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ON THE SPECTRUM OF A MORPHISM OF QUOTIENT HILBERT SPACES

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Abstract. In this paper we define the notion of spectrum for a morphism of quotient Hilbert spaces. The definition is the same with the one given by L.Waelbroeck but the proofs that the spectrum is a compact and nonempty set are different. In this context we also make some remarks concerning the resolvent function and the spectral radius.

1 Introduction

The starting point of the theory of quotient spaces was a series of papers of L. Waelbroeck (see [4], [5], [5]). He defines quotient Banach space as a linear space of the form X/X_0, where X is a Banach space and X_0 is a Banach subspace of X, namely X_0 has its own structure of a Banach space which makes the inclusion $X_0 \to X$ be continuous. First, L. Waelbroeck defines the notion of strict morphism between two quotient Banach spaces. This is a linear mapping $T : X/X_0 \to Y/Y_0$ induced by an operator $T_1 \in B(X, Y)$ such that $T_1X_0 \subset Y_0$, meaning $T(x + X_0) = T_1x + Y_0$. If the strict morphism T is induced by a surjective operator $T_1 \in B(X, Y)$ such that $T_1^{-1}(Y_0) = X_0$ then T is called pseudo-isomorphism. In the end L. Waelbroeck defines the morphism as a composition of strict morphisms and inverses of pseudo-isomorphisms.

By a morphism (see [3]) F.-H. Vasilescu understands a linear mapping $T : X/X_0 \to Y/Y_0$ such that

$$G_0(T) := \{(x, y) \in X \times Y \ : \ y \in T(x + X_0)\}$$

is a Banach subspace in $X \times Y$. His definition looks different but their categories are isomorphic.

The reason why I work in quotient Hilbert spaces is simply because each morphism is strict (see [2]). The results are also true in the more general case of quotient Banach spaces as L.Waelbroeck shows in [5], [6], but in our context the proofs are different.

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In this context we also make some remarks concerning the resolvent function and the spectral radius. I have to mention that some of ideas in the proofs come from M.Akkar and H.Arroub (see [1]).

2 The spectrum of a morphism

Let H/H_0 be a quotient Banach space and $B(H/H_0)$ the family of all morphism from H/H_0 into H/H_0. First, we mention that $B(H/H_0)$ can be regarded as a quotient Banach algebra A/α where

$$A = \{T_1 \in B(H) \ : \ T_1 H_0 \subset H_0\} \ ;$$

$$\alpha = \{T_1 \in B(H) \ : \ T_1 H \subset H_0\} = B(H, H_0) \ .$$

A is a Banach algebra with the norm

$$\| \ T_1 \ \|_A = max\{\| \ T_1 \ \|_{B(H)} \ , \ \| \ T_1 \ |_{H_0} \|_{B(H_0)}\} \ .$$

α is a Banach algebra with the norm

$$\| \ T_1 \ \|_\alpha = \| \ T_1 \ \|_{B(H,H_0)} \ .$$

We denote by T_0 the restriction of $T_1 \in A$ to H_0.
If $T \in B(H/H_0)$ we can define the spectrum of T in the classic way

$$\sigma(T) := \{z \in \mathbb{C} \ : \ z - T \text{ is not bijective}\}.$$

If $T \in B(H/H_0)$ is induced by $T_1 \in A$, we can speak about the spectrum of T as being the spectrum of $\hat{T} + \alpha \in A/\alpha$ in quotient Banach algebra A/α and we denote it by $\sigma_{A/\alpha}(\hat{T})$.
Obviously, we can consider the quotient Banach algebra $A/\overline{\alpha}$. We also denote by $\hat{T}$ the element $\hat{T} = T_1 + \overline{\alpha} \in A/\overline{\alpha}$, in hope of no ambiguity. Therefore we can speak about the spectrum $\sigma_{A/\overline{\alpha}}(\hat{T})$.
Another approach of spectrum is obtained by using the complexes. We denote this spectrum by $\sigma(T; H/H_0)$. Namely, we say that $z \in \mathbb{C} \setminus \sigma(T; H/H_0)$ if the sequence

$$0 \to H_0 \xrightarrow{i_z} H \times H_0 \xrightarrow{j_z} H \to 0$$

is exact, where

$$i_z(x_0) = (x_0, (z - T_1)x_0)$$

$$j_z(x, x_0) = (T_1 - z)x + x_0$$

for $x_0 \in H_0 \ , \ x \in H$.

