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GLOBAL EXISTENCE FOR VOLTERRA–FREDHOLM TYPE NEUTRAL IMPULSIVE FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the global existence of solutions for the initial value problems for Volterra-Fredholm type neutral impulsive functional integrodifferential equations. Using the Leray-Schauder's Alternative theorem, we derive conditions under which a solution exists globally. An application is provided to illustrate the theory.

1 Introduction

In recent years, a great deal of work has been done in the study of the existence of solutions for impulsive differential equations, by which a number of chemotherapy, population dynamics, optimal control, industrial robotics and physics phenomena are described. For the general aspects of impulsive differential equations, we refer the reader to the classical monographs Bainov et al. [1, 2], Lakshmikantham et al. [20], Samoilenko and Perestyuk [26] and Hernández et al. [12, 11] for the case of ordinary and partial differential functional differential equations with impulses. For some general and recent works on the theory of impulsive differential and integrodifferential equations, we refer the reader to [3, 4, 5, 6, 8, 7, 9, 19, 31].

Recently, in [21], Ntouyas studied the global existence for first order neutral impulsive functional integrodifferential equations of the form

$$\frac{d}{dt}[x(t) - g(t, x_t)] = f\Big(t, x_t, \int_0^t k(t,s)h(s,x_s)ds\Big),\ t \in I = [0,T],$$
$$x_0 = \phi$$

and also studied the global existence for second order neutral impulsive functional

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integrodifferential equations of the form

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = F\Big(t, x_t, \int_0^t k(t,s)H(s, x_s, x'(s))ds, x'(t)\Big),\ t \in I = [0,T],$$
$$x_0 = \phi,\ x'(0) = \eta$$

by using Leray-Schauder's Alternative theorem. Recent results on global existence and global solutions for ordinary and partial neutral impulsive integrodifferential equations can be foun in [13, 14, 15, 16, 17, 18, 22, 24, 23, 25, 27, 28, 29, 30].

In this paper, we study the global existence of solutions for the initial value problems for the first and second order Volterra-Fredholm type neutral impulsive functional integrodifferential equations. In Section 3, we study the following initial value problem

$$\frac{d}{dt}[x(t) - g(t, x_t)] = f\Big(t, x_t, \int_0^t a(t,s)w(s, x_s)ds, \int_0^T b(t,s)h(s, x_s)ds\Big),$$
$$t \in I = [0,T] \setminus \{t_1, ..., t_m\}, \quad (1.1)$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, ..., m, \quad (1.2)$$
$$x_0 = \phi \quad (1.3)$$

where $f : I \times D \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : I \times D \to \mathbb{R}^n$, $D = \{\psi : [-r, 0] \to \mathbb{R}^n;\ \psi$ is continuous everywhere except for a finite number of points $\tilde{t}$ at which $\psi(\tilde{t}^-)$ and $\psi(\tilde{t}^+)$ are exist with $\psi(\tilde{t}^-) = \psi(\tilde{t})\}$, $w : I \times D \to \mathbb{R}^n$, $h : I \times D \to \mathbb{R}^n$, $a : I \times I \to \mathbb{R}$, $b : I \times I \to \mathbb{R}$ are continuous functions, $\phi \in D$, $0 < r < \infty$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $(k = 1, ..., m)$. $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$.

In Section 4, we study the global existence of solutions for the initial value problems for the second order Volterra-Fredholm type neutral impulsive functional integrodifferential equations of the form

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = F\Big(t, x_t, x'(t), \int_0^t a(t,s)W(s, x_s, x'(s))ds,$$
$$\int_0^T b(t,s)H(s, x_s, x'(s))ds\Big), \quad t \in I = [0,T] \setminus \{t_1, ..., t_m\}, \quad (1.4)$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, ..., m, \quad (1.5)$$
$$\Delta x'|_{t=t_k} = J_k(x(t_k^-)), \quad k = 1, ..., m, \text{ and} \quad (1.6)$$
$$x_0 = \phi,\ x'(0) = \eta, \quad (1.7)$$

where $F : I \times D \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : I \times D \to \mathbb{R}^n$, $D = \{\psi : [-r, 0] \to \mathbb{R}^n;\ \psi$ is continuous everywhere except for a finite number of points $\tilde{t}$ at which $\psi(\tilde{t}^-)$ and $\psi(\tilde{t}^+)$ are exist with $\psi(\tilde{t}^-) = \psi(\tilde{t})\}$, $W : I \times D \times \mathbb{R}^n \to \mathbb{R}^n$, $H : I \times D \times \mathbb{R}^n \to \mathbb{R}^n$,

$a : I \times I \to \mathbb{R}$, $b : I \times I \to \mathbb{R}$ are continuous functions, $\phi \in D$ and $\eta \in \mathbb{R}^n$, $0 < r < \infty$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k, J_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $(k = 1, ..., m)$. $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$.

