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FINITE RANK INTERMEDIATE HANKEL OPERATORS ON THE BERGMAN SPACE

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Abstract. In this paper we characterize the kernel of an intermediate Hankel operator on the Bergman space in terms of the inner divisors and obtain a characterization for finite rank intermediate Hankel operators.

1 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, and $L^2_a(\mathbb{D})$ the Bergman space, consisting of those analytic functions on $\mathbb{D}$ that are square integrable on $\mathbb{D}$ with respect to area measure. The Bergman space is a closed subspace of the Hilbert space $L^2(\mathbb{D})$ of all square integrable complex-valued functions on $\mathbb{D}$. The inner product in $L^2(\mathbb{D})$, and hence in $L^2_a(\mathbb{D})$, is given by the formula

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), f, g \in L^2(\mathbb{D}),$$

where $dA(z) = \frac{1}{\pi}dxdy$, the normalized area measure on $\mathbb{D}$. The associated norm is denoted by $\|.\|_2$. Let $L^\infty(\mathbb{D}, dA)$ denote the Banach space of essentially bounded measurable functions on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\}.$$

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. The function $K(z, w) = \frac{1}{(1-z\bar{w})^2}$ is the reproducing kernel [7] for the Hilbert space $L^2_a(\mathbb{D})$. Let $K_z(w) = \overline{K(z, w)}$. Let $\overline{L^2_a(\mathbb{D})}$ be the subspace of $L^2(\mathbb{D})$ consisting of complex conjugates of functions in $L^2_a(\mathbb{D})$. For $p \geq 0$, let

$$E_p = \overline{\text{span}}\{|z|^{2k}\bar{z}^n, k = 0, \cdots, p\ ;\ n = 0, 1, 2, \cdots\}.$$

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For $\phi \in L^\infty(\mathbb{D})$, we define the intermediate Hankel operator $H_\phi^{E_p} : L_a^2 \to E_p$ by $H_\phi^{E_p}(f) = P_p(\phi f), f \in L_a^2$ where P_p is the orthogonal projection from $L^2(\mathbb{D})$ onto E_p. Note $\overline{L_a^2} \subseteq E_p \subseteq ((L_a^2)_0)^\perp$ where $(L_a^2)_0 = \{g \in L_a^2 : g(0) = 0\}$.
In this paper we characterize the kernel of an intermediate Hankel operator in terms of the inner divisors of the Bergman space and obtain a characterization for finite rank intermediate Hankel operators. Similar characterizations for finite rank intermediate Hankel operators were also obtained by E. Strouse [6] using different techniques. We use the invariant subspace theory for the Bergman space developed in [2],[3] and [4].

2 Intermediate Hankel operators

For $p \geq 0$, let E_p be the closed subspace of $L^2(\mathbb{D})$ described above. For $n > m$ and $j \in \{0, \cdots, p\}$, let

$$A_j^{n,m} = \frac{\prod_{1\leq l\leq p+1}(n-m+l+j)}{\prod_{1\leq l\leq p+1}(n+l)} \frac{1}{j!(p-j)!(-1)^{p-j}} \prod_{\substack{0\leq l\leq p\\ l\neq j}} (m-l).$$

It is not so difficult to check that

$$P_p(\bar{z}^n z^m) = \begin{cases} 0 & \text{if} \quad n < m; \\ \bar{z}^n z^m & \text{if} \quad n \geq m, 0 \leq m \leq p; \\ A_0^{n,m}\bar{z}^{n-m} + A_1^{n,m}\bar{z}^{n-m+1}z + \\ \quad \cdots + A_p^{n,m}\bar{z}^{n-m+p}z^p & \text{if} \quad n \geq m, m > p. \end{cases}$$

The details are given in [6, Lemma 1].

Lemma 1. *Suppose $\phi \in L^\infty(\mathbb{D})$. The operator $H_\phi^{E_p} \equiv 0$ if and only if $\phi \in E_p^\perp$.*