Theorem 1. $\sigma\left(T\right) = \sigma\left(T; H/H_0\right) = \sigma_{A/\alpha}\left(\hat{T}\right) = \sigma_{A/\overline{\alpha}}\left(\hat{T}\right)$.

Proof. We begin by proving the first equality. It is obvious that the considered sequence is a complex of Hilbert spaces. This means

$$R(i_z) \subset Ker(j_z) \Leftrightarrow j_z(x_0, (z - T_1)x_0) = 0 \Leftrightarrow (T_1 - z)x_0 + (z - T_1)x_0 = 0$$

which is true. Obviously i_z is always injective. Let's note that:
(1) z - T is surjective $\Leftrightarrow R(j_z) = H$;
(2) z - T is injective $\Leftrightarrow Ker(j_z) \subset R(i_z)$.
Indeed, z - T is injective

$$\Leftrightarrow [(z - T)(x + H_0) = H_0 \Rightarrow x \in H_0]$$

$$\Leftrightarrow [(z - T_1)x + H_0 = H_0 \Rightarrow x \in H_0] \Leftrightarrow [(z - T_1)x \in H_0 \Rightarrow x \in H_0]\ .$$

On the other hand

$$Ker(j_z) \subset R(i_z) \Leftrightarrow [(x, x_0) \in Ker(j_z) \Rightarrow (x, x_0) \in R(i_z)] \Leftrightarrow$$

$$[(T_1 - z)x + x_0 = 0 \Rightarrow (x, x_0) \in R(i_z)] \Leftrightarrow$$

$$[x_0 = (z - T_1)x \Rightarrow (x, (z - T_1)x) \in R(i_z)] \Leftrightarrow$$

$$[(z - T_1)x \in H_0 \Rightarrow x \in H_0]\ .$$

(1) and (2) show us that $z \in \mathbb{C} \setminus \sigma(T)$ if and only if $z \in \mathbb{C} \setminus \sigma(T; H/H_0)$.
The next is to prove the equality $\sigma(T) = \sigma_{A/\alpha}(\hat{T})$. In fact, we have to prove that

$$z \in \mathbb{C} \setminus \sigma(T),$$

namely

$$S = z - T$$

is bijective and $S^{-1} \in B(H/H_0)$, is equivalent to $\hat{S} = S_1 + \alpha \in A/\alpha$ is invertible in A/α, which means that $z \in \mathbb{C} \setminus \sigma_{A/\alpha}(\hat{T})$.
" $\Rightarrow$ " If S is invertible, let $S^{-1} \in B(H/H_0)$ its invers induced by $V_1 \in A$. Let $\hat{V} = V_1 + \alpha \in A/\alpha$. We have

$$SS^{-1} = I_{H/H_0} \Leftrightarrow SS^{-1}(x + H_0) = x + H_0 \Leftrightarrow S_1(V_1 x + H_0) = x + H_0 \Leftrightarrow$$

$$S_1 V_1 x + H_0 = x + H_0 \Leftrightarrow (S_1 V_1 - I)x \in H_0\ ,\ (\forall)x \in H \Leftrightarrow$$

$$S_1 V_1 - I \in \alpha \Leftrightarrow \hat{S}\hat{V} = I_{A/\alpha}\ .$$

In the same way we obtain

$$S^{-1}S = I_{H/H_0} \Leftrightarrow V_1 S_1 - I \in \alpha \Leftrightarrow \hat{V}\hat{S} = I_{A/\alpha}\ .$$

Hence, $\hat{S}$ is invertible in A/α.
" $\Leftarrow$ " If $\hat{S} = S_1 + \alpha$ is invertible in A/α let $\hat{V} = V_1 + \alpha$ its inverse. Let $S \in B(H/H_0)$ induced by S_1 and $V \in B(H/H_0)$ induced by V_1. Then

$$\hat{S}\hat{V} = I_{A/\alpha} \Leftrightarrow S_1 V_1 - I \in \alpha \Leftrightarrow SV = I_{H/H_0}$$

$$\hat{V}\hat{S} = I_{A/\alpha} \Leftrightarrow V_1 S_1 - I \in \alpha \Leftrightarrow VS = I_{H/H_0}$$

and therefore S is invertible in $B(H/H_0)$.
Finally, we prove the last equality $\sigma_{A/\alpha}(\hat{T}) = \sigma_{A/\overline{\alpha}}(\hat{T})$.
Let $hull(\alpha)$ the set of maximal ideals of A which contains α, or, in other words, the set of all characters χ of A with the property $\alpha \subset Ker(\chi)$. This set is compact and non empty.
We denote by $\hat{T}_G$ the Gelfand transform of $\hat{T}$, namely

$$\hat{T}_G : hull(\alpha) \to \mathbb{C} \quad , \quad \hat{T}_G(\chi) = \chi(\hat{T}) \ .$$

