i - q_j) = 4m - 3$, $\mathcal{R}_{2m} \cong \mathbb{C}^N$ and $\mathcal{M}_{2m}$ has dimension $1 + 4m - 3$. The parameter space $\mathcal{P}_{2m}$ for the monodromy space consists of the characteristic polynomials of the monodromy at $z = 0$. Since $\Lambda^{2m}M$ is the trivial differential module, this monodromy has determinant 1. Thus $\dim \mathcal{P}_{2m} = 2m - 1$. A similar explicit computation as the one mentioned in Section 3.1 shows: $\mathcal{R}_{2m} \to \mathcal{P}_{2m}$ *is surjective and the fibres have dimension* $2m - 2$. We make the method of construction a differential operator, a Lax pair and Painlevé type equations explicit for $n = 4$. The general case is discussed after that.

3.2.1 The case $n = 4$

Following Section 1, we may assume that the differential operator D has on the basis e_0, e_1, e_2, e_3 the formula

$$D(e_0) = \left(z^{1/2} + \frac{t}{4}z^{1/4} - 3/8\right)e_0 + \left(a_1 + b_1 z^{1/4}\right)e_1 + a_2 e_2 + \left(a_3 + b_3 z^{1/4}\right)e_3,$$

and D commutes with σ defined by $\sigma e_j = e_{j+1}$ for $j = 0, 1, 2$ and $\sigma e_3 = e_0$ and $\sigma z^{1/4} = \mathrm{i} z^{1/4}$. Consider the following basis of invariants:

$$B_0 = e_0 + e_1 + e_2 + e_3, \qquad\qquad B_1 = z^{1/4}\left(e_0 + \mathrm{i}e_1 + \mathrm{i}^2 e_2 + \mathrm{i}^3 e_3\right),$$
$$B_2 = z^{1/2}(e_0 - e_1 + e_2 - e_3), \qquad B_3 = z^{3/4}(e_0 - \mathrm{i}e_1 - e_2 + \mathrm{i}e_3).$$

The matrix of D with respect to this basis is

$$\begin{pmatrix} -\frac{3}{8} + a_1 + a_2 + a_3 & 0 & z & z\left(\frac{t}{4} + b_1 + b_3\right) \\ \frac{t}{4} - \mathrm{i}b_1 + \mathrm{i}b_3 & -\frac{1}{8} - \mathrm{i}a_1 - a_2 + \mathrm{i}a_3 & 0 & z \\ 1 & \frac{t}{4} - b_1 - b_3 & \frac{1}{8} - a_1 + a_2 - a_3 & 0 \\ 0 & 1 & \frac{t}{4} + \mathrm{i}b_1 - \mathrm{i}b_3 & \frac{3}{8} + \mathrm{i}a_1 - a_2 - \mathrm{i}a_3 \end{pmatrix}$$

and D is equal to the differential operator

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} \epsilon_0 & 0 & z & zf_0 \\ f_1 & \epsilon_1 & 0 & z \\ 1 & f_2 & \epsilon_2 & 0 \\ 0 & 1 & f_3 & \epsilon_3 \end{pmatrix}$$

with $\sum \epsilon_j = 0$, $f_0 + f_2 = f_1 + f_3 = \frac{t}{2}$. The $\epsilon_0, \ldots, \epsilon_3$ are parameters.

The operator D is completed to a Lax pair by the differential operator E with respect to $\frac{\mathrm{d}}{\mathrm{d}t}$. This operator, written on the basis e_0, e_1, e_2, e_3 is σ-invariant and has the form $E(e_0) = z^{1/4}e_0 + \sum_{j=1}^{3} h_j e_j$ for suitable functions h_1, h_2, h_3 of t. On the basis B_0, B_1, B_2 one obtains

$$E := \frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} g_0 & 0 & 0 & z \\ 1 & g_1 & 0 & 0 \\ 0 & 1 & g_2 & 0 \\ 0 & 0 & 1 & g_3 \end{pmatrix}$$

with $\sum g_j = 0$. The assumption that E commutes with D produces equations for the derivatives of f_0, f_1, f_2, f_3, seen as functions of t. These formulas are similar to those derived by Noumi–Yamada. Moreover, combining the differential equations for f_0 and f_1 leads to the standard P_5 equation, see [20, 26] for details.

An alternative computation is a consequence of the observation that isomonodromy is given by $DE(e_0) = ED(e_0)$ and $a_1, a_2, a_3 \in \mathbb{C}$. This produces equations with parameters a_1, a_2, a_3

$$4t \cdot \frac{db_1}{dt} = -16ib_1^2 b_3 + t^2 b_1 i + 16ib_3^3 - 4ia_1 t + 4a_1 t - 32a_2 b_3$$

$$4t \cdot \frac{db_3}{dt} = -16ib_1^3 + 16ib_1 b_3^2 - t^2 b_3 i + 4ia_3 t - 32a_2 b_1 + 4a_3 t,$$

$$h_1 = -ib_1/2 + b_1/2, \qquad t \cdot h_2 = 2b_1^2 i - 2b_3^2 i + 4a_2, \qquad h_3 = b_3 i/2 + b_3/2.$$

One observes that the equations for b_1, b_3 form a Hamiltonian system with

$$t\frac{db_1}{dt} = \frac{\partial H}{\partial b_3}, \qquad t\frac{db_3}{dt} = -\frac{\partial H}{\partial b_1},$$

$$H = \frac{i(b_1^2 - b_3^2)^2}{t} + \frac{itb_1 b_3}{4} + \frac{4a_2(b_1^2 - b_3^2)}{t} - (1+i)a_3 b_1 + (1-i)a_1 b_3.$$

Comments. The above Hamiltonian H and the differential equations for b_1, b_3 coincide, after a linear change of variables, with Okamoto's standard polynomial Hamiltonian for P_5, see [22, p. 265].

The fibers of $\mathcal{R}_4 \to \mathcal{P}_4$. The eigenvalues at $z = \infty$ are: $q_0 = z^{1/2} + tz^{1/4}$, $q_1 = -z^{1/2} + itz^{1/4}$, $q_2 = z^{1/2} - tz^{1/4}$, $q_3 = -z^{1/2} - itz^{1/4}$. The differences $q_0 - q_1$, $q_0 - q_3$, $q_2 - q_1$, $q_2 - q_3$ have the form $2z^{1/2} + \cdots$ and further $q_0 - q_2 = 2tz^{1/4}$ and $q_1 - q_3 = 2itz^{1/4}$. There is one singular direction in $[0,1)$ for the terms $\pm 2z^{1/2}$. For the terms $\pm 2tz^{1/4}$, $\pm 2itz^{1/4}$ there is only one singular direction in $[0,1)$. For a suitable choice of t, this leads to the monodromy identity

$$\mathrm{mon}_\infty = \begin{pmatrix} & & -1 \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & y \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ x_1 & 1 & x_2 \\ & & 1 \\ x_3 & & x_4 & 1 \end{pmatrix}.$$

One observes that $\mathrm{mon}_0 = \mathrm{mon}_\infty^{-1}$ cannot be the identity. Thus for the data considered here, $z = \infty$ singular with eigenvalues $z^{1/2} + tz^{1/4}$ and its conjugates and $z = 0$ *regular* cannot be combined.

The space $\mathcal{P}_4$ is parametrized by p_1, p_2, p_3, where the characteristic polynomial of mon_∞ is written as $T^4 + p_3 T^3 + p_2 T^2 + p_1 T + 1$. For a suitable choice of elimination of two variables (e.g., x_1, x_2), the fibres are described by a cubic equation in three variables y, x_3, x_4 and parameters p_1, p_2, p_3. The equation reads $v_1 v_2 v_3 + *v_1^2 + *v_2^2 + *v_1 + *v_2 + *v_3 + * = 0$ for suitable affine expressions $*$'s in the parameters p_1, p_2, p_3. This is expected, since the related Painlevé equation P_5 has the same cubic equation for its monodromy.

3.2.2 The general case with $n = 2m$

Consider the $\mathbb{C}(t)\left[z^{1/2m}\right]$-lattice with basis $e_0, \ldots, e_{2m-1}$, provided with the action of σ given by the formulas: $\sigma z^\lambda = e^{2\pi i\lambda} z^\lambda$ and $\sigma e_j = e_{j+1}$ for $j = 0, \ldots, 2m-2$ and $\sigma e_{2m-1} = e_0$. The operator D (with respect to the derivation $z\frac{d}{dz}$) representing $\mathcal{M}_{2m}$, is σ-invariant and is given by

$$D(e_0) = \left(z^{1/m} + \frac{t}{2m} z^{1/2m} - \frac{2m-1}{4m}\right) e_0 + \sum_{j=1}^{2m-1} (a_j + b_j z^{1/2m}) e_j,$$

with varying $a_j, b_j \in \mathbb{C}$ and where $b_m = 0$. On the basis $B_0, \ldots, B_{2m-1}$ of invariants, D has the form $z\frac{\mathrm{d}}{\mathrm{d}z} + A_0 + zA_1$. We make this explicit for $n = 6$, $m = 3$. The general $n = 2m$ case is similar,

$$D = z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} \epsilon_0 & 0 & 0 & 0 & z & z\left(\frac{t}{6} + f_0\right) \\ \frac{t}{6} + f_1 & \epsilon_1 & 0 & 0 & 0 & z \\ 1 & \frac{t}{6} + f_2 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 1 & \frac{t}{6} + f_3 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 1 & \frac{t}{6} + f_4 & \epsilon_4 & 0 \\ 0 & 0 & 0 & 1 & \frac{t}{6} + f_5 & \epsilon_5 \end{pmatrix}.$$

The ϵ_j are linear combinations of $a_1, \ldots, a_5$ satisfying $\sum \epsilon_j = 0$, and the $f_0, \ldots, f_5$ are linear combinations of b_1, b_2, b_4, b_5 such that the relations $f_0 + f_2 + f_4 = f_1 + f_3 + f_5 = 0$ hold.

One observes that the data of the eigenvalues of A_0 are equivalent to $a_1, \ldots, a_{2m-1}$. Thus for an isomonodromic family the a_j are constant and the $b_1, \ldots, b_{2m-1}$ (with the condition $b_m = 0$) are functions of t. The differential equations for the b_j are derived from a Lax pair $z\frac{\mathrm{d}}{\mathrm{d}z} + A_0 + zA_1$, $\frac{\mathrm{d}}{\mathrm{d}t} + B$ with an, a priori, unknown matrix B depending on t and z. The action of the operator $\frac{\mathrm{d}}{\mathrm{d}t} + B$ on the $\mathbb{C}(t)[z^{1/2m}]$-lattice with basis $e_0, \ldots, e_{2m-1}$ is called E. It is σ-invariant and one can prove that E is given by

$$E(e_0) = z^{1/2m}e_0 + \sum_{j=1}^{2m-1} h_j e_j \qquad \text{for certain functions } h_1, \ldots, h_{2m-1} \text{ of } t.$$

We make this explicit for $n = 6$, $m = 3$ (again, the general case $n = 2m$ is similar),

$$E = \frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} g_0 & 0 & 0 & 0 & 0 & z \\ 1 & g_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & g_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & g_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & g_5 \end{pmatrix} \qquad \text{and} \qquad \sum g_j = 0.$$

The Painlevé type equations are similar to those in Section 3.2.1.