For any continuous function x defined on $[-r, T] \setminus \{t_1, ..., t_m\}$ and any $t \in [0, T]$ we denote by x_t the element of D defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, where $x_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

2 Preliminaries

In this section, we introduce some basic definitions, notations and preliminary facts which are used throughout this paper.

Let $C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions from $[-r, 0]$ into $\mathbb{R}^n$ endowed with the norm

$$\|\phi\| = \sup\left\{|\phi(0)| : -r \le \theta \le 0\right\}$$

and $C([0, T], \mathbb{R}^n)$ denote the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}^n$ normed by

$$\begin{aligned}
\|x\|_r &= \sup\{|x(t)| : -r \le t \le T\},\\
\|x\|_0 &= \sup\{|x(t) : t \in I\},\\
\|x\|_1 &= \sup\{|x'(t) : t \in I\},\\
\|x\|^* &= \max\{\|x\|_r, \|x\|_1\},\\
\|x\|_T &= \max\{\|x\|_0, \|x\|_1\}.
\end{aligned}$$

In order to define the solutions of (1.1)-(1.3) and (1.4)-(1.7), we introduce the following space:

$$PC = \Big\{x : [0, T] \to \mathbb{R}^n : x_k \in C([t_k, t_{k+1}], \mathbb{R}^n),\ k = 0, ..., m \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k^-) = x(t_k^+),\ k = 1, ..., m\Big\}$$

which is a Banach space with the norm

$$\|x\|_{PC} = \max\{\|x_k\|_{(t_k, t_{k+1}]},\ k = 0, ..., m\},$$

where x_k is the restriction of x to $(t_k, t_{k+1}]$, $k = 0, ..., m$.

Set $\Omega = D \cup PC$. Then Ω is a Banach space normed by

$$\|x\|_\Omega = \max\{\|x\|_D, \|x\|_{PC}\}, \text{ for each } x \in \Omega.$$

and

$$PC^1 = \Big\{x : [0,T] \to \mathbb{R}^n : x_k \in C^1([t_k, t_{k+1}], \mathbb{R}^n),\ k = 0, ..., m \text{ and there exist } x'(t_k^-) \text{ and } x'(t_k^+) \text{ with } x'(t_k^-) = x'(t_k^+),\ k = 1, ..., m\Big\}$$

which is a Banach space with the norm

$$\|x\|_{PC^1} = \max\{\|x_k\|_{(t_k,t_{k+1}]},\ k = 0, ..., m\},$$

where x_k is the restriction of x to $(t_k, t_{k+1}]$, $k = 0, ..., m$.

Set $\Omega^1 = D \cup PC^1$. Then Ω^1 is a Banach space normed by

$$\|x\|_{\Omega^1} = \max\{\|x\|_D, \|x\|_{PC^1}\}, \text{ for each } x \in \Omega^1.$$

The considerations of this paper are based on the following fixed point result [10].

Lemma 1 (Leray-Schauder's Alternative Theorem)**.** *Let S be a closed convex subset of a normed linear space E and assume that $0 \in S$. If $F : S \to S$ be a completely continuous operator, i.e. it is continuous and the image of any bounded set is included in a compact set and let*

$$\Phi(F) = \{x \in S : x = \lambda Fx,\ \textit{for some } 0 < \lambda < 1\}.$$

Then either $\Phi(F)$ is unbounded or F has a fixed point.

3 Global Existence for First order IVP

In this section, we present the global existence results for the IVP (Initial value problem) (1.1)-(1.3).