It is obvious that this mapping is well defined, namely it does not depends on the choice of T_1 for $\hat{T}$, because $\alpha \subset Ker(\chi)$.
Let $z \in \mathbb{C}$. Then $z \in \sigma_{A/\alpha}(\hat{T}) \Leftrightarrow (z - T_1)A + \alpha \neq A \Leftrightarrow (\exists)\alpha'$ a maximal ideal of A which contains α and $z - T_1$

$$\Leftrightarrow (\exists)\chi \in hull(\alpha) \ : \ z - T_1 \in Ker(\chi)$$

$$\Leftrightarrow (\exists)\chi \in hull(\alpha) \ : \ z = \hat{T}_G(\chi).$$

Thus we obtain that $\sigma_{A/\alpha}(\hat{T}) = \hat{T}_G(hull(\alpha))$.
In the same way we obtain that $\sigma_{A/\overline{\alpha}}(\hat{T}) = \hat{T}_G(hull(\overline{\alpha}))$.
If we show that $hull(\alpha) = hull(\overline{\alpha})$ the proof will be complete. Let $M \in hull(\alpha)$. As every maximal ideal of A is closed, the fact that $\alpha \subset M$ implies that $\overline{\alpha} \subset M$, hence $M \in hull(\overline{\alpha})$. As the converse inclusion is evident we obtain the desired result. □

Remark 2. *The first equality can be used to proof that the spectrum of a morphism is a compact set.*

Let $z \in \mathbb{C} - \sigma(T)$. Then the sequence

$$0 \to H_0 \xrightarrow{i_z} H \times H_0 \xrightarrow{j_z} H \to 0$$

is exact.
As the exactness is invariant under small perturbations, it exists $\epsilon > 0$ such that for all $z' \in \mathbb{C} \ : \mid z' - z \mid < \epsilon$ we have that the sequence

$$0 \to H_0 \xrightarrow{i_{z'}} H \times H_0 \xrightarrow{j_{z'}} H \to 0$$

is exact. Hence $D(z, \epsilon) \subset \rho(T) = \mathbb{C} \setminus \sigma(T)$ and therefore $\rho(T)$ is open, namely $\sigma(T)$ is closed.

Proposition 3. $\sigma_A(T_1) = \sigma_{B(H)}(T_1) \cup \sigma_{B(H_0)}(T_0)$.

Proof. If $z \notin \sigma_A(T_1)$ then $z - T_1$ is invertible in A, namely it exists $S_1 \in A$ such that $(z - T_1)S_1 = S_1(z - T_1) = I_H$. Hence, $z - T_1$ is invertible in $B(H)$ and so $z \notin \sigma_{B(H)}(T_1)$. Let $S_0 = S_1 |_{H_0} \in B(H_0)$. The previous equality tells us that

$$(z - T_0)S_0 = S_0(z - T_0) = I_{H_0}$$

and so $z - T_0$ is invertible in $B(H_0)$, namely $z \notin \sigma_{B(H_0)}(T_0)$.

Conversely, if $z \notin \sigma_{B(H)}(T_1) \cup \sigma_{B(H_0)}(T_0)$ then $U_1 = z - T_1$ is invertible in $B(H)$, namely it exists $S_1 \in B(H)$ such that $U_1S_1 = S_1U_1 = I_H$ and $U_0 = z - T_0$ is invertible in $B(H_0)$ and therefore it exists $V_0 \in B(H_0)$ such that $U_0V_0 = V_0U_0 = I_{H_0}$. It results that $U_1S_1x = U_0V_0x$ for all $x \in H_0$, hence $U_1S_1x = U_1V_0x$. As U_1 is injective we obtain $S_1x = V_0x$ and so $S_1H_0 \subset H_0$, hence $S_1 \in A$ and $z - T_1$ is invertible in A, namely $z \notin \sigma_A(T_1)$. □