An alternative computation. From the equation $DE(e_0) = ED(e_0)$ and all a_j are constants, the differential equations for $b_1, \ldots, b_{2m-1}$ follow. For the case $n = 6$, $m = 3$, one obtains the following Painlevé type equations for b_1, b_2, b_4, b_5 ($b_3 = 0$, $w = \mathrm{e}^{2\pi \mathrm{i}/6}$)

$$-18t\frac{\mathrm{d}b_1}{\mathrm{d}t} = 288b_1b_2b_4w - 2b_1t^2w - 24b_2b_5tw - 288b_4^2b_5w + 18a_1tw$$
$$- 144b_1b_2b_4 + b_1t^2 + 12b_2b_5t + 144b_4^2b_5 - 18a_1t + 216a_3b_4,$$

$$-6t\frac{\mathrm{d}b_2}{\mathrm{d}t} = -4b_1^2tw + 96b_1b_2b_5w - 2b_2t^2w + 12b_4^2tw - 96b_4b_5^2w + 6a_2tw$$
$$+ 2b_1^2t - 48b_1b_2b_5 + b_2t^2 - 6b_4^2t + 48b_4b_5^2 - 12a_2t + 72a_3b_5,$$

$$-6t\frac{\mathrm{d}b_4}{\mathrm{d}t} = 96b_1^2b_2w - 96b_1b_4b_5w - 12b_2^2tw + 2b_4t^2w + 4b_5^2tw - 6a_4tw$$
$$- 48b_1^2b_2 + 48b_1b_4b_5 + 6b_2^2t - b_4t^2 - 2b_5^2t + 72a_3b_1 - 6a_4t,$$

$$-18t\frac{\mathrm{d}b_5}{\mathrm{d}t} = 88b_1b_2^2w + 24b_1b_4tw - 288b_2b_4b_5w + 2b_5t^2w - 18a_5tw$$
$$- 144b_1b_2^2 - 12b_1b_4t + 144b_2b_4b_5 - b_5t^2 + 216a_3b_2.$$

There is a Hamiltonian function H such that

$$\frac{\mathrm{d}b_5}{\mathrm{d}t} = -\frac{\partial H}{\partial b_1}, \qquad \frac{\mathrm{d}b_1}{\mathrm{d}t} = \frac{\partial H}{\partial b_5}, \qquad \frac{\mathrm{d}b_4}{\mathrm{d}t} = -\frac{\partial H}{\partial b_2}, \qquad \frac{\mathrm{d}b_2}{\mathrm{d}t} = \frac{\partial H}{\partial b_4},$$

where H is defined by

$$
\begin{aligned}
3tH = {} & (w - 1/2)(b_2 b_4 + b_1 b_5/3)t^2 + \big((-2w+1)b_2^3 + \big((2b_5^2 - 3a_4)w - b_5^2 - 3a_4\big)b_2 \\
& + (1 - 2w)b_4^3 + \big((2b_1^2 - 3a_2)w - b_1^2 + 6a_2\big)b_4 - (3a_1 b_5 + 3a_5 b_1)w + 3a_1 b_5\big)t \\
& + 12(-b_1 b_2 + b_4 b_5)\big((-b_1(2w-1)b_2 + b_5(2w-1)b_4 - 3a_3\big),
\end{aligned}
$$

with $w = e^{2\pi i/6}$. This is related to the equations studied in [12].

4 $z^{4/3} + tz^{2/3}$ and regular $z = 0$

The $q_0 = z^{4/3} + tz^{2/3}$, $q_1 = \omega z^{4/3} + \omega^2 tz^{2/3}$, $q_2 = \omega^2 z^{4/3} + \omega tz^{2/3}$ with $\omega = e^{2\pi i/3}$ are the eigenvalues. First, we assume that $z = 0$ is a regular singular point. Then, the monodromy space $\mathcal{R}$ is isomorphic to $\mathbb{C}^8$. The monodromy mon at $z = \infty$ (or equivalently at $z = 0$) is a product of the formal monodromy and 8 Stokes matrices. The singular directions in $[0,1)$ are $\frac{15}{16}, \frac{13}{16}, \frac{11}{16}, \frac{9}{16}, \frac{7}{16}, \frac{5}{16}, \frac{3}{16}, \frac{1}{16}$ for $q_1 - q_2$, $q_1 - q_0$, $q_2 - q_0$, $q_2 - q_1$, $q_0 - q_1$, $q_0 - q_2$, $q_1 - q_2$, $q_1 - q_0$. Each Stokes matrix has one nontrivial entry and these are in the same order x_{12}, x_{10}, x_{20}, x_{21}, x_{01}, x_{02}, y_{12}, y_{10}. A computation shows that the map $\mathcal{R} \to \mathrm{SL}_3(\mathbb{C})$, which send the Stokes data to mon, is birational. Moreover, the preimage of $\mathbf{1} \in \mathrm{SL}_3(\mathbb{C})$ is one point, namely $x_{01} = -1$, $x_{02} = 1$, $x_{10} = 1$, $x_{12} = -1$, $x_{20} = -1$, $x_{21} = 1$, $y_{10} = 1$, $y_{12} = -1$.

The rather curious conclusion is that the monodromy space, for the case that $z = 0$ is regular, consists of a single point.

The formal matrix differential operator is

$$
z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} -\frac{1}{3} & tz & z^2 \\ z & 0 & tz \\ t & z & \frac{1}{3} \end{pmatrix}.
$$

The *guess* that $\mathcal{M}$ is represented by the family of operators of the form

$$
\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} 0 & \frac{3t}{2} & z \\ 1 & 0 & \frac{3t}{2} \\ 0 & 1 & 0 \end{pmatrix}
$$

is *confirmed* by a Lax pair computation. The corresponding scalar differential equation is $y^{(3)} - 3ty^{(1)} - zy = 0$.

One concludes that the Stokes matrices, which are nontrivial, in this family do not depend on t.

5 $z + t^{1/2}z^{1/2}, z - t^{1/2}z^{1/2}, -2z$ and regular singular $z = 0$

The above formulas are the eigenvalues q_0, q_1, q_2 at $z = \infty$. For $t \in \mathbb{R}$, $t > 0$, the singular directions in $[0,1)$ are $1/2$ for $q_0 - q_2$, $q_1 - q_2$ and 0 for $q_1 - q_0$, $q_2 - q_0$, $q_2 - q_1$. The monodromy identity is

$$
\mathrm{mon} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_{02} & x_{12} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{10} & x_{20} \\ 0 & 1 & x_{21} \\ 0 & 0 & 1 \end{pmatrix}.
$$

This presentation is unique up to scaling the third basis vector. One has $\dim \mathcal{R} = 4$, $\dim \mathcal{P} = 2$ and the fibers are affine cubic surfaces with an equation of the form $x_1 x_2 x_3 + x_1^2 + *x_1 + *x_2 + *x_3 + * = 0$. The data do not match with $z = 0$ regular, since mon $= \mathbf{1}$ has no solution.

In order to simplify the computation, the eigenvalues at $z = \infty$ are replaced by $z^{1/2}$, $-z^{1/2}$, tz, obtained by shifting and scaling the given $z \pm t^{1/2} z^{1/2}$, $-2z$. This has no effect on monodromy and Lax pairs. The method of [24, Section 12.5], explained in Section 1, produces an explicit formula for the universal family with these data. It depends on 5 variables. By scaling the basis vectors, one of the variables is normalized to 1 and the resulting operator has the form $z \frac{\mathrm{d}}{\mathrm{d}z} + A$ with

$$A = \begin{pmatrix} -2a_1 & z & a_4 t - a_3 \\ 1 & 2a_1 + a_3 t & a_3 t \\ -(a_1 + 2a_8)t - 1 & tz & tz - a_3 t \end{pmatrix}.$$

The further computations are described as follows. The Lax pair formalism produces a set of differential equations $\frac{\mathrm{d}a_i}{\mathrm{d}t} = R_i$ for $i = 1, 3, 4, 8$ where the R_i are rational functions in a_1, a_3, a_4, a_8, t. The eigenvalues of the residue matrix of A at $z = 0$ are independent of t. One of the eigenvalues is $2a_1 + a_3 t$. Adding the equation $\frac{\mathrm{d}(2a_1 + a_3 t)}{\mathrm{d}t} = 0$ to the above system of equations eliminates a_4 and produces: $a_1 = c_1$, $a_3 = c_2/t$ with constants c_1, c_2 and a Riccati equation for a_8. In particular, *The Painlevé equation of this family is solvable by classical functions.*

We remark that the monodromy space for the case $\pm z^{1/2}$, tz produces again a 2-dimensional family of affine cubic surfaces of the type: $x_1 x_2 x_3 + x_1^2 + *x_1 + *x_2 + *x_3 + * = 0$.

However, in this case, there seems to be no relation with P_4.

6 z^{1/e_1}, tz^{1/e_2}, $e_1 \geq e_2 \geq 2$ and regular singular $z = 0$

For $e_1 > e_2 \geq 2$, one has $\dim \mathcal{R} = 3e_1 + e_2 - 3$ and $\dim \mathcal{P} = e_1 + e_2 - 1$. For the smallest case $e_1 = 3$, $e_2 = 2$, one has $\dim \mathcal{R} = 8$, $\dim \mathcal{P} = 4$, $\mathcal{R} \to \mathcal{P}$ is (generically) surjective and the fibres have dimension 4. The formulas for the fibres are complicated. This makes the computation of $\mathcal{M}$ and the Lax pair nearly impossible.

For $e_1 = e_2 = m \geq 2$, $n = 2m$, one has $\dim \mathcal{R} = \frac{n(n-1)}{m} - 1 = 2n - 3$, $\dim \mathcal{P} = n - 1$. For the smallest case $m = 2$, one has $\dim \mathcal{R} - \dim \mathcal{P} = 2$. The computation in Section 6.1 below produces a second-order Painlevé equation which is probably a pull back of the classical P_4 equation.

6.1 The case $z^{1/2}$, $tz^{1/2}$. The monodromy space $\mathcal{R}$

The eigenvectors are $q_1 = z^{1/2}$, $q_2 = -z^{1/2}$, $q_3 = tz^{1/2}$, $q_4 = -tz^{1/2}$ with a basis f_1, f_2, f_3, f_4 of eigenvectors is such that γ permutes the two pairs $\{f_1, f_2\}$ and $\{f_3, f_4\}$. There are 6 variables present in the Stokes matrices. Now f_1 and f_3 can be scaled independently and so $\dim \mathcal{R} = 6 - 1 = 5$. Further $\mathcal{P} \cong \mathbb{C}^3$ is the space of the characteristic polynomials $T^4 - p_3 T^3 + p_2 T^2 - p_1 T + 1$ of the elements of SL_4.