Definition 2. *A function $x \in \Omega$ is called solution of the initial value problem (1.1)-(1.3) if x satisfies the following integral equation*

$$x(t) = \phi(0) - g(0, \phi) + g(t, x_t) + \int_0^t f\Big(s, x_s, \int_0^s a(s,\sigma)w(\sigma, x_\sigma)d\sigma, \int_0^T b(s,\sigma)h(\sigma, x_\sigma)d\sigma\Big)ds + \sum_{0<t_k<t} I_k(x(t_k^-)),\ t \in I.$$

Theorem 3. *Let $f : I \times D \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : I \times D \to \mathbb{R}^n$, $w : I \times D \to \mathbb{R}^n$, $h : I \times D \to \mathbb{R}^n$, $a : I \times I \to \mathbb{R}$, $b : I \times I \to \mathbb{R}$ are continuous functions.*

Assume that

$\mathbf{H_{g1}}$ *There exists constants* c_1 *and* c_2 *such that*

$$|g(t,\phi)| \leq c_1\|\phi\| + c_2,\ \ t \in I,\ \ \phi \in D.$$

$\mathbf{H_{g2}}$ *The function* g *is completely continuous and such that the operator*

$$G : D \to PC([0,T],\mathbb{R}^n)$$

defined by $(G\phi)(t) = g(t,\phi)$ *is compact.*

$\mathbf{H_w}$ *There exists a continuous function* $m_1 : I \to [0,\infty)$ *such that*

$$|w(t,\phi)| \leq m_1(t)\Omega_1(\|\phi\|),\ \ t \in I,\ \ \phi \in D,$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_h}$ *There exists a continuous function* $m_2 : I \to [0,\infty)$ *such that*

$$|h(t,\phi)| \leq m_2(t)\Omega_2(\|\phi\|),\ \ t \in I,\ \ \phi \in D,$$

where $\Omega_2 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_f}$ *There exists an integrable function* $p : I \to [0,\infty)$ *such that*

$$|f(t,u,v,w)| \leq p(t)\Omega_3(\|u\| + |v| + |w|),\ \ t \in I,\ \ u \in D,\ \ v,w \in \mathbb{R}^n,$$

where $\Omega_3 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_a}$ *There exists a constant* L_1 *such that*

$$|a(t,s)| \leq L_1 \text{ for } t \geq s \geq 0.$$

$\mathbf{H_b}$ *There exists a constant* L_2 *such that*

$$|b(t,s)| \leq L_2 \text{ for } t \geq s \geq 0.$$

$\mathbf{H_I}$ *There exist constants* c_k *such that* $|I_k(x)| \leq c_k|x|$, $k = 1,...,m$ *for each* $x \in \mathbb{R}^n$.

Then if

$$\int_0^T \widehat{m}(s)ds < \int_c^{+\infty} \frac{ds}{\Omega_3(s) + \Omega_1(s) + \Omega_2(s)}$$

where

$$\widehat{m}(t) = \max\left\{\frac{1}{(1-c_1)}p(t),\ L_1 m_1(t),\ L_2 m_2(t)\right\} \text{ and}$$

$$c = \frac{1}{(1-c_1)}\left\{(1+c_1)\|\phi\| + 2c_2 + \sum_{k=1}^{n} c_k|x(t_k^-)|\right\},$$

the initial value problem (1.1)-(1.3) has at least one solution on $[-r,T]$.

Proof. To prove the existence of a solution of the initial value problem (1.1)-(1.3) we apply *Lemma 1*. First we obtain the *a priori bounds* for the solutions of the initial value problem $(1.1)_\lambda - (1.3)$, $\lambda \in (0,1)$, where $(1.1)_\lambda$ stands for the equation

$$\frac{d}{dt}[x(t) - g(t, x_t)] = \lambda f\Big(t, x_t, \int_0^t a(t,s)w(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\Big),\ t \in I.$$

Let x be a solution of the initial value problem $(1.1)_\lambda - (1.3)$. From

$$x(t) = \lambda\phi(0) - \lambda g(0,\phi) + \lambda g(t,x_t) + \lambda \int_0^t f\Big(s, x_s, \int_0^s a(s,\sigma)w(\sigma,x_\sigma)d\sigma,$$
$$\int_0^T b(s,\sigma)h(\sigma,x_\sigma)d\sigma\Big)ds + \lambda \sum_{0<t_k<t} I_k(x(t_k^-)),\ t \in I.$$

we have, for every $t \in I$,

$$|x(t)| \leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|x_t\| + \int_0^t p(s)\Omega_3\Big(\|x_s\| + L_1 \int_0^s m_1(\sigma)\Omega_1(\|x_\sigma\|)d\sigma$$
$$+ L_2 \int_0^T m_2(\sigma)\Omega_2(\|x_\sigma\|)d\sigma\Big)ds + \sum_{k=1}^n c_k|x(t_k^-)|.$$

We consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\},\ t \in I.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have, for every $t \in I$,

$$\mu(t) \leq (1+c_1)\|\phi\| + 2c_2 + c_1\mu(t) + \int_0^t p(s)\Omega_3\Big(\mu(s) + L_1 \int_0^s m_1(\sigma)\Omega_1(\mu(\sigma))d\sigma$$
$$+ L_2 \int_0^T m_2(\sigma)\Omega_2(\mu(\sigma))d\sigma\Big)ds + \sum_{k=1}^n c_k|x(t_k^-)|$$

or

$$\mu(t) \leq \frac{1}{(1-c_1)}\Big\{(1+c_1)\|\phi\| + 2c_2 + \int_0^t p(s)\Omega_3\Big(\mu(s) + L_1 \int_0^s m_1(\sigma)\Omega_1(\mu(\sigma))d\sigma$$
$$+ L_2 \int_0^T m_2(\sigma)\Omega_2(\mu(\sigma))d\sigma\Big)ds + \sum_{k=1}^n c_k|x(t_k^-)|\Big\}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the previous inequality obvious holds.

Denoting by $u(t)$ the right-hand side of the above inequality we have,

$$u(0) = \frac{1}{(1-c_1)}\Big\{(1+c_1)\|\phi\| + 2c_2 + \sum_{k=1}^n c_k|x(t_k^-)|\Big\} \equiv c,\ \mu(t) \leq u(t),\ t \in I,$$

and for every $t \in I$

$$\begin{aligned} u'(t) &= \frac{1}{(1-c_1)} p(t)\Omega_3\Big(\mu(t) + L_1 \int_0^t m_1(\sigma)\Omega_1(\mu(\sigma))d\sigma \\ &\quad + L_2 \int_0^T m_2(\sigma)\Omega_2(\mu(\sigma))d\sigma\Big) \\ &\leq \frac{1}{(1-c_1)} p(t)\Omega_3\Big(u(t) + L_1 \int_0^t m_1(\sigma)\Omega_1(u(\sigma))d\sigma \\ &\quad + L_2 \int_0^T m_2(\sigma)\Omega_2(u(\sigma))d\sigma\Big). \end{aligned}$$

Set

$$v(t) = u(t) + L_1 \int_0^t m_1(s)\Omega_1(u(s))ds + L_2 \int_0^T m_2(s)\Omega_2(u(s))ds, \quad t \in I.$$

Then

$$v(0) = u(0) \equiv c, \quad u(t) \leq v(t),\ t \in I, \quad u'(t) \leq \frac{1}{(1-c_1)} p(t)\Omega_3(v(t)),\ t \in I$$

and

$$\begin{aligned} v'(t) &= u'(t) + L_1 m_1(t)\Omega_1(u(t)) + L_2 m_2(t)\Omega_2(u(t)) \\ &\leq \frac{1}{(1-c_1)} p(t)\Omega_3(v(t)) + L_1 m_1(t)\Omega_1(v(t)) + L_2 m_2(t)\Omega_2(v(t)) \\ &\leq \widehat{m}(t)\big[\Omega_3(v(t)) + \Omega_1(v(t)) + \Omega_2(v(t))\big],\ t \in I. \end{aligned}$$

or

$$\frac{v'(t)}{\Omega_3(v(t)) + \Omega_1(v(t)) + \Omega_2(v(t))} \leq \widetilde{m}(t),\ t \in I.$$

Integrating this inequality from 0 to t and using the change of variables formula we get

$$\int_{v(0)}^{v(t)} \frac{ds}{\Omega_3(s) + \Omega_1(s) + \Omega_2(s)} \leq \int_0^T \widehat{m}(s)ds < \int_c^{+\infty} \frac{ds}{\Omega_3(s) + \Omega_1(s) + \Omega_2(s)},\ t \in I.$$

This inequality implies that there is a constant K such that $u(t) \leq K$, $t \in I$ and hence $\mu(t) \leq K$, $t \in I$. Since for every $t \in I$, $\|x_t\| \leq \mu(t)$ we have

$$\|x\|_r = \sup\{|x(t)| : -r \leq t \leq T\} \leq K,$$

where K depends only on the functions $m_1, m_2, p, \Omega_1, \Omega_2$ and Ω_3. We will rewrite the initial value problem (1.1)-(1.3) as follows. For $\phi \in D$ define $\widetilde{\phi} \in B$, $B = \Omega$ by

$$\widetilde{\phi} = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ \phi(0), & 0 \leq t \leq T. \end{cases}$$

