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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTION TO A SINGULAR ELLIPTIC PROBLEM

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Abstract. In this paper we obtain existence results for the positive solution of a singular elliptic boundary value problem. To prove the main results we use comparison arguments and the method of sub-super solutions combined with a procedure which truncates the singularity.

1 Introduction

This paper contains contribution of a technical nature to the study of positive solutions of the equations

$$-\Delta u+c(x)u^{-1}\left|\nabla u\right|^2 = a(x) \text{ for } x\in\mathbb{R}^N,\ u>0 \text{ in } \mathbb{R}^N,\ u(x)\to 0 \text{ as } |x|\to\infty \quad (1.1)$$

where $N>2$, $a: R^N \to R$ is a function satisfying the following conditions

AC1) $a, c \in C^{0,\alpha}_{loc}(R^N)$ for some $\alpha\in(0,1)$;

AC2) $a(x)>0, c(x)>0$ for all $x\in R^N$;

A3) for $\varphi(r)=\max_{|x|=r} a(x)$ we have

$$\int_0^\infty r\varphi(r)dr<\infty.$$

Problems like (1.1) has been intensively studied. Our study is motivated by the works of Shu [17], Arcoya, Carmona, Leonori, Aparicio, Orsina and Petitta [2], Arcoya, Barile and Aparicio [3] where the existence, non-existence and uniqueness of solution for the problem like (1.1) are solved.

In this article we present a new argument in the study of the problem (1.1) more simple that used in [2], [3], [17] and where the problem is considered just in the case when $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary.

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The above equation contains different quantities, such as: singular nonlinear term (like u^{-1}), convection nonlinearity (denoted by $|\nabla u|^2$), as well as potentials (c and a). The principal difficulty in the treatment of (1.1) is due to the singular character of the equation combined with the nonlinear gradient term.

The importance of the problem (1.1) is given considering the well know problem

$$\Delta u = a(x)h(u),\ u > 0 \ in \ \ \Omega, \qquad u(x) = \infty \ \ as \ \ x \to \partial\Omega, \tag{1.2}$$

because we can easily deduce the following two remarks:

Remark 1. *When $h(u) = e^u$, by a transformation of the form $w = e^{-u}$ the problem (1.2) becomes*

$$-\Delta w + \frac{|\nabla w|^2}{w} = a(x),\ w > 0 \ in \ \Omega,\ w\,(x) \to 0 \ as \ \ x \to \partial\Omega, \tag{1.3}$$

but this is the problem (1.1) when $c(x) = 1$.

Remark 2. *For $h(u) = u^\delta$ $(\delta > 1)$ and $w = C[u]^{-C^{-1}}, (C := 1/(\delta - 1))$ in (1.2) we have*

$$-\Delta w + \delta C\frac{|\nabla w|^2}{w} = a(x),\ w > 0,\ in \ \Omega,\ w \to 0 \ as \ \ x \to \partial\Omega, \tag{1.4}$$

which is the problem (1.1) when $c(x) = \delta C$.

This finish the motivation of our work.

The main results of the article are:

Theorem 3. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$ and $a,\ c \in C^{0,\alpha}(\overline{\Omega})$, $a(x) > 0$, $c(x) > 0$ for any $x \in \overline{\Omega}$, then the problem*

$$-\Delta u + c(x)u^{-1}\left|\nabla u\right|^2 = a(x) \ in \ \ \Omega,\ u_{|\partial\Omega} = 0, \tag{1.5}$$

has at least a positive solution $u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

In the next result we establish sufficient condition for the existence of solution to the problem (1.1) in the case when $\Omega = \mathbb{R}^N$.

Theorem 4. *We suppose that hypotheses AC1), AC2), A3) are satisfied. Then, the problem (1.1) has a $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ positive solution vanishing at infinity. If, in addition,*

$$\lim_{|x|\to\infty} |x|^\mu \varphi(|x|) < \infty, \tag{1.6}$$

for some $\mu \in (2, N)$, then

$$u(x) = O(|x|^{2-\mu}) \ as \ \ |x| \to \infty. \tag{1.7}$$

To prove the existence of such a solution to (1.1) we establish some preliminary results.