For the case $t = \mathrm{i}$, the singular directions in $[0, 1)$ are 0 for $q_2 - q_1$, $\frac{1}{4}$ for $q_2 - q_4$ and for $q_3 - q_1$, $\frac{1}{2}$ for $q_3 - q_4$, $\frac{3}{4}$ for $q_1 - q_4$ and for $q_3 - q_2$.

The topological monodromy mon, which has the above characteristic polynomial, is equal to the product

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & x_{14} \\ & 1 & x_{23} & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & x_{43} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_{13} & \\ & 1 & & \\ & & 1 & \\ x_{42} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The fibres of $\mathcal{R} \to \mathcal{P}$ have the following data. Assume that $x_{42} \neq 0$ (we note that $x_{42} = 0$ implies reducibility). After eliminating x_{12}, x_{23}, there remains one cubic equation in the variables x_{14}, x_{42}, x_{43}, x_{13}, namely $x_{14} x_{42} x_{43} - p_3 x_{43} + x_{13} x_{42} + x_{43}^2 + p_2 + 2 = 0$. After normalizing x_{13} to 1

(we note that $x_{13} = 0$ produces a similar cubic equation), the cubic equation is almost identical to the one for P_4 (see [23]) which is $x_1 x_2 x_3 + x_1^2 - (s_2^2 + s_1 s_3)x_1 - s_2^2 x_2 - s_2^2 x_3 + s_2^3 + s_1 s_2^3 = 0$.

Observation. The monodromy identity depends strongly on t. For instance, if $t \in \mathbb{R}_{>0}$, $t \neq 1$, then there is only one singular direction in $[0, 1)$. The fibres of $\mathcal{R} \to \mathcal{P}$ are again rational surfaces. The dependence of these surfaces on t is somewhat mysterious.

The space of connections $\mathcal{M}$. A differential module M over $\mathbb{C}(z)$ in this moduli space can be considered as a differential module N over $\mathbb{C}(z^{1/2})$ with an automorphism σ satisfying $\sigma \circ z^{1/2} = -z^{1/2} \circ \sigma$ and $\sigma^2 = 1$. Now N can be given a basis e_1, e_2, e_3, e_4 such that σ permutes the two pairs $\{e_1, e_2\}$ and $\{e_3, e_4\}$. The corresponding module M over $\mathbb{C}(z)$ has the basis

$$B_1 = e_1 + e_2, \qquad B_2 := z^{1/2}(e_1 - e_2), \qquad B_3 = e_3 + e_4, \qquad B_4 = z^{1/2}(e_3 - e_4).$$

Let D denote the operator of the form $z\frac{\mathrm{d}}{\mathrm{d}z} + (a\ matrix)$ acting upon N. This operator commutes with σ and is determined by De_1, De_3. The formal part $\hat{D}$ of D is given by $\hat{D}e_1 = z^{1/2}e_1$, $\hat{D}e_3 = tz^{1/2}e_3$. Using [24, Section 12], one concludes that D is given by the formulas

$$De_1 = z^{1/2}e_1 + a_1 e_2 + a_2 e_3 + a_3 e_4, \qquad De_3 = tz^{1/2}e_3 + a_4 e_1 + a_5 e_2 + a_6 e_4$$

with constants $a_1, \dots, a_6$. For the generic case, one can normalize to $a_5 = 1$. A computation of D on the basis $B_1, \dots, B_4$ produces the operator (normalized to trace equal to zero) $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ with

$$A = \begin{pmatrix} a_1 - 1/4 & z & a_4 + a_5 & 0 \\ 1 & -a_1 + 1/4 & 0 & a_4 - a_5 \\ a_2 + a_3 & 0 & a_6 - 1/4 & tz \\ 0 & a_2 - a_3 & t & -a_6 + 1/4 \end{pmatrix}.$$

As mentioned above, one may normalize to $a_5 = 1$. Further the coefficients of the characteristic polynomial of residue matrix at $z = 0$ are the parameters. We conclude that for fixed parameters, the above family of operators has dimension 2 (not counting the variable t). This is in agreement with the computation of the fibers of $\mathcal{R} \to \mathcal{P}$.

The operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ is extended to a Lax pair by an operator of the form $\frac{\mathrm{d}}{\mathrm{d}t} + B$ with $B = B_0 + B_1 z$ for matrices B_0, B_1 depending on t only. The property $\frac{\mathrm{d}}{\mathrm{d}t}(A) = z\frac{\mathrm{d}}{\mathrm{d}z}(B) + [A, B]$ yields a vector field, represented by differential equations $\frac{\mathrm{d}}{\mathrm{d}t}a_j = R_j$, $j = 1, 2, 3, 4, 6$ with R_j rational expressions in $a_1, \dots, a_4, a_6, t$. The characteristic polynomial of the residue matrix at $z = 0$ is written as $T^4 + P_2 T^2 + P_1 T + P_0$. We have used the formulas for P_1, P_2 and another invariant $P_0 = a_1 + a_6$ to eliminate (stepwise) the functions a_3, a_6, a_2. For the remaining a_1, a_4 one obtains the equations

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\left(2P_1 a_4^2 + \left(-4P_0^2 + (8a_1 + 2)P_0 - 8a_1^2 - 4P_2\right)a_4 + 2P_1\right)}{\left(a_4^2 - 1\right)\left(t^2 - 1\right)},$$

$$\frac{\mathrm{d}a_4}{\mathrm{d}t} = \frac{\left(\left(-2a_4^2 + 2\right)a_1 t^2 + 2\left(a_4^2 + 1\right)(P_0 - 2a_1)t + 2(a_4 - 1)(a_4 + 1)(-a_1 + P_0)\right)}{\left(t^3 - t\right)}.$$

The second equation can be used to write a_1 as an expression in a_4 and $\frac{\mathrm{d}a_4}{\mathrm{d}t}$. Substitution in the first equation yields an explicit (and rather long) second-order differential equation for a_4. The poles with respect to t are 0, 1, -1, ∞. The cubic form of the fibres of $\mathcal{R} \to \mathcal{P}$ suggested a relation with P_4.

Identifying Painlevé equations is in general an almost impossible task; the present manuscript focuses on constructing Painlevé-type equations rather than identifying them. However, Dzhamay [5] succeeded in using the geometry of Okamoto–Painlevé spaces and the algorithms of [6],

to identify the above equation with the standard Okamoto form $P_{\mathrm{VI}}(q(t), p(t), t; \kappa_0, \kappa_1, \kappa_8, \theta)$ of the sixth Painlevé equation, where

$$a_4(s) = (s \cdot q(t) - 1)/(s \cdot q(t) + 1), \qquad a_1(s) = 1/2(-2p(t) + P_0 - \kappa_1), \qquad t = 1/s^2,$$
$$P_0 = 1/2 - \kappa_1, \qquad P_1 = (\kappa_8^2 - \kappa_0^2)/4, \qquad P_2 = (2P_0 - 2P_0^2 - \kappa_0^2 - \kappa_8^2)/4.$$

7 $\left(z^3 + tz\right)_{m_1}$, $\left(\frac{-m_1}{m_2}(z^3 + tz)\right)_{m_2}$ and regular $z = 0$

The case $m_1 = m_2 = 1$. This is in fact the standard family for P_2. One has $q_1 = z^3 + tz$, $q_2 = -(z^3 + tz)$; the singular directions for $q_1 - q_2$ are $\frac{1,3,5}{6}$; the singular directions for $q_2 - q_1$ are $\frac{0,2,4}{6}$. The monodromy identity reads

$$\mathrm{mon} = \begin{pmatrix} g & 0 \\ 0 & 1/g \end{pmatrix} \begin{pmatrix} 1 & x_5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}.$$

The equation $\mathrm{mon} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and division by $\mathbb{G}_m$ (made explicit by normalizing $x_5 = 1$) produce an explicit $\mathcal{R}$ of dimension 3. Furthermore, $\dim \mathcal{P} = 1$ (parameter g). See [23, Section 3.9] for more details.

Case $m_1 = 2$, $m_2 = 1$, $z^3 + tz$, $z^3 + tz$, $-2z^3 - 2tz$ and $z = 0$ is regular.

Description of $\mathcal{R}$. The Stokes matrices are described by 12 variables, there are 2 variables describing the formal monodromy. The topological monodromy is supposed to be the identity. This produces 8 equations.

Actual computation, using the monodromy identity produces a space of dimension 6. Dividing by the action of $\mathbb{G}_m^2$, due to scaling the basis vectors, produces $\dim \mathcal{R} = 4$. Moreover, $\dim \mathcal{P} = 2$ and the fibers of $\mathcal{R} \to \mathcal{P}$ have dimension 2. One expects a relation with a classical Painlevé equation with at most two singularities.

We follow the discussion of the cases (i) and (ii) on pages 7–8 for the construction of the matrix differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$. The formal operator

$$\mathrm{ST} := z\frac{\mathrm{d}}{\mathrm{d}z} + \mathrm{diag}\left(z^3 + tz + a_1, z^3 + tz + a_2, -2z^3 - 2tz - a_1 - a_2\right)$$

is conjugated with the matrix $\mathbf{1} + M$, where

$$M = \begin{pmatrix} 0 & 0 & b_1/z + b_2/z^2 + b_3/z^3 \\ 0 & 0 & b_4/z + b_5/z^2 + b_6/z^3 \\ b_7/z + b_8/z^2 + b_9/z^3 & b_{10}/z + b_{11}/z^2 + b_{12}/z^3 & 0 \end{pmatrix}.$$

This produces an operator

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \tilde{A} = \mathrm{Prin}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + (\mathbf{1} + M)\mathrm{ST}(\mathbf{1} + M)^{-1}\right).$$

Then A is obtained from $\tilde{A}$ by adding the eight equations given by $\tilde{A}(0) = 0$. After scaling the basis vectors one has $b_2 = 1$, $b_{11} = 1$ and the matrix A depends only on the variables b_1, b_5, b_7, b_8. The substitution $b_5 = B_5 b_8$ removes some denominators. The characteristic polynomial of the formal monodromy at $z = \infty$ is written as $T^3 + pT + q$. The Lax pair equation together with the equations $\frac{\mathrm{d}p}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}q}{\mathrm{d}t} = 0$ produce a differential system

$$\frac{\mathrm{d}B_5}{\mathrm{d}t} = 0, \frac{\mathrm{d}b_1}{\mathrm{d}t} = -6B_5 b_1^3 b_7 - 6b_1^3 b_7 - 3b_1^2 t - 3,$$
$$\frac{\mathrm{d}b_7}{\mathrm{d}t} = \left(27b_7^2(B_5 + 1)^2 b_1^2 + 18tb_7(B_5 + 1)b_1 + 3t^2 - p\right)/(3B_5 + 3).$$

This is a Hamiltonian system $\frac{db_1}{dt} = \frac{\partial H}{\partial b_7}$, $\frac{db_7}{dt} = -\frac{\partial H}{\partial b_1}$, depending on parameters B_5, p, with

$$H = -3(B_5 + 1)b_7^2 b_1^3 - 3tb_7 b_1^2 + \frac{-3t^2 + p}{3B_5 + 3}b_1 - 3b_7.$$

There is no evident relation between this Hamiltonian and Okamoto's list of polynomial Hamiltonians in [22]. Possibly the method described in [6] can be applied here.