If $x(t) = y(t) + \widetilde{\phi}(t)$, $t \in [-r, T]$ it is easy to see that y satisfies

$$y_0 = 0$$

$$y(t) = -g(0,\phi) + g(t, y_t + \widetilde{\phi}_t) + \int_0^t f\Big(s, y_s + \widetilde{\phi}_s, \int_0^s a(s,\tau) w(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau,$$
$$\int_0^T b(s,\tau) h(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau\Big) ds + \sum_{0<t_k<t} I_k(x(t_k^-)), \ t \in I.$$

Define $B_0 = \{y \in B : y_0 = 0\}$ and $S : B_0 \to B_0$, by

Now define an operator $Q : E_0 \to E$ by

$$(Sy)(t) = \begin{cases} 0, \quad t \in [-r, 0] \\ -g(0,\phi) + g(t, y_t + \widetilde{\phi}_t) + \int_0^t f\Big(s, y_s + \widetilde{\phi}_s, \int_0^s a(s,\tau) w(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau, \\ \int_0^T b(s,\tau) h(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau\Big) ds + \sum_{0<t_k<t} I_k(x(t_k^-)), \quad t \in I. \end{cases}$$

S is clearly continuous. We shall prove that S is completely continuous.

By $\mathbf{H_{g2}}$ it suffices to prove that the operator $S_1 : B_0 \to B_0$ defined by

$$(S_1 y)(t) = \begin{cases} 0, \quad t \in [-r, 0] \\ \int_0^t f\Big(\tau, y_s + \widetilde{\phi}_s, \int_0^s a(s,\tau) w(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau, \\ \int_0^T b(s,\tau) h(\tau, y_\tau + \widetilde{\phi}_\tau) d\tau\Big) ds, \quad t \in I. \end{cases}$$

is completely continuous.

Let $\{u_v\}$ be a bounded sequence in B_0, i.e.

$$\|u_v\|_r \leq b, \quad \text{for all} \quad v,$$

where b is a positive constant. We obviously have $\|u_{vt}\| \leq b$, $t \in I$, for all v. Hence, if $M^* = \sup\{\widetilde{m} : t \in I\}$, $p_0 = \sup\{p(t) : t \in I\}$, we obtain

$$\|S_1 u_v\|_r \leq p_0 \Omega_3 \Big[(b + \|\phi\|)T + \frac{M^* T^2}{2} \Omega_1(b + \|\phi\|) + \frac{M^* T^2}{2} \Omega_2(b + \|\phi\|)\Big].$$

This means that $\{S_1 u_v\}$ is uniformly bounded.

Moreover, the sequence $\{S_1 u_v\}$ is equicontinuous, since for $t_1, t_2 \in [-r, T]$ we have:

(i) if $0 \le t_1 \le t_2$ then

$$|S_1u_v(t_1) - S_1u_v(t_2)| \le p_0\Omega_3\big[(b+\|\phi\|) + M^*T(\Omega_1(b+\|\phi\|) + \Omega_2(b+\|\phi\|)\big]|t_1 - t_2|.$$

(ii) if $t_1 \le 0 \le t_2$ then

$$|S_1u_v(t_1) - S_1u_v(t_2)| \le p_0\Omega_3\big[(b+\|\phi\|) + M^*T(\Omega_1(b+\|\phi\|) + \Omega_2(b+\|\phi\|)\big]|0 - t_2|.$$

(iii) if $t_1 \le t_2 \le 0$ then

$$|S_1u_v(t_1) - S_1u_v(t_2)| = 0.$$

Thus, by Arzela-Ascoli theorem the operator S_1 is completely continuous.

Consequently the operator S is completely continuous.

Finally, the set $\Phi(S) = \{y \in B_0 : y = \lambda Sy, \lambda \in (0,1)\}$ is bounded, since for every solution $y \in B_0$ the function $x = y + \phi$ is a solution of the initial value problem (1.1)-(1.3), for which we have proved that $\|x\|_r \le K$ and hence $\|y\|_r \le K + \|\phi\|$.