Proof. Note $H_\phi^{E_p} = 0$ implies $\phi f \in E_p^\perp$ for all $f \in L_a^2(\mathbb{D})$ and hence in particular $\phi \in E_p^\perp$. Conversely, if $\phi \in E_p^\perp$ then $\langle \phi, |z|^{2k}\bar{z}^n \rangle = 0$ for all $n \in \mathbb{Z}, n \geq 0$, and $k = 0, 1, \cdots, p$.
Let $f \in L_a^2(\mathbb{D})$ and $g \in E_p$ and $g(z) = |z|^{2k}\bar{z}^n, n = 0, 1, 2, \cdots; k = 0, 1, \cdots, p$. Then $\langle H_\phi^{E_p} f, g \rangle = \langle P_p(\phi f), g \rangle = \langle \phi f, g \rangle = \langle \phi, \bar{f}g \rangle = 0$ as $\bar{f}g \in E_p$. This implies $H_\phi^{E_p} f = 0$ for all $f \in L_a^2(\mathbb{D})$ and thus $H_\phi^{E_p} \equiv 0$. □

Proposition 2. *If $Q : L^2 \to L_a^2$ is the Bergman projection, then $(H_\phi^{E_p})^* = Q(\bar{\phi} f)$.*

Proof. If $f \in E_p$, $g \in L_a^2$ then $\langle (H_\phi^{E_p})^* f, g \rangle = \langle f, H_\phi^{E_p} g \rangle = \langle f, P_p(\phi g) \rangle = \langle f, \phi g \rangle = \langle \bar{\phi} f, g \rangle = \langle Q(\bar{\phi} f), g \rangle$. Thus $(H_\phi^{E_p})^* : E_p \to L_a^2$ such that $(H_\phi^{E_p})^* f = Q(\bar{\phi} f)$. □

3 Inner functions and kernel of a finite rank intermediate Hankel operator

Definition 3. *An invariant subspace of* $L^2_a(\mathbb{D})$ *is a closed subspace* I *such that* $zI \subset I$; *in other words* $zf(z)$ *is in* I *whenever* f *is in* I.

Definition 4. *A function* $G \in L^2_a(\mathbb{D})$ $(G \in H^2)$ *is called an inner function in* $L^2_a(\mathbb{D})$ *(respectively,* H^2*) if* $|G|^2 - 1$ *is orthogonal to* H^∞.

This definition of inner function in a Bergman space was given by Korenblum and Stessin [5]. If N is a subspace of $L^2_a(\mathbb{D})$, let $Z(N) = \{z \in \mathbb{D} : f(z) = 0 \text{ for all } f \in N\}$, which is called the common zero set of functions in N. Hence if z_1 is a zero of multiplicity at most n of all functions in N, then z_1 appears n times in the set $Z(N)$, and each z_1 is treated as a distinct element of $Z(N)$.

Lemma 5. *If* $\mathcal{I}$ *is an invariant subspace of* $L^2_a(\mathbb{D})$ *of finite codimension and* $Z(\mathcal{I}) = \{z \in \mathbb{D} : f(z) = 0$ *for all* $f \in \mathcal{I}\}$ *then* $Z(\mathcal{I})$ *is a finite set and* $\mathcal{I} = I(Z(\mathcal{I})) = \{f \in L^2_a(\mathbb{D}) : f(z) = 0$ *for all* $z \in Z(\mathcal{I})\}$.

Proof. For proof see [1]. □

For notational convenience, henceforth we shall assume that p is a fixed positive integer.

Theorem 6. *Let* $\phi \in L^\infty(\mathbb{D})$ *and* $H^{E_p}_\phi$ *be a finite rank intermediate Hankel operator on* $L^2_a(\mathbb{D})$. *Then* $\ker H^{E_p}_\phi = GL^2_a(\mathbb{D})$ *for some inner function* $G \in L^2_a(\mathbb{D})$ *and the following hold.*

(i) If $\mathbf{a} = \{a_j\}_{j=1}^N = Z(\ker H^{E_p}_\phi)$ *then* G *vanishes on* $\mathbf{a}$.

(ii) $\|G\|_{L^2} = 1$ *and* G *is equal to a constant plus a linear combination of the Bergman kernel functions* $K(z, a_1), K(z, a_2), \cdots, K(z, a_n)$ *and certain of their derivatives.*

(iii) $|G|^2 - 1 \perp L^1_h$ *where* L^1_h *is the class of harmonic functions in* L^1 *of the disc.*