8　z, tz, $(-1-t)z$, Harnad's case

The moduli space $\mathcal{M}$ for these data is a family with irregular singularities and its isomonodromy produces the Painlevé VI equation. Classically, P_6 is derived from isomonodromy with four regular singularities. The construction and formulas for this new family are introduced by Harnad [11]. A more detailed investigation is given by Mazzocco [18]. A formula for the Stokes data of this new family in terms of invariants for the classical family is computed in [4].

In this section, we compute the Stokes data, the monodromy space $\mathcal{R}$, a matrix differential operator representing $\mathcal{M}$, its identification with the data in [4, 11, 18] and the Lax pair. A direct identification of the resulting Painlevé type equations with P_6 seems difficult to find. However this identification is explicitly present in [11, 18].

8.1　Computation of Stokes data and monodromy space $\mathcal{R}$

The formal solution space V at $z = \infty$ has a basis e_0, e_1, e_2 such that the formal differential operator has the form

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} z + a_0 & & \\ & tz + a_1 & \\ & & (-1-t)z - a_0 - a_1 \end{pmatrix}.$$

The formal monodromy and the Stokes matrices are given with respect to this natural basis. The basis is unique up to multiplying each e_j by a scalar. Since $z = 0$ is regular singular, one has $\dim \mathcal{R} = 6$. The formal monodromy at $z = \infty$ is the diagonal matrix $\mathrm{diag}(g_1, g_2, g_3)$ with $g_1 g_2 g_3 = 1$ and $g_1 = \mathrm{e}^{2\pi \mathrm{i} a_0}$, $g_2 = \mathrm{e}^{2\pi \mathrm{i} a_1}$.

The singular directions depend on t. For t close to i, the singular direction $d_{kl} \in [0, 1)$ for $q_k - q_l$ are approximated by 0.93, 0.83, 0.62, 0.43, 0.33, 0.12 for d_{20}, d_{21}, d_{01}, d_{02}, d_{12}, d_{10}. This determines the order of the six Stokes matrices in the monodromy identity, which states that the topological monodromy mon at $z = 0$ is, up to conjugation, equal to the product

$$\begin{pmatrix} g_1 & & \\ & g_2 & \\ & & \frac{1}{g_1 g_2} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ x_{20} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & x_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{01} & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & x_{02} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & x_{12} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ x_{10} & 1 & \\ & & 1 \end{pmatrix}.$$

If $z = 0$ is regular (i.e., mon $= \mathbf{1}$), then the Stokes matrices and the formal monodromy are equal to the identity. This case is uninteresting. We suppose that $z = 0$ is any regular singularity.

Since the basis e_0, e_1, e_2 is unique up to multiplying each e_j by a constant, the monodromy space $\mathcal{R}$ is the quotient of the space of tuples $\{g_1, g_2, x_{20}, x_{21}, x_{01}, x_{02}, x_{12}, x_{10}\}$ by the action of $\mathbb{G}_m^2$. A dense affine subspace of $\mathcal{R}$ is obtained by normalization two of the x_{**} to 1, for

instance, $x_{20} = x_{12} = 1$. It can be shown that $\mathcal{R} \to \mathcal{P}$ is surjective. The space $\mathcal{P}$ is given by the tuples (g_1, g_2, c_1, c_0) where c_0, c_1 are the nontrivial coefficients of the characteristic polynomial of mon. Now $\dim \mathcal{P} = 4$ and a computation shows that the fibers of $\mathcal{R} \to \mathcal{P}$ are affine cubic surfaces with equation $xyz + x^2 + y^2 + z^2 + p_1 x + p_2 y + p_3 z + p_4 = 0$ with $p_1, \ldots, p_4$ expressions in the parameters g_1, g_2, c_1, c_0.

This is the *expected* structure of the monodromy space if one admits the equivalence with the classical isomonodromy for P_6 (see the list in [23, Section 2.2]).

8.2 Constructing the connection and the Lax pair

A Zariski open, dense subspace of the moduli space $\mathcal{M}$ is obtained from [24, Theorem 12.4] (see also Section 1) by considering the family of differential operators of the form

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} a_0 & m_1 & m_2 \\ m_3 & a_1 & m_4 \\ m_5 & m_6 & -a_0 - a_1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -1-t \end{pmatrix}.$$

Changing the eigenvalues 1, t, $-1 - t$ of the irregular part of the operator into 0, 1, t has no effect on the monodromy and the Lax pair. One obtains in this way the [11, formula (3.62)] proposed by Harnad and the formula on [18, p. 3].

The Zariski open subspace of $\mathcal{M}$ is obtained by taking equivalence classes of the above family. Indeed, each of the basis vectors for this presentation can be multiplied by nonzero elements and therefore the family has to be divided by this action of $\mathbb{G}_m^2$. A Zariski open part of the quotient space is obtained by assuming $m_3 m_4 \neq 0$ and normalizing $m_3 = m_4 = 1$. In this way, one obtains the following explicit description of an open part of $\mathcal{M}$ in terms of

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} z + a_0 & v_1 & v_2 \\ 1 & tz + a_1 & 1 \\ v_3 & v_4 & (-1-t)z - a_0 - a_1 \end{pmatrix}.$$

This operator, with $v_1, \ldots, v_4$ as functions of t and a_0, a_1 constants, is completed to a Lax pair with the operator $\frac{\mathrm{d}}{\mathrm{d}t} + B_0(t) + zB_1(t)$. The assumption that the two operators commute leads to a set of differential equations for $v_1, \ldots, v_4$, namely

$$v_1' = \frac{-3v_3 v_1 + 3v_2 v_4}{2t^2 + 5t + 2}, \qquad v_2' = \frac{(6t+3)v_2^2 - 3v_3(t-1)v_2 - 3v_1(t+2) + (-9a_0 t + 9a_1)v_2}{2t^3 + 3t^2 - 3t - 2},$$

$$v_3' = \frac{(3t-3)v_3^2 + (-6t-3)v_2 v_3 + (9a_0 t - 9a_1)v_3 + 3v_4(t+2)}{2t^3 + 3t^2 - 3t - 2}, \qquad v_4' = \frac{3v_3 v_1 - 3v_2 v_4}{t^2 + t - 2}.$$

In a monodromic family the topological monodromy is constant and then also the characteristic polynomial of the residue matrix is constant. This means that there are constants δ_0, δ_1, explicitly

$$-a_0^2 - a_0 a_1 - a_1^2 - v_2 v_3 - v_1 - v_4 = \delta_1,$$
$$a_0^2 a_1 + a_0 a_1^2 + a_1 v_2 v_3 - a_0 v_1 + a_0 v_4 - a_1 v_1 - v_1 v_3 - v_2 v_4 = \delta_0.$$

The algebra of functions on the parameter space $\mathcal{P}^+$ for the connection is generated by a_0, a_1, δ_0, δ_1. They correspond to the 4 parameters for the moduli space $\mathcal{R}$ of the analytic data.

We know by [11, 18], that reduction of the above equations and parameters to the same for P_6 is possible. However, we have no explicit computation.

Remark 8.1. There are explicit formulas for reducible loci and the corresponding Riccati equations. These turn out to be hypergeometric differential equations. This is also expected if one admits that the Painlevé type system is equivalent to P_6.

Example 8.2. $v_3 = v_4 = 0$ and $v_1' = 0$,

$$v_2' = \frac{3(2t+1)v_2^2 - 3(t+2)v_1 + 9(-a_0t + a_1)v_2}{(t-1)(2t+1)(t+2)}.$$

8.3 The hierarchy $(z)_{m_1}$, $(tz)_{m_2}\left(\frac{-m_1 - tm_2}{m_3}z\right)_{m_3}$

The $m_1, m_2, m_3 \geq 1$ stand for multiplicities. A computation shows that "$z = 0$ is regular" leads to trivial Stokes matrices and formal monodromy. Consider the case $m_1 = 2$, $m_2 = m_3 = 1$. Then $\dim \mathcal{R} = 10$ and $\dim \mathcal{P} = 6$. According to [24, Exercise 12.5, p. 300], the universal family is represented by the operator

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} z + a_1 & 0 & x_1 & x_2 \\ 0 & z + a_2 & x_3 & x_4 \\ x_5 & x_6 & tz + a_3 & x_7 \\ x_8 & x_9 & x_{10} & (-2-t)z - a_1 - a_2 - a_3 \end{pmatrix}.$$

Consider a normalization, say, $x_8 = x_9 = x_{10} = 1$, obtained by restricting to the open subspace defined by $x_8 x_9 x_{10} \neq 0$ and multiplying the base vectors by scalars. The Lax pair consists of the above operator and $\frac{\mathrm{d}}{\mathrm{d}t} + B_0 + zB_1$. An easy computation produces a system of differential equations (or vector field) $\frac{\mathrm{d}x_j}{\mathrm{d}t} \in \mathbb{C}(a_1, \ldots, a_4, x_1, \ldots, x_7)$, $j = 1, \ldots, 7$. Using three coefficients of the characteristic polynomial of the residue matrix at $z = 0$, one can eliminate x_2, x_4, x_7 in a rational way. The resulting vector field of rank 4 is a Painlevé type system. It is unfortunately too complicated for presentation here. This system contains, of course, many closed subsystems corresponding to z, tz, $(-1-t)z$, that produce equations related to P_6.

9 $z^{1/e}$, tz, $-tz$, $e > 1$ and regular singular $z = 0$

Write $q_1, \ldots, q_e$ for the conjugates of $z^{1/e}$; write r and s for tz and $-tz$. The value of the integer N (i.e., the dimension of the Stokes data) is equal to $e(e-1)\frac{1}{e}$ (for the $q_k - q_l$) plus $e + e + e + e$ (for $q_k - r$, $r - q_k$, $q_k - s$, $s - q_k$) plus 2 (for $r - s$, $s - r$) and sums up to $5e + 1$. The formal monodromy γ depends on 1 parameter. Normalization by the action of $\mathbb{G}_m^2$ results in $\dim \mathcal{R} = 5e$. The parameters are the $e + 1$ coefficients of the characteristic polynomial of the monodromy at $z = 0$ and the eigenvalues of γ. This leads to $\dim \mathcal{P} = e + 2$ and $\dim \mathcal{R} - \dim \mathcal{P} = 4e - 2$.