2 Preliminary results

Since we apply sub and super solution method due to Amann [1], we recall the following definition of sub and super solution which are our main tools in the proof of the solvability of problem (1.1).

For $f_1(x,\eta,\xi) : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $g_1 : \partial\Omega \to \mathbb{R}$, Amann introduce the following definitions:

Definition 5. *A function $\underline{u} \in C^{2,\alpha}(\overline{\Omega})$ is called a sub solution for the problem*

$$-\Delta u = f_1(x,u,\nabla u) \text{ in } \Omega,\ u = g \text{ on } \partial\Omega, \tag{2.1}$$

if

$$-\Delta \underline{u} \le f_1(x,\underline{u},\nabla \underline{u}) \text{ in } \Omega,\ \underline{u} = g \text{ on } \partial\Omega.$$

Definition 6. *A function $\overline{u} \in C^{2,\alpha}(\overline{\Omega})$ is called a super solution of the problem (2.1) if*

$$-\Delta \overline{u} \ge f_1(x,\overline{u},\nabla \overline{u}) \text{ in } \Omega,\ \overline{u} = g \text{ on } \partial\Omega.$$

One of the important results from [1] is:

Lemma 7. *Let Ω be a bounded domain from $\mathbb{R}^N$, with boundary $\partial\Omega$ of class $C^{2,\alpha}$ for some $\alpha\in(0,1)$, $g\in C^{2,\alpha}(\partial\Omega)$ and f_1 be a continuous function with the property that $\partial f_1/\partial\eta$, $\partial f_1/\partial \xi^i$, $i=\overline{1,N}$ exists and are continuous on $\overline{\Omega}\times\mathbb{R}^{N+1}$ and such that*

AM1) $f_1(\cdot,\eta,\xi)\in C^{\alpha}(\overline{\Omega})$, uniformly for (η,ξ) in bounded subsets of $\mathbb{R}\times\mathbb{R}^N$;

AM2)there exists a function $f_2 : \mathbb{R}_+ \to \mathbb{R}_+ := [0,\infty)$ such that

$$|f_1(x,\eta,\xi)| \le f_2(\rho)(1+|\xi|^2), \tag{2.2}$$

for every $\rho\ge 0$ and $(x,\eta,\xi)\in\overline{\Omega}\times[-\rho,\rho]\times\mathbb{R}^N$.

Under these assumption, if the problem (2.1) has a sub solution $\underline{u}$ and a super solution $\overline{u}$ such that $\underline{u}(x)\le\overline{u}(x)$, $\forall x\in\overline{\Omega}$ then there exists at least a function $u(x)\in C^{2+\alpha}(\overline{\Omega})$ which satisfies $\underline{u}(x)\le u(x)\le\overline{u}(x)$ for all $x\in\overline{\Omega}$ and satisfying (2.1) pointwise. More precisely, there exist a minimal solution $\widetilde{u}(x)\in[\underline{u}(x),\overline{u}(x)]$ and a maximal solution $\widetilde{\widetilde{u}}(x)\in[\underline{u}(x),\overline{u}(x)]$, in the sense that every solution $u(x)\in[\underline{u}(x),\overline{u}(x)]$ satisfies $\widetilde{u}(x)\le u(x)\le\widetilde{\widetilde{u}}(x)$.

We will need the following variant of the maximum principle:

Lemma 8. *Assume that Ω is a bounded open set in $\mathbb{R}^N$. If $u:\overline{\Omega}\to\mathbb{R}$ is a smooth function such that*

$$\begin{cases} -\Delta u \ge 0 & \text{in } \Omega,\\ u\ge 0 & \text{on } \partial\Omega,\end{cases}$$

then $u\ge 0$ in Ω.

This finishes the auxiliary results. Now we prove the announced Theorems.