Proof. Note $\ker H^{E_p}_\phi = \{f \in L^2_a(\mathbb{D}) : H^{E_p}_\phi f = 0\} = \{f \in L^2_a(\mathbb{D}) : P_p(\phi f) = 0\} = \{f \in L^2_a(\mathbb{D}) : \phi f \in E^\perp_p\} = \{f \in L^2_a(\mathbb{D}) : \langle \phi f, |z|^{2k}\bar{z}^n\rangle = 0$ for all $n \in \mathbb{Z}, n \geq 0$ and $k = 0, 1, \cdots, p\}$.
Now if $f \in \ker H^{E_p}_\phi$ then $\langle \phi f, |z|^{2k}\bar{z}^n\rangle = 0$ for all $n \in \mathbb{Z}, n \geq 0$ and $k = 0, 1, \cdots, p$ and therefore $\langle z\phi f, |z|^{2k}\bar{z}^n\rangle = \langle \phi f, |z|^{2k}\bar{z}^{n+1}\rangle = 0$ for all $n \in \mathbb{Z}, n \geq 0$ and $k = 0, 1, \cdots, p$. Hence $z\phi f \in E^\perp_p$ and then $zf \in \ker H^{E_p}_\phi$. Thus $\ker H^{E_p}_\phi \subset L^2_a$ is invariant under

multiplication by z, and $\ker H_\phi^{E_p}$ has finite codimension since $H_\phi^{E_p}$ is of finite rank. Let $Z(\ker H_\phi^{E_p}) = \mathbf{a} = \{a_j\}_{j=1}^N$. Let G be the extremal function for the problem

$$\sup\{Re f^{(k)}(0) : f \in L_a^2, \|f\|_{L^2} \le 1, f = 0 \text{ on } \mathbf{a}\},$$

where k is the multiplicity of the number of times zero appears in $\mathbf{a} = \{a_j\}_{j=1}^N$ ($k = 0$ if $0 \notin \{a_j\}_{j=1}^N$). It is clear from [2, 3, 4] that G satisfies conditions (i)-(iii), and G vanishes precisely on $\mathbf{a}$ in $\overline{\mathbb{D}}$, counting multiplicities. Moreover, for every function $f \in L_a^2(\mathbb{D})$ that vanishes on $\mathbf{a} = \{a_j\}_{j=1}^N$, there exists $g \in L_a^2(\mathbb{D})$ such that $f = Gg$. Thus $\ker H_\phi^{E_p} = GL_a^2(\mathbb{D})$. □

If $H_\phi^{E_p}$ is of finite rank, then $\operatorname{rank} H_\phi^{E_p}$ = number of zeroes of G counting multiplicities. We now make the link between inner functions and finite rank Hankel operators as follows.

Proposition 7. *Suppose $\Psi \in L^\infty(\mathbb{D})$ and $H_\Psi^{E_p}$ is a finite rank intermediate Hankel operator. Then there exist functions ϕ and χ such that $\Psi = \phi + \chi$, where $\chi \in E_p^\perp$ and $\bar{\phi} z^k \in \overline{E_p} \cap (GL_a^2)^\perp$, for all $k = 0, 1, \cdots, p$ and for some inner function $G \in H^\infty$.*

Proof. Suppose $\Psi \in L^\infty(\mathbb{D})$ and $H_\Psi^{E_p}$ is a finite rank intermediate Hankel operator. Let $\Psi = \phi + \chi$, where $\chi \in E_p^\perp$ and $\phi \in E_p$. By Lemma 1, $H_\chi^{E_p} \equiv 0$. Hence $H_\Psi^{E_p} \equiv H_\phi^{E_p}$ and therefore, $H_\phi^{E_p}$ is a finite rank intermediate Hankel operator.

By Theorem 6, there exists an inner function $G \in L_a^2(\mathbb{D})$ such that $\ker H_\phi^{E_p} = GL_a^2(\mathbb{D})$. Thus $\phi G \in E_p^\perp$. So $\langle \phi G, h\rangle = 0$ for all $h \in E_p$. That is, $\langle G\bar{h}, \bar{\phi}\rangle = 0$ for all $h \in E_p$, and so $\bar{\Psi} = \bar{\phi} + \bar{\chi}$, where $\bar{\chi} \in \overline{E_p}^\perp$ and $\bar{\phi} \in \overline{E_p} \cap (G\overline{E_p})^\perp$. By Theorem 6, G vanishes precisely at $\mathbf{a} = \{a_j\}_{j=1}^N$, a finite sequence of points in $\mathbb{D}$, counting multiplicities. Now $\bar{\phi} \in \overline{E_p} \cap (G\overline{E_p})^\perp$ implies $\langle \bar{\phi}, G|z|^{2k} z^n\rangle = 0$ for all $k = 0, 1, \cdots p, n \in \mathbb{Z}, n \ge 0$. Hence $\langle \bar{\phi} z^k, G z^{k+n}\rangle = 0$ for all $k = 0, 1, \cdots p, n \in \mathbb{Z}, n \ge 0$. Thus $\bar{\phi} z^k \in \overline{E_p} \cap (GL_a^2)^\perp$ for all $k = 0, 1, \cdots p$. □