The case $e = 2$. For $t = 1$, the singular directions in $[0, 1)$ are $d = 1/2$ for $r - q_1$, $r - q_2$, $q_1 - s$, $q_2 - s$, $r - s$ and $q_2 - q_1$, $q_1 - r$, $q_2 - r$, $s - q_1$, $s - q_2$, $s - r$ for $d = 0$. This leads to a matrix formula for the topological monodromy

$$\text{mon}_0 = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & g & \\ & & & 1/g \end{pmatrix} \begin{pmatrix} 1 & & x_1 \\ & 1 & x_2 \\ & & 1 \\ x_3 & x_4 & x_5 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_6 & & x_7 \\ & 1 & & x_8 \\ x_9 & x_{10} & 1 & x_{11} \\ & & & 1 \end{pmatrix}.$$

One may normalize to $x_{10} = x_{11} = 1$. The parameters are g and three coefficients of the characteristic polynomial of mon_0. A Maple computation verifies that the fibres of $\mathcal{R} \to \mathcal{P}$ are birational to $\mathbb{A}^6$.

The form of the differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ is found, using the method explained in Section 1, namely

$$A = \begin{pmatrix} a & z & * & * \\ 1 & -a & * & * \\ * & * & tz + b & * \\ * & * & * & -tz - b \end{pmatrix},$$

where a, b and $*$ are variables. Since two of the constants can be normalized to 0, this is a family of dimension 10 (over the field $\mathbb{C}(t)$). The invariants are three coefficients of the characteristic polynomial of the residue matrix at $z = 0$ and b at $z = \infty$.

The Lax pair $\{z\frac{d}{dz} + A, \frac{d}{dt} + B\}$ has the form $B = B_0 + B_1 z$, where B_0, B_1 are traceless matrices depending on t only. The Lax pair calculations for the case $e = 2$ produces a rational Painlevé vector field of dimension 6 which is too large to be presented here.

10 z^2, $-z^2 - tz$, tz and $z = 0$ regular

The assumptions: $z = 0$ is regular and $z = \infty$ is unramified and has Katz invariant 2 produces the eigenvalues $(z^2)_{m_1}$, $(a_2 z^2 + a_1 z)_{m_2}$, $(b_1 z)_{m_3}$ such that the m_1, m_2, m_3 satisfy

$$m_1 z^2 + m_2(a_2 z^2 + a_1 z) + m_3 b_1 z = 0.$$

Then $\dim \mathcal{R} = (n-1)^2$, where $n = m_1 + m_2 + m_3 \geq 3$.

We present computations for the case $m_1 = m_2 = m_3 = 1$. Then $\dim \mathcal{R} = 4$ and $\dim \mathcal{P} = 2$. The fibers of $\mathcal{R} \to \mathcal{P}$ are affine cubic surfaces, which have, after an affine linear change of the variables, the equation $xyz + x + y + 1 = 0$. We note that the monodromy space of standard family which produces the first Painlevé equation P_1 is the affine cubic surface with the same equation (see [23, Section 3.10]). Despite this similarity, we did not find a relation between the case under consideration and P_1.

We *propose* a normalized differential operator

$$\frac{d}{dz} + \begin{pmatrix} z & a_1 & 1 \\ 1 & -z - t & a_2 \\ a_3 & a_4 & t \end{pmatrix}.$$

The reasoning for this proposal is the following. We observe that

$$z\frac{d}{dz} + \begin{pmatrix} z^2 + c_1 & 0 & 0 \\ 0 & -z^2 - tz + c_2 & 0 \\ 0 & 0 & tz - c_1 - c_2 \end{pmatrix}$$

has at $z = \infty$ the universal deformation

$$z\frac{d}{dz} + \begin{pmatrix} z^2 + c_1 & * & * \\ * & -z^2 - tz + c_2 & * \\ * & * & tz - c_1 - c_2 \end{pmatrix}.$$

Here the $*$'s are arbitrary polynomials in z of degree ≤ 1. The assumption that $z = 0$ is regular implies that $c_1 = c_2 = 0$ and the $*$ are elements of $\mathbb{C}z$. Finally, in the general case one can, by a change of the basis, arrive at two entries being z. Dividing by z produces the above proposal.

For the Lax pair situation, a_1, a_2, a_3, a_4 are functions of t and the above operator is supposed to commute with $\frac{d}{dt} + B(z, t)$, where $B(z, t)$ has degree 1 in the variable z. The resulting differential equations are

$$a_1' = -3a_1 a_2 a_3 + 3a_4, \, a_2' = -\frac{3}{2}a_1 a_2^2 + 3a_2^2 a_3 - \frac{9}{2}a_2 t - 3/2,$$

$$a_3' = \frac{3}{2}a_3 a_1 a_2 - \frac{3}{2}a_4, \, a_4' = \frac{3}{2}a_4 a_1 a_2 - 3a_4 a_2 a_3 + \frac{3}{2}a_1 a_3 + \frac{9}{2}a_4 t.$$

The two parameters describing the parameter space $\mathcal{P}$ are $p_1 = a_1 + 2a_3$ and $p_2 := a_1 + a_3 + a_2 a_4$ (thus $p_1' = 0$, $p_2' = 0$). Elimination of a_1, a_3 leads to the system of differential equations

$$a_2' = -\frac{3}{2} + 6a_2^3 a_4 + \frac{9p_1 - 12p_2}{2}a_2^2 - \frac{9a_2 t}{2},$$

$$a_4' = \frac{3(-p_1 + 2p_2)(-p_2 + p_1)}{2} + \frac{9t}{2}a_4 + (-9p_1 + 12p_2)a_2 a_4 - 9a_2^2 a_4^2.$$

The first equation can be used to write a_4 as rational expression in a_2, a_2'. Substitution in the second equation yields an explicit second-order equation. The Hamiltonian H is equal to

$$-3a_4^2 a_2^3 - \frac{9p_1 - 12p_2}{2} a_4 a_2^2 - \frac{3p_1^2 - 9p_1 p_2 + 6p_2^2}{2} a_2 + \frac{3}{2} a_4 + \frac{9a_2 a_4}{2} t,$$

where $a_4' = \frac{\partial H}{\partial a_2}$, $a_2' = -\frac{\partial H}{\partial a_4}$ and p_1, p_2 are parameters (constants).

We did not find a relation with a classical Painlevé equation, but Dzhamay [5, Section 2], using the geometry of Okamoto–Painlevé spaces and the algorithms of [6], computed that the case at hand corresponds to a P_4 equation. More precisely, the above equation is equivalent to the standard Okamoto form $P_{\mathrm{IV}}(q(t), p(t), t; \kappa_0, \theta_8)$ of the fourth Painlevé equation, via $a_2(s) = -\mathrm{i}/q(t)$, $a_4(s) = -\mathrm{i}q(t)(q(t)p(t) - \theta_8)$, $s = 2\mathrm{i}t/3$, and $p_1 = -2\kappa_0$, $p_2 = \theta_8 - 2\kappa_0$.

11 $1/z, -1/z$ at $z = 0$ and $tz, -tz$ at $z = \infty$

The data above is the $m = 1$ case of the hierarchy $\mathcal{M}_m$ defined for $m \geq 1$ by the eigenvalues $\left(z^{-1}\right)_m$, $-mz^{-1}$ at $z = 0$ and $(tz)_m$, $-mtz$ at $z = \infty$. One shows that $\dim \mathcal{R}_m = 4m$ and $\dim \mathcal{P} = 2m$. The case $m = 1$ is the family for $P_3(D_6)$.

For $m = 2$, a computation reveals that $\mathcal{R} \to \mathcal{P}$ is surjective with fibres of dimension 4. A computation of $\mathcal{M}$, indicated in Section 1, and a choice of normalizations give rise to an operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ representing an open affine subset of $\mathcal{M}$, with

$$A = z^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} c_1 & 0 & -3m_1 \\ 0 & c_2 & -3 \\ 3 & 3m_3 & -c_1 - c_2 \end{pmatrix} + tzP \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} P^{-1},$$

where

$$P = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_3 & x_4 & 1 \end{pmatrix}.$$

A straightforward Lax pair computation produces formulas for $\frac{\mathrm{d}x_i}{\mathrm{d}t}$, $i = 1, \ldots, 4$, as rational functions in x_1, x_2, x_3, x_4, t. These are however too large to be displayed here. A computation verifies that in an isomonodromic family the c_1, c_2, m_3, m_1 are constant.

12 $z^{-1/2}$ at $z = 0$ and $tz, -tz$ at $z = \infty$

The data above defines the usual family for $P_3(D_7)$. By attaching multiplicities, e.g., $\left(z^{-1/2}\right)_m$, $(tz)_m$, $(-tz)_m$ and $m \geq 1$, one obtains a hierarchy. For the case $m = 2$, the space $\mathcal{R}$ is given by the equation $L \circ \mathrm{mon}_0 = \mathrm{mon}_\infty \circ L$ with $L \colon V(0) \to V(\infty)$ a linear bijection where $\mathrm{mon}_0 \colon V(0) \to V(0)$, $\mathrm{mon}_\infty \colon V(\infty) \to V(\infty)$ are the topological monodromies. The link L is considered up to multiplication by $\mathbb{C}^*$ and the matrices of mon_0 and mon_∞ have the form

$$\mathrm{mon}_0 = \begin{pmatrix} & & * & \\ & & & * \\ 1 & & & \\ & 1 & & \end{pmatrix} \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$\mathrm{mon}_\infty = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix} \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ * & & 1 & \\ * & * & & 1 \end{pmatrix}.$$

After normalization, one obtains $\dim \mathcal{R} = 12$ and $\dim \mathcal{P} = 4$. Therefore, the relative dimension of $\mathcal{M}$ over $\mathbb{C}(t)$ is also 12. The construction of a differential operator for $\mathcal{M}$ and the Lax pair computations seem to be out of reach.

13 $z^{-1/n}$ and $tz^{1/n}$, a hierarchy related to $P_3(D_8)$

The assumption that $z = 0$ and $z = \infty$ are both irregular singular and totally ramified leads, after normalization, to the Galois orbit of $z^{-1/n}$ at $z = 0$ and the Galois orbit $tz^{1/n}$ at $z = \infty$. In the sequel, we replace t by $t^{1/n}$. The moduli spaces will be denoted by $\mathcal{M}_n$ and $\mathcal{R}_n$. The standard isomonodromic family for $P_3(D_8)$ is derived from $\mathcal{M}_2$. First we study the structure of $\mathcal{R}_n$.

13.1   The structure of the monodromy space $\mathcal{R}_n$

We refer to Section 1 and [24, Sections 8 and 9] for notation and results. For a connection $M \in \mathcal{M}_n$, the solution space $V(\infty)$ at $z = \infty$ has the structure: $V(\infty) = \bigoplus_{j=0}^{n-1} \mathbb{C} e_j$ with $\mathbb{C} e_j = V(\infty)_{q_j}$, $q_j = \sigma^j\big(t^{1/n} z^{1/n}\big) = \zeta_n^j t^{1/n} z^{1/n}$, where $\zeta_n = e^{2\pi i/n}$. The basis $\{e_j\}$ is chosen such that the formal monodromy $\gamma_{V(\infty)}$ acts by $e_0 \mapsto e_1 \mapsto \cdots \mapsto e_{n-1} \mapsto (-1)^{n-1} e_0$.