3 Proof of the Theorem 3

In the following will we use similarly argument that were used by Crandall, Rabinowitz and Tartar [7], Noussair [15] and the author [6].

Let $\varepsilon \in (0,1)$. The existence will be established by solving the approximate problems

$$\begin{cases} -\Delta u + c(x)u^{-1}\left|\nabla u\right|^2 = a(x), & in \quad \Omega, \; u > \varepsilon \; in \quad \Omega, \\ \qquad\qquad\qquad\qquad u = \varepsilon, & \qquad\qquad\qquad on \; \partial\Omega. \end{cases} \tag{3.1}$$

For this, let φ_1 be the first positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$-\Delta u(x) = \lambda u(x), \; in \quad \Omega, \; u_{|\partial\Omega}(x) = 0. \tag{3.2}$$

It is well known that $\varphi_1 \in C^{2+\alpha}(\overline{\Omega})$. We note by $m_2 := \min_{x\in\overline{\Omega}} a(x)$ and $M_1 := \max_{x\in\overline{\Omega}} c(x)$ to prove that the function $\underline{u}(x) = \sigma_1\varphi_1^2 + \varepsilon$, where

$$0 < \sigma_1 \leq \min\left\{\frac{m_2}{2\lambda_1 \max_{x\in\overline{\Omega}} \varphi_1^2 + 4M_1 \max_{x\in\overline{\Omega}} \left|\nabla\varphi_1\right|^2}, 1\right\} \tag{3.3}$$

is a sub solution of (3.1) in the sense of Lemma 7. Indeed, by (3.3) we have

$$\begin{aligned} &-\Delta\underline{u} + c(x)\underline{u}^{-1}\left|\nabla\underline{u}\right|^2 - a(x) \leq -\Delta\underline{u} + M_1\underline{u}^{-1}\left|\nabla\underline{u}\right|^2 - m_2 \\ &\leq -2\sigma_1\varphi_1\Delta\varphi_1 - 2\sigma_1\left|\nabla\varphi_1\right|^2 + 4M_1\sigma_1\left|\nabla\varphi_1\right|^2 - m_2 \\ &= 2\sigma_1\lambda_1\varphi_1^2 - 2\sigma_1\left|\nabla\varphi_1\right|^2 + 4M_1\sigma_1\left|\nabla\varphi_1\right|^2 - m_2 \\ &\leq 2\sigma_1\lambda_1\varphi_1^2 + 4M_1\sigma_1\left|\nabla\varphi_1\right|^2 - m_2 \leq 0. \end{aligned}$$

In the next step we prove the existence of a super solution to the problem (3.1). For this, let $v \in C^{2+\alpha}(\overline{\Omega})$ be the unique solution of the problem

$$-\Delta y = a(x) \; in \quad \Omega, \; y(x) = 0 \text{ for } x \in \partial\Omega. \tag{3.4}$$

We observe that, $\overline{u} = v + \varepsilon \in C^{2+\alpha}(\overline{\Omega})$, fulfils

$$-\Delta\overline{u}(x) + c(x)\overline{u}^{-1}(x)\left|\nabla\overline{u}(x)\right|^2 = a(x) + c(x)\overline{u}^{-1}(x)\left|\nabla\overline{u}(x)\right|^2 \geq a(x) \text{ for } x \in \Omega.$$

Clearly, $\overline{u}$ is a super solution to (3.1). Now, since

$$\begin{cases} -\Delta[\overline{u} - \underline{u}] & \geq & a(x) + c(x)\underline{u}^{-1}\left|\nabla\underline{u}\right|^2 - a(x) \geq 0, & \text{in} \quad \Omega, \\ \overline{u} - \underline{u} & = & 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

follows from the maximum principle, *Lemma 8,* that $\underline{u}(x) \leq \overline{u}(x)$, $x \in \overline{\Omega}$.