Corollary 8. *If $\Psi \in \overline{H^\infty}$ and $H_\Psi^{E_p}$ is of finite rank then for all $k = 0, 1, \cdots p$,*

$$\bar{\Psi} z^k = \sum_{j=1}^{N} \sum_{\nu=0}^{m_j - 1} c_{j\nu}^{(k)} \frac{\partial^\nu}{\partial \bar{b}_j^\nu} K_{b_j}(z)$$

where $c_{j\nu}^{(k)}$ are constants for all $k = 0, 1, \cdots p$ and $j = 1, \cdots, N$ and $\nu = 0, \cdots m_j - 1$. Here $\mathbf{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$ and m_j is the number of times b_j appears in $\mathbf{b}$.

Proof. By Proposition 7, $\Psi = \phi + \chi$, where $\chi \in E_p^{\perp}$ and $\bar{\phi}z^k \in \overline{E_p} \cap (GL_a^2)^{\perp}$, for all $k = 0, 1, \cdots, p$ and for some inner function $G \in H^{\infty}$. Since $\Psi \in \overline{H^{\infty}}, \chi \equiv 0$. Thus $H_{\Psi} \equiv H_{\phi}$ is a finite rank operator and $\overline{\Psi}z^k = \bar{\phi}z^k \in \overline{E_p} \cap (GL_a^2)^{\perp}$, for all $k = 0, 1, \cdots, p$ and for some inner function $G \in H^{\infty}$. Further, $\ker H_{\Psi}^{E_p} = GL_a^2(\mathbb{D})$. Now $\overline{\Psi}z^k \in L_a^2 \subset \overline{E_p}$. Thus $\overline{\Psi}z^k \in L_a^2 \cap \overline{E_p} \cap (GL_a^2)^{\perp} = L_a^2 \ominus GL_a^2$. Let $\mathbf{b} = \{b_j\}_{j=1}^N$ be the zeros of G (counting multiplicities). From [2, 4], it follows that

$$\{K_{b_1}, \cdots, \frac{\partial^{m_1-1}}{\partial \bar{b}_1^{m_1-1}} K_{b_1}, \cdots, K_{b_N}, \cdots, \frac{\partial^{m_N-1}}{\partial \bar{b}_N^{m_N-1}} K_{b_N}\}$$

form a basis for $(GL_a^2(\mathbb{D}))^{\perp}$, hence the result follows. □

Notice that $\overline{\Psi}$ is a polynomial of degree $\leq p$ if and only if $\text{rank} H_{\Psi}^{E_p} \leq p$. The proof of this fact is given in [6]. For the sake of completeness, we are presenting the proof of [6] here: If $\overline{\Psi}$ is a polynomial of degree less than or equal to p then $\text{rank} H_{\Psi}^{E_p} \leq p$. This is so because if $\overline{\Psi}(z) = a_0 + a_1 z + \cdots + a_k z^k$, $k \leq p, a_k \neq 0$ then for $m > k$, $H_{\Psi}^{E_p}(z^m) = P_p(\Psi z^m) = P_p((\bar{a_0} + \bar{a_1}\bar{z} + \cdots + \bar{a_k}\bar{z}^k)z^m) = 0$. If $\Psi \in \overline{L_a^2}$ then (see [7]), $\Psi(z) = \sum_{n=0}^{\infty} \hat{\Psi}(n)\bar{z}^n, \hat{\Psi}(n) \in \mathbb{C}$ and $\sum_{n=0}^{\infty} \frac{|\hat{\Psi}(n)|^2}{n+1} < \infty$. Now if $H_{\Psi}^{E_p}$ is of rank $\leq p$ and $\overline{\Psi}$ is not a polynomial then the functions $H_{\Psi}^{E_p}(1) = \Psi, H_{\Psi}^{E_p}(z) = z(\Psi - \hat{\Psi}(0)), \cdots, H_{\Psi}^{E_p}(z^p) = z^p(\Psi - \sum_{n=0}^{p-1} \hat{\Psi}(n)\bar{z}^n)$ are linearly independent and $\text{rank} H_{\Psi}^{E_p} \geq p + 1$ which is a contradiction. Let $v_k = \sum_{j=o}^{p} A_j^{m+k,m} \bar{z}^{k+j} z^j$. Notice that $v_k \perp v_l$ for $k \neq l$. Now if for some $m \geq 0, H_{\Psi}^{E_p}(z^m) = 0$ then since $H_{\Psi}^{E_p}(z^m) = \sum_{k=o}^{\infty} \hat{\Psi}(m + k)(\sum_{j=o}^{p} A_j^{m+k,m} \bar{z}^{k+j} z^j)$; hence $\hat{\Psi}(m + k) = 0$ for all $k = 0, 1, 2, \cdots$. This implies $\overline{\Psi}$ is a polynomial of degree $\leq m$ and in which case $H_{\Psi}^{E_p}(z^n) = 0$ for all $n \geq m$. Thus $\text{rank} H_{\Psi}^{E_p} \leq p$ implies $\overline{\Psi}$ is a polynomial of degree $\leq p$.