By Lemma 1.2, the space of the Stokes data at $z = \infty$ can be identified with $\mathbb{C}^{n-1}$. The monodromy identity for the topological monodromy mon_∞ at $z = \infty$ has been studied in detail in [3, pp. 146–147]. The *surprising property* is:

Let the Stokes data be $(x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$. *Then the characteristic polynomial of the topological monodromy* mon_∞ *is*

$$T^n + x_{n-1} T^{n-1} + \cdots + x_1 T + (-1)^n.$$

Thus the map from the Stokes data to the characteristic polynomial of mon_∞ is bijective.

The local solution space $V(0)$ at $z = 0$ has a similar description. The map from the space of the Stokes matrices to the (nontrivial) coefficients of the characteristic polynomial of the topological monodromy mon_0 at $z = 0$, is bijective.

The monodromy space $\mathcal{R}_n$ consists of the local analytic data at $z = \infty$ and $z = 0$ together with a link which glues the solution space above $\mathbb{P}^1 \setminus \{\infty\}$ to the solution space above $\mathbb{P}^1 \setminus \{0\}$. More precisely, the link $L \colon V(0) \to V(\infty)$ is a linear bijection such that $L \circ (\mathrm{mon}_0)^{-1} = \mathrm{mon}_\infty \circ L$. The "inverse sign" reflects the difference in directions of the paths for mon_0 and mon_∞.

It follows that all the structure of $V(0)$ is determined by mon_∞ and the link. In particular, mon_0^{-1} has the same characteristic polynomial as mon_∞. Furthermore, the Stokes matrices at $z = 0$ are the same as those at $z = \infty$, however taken in the opposite order.

Let L_0 be a fixed choice for the link. Any other link has the form $M \circ L_0$ where $M = (m_{i,j}) \in \mathrm{GL}(V(\infty))$ commutes with mon_∞. Then $\mathcal{R}_n$ can be identified with the tuples $(M, x_1, \ldots, x_{n-1})$ as above and M taken modulo multiplication by a scalar (since the basis of $V(0)$ and $V(\infty)$ can be scaled).

A computation, using the formulas in [3, Sections 3.3 and 3.4] for mon_∞, shows that M is determined by its last row $(m_{n,1}, \ldots, m_{n,n})$ and that this space can be identified with the open subspace of $\mathbb{P}^{n-1} \times \mathbb{A}^{n-1}$ consisting of the tuples

$$((m_{n,1} : \cdots : m_{n,n}), (x_1, \ldots, x_{n-1})) \in \mathbb{P}^{n-1} \times \mathbb{A}^{n-1}$$

such that the determinant F of the matrix M is not zero. One easily sees that F is homogeneous of degree n in the n variables $m_{n,1}, \ldots, m_{n,n}$ and its coefficients are polynomials in $x_1, \ldots, x_{n-1}$. In particular, $\mathcal{R}_n$ *is smooth, connected, quasi projective of dimension* $2(n-1)$.

Example 13.1 (example $\mathcal{R}_2$). The local analytic data at $z = \infty$ are

$$V(\infty) = V(\infty)_{\sqrt{tz}} \oplus V(\infty)_{-\sqrt{tz}} = \mathbb{C}e_0 + \mathbb{C}e_1, \qquad \gamma\colon e_0 \mapsto e_1 \mapsto -e_0.$$

The singular directions depend on $t^{1/2}$. For $t^{1/2}$ in a neighbourhood of 1, the monodromy identity is

$$\mathrm{mon}_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} -x & -1 \\ 1 & 0 \end{pmatrix}.$$

As above, there is a surjective morphism $\mathcal{R}_2 \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$. The fibres consist of the $(b_3 : b_4) \in \mathbb{P}^1$ such that the determinant $F = -b_3 b_4 x + b_3^2 + b_4^2$ of the matrix $M := \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right)$, commuting with mon_∞, is nonzero. Thus $\mathcal{R}_2 \subset \mathbb{P}^1 \times \mathbb{A}^1$ is the complement of the quadratic curve $F = 0$ over $\mathbb{A}^1$.

We note that the description in [23, Section 3.6] of monodromy space for the classical case $P_3(D_8)$ is slightly different. There the link L is normalized by the assumption $\det L = 1$.

Example 13.2 (example $\mathcal{R}_3$). The local analytic data at $z = \infty$ are $V(\infty) = \mathbb{C}e_0 + \mathbb{C}e_1 + \mathbb{C}e_2$ with $\mathbb{C}e_j = V(\infty)_{q_j}$ for $j = 0, 1, 2$ and $q_0 = t^{1/3} z^{1/3}$, $q_1 = \zeta_3 t^{1/3} z^{1/3}$, $q_2 = \zeta_3^2 t^{1/3} z^{1/3}$ and $\zeta_3 = e^{2\pi i/3}$. The basis vectors e_0, e_1, e_2 are chosen such that the formal monodromy γ satisfies $e_0 \mapsto e_1 \mapsto e_2 \mapsto e_0$. The basis e_0, e_1, e_2 is unique up to a simultaneous multiplication by a scalar.

For $t^{1/3}$ equal to 1, the topological monodromy mon_∞ at $z = \infty$ is $\mathrm{mon}_\infty = \gamma \mathrm{St}_{3/4}\mathrm{St}_{1/4}$ which is explicitly

$$\mathrm{mon}_\infty = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x_{21} & 1 \end{pmatrix}\begin{pmatrix} 1 & x_{01} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{21} & 1 \\ 1 & x_{01} & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of mon_∞ is $X^3 - x_{01}X^2 - x_{21}X - 1$.

The space $\mathbb{A}^2$ of the topological monodromies at $z = \infty$, consists of the pairs $(x_{01}, x_{21}) \in \mathbb{C}^2$. The fibres of the obvious map $\mathcal{R}_3 \to \mathbb{A}^2$ consist of the elements $(m_{3,1} : m_{3,2} : m_{3,3}) \in \mathbb{P}^2$ such that the determinant F of M is invertible. Explicitly,

$$\begin{aligned} F = {}& a_7^3 x_1 x_2 + a_7^2 a_8 x_1^2 + a_7^2 a_9 x_2^2 + a_7 a_8 a_9 x_1 x_2 - a_7^2 a_8 x_2 + a_7^2 a_9 x_1 \\ & - 2a_7 a_8^2 x_1 + 2a_7 a_9^2 x_2 - a_8^2 a_9 x_2 + a_8 a_9^2 x_1 + a_7^3 - 3a_7 a_8 a_9 + a_8^3 + a_9^3, \end{aligned}$$

where $(a_7, a_8, a_9) = (m_{3,1}, m_{3,2}, m_{3,3})$ and $x_1 = x_{0,1}$, $x_2 = x_{2,1}$. Thus $\mathcal{R}_3 \subset \mathbb{P}^2 \times \mathbb{A}^2$ is the complement of the cubic curve over $\mathbb{A}^2$ with equation $F = 0$.

13.2 Construction of $\mathcal{M}_n$ by using a cyclic covering

The space $\mathcal{M}_n$ will be represented by the "universal" matrix differential operator $L = z\frac{\mathrm{d}}{\mathrm{d}z} + A$ of size $n \times n$ over $\mathbb{C}(z)$. This operator has only at $z = 0$ and $z = \infty$ singularities and these are given by the Galois orbits of $z^{-1/n}$ at $z = 0$ and $tz^{1/n}$ at $z = \infty$. As in Section 3, we use the n-cyclic covering of $\mathbb{P}^1$ to produce explicit formulas.

Now L is seen as a map on a vector space V of dimension n over $\mathbb{C}(z)$. Let σ denote the automorphism of $\mathbb{C}(z^{1/n})$ over $\mathbb{C}(z)$, given by $\sigma(z^{1/n}) = \omega z^{1/n}$ with $\omega = e^{2\pi i/n}$. Then σ acts as semi-linear map on $W := \mathbb{C}(z^{1/n}) \otimes V$ and L extends uniquely to a derivation D on W. The operator D has no ramification. Define the trace $\mathrm{tr}\colon W \to V$ by $tr(w) = \sum_{j=0}^{n-1} \sigma^j(w)$.

We apply the method of [23, Chapter 12], *to construct a universal family* (see also Section 1). One considers a basis $e_0, e_1, \ldots, e_{n-1}$ of W over $\mathbb{C}(z^{1/n})$ such that σ acts by $e_0 \mapsto e_1 \mapsto \cdots \mapsto$

$e_{n-1} \mapsto e_0$ and D has, with respect to this basis, poles of order 1 at $z^{1/n} = 0$ and at $z^{1/n} = \infty$ and no further singularities. By construction, D commutes with σ. In particular, $D(e_0)$ determine D and $D(e_0)$ has the form $\sum_{j=0}^{n-1} (a_j z^{-1/n} + b_j + c_j z^{1/n}) e_j$ where the a_j, b_j, c_j are variables, parametrizing the family.

Define the basis $\{B_0, B_1, \ldots, B_{n-1}\}$ of V by $B_j = \mathrm{tr}(z^{j/n} e_0)$ for all j. In the computations, we change B_{n-1} into $z^{-1} B_{n-1}$. The given data for $D(e_0)$ induces a formula $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ for D on the basis $B_0, \ldots, B_{n-1}$.

It is seen that the matrix A has at most singularities at $z = 0$ and $z = \infty$ (in fact poles of order at most 1). The characteristic polynomial of A is seen to have the form

$$T^n + p_{n-1} T^{n-1} + \cdots + p_1 T + p_0 - (\alpha z^{-1} + \beta z)$$

with all entries $p_0, \ldots, p_{n-1}, \alpha, \beta$ are in $\mathbb{C}[a_0, b_0, c_0, \ldots, b_{n-1}, c_{n-1}]$. In particular, there are explicit expressions $\neq 0$ for α and β. In the family given by A, we *require* that α and β are invertible. Indeed, this follows from the assumption that $z = 0$ and $z = \infty$ are totally ramified and have Katz invariant $\frac{1}{n}$ for the operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$.

The resulting affine family of operators $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ is parametrized by

$$\mathrm{Spec}\left(\mathbb{C}\left[a_0, b_0, \ldots, c_{n-1}, \frac{1}{\alpha}, \frac{1}{\beta}\right]\right).$$

Next, we make the following restrictions and normalizations. We *require* that A has trace zero. This is equivalent to giving b_0 the value $\frac{3-n}{2n}$. The variable z is scaled such that $\alpha = 1$ and we write t for β. The next step is to divide by the action, by conjugation, of the group of the (constant) diagonal matrices on the differential operator $z\frac{\mathrm{d}}{\mathrm{d}z} + A$. In examples this is done by replacing $(n-1)$ suitable entries of A by 1 (for example, resulting in $a_1 = 1$, $a_2 = \cdots = a_{n-1} = 0$). One sees that the dimension of the final family (not counting t) is $3n - 1 - 2 - (n-1) = 2n - 2$ (with -1 for b_0 and -2 for α, β and $-(n-1)$ for conjugation). This is in accordance with $\dim \mathcal{R}_n = 2(n-1)$.