We have obtained a sub solution $\underline{u} \in C^{2,\alpha}(\overline{\Omega})$ and a super solution $\overline{u} \in C^{2,\alpha}(\overline{\Omega})$ for the problem (3.1) such that $\underline{u} \leq \overline{u}$ in $\overline{\Omega}$ with the property from *Lemma 7*. Then, there exists $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\underline{u}(x) \quad \leq \quad u_\varepsilon(x) \quad \leq \quad \overline{u}(x), \qquad x \in \overline{\Omega}. \tag{3.6}$$

and satisfying (pointwisely) the problem (3.1).

The relation (3.6) shows that $u > 0$ in Ω. We remark that $\underline{u} = \sigma_1 v^2 + \varepsilon$, where σ_1 is a positive constant such that

$$0 < \sigma_1 \leq \min\left\{\frac{m_2}{\max_{x\in\overline{\Omega}}[2v + 4M_1 |\nabla v|^2]}, 1\right\}, \tag{3.7}$$

is again a sub solution of (3.1) with the same property from *Lemma 7*.

In this time we have obtained a function $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ that satisfies pointwisely the equivalently form of (3.1):

$$\begin{cases} -\Delta u + c(x)\,(u+\varepsilon)^{-1}\,|\nabla u|^2 = a(x), & in\ \Omega, \\ u > 0, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{3.8}$$

Moreover $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ is unique. Indeed, assume that the problem (3.8) has more that one solution and let v_ε the second solution. Let us show that $u_\varepsilon \leq v_\varepsilon$ or, equivalently, $u_\varepsilon\,(x) + \varepsilon \leq v_\varepsilon\,(x) + \varepsilon$ for any $x \in \overline{\Omega}$. Assume the contrary. Set

$$\alpha(x) := \frac{u_\varepsilon\,(x) + \varepsilon}{v_\varepsilon\,(x) + \varepsilon} - 1.$$

Since we have $[\alpha\,(x)]|_{\partial\Omega} = 0$ we deduce that $\max_{\overline{\Omega}} \alpha\,(x)$, exists and is positive. At that point, say x_0, we have $\nabla\alpha(x_0) = 0$ *and* $\Delta\alpha(x_0) \leq 0$, which implies

$$\Big(-(v_\varepsilon+\varepsilon)\,\Delta u_\varepsilon + (u_\varepsilon+\varepsilon)\,\Delta v_\varepsilon\Big)(x_0) \geq 0, \tag{3.9}$$

and

$$\frac{|\nabla u_\varepsilon(x_0)|^2}{(u_\varepsilon(x_0)+\varepsilon)^2} = \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon(x_0)+\varepsilon)^2}. \tag{3.10}$$

By (3.9) and (3.10) we have

$$\frac{a\,(x_0)}{u_\varepsilon(x_0)+\varepsilon} - \frac{a\,(x_0)}{v_\varepsilon(x_0)+\varepsilon} + c(x_0)\left(\frac{(v_\varepsilon+\varepsilon)^{-1}\,|\nabla v_\varepsilon|^2}{v_\varepsilon+\varepsilon} - \frac{(u_\varepsilon+\varepsilon)^{-1}\,|\nabla u|^2}{u_\varepsilon+\varepsilon}\right)(x_0) \geq 0, \tag{3.11}$$

or, equivalently

$$a\,(x_0)\,\frac{v_\varepsilon(x_0) - u_\varepsilon(x_0)}{(u_\varepsilon(x_0)+\varepsilon)\,(v_\varepsilon(x_0)+\varepsilon)} \geq 0. \tag{3.12}$$

which is a contradiction with $u_\varepsilon(x_0) > v_\varepsilon(x_0)$. So $u_\varepsilon(x) \leq v_\varepsilon(x)$ in $\overline{\Omega}$. A similar argument can be made to produce $v_\varepsilon(x) \leq u_\varepsilon(x)$ forcing $u_\varepsilon(x) = v_\varepsilon(x)$.