Theorem 9. *If $\Psi \in (E_p)^{\perp} \oplus \overline{H^{\infty}}$ and $H_{\Psi}^{E_p}$ is a finite rank operator of rank $p+r$ then $\overline{\Psi} = \chi + \overline{\Theta} + \overline{\phi}$ where $\chi \in (\overline{E_p})^{\perp}$, $\overline{\Theta}$ is a polynomial of degree $\leq p$, and $\text{rank} H_{\phi G_1}^{E_p} \leq r$ for some inner function G_1.*

Proof. Suppose $\Psi \in (E_p)^{\perp} \oplus \overline{H^{\infty}}$ and $H_{\Psi}^{E_p}$ is a finite rank operator of rank $p+r$. Then $\Psi = \overline{\chi} + \Omega$ where $\overline{\chi} \in (E_p)^{\perp}$ and $\Omega \in \overline{H^{\infty}}$. Since $H_{\overline{\chi}} \equiv 0$ if and only if $\overline{\chi} \in (E_p)^{\perp}$, hence $H_{\Psi}^{E_p} = H_{\Omega}^{E_p}$ is a finite rank operator of rank $p + r$. By Theorem 6 this implies there exists an inner function (a finite zero divisor) $G \in H^{\infty}$ such that $\ker H_{\Omega}^{E_p} = GL_a^2(\mathbb{D})$. Let $Z(\ker H_{\Omega}^{E_p}) = \{\xi_j\}_{j=1}^N$ repeated according to their multiplicities. From [2, 3, 4], it follows that $G(z) = J(0,0)^{-\frac{1}{2}} B(z) J(z, 0)$, where $J(z, \zeta)$ is the kernel function of the Bergman space $L_a^2(w(z)dA(z))$ with weight $w = |B|^p$, and B is the finite Blaschke product associated with $\{\xi_j\}_{j=1}^N$. Without loss of generality assume that G has no zeros at the origin. That is, $B(z) = \prod_{n=1}^N \frac{|\xi_n|}{\xi_n} \frac{\xi_n - z}{1 - \overline{\xi_n} z}$. Let $B_1(z) =$

$\prod_{n=1}^{p} \frac{|\xi_n|}{\xi_n} \frac{\xi_n - z}{1-\overline{\xi_n} z}$ and $B_2(z) = \prod_{n=p+1}^{N} \frac{|\xi_n|}{\xi_n} \frac{\xi_n - z}{1-\overline{\xi_n} z}$. Then $G(z) = J(0,0)^{-\frac{1}{2}} B(z) J(z,0) = J(0,0)^{-\frac{1}{2}} B_1(z) J(z,0) B_2(z) = G_1(z) B_2(z)$ where $G_1(z)$ is an inner function in the Bergman space $L^2_a(\mathbb{D})$ and $B_2(z)$ is a classical inner function , in fact a finite Blaschke product. Notice that G_1 has p zeros and B_2 has $N-p$ zeros counting multiplicities. Now $\ker H^{E_p}_\Omega = GL^2_a(\mathbb{D})$ implies $H^{E_p}_\Omega(GL^2_a) = \{0\}$. Hence, $\Omega G \in (E_p)^\perp$. That is, $\Omega \in (\overline{G}E_p)^\perp$ or $\overline{\Omega} \in (G\overline{E_p})^\perp$. But observe that $(G\overline{E_p})^\perp = (G_1\overline{E_p})^\perp \oplus [(G\overline{E_p})^\perp \ominus (G_1\overline{E_p})^\perp] = (G_1\overline{E_p})^\perp \oplus [(G\overline{E_p})^\perp \cap G_1\overline{E_p}]$. Thus, $\overline{\Omega} = \overline{\Theta} + \overline{\phi}$ where $\overline{\Theta} \in (G_1\overline{E_p})^\perp$ and $\overline{\phi} \in (G\overline{E_p})^\perp \cap G_1\overline{E_p}$. Hence $H^{E_p}_\Omega = H^{E_p}_\Theta + H^{E_p}_\phi$. We shall now verify that $H^{E_p}_\Theta$ is a finite rank operator of rank $\leq p$ and rank$H^{E_p}_{\phi G_1} \leq r$.