We do not attempt to describe the full moduli space $\mathcal{M}_n$, but claim that the constructed family $D = z\frac{\mathrm{d}}{\mathrm{d}z} + A$ describes an affine open subset. The operator $E := \frac{\mathrm{d}}{\mathrm{d}t} + B$ such that $\{D, E\}$ forms a Lax pair is also considered as σ-equivariant operator on W and is determined by $E(e_0)$. One computes that $E(e_0) = z^{1/n} \sum_{j=0}^{n-1} c_j e_j$ holds, under the assumption that $a_0 = 1$, $a_1 = \cdots = a_{n-1} = 0$. Below, the above construction is made explicit for $n = 3$, extended to $n = 4$ and to general $n \geq 3$. For $n = 2$, it is compared to the classical formula.

13.2.1 Case $n = 3$

The matrix of D with respect to the basis B_0, B_1, $z^{-1} B_2$ is

$$\begin{pmatrix} b_0 + b_1 + b_2 & a_0 + a_1 + a_2 & c_0 + c_1 + c_2 \\ c_0 + c_1\omega^2 + c_2\omega & \frac{1}{3} + b_0 + b_1\omega^2 + b_2\omega & z^{-1}(a_0 + a_1\omega^2 + a_2\omega) \\ a_0 + a_1\omega + a_2\omega^2 & z(c_0 + c_1\omega + c_2\omega^2) & -1/3 + b_0 + b_1\omega + b_2\omega^2 \end{pmatrix}$$

with $\alpha = (a_0 + a_1 + a_2)(a_0 + a_1\omega + a_2\omega^2)(a_0 + a_1\omega^2 + a_2\omega)$ and $\beta = (c_0 + c_1 + c_2)(c_0 + c_1\omega + c_2\omega^2)(c_0 + c_1\omega^2 + c_2\omega)$.

Normalization: $(a_0 + a_1 + a_2) = (a_0 + a_1\omega + a_2\omega^2) = (a_0 + a_1\omega^2 + a_2\omega) = 1$ (equivalently $a_0 = 1$, $a_1 = a_2 = 0$), $b_0 = 0$ and $\beta = t$. This produces

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} d_0 & 1 & f_0 \\ f_1 & d_1 & \frac{1}{z} \\ 1 & f_2 z & d_2 \end{pmatrix} \qquad \text{with} \quad f_0 f_1 f_2 = t \quad \text{and} \quad d_0 + d_1 + d_2 = 0.$$

It is completed to a Lax pair by

$$t\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} 0 & 0 & f_0 \\ f_1 & 0 & 0 \\ 0 & f_2 z & 0 \end{pmatrix}.$$

The Painlevé type equations are

$$t\frac{f_0'}{f_0} = d_0 - d_2, \qquad t\frac{f_1'}{f_1} = d_1 - d_0, \qquad t\frac{f_2'}{f_2} = d_2 - d_1 + 1,$$
$$td_0' = f_1 - f_0, \qquad td_1' = f_2 - f_1, \qquad td_2' = f_0 - f_2.$$

The system of equations has symmetries ρ and σ defined by ρ: $f_0, f_1, f_2 \mapsto f_1, f_2, f_0$ and $d_0, d_1, d_2 \mapsto d_1 - 1/3, d_2 + 2/3, d_0 - 1/3$, σ: $f_0, f_1, f_2 \mapsto f_1, f_0, f_2$ and $d_0, d_1, d_2 \mapsto -d_0, -d_2, -d_1$, and generating $D_3 = S_3$.

The above formulas are almost identical to the ones of Kawakami [14, p. 35]. In the latter the trace of the operator is $+1$ instead of 0 and the entries of the matrix are written in terms of canonical variables p_1, p_2, q_1, q_2 for a certain Hamiltonian.

One substitutes $F = f_0$, $G = f_1$ and obtains the equivalent system

$$F'' = \frac{(F')^2}{F} - \frac{F'}{t} + \frac{FG - 2F^2}{t^2} + \frac{1}{tG},$$

$$G'' = \frac{(G')^2}{G} - \frac{G'}{t} + \frac{FG - 2G^2}{t^2} + \frac{1}{tF}.$$

For a solution $(f_0, f_1, f_2, d_0, d_1, d_2)$, invariant under the symmetry $f_0 \leftrightarrow f_1$, one has $F = G$ and resulting equation

$$F'' = \frac{(F')^2}{F} - \frac{F'}{t} + \frac{-F^2}{t^2} + \frac{1}{tF}.$$

The substitution $t = x^4$, $F(t) = xf(x)$ produces a solution f of $P_3(D_8)$, i.e., P_3 with parameters $(\alpha, \beta, \gamma, \delta) = (-16, 0, 0, 16)$. The invariant solutions under $D_3 = S_3$ are $F = G = \zeta t^{1/3}$ with $\zeta^3 = 1$.

13.2.2 The general case

We make the following normalization:

$$D(e_0) = \left(z^{-1/n} + b_0 + c_0 z^{1/n}\right)e_0 + \sum_{j=1}^{n-1}\left(b_j + c_j z^{1/n}\right)e_{n-1}, \qquad b_0 = \frac{3-n}{2n}, \qquad \beta = t$$

and $E(e_0) = z^{1/n}\sum_{j=0}^{n-1} c_j e_j$ and basis $B_0, \ldots, B_{n-2}, z^{-1}B_{n-1}$.

For general $n \geq 3$, the formulas for the Lax pair are

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} d_0 & 1 & 0 & . & 0 & f_0 \\ f_1 & d_1 & 1 & . & 0 & 0 \\ 0 & f_2 & d_2 & . & 0 & 0 \\ . & . & . & . & 1 & . \\ 0 & . & . & f_{n-2} & d_{n-2} & \frac{1}{z} \\ 1 & 0 & . & 0 & f_{n-1}z & d_{n-1} \end{pmatrix}, \qquad t\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} 0 & 0 & 0 & . & 0 & f_0 \\ f_1 & 0 & 0 & . & 0 & 0 \\ 0 & f_2 & 0 & . & 0 & 0 \\ . & . & . & . & 0 & . \\ 0 & . & . & f_{n-2} & 0 & 0 \\ 0 & 0 & . & 0 & f_{n-1}z & 0 \end{pmatrix}$$

with $\sum d_j = 0$, $\prod f_j = t$. *The Painlevé type equations are*

$$t\frac{f_0'}{f_0} = d_0 - d_{n-1}, \qquad t\frac{f_1'}{f_1} = d_1 - d_0, \qquad \dots, \qquad t\frac{f_{n-1}'}{f_{n-1}} = d_{n-1} - d_{n-2} + 1,$$
$$td_0' = f_1 - f_0, \qquad td_1' = f_2 - f_1, \qquad \dots, \qquad td_{n-1}' = f_0 - f_{n-1}.$$

The symmetries observed for $n = 3$ generalize to $n \geq 3$ as follows

$$\rho: \ (f_0, f_1, \dots, f_{n-1}) \mapsto (f_1, f_2, \dots, f_{n-1}, f_0),$$
$$(d_0, d_1, \dots, d_{n-1}) \mapsto \left(d_1 - \frac{1}{n}, \dots, d_{n-2} - \frac{1}{n}, d_{n-1} + \frac{n-1}{n}, d_0 - \frac{1}{n} \right)$$

and

$$\sigma: \ (f_0, f_1, \dots, f_{n-1}) \mapsto (f_{n-1}, f_{n-2}, \dots, f_1, f_0),$$

combined with $\sigma(d_j) = -d_{\pi(j)} + c_j$ for a permutation π satisfying $\pi^2 = 1$ and constants $c_j \in \{-\frac{1}{n}, \frac{n-1}{n}\}$ such that $\sum c_j = 0$. These symmetries generate the dihedral group D_n of order $2n$.

Taking D_n-invariants $f_0 = \cdots = f_{n-1} := t^{1/n}$ and corresponding d_j's produces algebraic solutions of the Painlevé type equations.

Case $n = 4$. For invariant solutions under $f_0 \leftrightarrow f_1$, $f_2 \leftrightarrow f_3$ one has $(f_0, f_1, f_2, f_4) = \left(f, f, \frac{\sqrt{t}}{f}, \frac{\sqrt{t}}{f} \right)$ and $d_0 = -1/8$, $d_1 = t\frac{f'}{f} - 1/8$, $d_2 = 3/8$, $d_3 = -t\frac{f'}{f} - 1/8$ and the equation

$$f'' = \frac{(f')^2}{f} - \frac{f'}{t} - \frac{f^2}{t^2} + \frac{1}{t^{3/2}}.$$

Substitution $t = x^4$, $f(t) = xF(x)$ yields the equation $P_3(D_8)$ for F.

The D_4-symmetric solutions are $f = f_0 = f_1 = f_2 = f_3 = \zeta t^{1/4}$ with $\zeta^4 = 1$.

Case $n = 5$. Consider solutions invariant under $f_0 \leftrightarrow f_4$, $f_1 \leftrightarrow f_3$. Then $(f_0, f_1, f_2, f_3, f_4) = \left(f, g, \frac{t}{f^2 g^2}, g, f \right)$. Moreover,

$$(d_0, d_1, d_2, d_3, d_4) = \left(t\frac{f'}{f} - \frac{2}{5}, t\frac{f'}{f} + t\frac{g'}{g} - \frac{2}{5}, -t\frac{f'}{f} - t\frac{g'}{g} + \frac{3}{5}, t\frac{f'}{f} + \frac{3}{5}, \frac{-2}{5} \right).$$

The system of equations for f, g reads

$$f'' = \frac{(f')^2}{f} - \frac{f'}{t} + \frac{fg - f^2}{t^2}, \qquad g'' = \frac{(g')^2}{g} - \frac{g'}{t} + \frac{fg - 2g^2}{t^2} + \frac{1}{f^2 gt}.$$

The D_5-invariant solutions are $f = g = \zeta t^{1/5}$ with $\zeta^5 = 1$.

The presented examples for small n suggest that subgroups of D_n produce interesting subsystems.

For completeness, we consider also the case $n = 2$. The Lax pair is

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} d_0 & \frac{1}{z} + f_0 \\ 1 + f_1 z & d_1 \end{pmatrix}, \qquad t\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} 0 & f_0 \\ f_1 z & 0 \end{pmatrix}$$

with $f_0 f_1 = t$ and $d_0 + d_1 = 0$. The equations are

$$t\frac{f_0'}{f_0} = d_1 - d_0, \qquad t\frac{f_1'}{f_1} = d_0 - d_1 + 1, \qquad td_0' = f_1 - f_0, \qquad td_1' = f_0 - f_1.$$

One observes that $q = f_0$ satisfies the classical equation for $P_3(D_8)$, namely

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} + \frac{2q^2}{t^2} - \frac{2}{t}.$$

There is a symmetry ρ, given by $f_0, f_1 \mapsto f_1, f_0$ and $d_0, d_1 \mapsto d_1 - 1/2, d_0 + 1/2$.