We will show that, for any smooth bounded subdomain Ω' of $\mathbb{R}^N$ there exists a constant $C_4 > 0$ such that

$$\|u_\varepsilon\|_{C^{2,\alpha}(\overline{\Omega}')} \leq C_4. \tag{3.13}$$

For any bounded $C^{2,\alpha}$-smooth domain $\Omega' \subset \mathbb{R}^N$, take Ω_1, Ω_2 and Ω_3 with $C^{2,\alpha}$-smooth boundaries, such that $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Note that

$$u_\varepsilon(x) \geq \underline{u}(x) > 0, \; \forall x \in \Omega_i, \; i = \overline{1,3}. \tag{3.14}$$

Let $h_\varepsilon(x) = a(x) - c(x)\left(u_\varepsilon(x) + \varepsilon\right)^{-1} \left|\nabla u_\varepsilon(x)\right|^2, x \in \overline{\Omega_3}$. Following, we use $C_{i=\overline{1,4}}$, to denote positive constants which are independent of ε.

Since $-\Delta u_\varepsilon(x) = h_\varepsilon(x)$, $x \in \overline{\Omega_3}$, we see by the interior gradient estimate theorem of Ladyzenskaya and Ural'tseva [11, Theorem 3.1, p. 266] that there exists a positive constant C_1 independent of ε such that

$$\max_{x \in \overline{\Omega}_2} \|\nabla u_\varepsilon(x)\| \leq C_1 \max_{x \in \overline{\Omega}_3} u_\varepsilon(x). \tag{3.15}$$

Using (3.6) and (3.15) we obtain that $\|\nabla u_\varepsilon\|$ is uniformly bounded on $\overline{\Omega}_2$. This final result, the property of a and c shows that $|h_\varepsilon|$ is uniformly bounded on $\overline{\Omega}_2$ and so $h_\varepsilon \in L^p(\Omega_2)$ for any $p > 1$.

Since $-\Delta u_\varepsilon(x) = h_\varepsilon(x)$ for $x \in \Omega_2$, we see from *[6]*, that there exists a positive constant C_2 independent of ε such that

$$\|u_\varepsilon\|_{W^{2,p}(\Omega_1)} \leq C_2(\|h_\varepsilon(x)\|_{L^p(\Omega_2)} + \|u_\varepsilon\|_{L^p(\Omega_2)}),$$

i.e. $\|u_\varepsilon\|_{W^{2,p}(\Omega_1)}$ is uniformly bounded.

Choose p such that $p > N$ and $p > N(1-\alpha)^{-1}$. Then by Sobolev's imbedding theorem, it follows that $\|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega}_1)}$ is uniformly bounded by a constant independent of ε.

Moreover, this say that $h_\varepsilon \in C^{0,\alpha}(\overline{\Omega}_1)$ and $\|h_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega}_1)}$, is uniformly bounded. Using this and the interior Schauder estimates (see *[6, 8]*), for solutions of elliptic equations (4.1) we have that there exists a positive constant C_3 independent of ε with the property

$$\|u_\varepsilon\|_{C^{2,\alpha}(\overline{\Omega}')} \leq C_3 \left(\|h_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega}_1)} + \sup_{\overline{\Omega}_1} u_\varepsilon \right). \tag{3.16}$$

Because $\|h_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega}_1)}$ is uniformly bounded, we see from (3.16) that

$$\|u_\varepsilon\|_{C^{2,\alpha}\left(\overline{\Omega}'\right)} \leq C_4. \tag{3.17}$$

Thus (3.13) is proved.