Consequently, by *Lemma 1*, the operator S has a fixed point y^* in B_0. Then $x^* = y^* + \widetilde{\phi}$ is a solution of the initial value problem (1.1)-(1.3).

Hence the proof of the theorem is complete. □

4 Global Existence for Second order IVP

In this section we study the global existence for the IVP (Initial value problem) (1.4)-(1.7).

Definition 4. *A function $x \in \Omega \cap \Omega^1$ is called solution of the initial value problem (1.4)-(1.7) if x satisfies the following integral equation*

$$x(t) = \phi(0) + [\eta - g(0,\phi)]t + \int_0^t g(s,x_s)ds + \int_0^t \int_0^s F\Big(\tau, x_\tau, x'(\tau), \int_0^\tau a(\tau,\sigma)W(\sigma,x_\sigma,x'(\sigma))d\sigma, \int_0^T b(\tau,\sigma)H(\sigma,x_\sigma,x'(\sigma))d\sigma\Big)d\tau ds + \sum_{0<t_k<t}[I_k(x(t_k^-)) + (T-t_k)J_k(x(t_k^-))],\ t \in I.$$

Theorem 5. *Let $F : I\times D\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $g : I\times D \to \mathbb{R}^n$, $W : I\times D\times\mathbb{R}^n \to \mathbb{R}^n$, $H : I\times D\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous functions and $a : I\times I \to \mathbb{R}$, $b : I\times I \to \mathbb{R}$ are measurable for $t \ge s \ge 0$ function.*

Assume that $\mathbf{H_I}$ is satisfied and g satisfies $\mathbf{H_{g1}}$, $\mathbf{H_{g2}}$, a, b satisfies $\mathbf{H_a}$ and $\mathbf{H_b}$. Moreover we assume that:

$\mathbf{H_W}$ *There exists a continuous function* $m_3 : I \to [0,\infty)$ *such that*

$$|W(t,\phi,\psi)| \le m_3(t)\Omega_4(\|\phi\| + |\psi|),\ t \in I,\ \phi \in D,\ \psi \in \mathbb{R}^n,$$

where $\Omega_4 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_H}$ *There exists a continuous function* $m_4 : I \to [0,\infty)$ *such that*

$$|H(t,\phi,\psi)| \le m_4(t)\Omega_5(\|\phi\| + |\psi|),\ t \in I,\ \phi \in D,\ \psi \in \mathbb{R}^n,$$

where $\Omega_5 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_F}$ *There exists an integrable function* $P : I \to [0,\infty)$ *such that*

$$|F(t,u,v,w,y)| \le P(t)\Omega_6(\|u\| + |v| + |w| + |y|),\ t \in I,\ u \in D,\ v,w,y \in \mathbb{R}^n,$$

where $\Omega_6 : [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

$\mathbf{H_J}$ *There exist constants* d_k *such that* $|J_k(x)| \le d_k|x|$, $k = 1,...,m$ *for each* $x \in \mathbb{R}^n$.

Then if

$$\int_0^T \widehat{M}(s)ds < \int_{c_0}^{\infty} \frac{ds}{s + 2\Omega_6(s) + \Omega_4(s) + \Omega_5(s)}$$

where

$$\widehat{M}(t) = \max\Big\{(1+c_1)c_1,\ (1+c_1)\int_0^t P(\tau)d\tau,\ P(t),\ L_1m_3(t),\ L_2m_4(t)\Big\},\ \textit{and}$$

$$c_0 = \|\phi\| + \big[|\eta| + c_1\|\phi\| + 2c_2)\big](1+T) + c_1u(0) + \sum_{k=1}^{n}\big[c_k|x(t_k^-)| + (T-t_k)d_k|x(t_k^-)|\big]$$

the initial value problem (1.4)-(1.7) has at least one solution on $[-r,T]$.