Proposition 4. *The union of any two of the sets* $\sigma(T), \sigma_{B(H)}(T_1)$, $\sigma_{B(H_0)}(T_0)$ *contains the third.*

Proof. (1) $\sigma(T) \subset \sigma_{B(H)}(T_1) \cup \sigma_{B(H_0)}(T_0)$.
Let $z \in \mathbb{C} \setminus \sigma_A(T_1)$. Then $S_1 = z - T_1$, is invertible in A. Let $V_1 \in A$ its inverse and $V \in B(H/H_0)$ the morphism induced by V_1. We have

$$(z - T)V(x + H_0) = (z - T)(V_1x + H_0) =$$

$$= (z - T_1)V_1x + H_0 = S_1V_1x + H_0 = x + H_0 \, .$$

Similarly, $V(z - T)(x + H_0) = x + H_0$. Hence $z - T$ is invertible in H/H_0, namely $z \in \mathbb{C} \setminus \sigma(T)$. The proof is complete, via previous proposition.
(2) $\sigma_{B(H)}(T_1) \subset \sigma_{B(H_0)}(T_1) \cup \sigma(T)$.
Let $z \notin \sigma_{B(H_0)}(T_0) \cup \sigma(T)$ and we assume that $(z - T_1)x = 0$. Then $(z - T)(x + H_0) = (z - T_1)x + H_0 = H_0$. As $z - T$ is injective it results that $x \in H_0$. Next $(z - T_1)x = 0$ it implies that $(z - T_0)x = 0$. But $z - T_0$ is injective and therefore x=0 and so $z - T_1$ is injective.
Let now $y \in H$. As z-T is surjective it exists $x + H_0 \in H/H_0$ such that

$$(z - T)(x + H_0) = y + H_0$$

But $(z - T)(x + H_0) = (z - T_1)x + H_0$. It results that $(z - T_1)x + h_0 = y$, where $h_0 \in H_0$. But $z - T_0$ is surjective and therefore it exists $k_0 \in H_0 \, : \, (z - T_0)k_0 = h_0$, or $(z - T_1)k_0 = h_0$. Then $y = (z - T_1)(x + k_0)$ and therefore $z - T_1$ is surjective.
(3) $\sigma_{B(H_0)}(T_0) \subset \sigma(T) \cup \sigma_{B(H)}(T_1)$.
We assume that $z \notin \sigma(T) \cup \sigma_{B(H)}(T_1)$. Let $x_0 \in H_0$ such that $(z - T_0)x_0 = 0$. Then $(z - T_1)x = 0$ and it results that $x_0 = 0$ because $z - T_1$

is injective. Hence $z - T_0$ is injective.
Let now $y_0 \in H_0$. We search for $x_0 \in H_0 \ : \ (z - T_0)x_0 = y_0$. As $z - T_1$ is surjective it exists $x_0 \in H \ : \ (z - T_1)x_0 = y_0$. We have

$$(z - T)(x_0 + H_0) = (z - T_1)x_0 + H_0 = y_0 + H_0 = H_0 \ .$$

But z-T is injective. Consequently $x_0 \in H_0$. We have found $x_0 \in H_0$ such that $(z - T_1)x_0 = y_0$. It results that $(z - T_0)x_0 = y_0$ and so $z - T_0$ is surjective. $\square$

Remark 5. $\sigma(T) \subset \sigma_A(T_1)$.

3 The resolvent function

As we have seen the spectrum of a morphism $\sigma(T)$ is a compact set and its complement $\rho(T)$ will be called the resolvent set. Of course, we can consider the mapping