Since $\overline{\Theta} \in (G_1\overline{E_p})^\perp$, we have $\Theta G_1 \in (E_p)^\perp$ and hence $\ker H^{E_p}_\Theta \supset G_1 L^2_a$. Thus $(\ker H^{E_p}_\Theta)^\perp = \text{range} H^{*E_p}_\Theta \subset (G_1 L^2_a)^\perp \cap L^2_a$. Since $G_1 L^2_a \subset L^2_a$ and $(G_1 L^2_a)^\perp$ has dimension p; the space $\ker H^{E_p}_\Theta$ has finite codimension and dim range $H^{E_p}_\Theta \leq p$. Hence $\overline{\Theta}$ is a polynomial of degree $\leq p$. Thus $\overline{\Theta} \in H^\infty$ and therefore $\bar{\phi} \in H^\infty$. Now $\bar{\phi} \in (G\overline{E_p})^\perp \cap G_1\overline{E_p}$. This implies $\bar{\phi} \in G_1\overline{E_p}$ and $\bar{\phi} \perp G\overline{E_p}$. That is, $\langle \bar{\phi}\overline{G_1}, B_2 g\rangle = \langle \bar{\phi}, G_1 B_2 g\rangle = \langle \bar{\phi}, Gg\rangle = 0$ for all $g \in \overline{E_p}$. Thus $\bar{\phi}\overline{G_1} \in (B_2\overline{E_p})^\perp$. That is, $\phi G_1 \in (\overline{B_2}E_p)^\perp$. Hence rank$H^{E_p}_{\phi G_1} \leq r$. □

Theorem 10. *If $H^{E_p}_\phi$ is an intermediate Hankel operator on $L^2_a(\mathbb{D})$, and $\ker H^{E_p}_\phi = \{f \in L^2_a(\mathbb{D}) : f = 0$ on $\mathbf{b}\}$ where $\mathbf{b} = \{b_j\}_{j=1}^\infty$ is an infinite sequence of points in $\mathbb{D}$, then there exists an inner function $G \in L^2_a(\mathbb{D})$ such that $\ker H^{E_p}_\phi = GL^2_a(\mathbb{D}) \cap L^2_a(\mathbb{D})$.*

Proof. The proof follows from the result of Hedenmalm [4] as $\ker H^{E_p}_\phi$ is an invariant subspace of the operator of multiplication by z. □

It is not known for the Bergman space whether the invariant subspaces determined by infinite zero sets are generated by the corresponding canonical divisors (see [2, 4]). Now let $\mathbf{b} = \{b_j\}_{j=1}^\infty$ be an infinite sequence of points in $\mathbb{D}$. Let $\mathcal{I} = I(\mathbf{b}) = \{f \in L^2_a(\mathbb{D}) : f = 0$ on $\mathbf{b}\}$. Let $G_\mathbf{b}$ be the solution of the extremal problem

$$\sup\{Re f^{(n)}(0) : f \in \mathcal{I}, \|f\|_{L^2} \leq 1\}, \tag{3.1}$$

where n is the number of times zero appears in the sequence $\mathbf{b}$(that is, the functions in $\mathcal{I}$ have a common zero of order n at the origin). The natural question that arises at this point is to see if it is possible to construct an intermediate Hankel operator $H^{E_p}_\phi$ whose kernel is $G_\mathbf{b} L^2_a \cap L^2_a$. In the case that $\mathbf{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$, it is possible to construct an intermediate Hankel operator $H^{E_p}_\phi$ such that $\ker H^{E_p}_\phi = G_\mathbf{b} L^2_a(\mathbb{D})$, as follows.