Set $\varepsilon := 1/n$ and $u_\varepsilon := u^n$. Since the sequence u^n is bounded in $C^{2,\alpha}\left(\overline{\Omega}'\right)$ for any bounded domain $\Omega' \subset\subset \Omega$ by (3.17), using the Ascoli-Arzela theorem and the standard diagonal process, we can find a subsequence of u^n, denote again by u^n and a function $u \in C^2\left(\overline{\Omega}'\right)$ such that $\|u^n - u\|_{C^2\left(\overline{\Omega}'\right)} \to 0$ for $n \to \infty$. In particular

$$\Delta u^n \textit{ respectively } a(x) - c(x)(u^n(x) + 1/n)^{-1} \left|\nabla u^n(x)\right|^2$$

converge for $n \to \infty$ in $\overline{\Omega}'$ to

$$\Delta u \textit{ respectively } a(x) - c(x)u(x)^{-1} |\nabla u(x)|^2 .$$

It follows that u is a solution of

$$-\Delta u = a(x) - c(x)u^{-1}(x) |\nabla u(x)|^2 , \text{ in } \overline{\Omega}', \tag{3.18}$$

of class $C^2(\overline{\Omega}')$, and hence of class $C^{2,\alpha}(\overline{\Omega}')$ by a standard regularity arguments based on Schauder estimates.

Since Ω' is arbitrary, we also see that $u \in C^{2,\alpha}(\Omega)$. We have obtained $u^n \overset{n\to\infty}{\to} u$ (pointwisely) in $C^{2,\alpha}(\Omega)$.

For $\varepsilon := 1/n \overset{n\to\infty}{\to} 0$ in (3.6) we have

$$\underline{u}_2(x) := \sigma_1 \varphi_1^2 \quad \leq \quad u(x) \quad \leq \quad \overline{u}^2(x) := v(x), \quad x \in \overline{\Omega}. \tag{3.19}$$

Moreover, by (3.18) and (3.19), we obtain

$$-\Delta u = a(x) - c(x)u^{-1} |\nabla u|^2 \textit{ a.e. in } \Omega, \ u > 0 \textit{ in } \Omega, \ u_{|\partial\Omega} = 0.$$

Thus $u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ is the solution of the problem (1.5).

4 Proof of the Theorem 4

To prove the existence of solution to (1.1) we consider the following boundary value problem

$$-\Delta u + c(x)u^{-1} |\nabla u|^2 = a(x), \quad u > 0 \text{ in } B_k, \ u = 0 \text{ on } \partial B_k, \tag{4.1}$$

where $B_k := \{x \in \mathbb{R}^N \,||x| < k\}$ is the ball of center 0 and radius $k = 1, 2, ...$ Put $\Omega = B_k$ in *Theorem 3*. Then the problem (4.1) has at least one solution $u_k \in C(\overline{B}_k) \cap C^{2,\alpha}(B_k)$, which satisfies

$$\underline{u}_2 \leq u_k \leq \overline{u}^2 \textit{ in } B_k, \tag{4.2}$$

for $\underline{u}_2$ (resp. $\overline{u}^2$) the corresponding functions from *Theorem 3* when $\Omega = B_k$. In outside of B_k we put $u_k = 0$. The resulting function is in $\mathbb{R}^N$. Now, we observe that

$$w(r) := \int_r^\infty \xi^{1-N} \int_0^\xi \sigma^{N-1} \varphi(\sigma) d\sigma d\xi, \ r := |x| \tag{4.3}$$

is the unique radial solution of the problem $-\Delta w = \varphi(| \ x \ |)$ in $\mathbb{R}^N$, $w > 0$ in $\mathbb{R}^N$, $w \overset{|x|\to\infty}{\to} 0$. We prove that w is bounded. Using integration by parts and L' Hôpital rule, we have