$$R : \rho(T) \to B(H/H_0) \quad , \qquad R(z) = (z - T)^{-1}$$

which we call resolvent function and we naturally question if, as in the classic case, we can speak about an analicity of the resolvent function. Unfortunately, the values of this function are in a quotient Banach space (using the identification $B(H/H_0) = A/\alpha$). A further idea to continue is given by a notation of L.Waelbroeck (see [6]): $O(U, A/\alpha) := O(U, A)/O(U, \alpha)$ where $O(U, A)$ is the Fr´echet space of analytic function $f : U \to A$ and $O(U, \alpha)$ is a Fr´echet subspace of this. For the moment we will not obtain an " analicity " of resolvent function, only the fact that it is of " class $\mathbb{C}^\infty$". Thus, in a similar way we consider $\mathbb{C}^\infty(U, A)$, the unitary algebra of functions $f : U \to A$ of class $\mathbb{C}^\infty$, in which $\mathbb{C}^\infty(U, \alpha)$ is a bilateral ideal. We define $\mathbb{C}^\infty(U, A/\alpha) = \mathbb{C}^\infty(U, A)/\mathbb{C}^\infty(U, \alpha)$. The result which we will obtain is: the element $z - T$ is invertible in algebra $\mathbb{C}^\infty(\rho(T), A/\alpha)$. This means that it exists $f : \rho(T) \to A$ of class $\mathbb{C}^\infty$ such that $R(z) = f\hat{(z)}$, where $f\hat{(z)}$ is the class of equivalence of f(z) modulo α, and the equalities $(z-T)R(z) = R(z)(z-T) = I_{H/H_0}$ are in $\mathbb{C}^\infty(\rho(T), A/\alpha)$, i.e. the functions

$$g : \rho(T) \to \alpha \quad , \quad g(z) = (z - T_1)f(z) - I_H$$

$$h : \rho(T) \to \alpha \quad , \quad h(z) = f(z)(z - T_1) - I_H$$

are from $\mathbb{C}^\infty(\rho(T), \alpha)$. The idea of the proof comes from the same article of L. Waelbroeck, with the necessary changes due to the fact that the algebra A which appears is noncommutative and also because we work with a single morphism.

Theorem 6. *z-T is invertible in* $\mathbb{C}^\infty(\rho(T), A/\alpha)$.

Proof. Let $\lambda \in \rho(T)$. Then

$$(\exists)S_\lambda \in B(H/H_0)\ :\ (\lambda - T)S_\lambda = S_\lambda(\lambda - T) = 1_{H/H_0} \quad (*)$$

But $S_\lambda \in B(H/H_0)$ implies that there exists $V_\lambda \in B(H)\ :\ V_\lambda H_0 \subset H_0$ which induces S_λ. Therefore, the equalities $(*)$ become

$$(\lambda - T_1)V_\lambda - I_H \in \alpha \qquad ; \qquad V_\lambda(\lambda - T_1) - I_H \in \alpha$$

namely, it exists $A_\lambda, B_\lambda \in \alpha$ such that

$$(\lambda - T_1)V_\lambda - A_\lambda = I_H \quad (1)$$

$$V_\lambda(\lambda - T_1) - B_\lambda = I_H \quad (2)$$

Let's note that

$$(z - T_1)V_\lambda - A_\lambda = (z-\lambda)V_\lambda + (\lambda - T_1)V_\lambda - A_\lambda = (z-\lambda)V_\lambda + I_H\ .$$

If we choose $r = \frac{1}{2\|V_\lambda\|_A}$, for $z \in B(\lambda, r)$ this element is invertible in algebra A. Moreover, the function

$$\Phi_\lambda : B(\lambda, r) \to A \quad , \quad \Phi_\lambda(z) = (I_H + (z-\lambda)V_\lambda)^{-1}$$

is analytic, in particular it is of class $\mathbb{C}^\infty$.
Thus, we can consider the following function of class $\mathbb{C}^\infty$:

$$f_\lambda : B(\lambda, r) \to A \quad , \quad f_\lambda(z) = V_\lambda \Phi_\lambda(z)\ ;$$

$$g_\lambda : B(\lambda, r) \to \alpha \quad , \quad g_\lambda(z) = A_\lambda \Phi_\lambda(z)\ ;$$

$$h_\lambda : B(\lambda, r) \to \alpha \quad , \quad h_\lambda(z) = \Phi_\lambda(z) B_\lambda\ .$$

We also note that

$$(z - T_1)f_\lambda(z) - g_\lambda(z) = (z - T_1)V_\lambda\Phi(z) - A_\lambda\Phi(z) =$$
$$= [(z - T_1)V_\lambda - A_\lambda]\Phi(z) = [(z-\lambda)V_\lambda + I_H]\Phi(z) = I_H;$$
$$f_\lambda(z)(z - T_1) - h_\lambda(z) = V_\lambda\Phi(z)(z - T_1) - \Phi(z)B_\lambda =$$
$$= \Phi(z)[V_\lambda(z - T_1) - B_\lambda] = \Phi(z)[V_\lambda(\lambda - T_1) + V_\lambda(z-\lambda) - B_\lambda] =$$
$$= \Phi(z)[I_H + V_\lambda(z-\lambda)] = I_H\ .$$