Theorem 11. *If $\mathbf{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}, \mathcal{I} = I(\mathbf{b}) = \{f \in L^2_a(\mathbb{D}) :$*

$f=0$ on $\mathbf{b}\}$ and $G_{\mathbf{b}}$ is the solution of the extremal problem (3.1),

$$\bar{\phi}z^k = \sum_{j=1}^{N}\sum_{\nu=0}^{m_j-1} c_{j\nu}^{(k)} \frac{\partial^\nu}{\partial \bar{b}_j^\nu} K_{b_j}(z)$$

where $c_{j\nu}^{(k)}$ are constants, $c_{j\nu}^{(k)} \neq 0$ for all $j,\nu,k=0,1,\cdots,p$ and m_j is the number of times b_j appears in $\mathbf{b}$, then $kerH_\phi^{E_p} = G_{\mathbf{b}}L_a^2(\mathbb{D})$.

Proof. $\{K_{b_1},\cdots,\frac{\partial^{m_1-1}}{\partial \bar{b}_1^{m_1-1}}K_{b_1},\cdots,K_{b_N},\cdots,\frac{\partial^{m_N-1}}{\partial \bar{b}_N^{m_N-1}}K_{b_N}\}$ forms a basis for $(G_{\mathbf{b}}L_a^2(\mathbb{D}))^\perp$. By the Gram-Schmidt orthogonalization process, we can obtain an orthonormal basis $\{\Psi_j\}_{j=1}^{l}$ for $(G_{\mathbf{b}}L_a^2)^\perp$. Since $\bar{\phi}z^k \in (G_{\mathbf{b}}L_a^2)^\perp$, hence $\langle \bar{\phi}z^k, G_{\mathbf{b}}z^n z^k\rangle = 0$ for all $k=0,1,\cdots,p, n\in\mathbb{Z}, n\geq 0$. This implies $\langle\bar{\phi}, G_{\mathbf{b}}|z|^{2k}z^n\rangle = 0$ for all $k=0,1,\cdots,p, n\in\mathbb{Z}, n\geq 0$. Therefore $\langle |z|^{2k}\bar{z}^n, \phi G_{\mathbf{b}}\rangle = 0$ for all $k=0,1,\cdots,p, n\in\mathbb{Z}, n\geq 0$. Thus $\phi G_{\mathbf{b}} \in E_p^\perp$ and $G_{\mathbf{b}} \in \ker H_\phi^{E_p}$. Since $\ker H_\phi^{E_p}$ is invariant under the operator of multiplication by z, hence

$$G_{\mathbf{b}}L_a^2 \subset \ker H_\phi^{E_p}. \tag{3.2}$$

Suppose $f\in\ker H_\phi^{E_p}$, then $\phi f \in E_p^\perp$. That is, $\langle \phi f, |z|^{2k}\bar{z}^n\rangle = 0$ for all $n\geq 0, n\in\mathbb{Z}, k=0,1,\cdots,p$. Hence $\langle |z|^{2k}\phi f, \bar{z}^n\rangle = 0$ for all $n\geq 0, n\in\mathbb{Z}, k=0,1,\cdots,p$ and therefore $\langle |z|^{2k}\phi f, \bar{g}\rangle = 0$ for all $g\in L_a^2$ and $k=0,1,\cdots,p$. So in particular, $\langle |z|^{2k}\phi f, \overline{K_{b_j}}\rangle = 0$ for all $j=1,2,\cdots,N; k=0,1,\cdots,p$. Thus $\overline{\phi(b_j)}|b_j|^{2k}\overline{f(b_j)} = 0$ for all $j=1,2,\cdots,N; k=0,1,\cdots,p$. In particular, $\overline{\phi(b_j)f(b_j)} = 0$ for all $j=1,2,\cdots,N$.
Since $\overline{\phi(b_j)} \neq 0$ for all $j=1,2,\cdots,N$ hence we have, $\overline{f(b_j)} = 0$ for all $j=1,2,\cdots,N$. Thus $f\in\mathcal{I}$. Since $G_{\mathbf{b}}$ is the solution of the extremal problem (1), $f\in G_{\mathbf{b}}L_a^2$. Hence

$$\ker H_\phi^{E_p} \subset G_{\mathbf{b}}L_a^2. \tag{3.3}$$

From (3.2) and (3.3), $\ker H_\phi^{E_p} = G_{\mathbf{b}}L_a^2 = \mathcal{I}$ as required. □

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