$$\begin{aligned}
&\int_r^\infty \xi^{1-N}\int_0^\xi \sigma^{N-1}\varphi(\sigma)d\sigma d\xi = -\frac{1}{N-2}\int_r^\infty \frac{d}{d\xi}\left(\xi^{2-N}\right)[\int_0^\xi \sigma^{N-1}\varphi(\sigma)d\sigma]d\xi \\
=\ & \frac{1}{N-2}\lim_{R\to\infty}\left\{\int_r^R \xi\varphi(\xi)d\xi - R^{2-N}\int_0^R \sigma^{N-1}\varphi(\sigma)d\sigma + r^{2-N}\int_0^r \sigma^{N-1}\varphi(\sigma)d\sigma\right\} \\
=\ & \frac{1}{N-2}\lim_{R\to\infty}\frac{R^{N-2}[\int_r^R \xi\varphi(\xi)d\xi + r^{2-N}\int_0^r \xi^{N-1}\varphi(\xi)d\xi] - \int_0^R \xi^{N-1}\varphi(\xi)d\xi}{R^{N-2}} \\
=\ & \frac{1}{N-2}\left[\int_r^\infty \xi\varphi(\xi)d\xi + r^{2-N}\int_0^r \xi^{N-1}\varphi(\xi)d\xi\right],\ R>r. \qquad (4.4)
\end{aligned}$$

Now, by the second mean value theorem for integrals follows that there exists $r_1 \in (0,r)$ such that

$$\begin{aligned}
\int_0^r \xi^{N-1}\varphi(\xi)d\xi &= \int_0^r \xi^{N-2}\xi\varphi(\xi)d\xi \\
&= r^{N-2}\int_{r_1}^r \xi\varphi(\xi)d\xi \le r^{N-2}\int_0^r \xi\varphi(\xi)d\xi \qquad (4.5)
\end{aligned}$$

for $N>2$. By (4.4)-(4.5) we obtain $w(r) \le K := \frac{1}{N-2}\int_0^\infty \xi\varphi(\xi)d\xi$. We observe, in addition, that w satisfies $-\Delta w(|x|) + c(x)w^{-1}(|x|)\left|\nabla w(|x|)\right|^2 \ge a(x)$, $x \in \mathbb{R}^N$, $0 < w \le K$ and $w(r) \to 0$ as $r \to \infty$.

We prove that

$$u_k \le w(|x|),\ \ x \in \mathbb{R}^N,\ k = 1,2,3,... \qquad (4.6)$$

Since $w(|x|) > 0$ in $\mathbb{R}^N$ and $u_k = 0$ in $\mathbb{R}^N \backslash B_k$ it is enough to prove that $u_k \le w$ in B_k, $k = 1,2,3,...$ To prove this we observe that $w \in C^2\left(\overline{B}_k\right)$ and

$$\begin{cases} -\Delta[w(x)-u_k(x)] & \ge & c(x)u_k^{-1}(x)\left|\nabla u_k(x)\right|^2 - a(x) + a(x) \ge 0, & in\ B_k, \\ w(x)-u_k(x) & > & 0, & on\ \partial B_k. \end{cases}$$

As a consequence of the maximum principle, *Lemma 8*, we have that $u_k \le w$ in B_k. So (4.6) holds.

To finish the proof, use the standard convergence procedure (see [6] or [15]) and so u_k has a subsequence, denoted again by u_k, such that $u_k \to u$ (pointwise) in $C_{loc}^{2,\alpha}(\mathbb{R}^N)$ and that u is a solution for the problem (1.5) that vanishing at infinity.

In order to show (1.7), from the above arguments we have

$$u \le w \text{ in } \mathbb{R}^N. \qquad (4.7)$$

On the other hand, using (4.3) we have

$$\begin{aligned}
\lim_{|x|\to\infty}\frac{w(|x|)}{|x|^{2-\mu}} &= \frac{1}{2-\mu}\lim_{|x|\to\infty}\frac{w'(x)}{|x|^{1-\mu}} = \frac{1}{\mu-2}\lim_{|x|\to\infty}\left[\int_0^{|x|}\sigma^{N-1}\varphi(\sigma)d\sigma/\left|x\right|^{N-\mu}\right] \\
&= \frac{1}{\mu-2}\lim_{|x|\to\infty}\left|x\right|^{\mu}\varphi(|x|) < \infty.
\end{aligned}$$

The above relation imply

$$w(x) = O(|x|^{2-\mu}) \text{ as } |x| \to \infty. \tag{4.8}$$

Now, (1.7) follows from (4.8) and (4.7). The proof of *Theorem 4* is completed.

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