Hence, the considered functions satisfy

$$(z - T_1)f_\lambda(z) - g_\lambda(z) = I_H \quad (1')$$

$$f_\lambda(z)(z - T_1) - h_\lambda(z) = I_H \quad (2')$$

We note that $\{B(\lambda, r)\}_{\lambda \in \rho(T)}$ is on open coverage of $\rho(T)$. The theorem of existance of a partition of unity leads to the existance of a family $\{\psi_\lambda\}_{\lambda \in \rho(T)}$ of positive functions of class $\mathbb{C}^\infty$ with the following properties:

1. $supp(\psi_\lambda) \subset B(\lambda, r) \quad , \quad (\forall)\lambda \in \rho(T);$

2. $(\forall)K \subset \rho(T)$ a compact set there exists a finite number of functions ψ_λ which are not identically nule on K;

3. $\sum\limits_{\lambda \in \rho(T)} \psi_\lambda(z) = 1 \quad , \quad (\forall)z \in \rho(T).$

We define

$$f'_\lambda : \rho(T) \to A \quad , \quad f'_\lambda(z) = \begin{cases} \psi_\lambda(z) f_\lambda(z), & \text{if } z \in B(\lambda, r) \\ 0, & \text{else} \end{cases}$$

$$g'_\lambda : \rho(T) \to \alpha \quad , \quad g'_\lambda(z) = \begin{cases} \psi_\lambda(z) g_\lambda(z), & \text{if } z \in B(\lambda, r) \\ 0, & \text{else} \end{cases}$$

$$h'_\lambda : \rho(T) \to \alpha \quad , \quad h'_\lambda(z) = \begin{cases} \psi_\lambda(z) h_\lambda(z), & \text{if } z \in B(\lambda, r) \\ 0, & \text{else} \end{cases}$$

Then, we define

$$f : \rho(T) \to A \quad , \quad f(z) = \sum_{\lambda \in \rho(T)} f'_\lambda(z);$$

$$g : \rho(T) \to \alpha \quad , \quad g(z) = \sum_{\lambda \in \rho(T)} g'_\lambda(z);$$

$$h : \rho(T) \to \alpha \quad , \quad h(z) = \sum_{\lambda \in \rho(T)} h'_\lambda(z).$$

By the previous construction they are of class $\mathbb{C}^\infty$ and they satisfy

$$(z - T_1)f(z) - g(z) = I_H \quad ; \quad f(z)(z - T_1) - h(z) = I_H, \quad (\forall)z \in \rho(T)\ .$$

These relations can also be written

$$(z - T_1)f(z) - I_H = g(z) \quad ; \quad f(z)(z - T_1) - I_H = g(z), \quad (\forall)z \in \rho(T)\ .$$

These relations show that z-T is invertible in $\mathbb{C}^\infty(\rho(T), A/\alpha)$ and its inverse is $R(z) = f\hat{(z)}$, $(\forall)z \in \rho(T)$. □

Remark 7. *In other words, the result of this theorem can be rewritten: it exists* $f \in \mathbb{C}^\infty(\rho(T), A)$*;* $g,\ h \in \mathbb{C}^\infty(\rho(T), \alpha)$ *such that*

$$(z - T_1)f(z) - g(z) = f(z)(z - T_1) - h(z) = I_H, \quad (\forall)z \in \rho(T).$$

We are asking if the functions f,g,h can be extended to functions of class $\mathbb{C}^\infty$ *on* $\mathbb{C}$ *and the equalities rest true.*

Theorem 8. *Let U be an open neighborhood for $\sigma(T)$. Then there exists $f' \in \mathbb{C}^\infty(\mathbb{C}, A)$; $g', h' \in \mathbb{C}^\infty(\mathbb{C}, \alpha)$ and $u' : \mathbb{C} \to [0,1]$ a function of class $\mathbb{C}^\infty$ with compact support contained in U, such that*

$$(z - T_1)f'(z) - g'(z) = f'(z)(z - T_1) - h'(z) = I_H - u'(z) \quad , \quad (\forall) z \in \mathbb{C} \, .$$