The symmetry means that if q is a solution, then so is $\frac{t}{q}$. Further the invariant element $r = q + \frac{t}{q}$ satisfies the equation

$$r'' = \frac{r}{r^2 - 4t}(r')^2 - \frac{r^2}{t(r^2 - 4t)}r' + \frac{2r^4 - 16tr^2 + tr + 32t^2}{t^2(r^2 - 4t)}.$$

The symmetric solutions are $q = \pm\sqrt{t}$ and $r = \pm 2\sqrt{t}$.

14   A companion of P_1

What we like to call the *companion of P_1* is the family $\mathcal{M}$ of connections, given by the set of differential modules M over $\mathbb{C}(z)$ with dimension 2, $\Lambda^2 M$ is trivial, $z = 0$ is regular singular and the generalized eigenvalues at $z = \infty$ are $\pm w$ with $w = z^{5/2} + \frac{t}{2}z^{1/2}$. This is the P_1 case except for allowing a regular singularity at $z = 0$.

Description of $\mathcal{R}$. The singular directions at $z = \infty$, lying in $[0, 1)$ are $\frac{1}{5}$, $\frac{3}{5}$ for the difference of eigenvalues $w - (-w)$, and 0, $\frac{2}{5}$, $\frac{4}{5}$ for $(-w) - w$. Thus $\mathcal{R} \cong \mathbb{A}^5$. Let mon: $\mathcal{R} \to \mathrm{SL}_2(\mathbb{C})$ denote the morphism which sends the Stokes matrices to the monodromy matrix at $z = 0$.

The fibre of mon above $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ is given by the monodromy identity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_5 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One eliminates x_1, x_2 by $x_1 = -c(x_3 + x_5 + x_3x_4x_5) - a(1 + x_3x_4)$ and $x_2 = d(1 + x_4x_5) + bx_4$. Since $ad - bc = 1$ we are left with the equation $d(x_3 + x_5 + x_3x_4x_5) + b(1 + x_3x_4) + 1 = 0$.

For $d \neq 0$, the fibre is (as often) an affine cubic surface with three lines at infinity. Its equation coincides with the one for the standard family which produces the classical P_1 (see [23, Section 3.10]).

For $d = 0$, the equation of the fibre reads $x_3x_4 = -b^{-1} - 1$ and x_5 has no relations. For $b \neq -1$ it is the surface $\mathbb{C}^* \times \mathbb{C}$. In particular, $\mathcal{R} \to \mathrm{SL}_2$ is surjective. Furthermore, the parameter space $\mathcal{P}$ has dimension 1.

Description of $\mathcal{M}$. Since the fibres of RH: $\mathcal{M} \to \mathcal{R}$ are parametrized by t, one has dim $\mathcal{M} = 6$. The isomorphic classes of the residue matrix at $z = 0$ form the parameter space. It can be shown that the family

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}z^3 + \begin{pmatrix} 0 & b_2 \\ 1 & 0 \end{pmatrix}z^2 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}z + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix},$$

with $t = b_1 - b_2^2 + c_0$, is the universal family of connections $\mathcal{M}$. We eliminate b_1 by $b_1 = t + b_2^2 - c_0$. Furthermore, $p_0 := a_0^2 + b_0c_0$ is the basic parameter (i.e., independent of t in an isomonodromic family).

For the Lax pair computation, we suppose that the above operator commutes with

$$\frac{\mathrm{d}}{\mathrm{d}t} + \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix} + z\begin{pmatrix} y_4 & y_5 \\ y_6 & -y_4 \end{pmatrix}$$

and that $\frac{d(a_0^2 + b_0c_0)}{dt} = 0$. This eliminates $y_1, \ldots, y_6$ and produces the equations $a_0' = 2b_2c_0 - b_0$, $b_0' = -4a_0b_2$, $a_1' = -3b_2^2 + 2c_0 - t$, $b_2' = -2a_1$, $c_0' = 2a_0$.

For $c_0 = 0$, and also for a fixed residue matrix (i.e., $a_0' = b_0' = c_0' = 0$), one obtains the P_1 equation. For $c_0 \neq 0$, one eliminates $b_0 = (p_0 - a_0^2)/c_0$ and $b_0' = (-2a_0a_0'c_0 - (p_0 - a_0^2)c_0')/c_0^2$. This results in the Painlevé type vector field

$$a_0' = 2b_2c_0 - \frac{p_0 - a_0^2}{c_0}, \qquad c_0' = 2a_0, \qquad a_1' = -3b_2^2 + 2c_0 - t, \qquad b_2' = -2a_1.$$

One eliminates a_0, a_1, and c_0 in the above equations by $a_1 = -\frac{1}{2}b_2'$, $c_0 = \frac{t}{2} + \frac{3}{2}b_2^2 - \frac{1}{4}b_2''$, $a_0 = \frac{1}{4} + \frac{3}{2}b_2b_2' - \frac{1}{8}b_2^{(3)}$. The remaining equation produces the following fourth-order explicit differential equation for $f := b_2$

$$-2\big(6f^2 - f^{(2)} + 2t\big)f^{(4)} = 288f^5 - 240f^3f^{(2)} + 192tf^3 - 24ff^{(1)}f^{(3)} + 32f\big(f^{(2)}\big)^2$$

$$-80tff^{(2)} + 32ft^2 + 24\big(f^{(1)}\big)^2f^{(2)} - 48\big(f^{(1)}\big)^2 + 48ff^{(1)} + \big(f^{(3)}\big)^2 - 4f^{(3)} + 64p_0 + 4$$

with $f = b_2$, $f^{(j)} := \big(\frac{\mathrm{d}}{\mathrm{d}t}\big)^j(b_2)$ for $j = 1, 2, 3, 4$. We note that the denominator of the formula for $f^{(4)}$ is the equation for P_1. It seems probable, but we have no proof, that the field $\mathbb{C}(t)\big(b_2, \frac{\mathrm{d}}{\mathrm{d}t}b_2, \big(\frac{\mathrm{d}}{\mathrm{d}t}\big)^2b_2, \big(\frac{\mathrm{d}}{\mathrm{d}t}\big)^3b_2\big)$ has, for generic p_0, transcendence degree 4 over $\mathbb{C}(t)$. Indeed, this would fit with the observation that the fibres of $\mathcal{R} \to \mathcal{P}$ have dimension 4.

Comments. There are two reasons why this "companion of P_1" is not in the classical list $P_1 - P_6$. The Painlevé type equations describe in fact a vector field of rank 4 (written above as explicit differential equation of order 4).

Secondly, the monodromic family is a subfamily of the natural monodromic family with "two time variables" given by the data: $z = 0$ is regular singular and $z = \infty$ is irregular singular with generalized eigenvalues $\pm\big(z^{5/2} + \frac{t_1}{2}z^{3/2} + \frac{t_2}{2}z^{1/2}\big)$ with time variables t_1, t_2. We extend our computations to this case.

Isomonodromy and Lax pairs for $q = \big(z^{5/2} + \frac{t_1}{2}z^{3/2} + \frac{t_2}{2}z^{1/2}\big)$. As in the case $q = z^{5/2} + \frac{t}{2}z^{1/2}$ the family of connections $z\frac{\mathrm{d}}{\mathrm{d}z} + A$ can be normalized to

$$z\frac{\mathrm{d}}{\mathrm{d}z} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}z^3 + \begin{pmatrix} 0 & b_2 \\ 1 & 0 \end{pmatrix}z^2 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}z + \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix},$$

where $c_0 = b_2^2 - b_2t_1 + \frac{1}{4}t_1^2 - b_1 + t_2$ and $c_1 = -b_2 + t_1$. The variables a_0, a_1, b_0, b_1, b_2 are seen as functions of t_1, t_2. The Lax pairs are expressed by $\big[z\frac{\mathrm{d}}{\mathrm{d}z} + A, \frac{\mathrm{d}}{\mathrm{d}t_i} + B_i\big] = 0$ for $i = 1, 2$ and B_i a matrix depending on t_1, t_2, z and polynomial in z of degree ≤ 2. One obtains in terms of closed one-forms $d(a_0), \ldots, d(b_2)$ the system

$$\mathrm{d}(a_0) = \big\{16b_2^4 - 16b_2^3t_1 + 4b_2t_1^3 - t_1^4 - 48b_1b_2^2 + 32b_1b_2t_1 - 4b_1t_1^2$$

$$+ 32b_2^2t_2 - 16b_2t_1t_2 - 16b_0b_2 + 8b_0t_1 + 32b_1^2 - 48b_1t_2 + 16t_2^2\big\}\frac{\mathrm{d}t_1}{48}$$

$$+ \Big\{3b_2^2t_1 - 2b_2^3 - \frac{3b_2t_1^2}{2} + \frac{t_1^3}{4} + 2b_1b_2 - b_1t_1 - 2b_2t_2 + t_1t_2 + b_0\Big\}\,\mathrm{d}t_2,$$

$$\mathrm{d}(a_1) = \big\{-16b_2^3 + 20b_2^2t_1 - 4b_2t_1^2 - t_1^3 + 16b_1b_2 - 16b_1t_1 - 16b_2t_2$$

$$+ 12t_1t_2 + 8b_0\big\}\frac{\mathrm{d}t_1}{24} + \Big\{b_2^2 - 2b_2t_1 + \frac{3}{4}\big(t_1^2\big) + 2b_1 - t_2\Big\}\,\mathrm{d}t_2,$$

$$\mathrm{d}(b_0) = \big\{a_0t_1^2 - 4a_0b_2^2 + 8a_0b_1 - 4a_0t_2 - 4a_1b_0\big\}\frac{\mathrm{d}t_1}{6} + \{(4b_2 - 2t_1)a_0\}\,\mathrm{d}t_2,$$

$$\mathrm{d}(b_1) = \big\{-4a_1b_2^2 + a_1t_1^2 + 4a_0b_2 - 2a_0t_1 + 4a_1b_1 - 4a_1t_2 + 2b_2 - t_1\big\}\frac{\mathrm{d}t_1}{6}$$

$$+ \{4a_1b_2 - 2a_1t_1 + 2a_0 + 1\}\,\mathrm{d}t_2,$$

$$\mathrm{d}(b_2) = \{-a_1t_1 + 2a_0 + 2\}\frac{\mathrm{d}t_1}{3} + 2a_1\,\mathrm{d}t_2.$$

Note that $p_0 := a_0^2 + b_0c_0$ satisfies $\mathrm{d}(p_0) = 0$ and p_0 is a generating parameter for this system.

Acknowledgements

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