Proof. To prove the existence of a solution of the initial value problem (1.4)-(1.7) we apply *Lemma 1*. First we obtain the *a priori bounds* for the solutions of the initial value problem $(1.4)_\lambda - (1.7)$, $\lambda \in (0,1)$, where $(1.4)_\lambda$ stands for the equation

$$\frac{d}{dt}[x'(t) - g(t,x_t)] = \lambda F\Big(t, x_t, x'(t), \int_0^t a(t,s)g(s,x_s,x'(s))ds, \int_0^T b(t,s)h(s,x_s,x'(s))ds\Big),\quad t \in I.$$

Let x be a solution of the initial value problem $(1.4)_\lambda - (1.7)$. From

$$x(t) = \lambda\phi(0) + \lambda[\eta - g(0,\phi)]t + \lambda\int_0^t g(s,x_s)ds + \lambda\int_0^t\int_0^s F\Big(\tau, x_\tau, x'(\tau), \int_0^\tau a(\tau,\sigma)w(\sigma,x_\sigma,x'(\sigma))d\sigma, \int_0^T b(\tau,\sigma)h(\sigma,x_\sigma,x'(\sigma))d\sigma\Big)d\tau ds + \lambda\sum_{0<t_k<t}[I_k(x(t_k^-)) + (T-t_k)J_k(x(t_k^-))],\ t\in I$$

we have, for every $t \in I$,

$$|x(t)| \le \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + c_1\int_0^t \|x_s\|ds + \int_0^t\int_0^s P(\tau)\Omega_6\Big(\|x_\tau\| + |x'(\tau)| + L_1\int_0^\tau m_3(\sigma)\Omega_4(\|x_\sigma\|)d\sigma + L_2\int_0^T m_4(\sigma)\Omega_5(\|x_\sigma\|)d\sigma\Big)d\tau ds + \sum_{k=1}^n[c_k|x(t_k^-)| + (T-t_k)d_k|x(t_k^-)|].$$

We consider the function μ given in the proof of *Theorem 3*. Then by the previous inequality we have,

$$\mu(t) = \sup\{|x(s)| : -r \le s \le t\},\ t \in I.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have, for every $t \in I$,

$$|\mu(t)| \le \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + c_1\int_0^t \mu(s)ds + \int_0^t\int_0^s P(\tau)\Omega_6\Big(\mu(\tau) + |x'(\tau)| + L_1\int_0^\tau m_3(\sigma)\Omega_4(\mu(\sigma) + |x'(\sigma)|)d\sigma + L_2\int_0^T m_4(\sigma)\Omega_5(\mu(\sigma) + |x'(\sigma)|)d\sigma\Big)d\tau ds + \sum_{k=1}^n[c_k|x(t_k^-)| + (T-t_k)d_k|x(t_k^-)|].$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the previous inequality obvious holds.

Denoting by $u(t)$ the right hand side of the above inequality we have,

$$u(0) = \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + \sum_{k=1}^n[c_k|x(t_k^-)| + (T-t_k)d_k|x(t_k^-)|],$$
$$\mu(t) \le u(t),\ t\in I.$$

and

$$\begin{aligned}u'(t) = c_1\mu(t) + \int_0^t P(\tau)\Omega_6\Big(\mu(\tau) + |x'(\tau)| + L_1\int_0^\tau m_3(\sigma)\Omega_4(\mu(\sigma) + |x'(\sigma)|)d\sigma\\ + L_2\int_0^T m_4(\sigma)\Omega_5(\mu(\sigma) + |x'(\sigma)|)d\sigma\Big)d\tau\\ \le c_1u(t) + \int_0^t P(\tau)\Omega_6\Big(u(\tau) + |x'(\tau)| + L_1\int_0^\tau m_3(\sigma)\Omega_4(u(\sigma) + |x'(\sigma)|)d\sigma\\ + L_2\int_0^T m_4(\sigma)\Omega_5(u(\sigma) + |x'(\sigma)|)d\sigma\Big)d\tau, \quad t\in I.\end{aligned}$$

Therefore if

$$v(t) = \sup\{|x'(s)| : s\in I\},\ t\in I$$

we obtain

$$\begin{aligned}u'(t) \le c_1u(t) + \int_0^t P(\tau)\Omega_6\Big(u(\tau) + v(\tau) + L_1\int_0^\tau m_3(\sigma)\Omega_4(u(\sigma) + v(\sigma))d\sigma\\ + L_2\int_0^T m_4(\sigma)\Omega_5(u(\sigma) + v(\sigma))d\sigma\Big)d\tau,\ t\in I.\end{aligned} \tag{4.1}$$