Proof. As U is an open neighbourhood for the compact $\sigma(T)$, the theorem of existance of a partition of unity yiels the function $u' : \mathbb{C} \to [0,1]$ of class $\mathbb{C}^\infty$ with the support contained in U, such that $u' = 1$ on a neighbourhood of $\sigma(T)$. We define

$$f' : \mathbb{C} \to A \quad , \quad f'(x) = \begin{cases} (I_H - u'(z))f(z), & \text{if } z \in \rho(T) \\ 0, & \text{else} \end{cases} \quad ;$$

$$g' : \mathbb{C} \to \alpha \quad , \quad g'(z) = \begin{cases} (I_H - u'(z))g(z), & \text{if } z \in \rho(T) \\ 0, & \text{else} \end{cases} \quad ;$$

$$h' : \mathbb{C} \to \alpha \quad , \quad h'(z) = \begin{cases} (I_H - u'(z))h(z), & \text{if } z \in \rho(T) \\ 0, & \text{else} \end{cases} \quad .$$

Obviously, they are functions of class $\mathbb{C}^\infty$ and we have

$$(z - T_1)f'(z) - g'(z) = f'(z)(z - T_1) - h'(z) = I_H - u'(z) \quad , \quad (\forall) z \in \mathbb{C}.$$

□

4 Spectral radius

Definition 9. *Let $T \in B(H/H_0)$. We define the spectral radius of T by*

$$r(T) = sup\{\mid z \mid \, ; \; z \in \sigma(T)\}.$$

As we can see this definition is the same as in the classic case, with the only mention that the spectral radius was enjoying the properties

$$r(T) = \lim_{n \to \infty} \parallel T^n \parallel^{1/n} ; \quad r(T) \leq \parallel T \parallel .$$

Let us see how we can establish a similar thing. For this we can consider on A/α a semi-norm, the one induced by the norm of A

$$\parallel T \parallel_{A/\alpha} = \inf_{T_0 \in \alpha} \parallel T_1 + T_0 \parallel_A .$$

where $T_1 \in A$ induces T.
As $\sigma(T) \subset \sigma_A(T_1)$, we have that $r(T) \leq r_A(T_1) \leq \parallel T_1 \parallel_A$. But the previous inclusion takes place for any operator T_1 which induces T and so

$$r(T) \leq \parallel T_1 + T_0 \parallel , (\forall) T_0 \in \alpha.$$

Consequently
$$r(T) \leq \inf_{T_0 \in \alpha} \| T_1 + T_0 \|_A$$
and therefore $r(T) \leq \| T \|_{A/\alpha}$.
We denote by B the completion of semi-normed algebras $(A/\alpha, \| \cdot \|_{A/\alpha})$. Then $\sigma_B(T) \subset \sigma(T)$ and therefore $r_B(T) \leq r(T)$. Hence
$$\lim_{n\to\infty} \| T^n \|_{A/\alpha}^{1/n} \leq r(T).$$
Concluding, we have:

Proposition 10. *Let $T \in B(H/H_0)$. Then*
$$\lim_{n\to\infty} \| T^n \|_{A/\alpha}^{1/n} \leq r(T) \leq \| T \|_{A/\alpha} .$$

References

[1] M.Akkar; H.Arroub, *Calcul fonctionnel harmonique dans une algebre de Banach involutive quotient*, Bull. Belg. Math. Soc **3**, No. 5 (1996), 521-535. MR1425280 (97k:46054). Zbl 0873.46027.

[2] A. Terescenco, F. Turcu, S. Nădăban, *A definition of the adjoint between quotient Hilbert spaces*, talk at the 17^{th} International Conference of Operator Theory, Timişoara, 1998.

[3] F.-H.Vasilescu, *Spectral Theory in Quotient Fr´echet Spaces I*, Revue Roumaine Mathematique Pures Applique, **32**(1987), 561-579. MR0900363 (88h:47004). Zbl 0665.46058.

[4] L.Waelbroeck, *Quotient Banach Spaces*, Banach Center Publications, Vol.8, Warszaw, (1982), 553-562. MR0738315 (85m:46072a). Zbl 0492.46013.

[5] L.Waelbroeck, *The Taylor Spectrum and Quotient Banach Spaces*, Banach Center Publications, Vol.8, Warszaw, (1982), 573-578. MR0738317 (85m:46072c). Zbl 0492.46014.

[6] L.Waelbroeck, *Holomorfic Functional Calculus and Quotient Banach Algebra*, Studia Math., **75**(1983), 273-286. MR0722252 (85j:46088a). Zbl 0492.46039.

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