On the other hand, by

$$\begin{aligned}x'(t) =& \lambda[\eta - g(0,\phi)] + \lambda g(t,x_t) + \lambda\int_0^t F\Big(\tau, x_\tau, x'(\tau),\\ &\int_0^\tau a(\tau,s)W(s,x_s,x'(s))ds, \int_0^T b(\tau,s)H(s,x_s,x'(s))ds\Big)d\tau\end{aligned}$$

for any $t\in I$ and every $s\in[0,t]$, we obtain

$$\begin{aligned}|x'(t)| \le& |\eta| + c_1\|\phi\| + 2c_2 + c_1\|x_t\| + \int_0^t P(\tau)\Omega_6\Big(u(\tau) + |x'(\tau)| + L_1\int_0^\tau m_3(\sigma)\\ &(\times)\Omega_4(u(\sigma) + |x'(\tau)|)d\sigma + L_2\int_0^T m_4(\sigma)\Omega_5(u(\sigma) + |x'(\tau)|)d\sigma\Big)d\tau,\end{aligned}$$

or

$$\begin{aligned}v(t) \le& |\eta| + c_1\|\phi\| + 2c_2 + c_1u(t) + \int_0^t P(\tau)\Omega_6\Big(u(\tau) + v(\tau) + L_1\int_0^\tau m_3(\sigma)\\ &(\times)\Omega_4(u(\sigma) + v(\tau))d\sigma + L_2\int_0^T m_4(\sigma)\Omega_5(u(\sigma) + v(\tau))d\sigma\Big)d\tau,\ t\in I.\end{aligned}$$

Denoting by $z(t)$ the right hand side of the above inequality we have:

$$z(0) = |\eta| + c_1\|\phi\| + 2c_2 + c_1u(0), \quad v(t)\le z(t),\ t\in I$$

and

$$\begin{aligned} z'(t) = c_1 u'(t) + P(t)\Omega_6\Big(u(t) + v(t) + L_1 \int_0^t m_3(\sigma)\Omega_4(u(\sigma) + v(\tau))d\sigma \\ + L_2 \int_0^T m_4(\sigma)\Omega_5(u(\sigma) + v(\tau))d\sigma\Big), \\ \le c_1 u'(t) + P(t)\Omega_6\Big(u(t) + z(t) + L_1 \int_0^t m_3(\sigma)\Omega_4(u(\sigma) + z(\tau))d\sigma \\ + L_2 \int_0^T m_4(\sigma)\Omega_5(u(\sigma) + z(\tau))d\sigma\Big), \ t \in I. \end{aligned}$$

From (4.1), since $v(t) \le z(t)$ we have

$$\begin{aligned} u'(t) \le c_1 u(t) + \int_0^t P(\tau)\Omega_6\Big(u(\tau) + z(\tau) + L_1 \int_0^\tau m_3(\sigma)\Omega_4(u(\sigma) + z(\sigma))d\sigma \\ + L_2 \int_0^T m_4(\sigma)\Omega_5(u(\sigma) + z(\sigma))d\sigma\Big)d\tau, \ t \in I. \end{aligned}$$

Let

$$\begin{aligned} w(t) = u(t) + z(t) + L_1 \int_0^t m_1(\sigma)\Omega_4(u(\sigma) + z(\sigma))d\sigma \\ + L_2 \int_0^T m_2(\sigma)\Omega_5(u(\sigma) + z(\sigma))d\sigma, \quad t \in I. \end{aligned}$$

Then

$$w(0) = u(0) + z(0) = c_0, \quad u(t) + z(t) \le w(t), \ t \in I$$

and

$$\begin{aligned} w'(t) &= u'(t) + z'(t) + L_1 m_3(t)\Omega_4(u(t) + z(t)) + L_2 m_4(t)\Omega_5(u(t) + z(t)) \\ &\le c_1 u(t) + \int_0^t P(\tau)\Omega_6(w(\tau))d\tau + c_1 u'(t) + P(t)\Omega_6(w(t)) \\ &\quad + L_1 m_3(t)\Omega_4(u(t) + z(t)) + L_2 m_4(t)\Omega_5(u(t) + z(t)) \\ &\le (1 + c_1)c_1 w(t) + (1 + c_1) \int_0^t P(\tau)\Omega_6(w(\tau))d\tau + P(t)\Omega_6(w(t)) \\ &\quad + L_1 m_3(t)\Omega_4(u(t) + z(t)) + L_2 m_4(t)\Omega_5(u(t) + z(t)) \\ &\le \widehat{m}(t)\big[w(t) + 2\Omega_6(w(t)) + \Omega_4(w(t)) + \Omega_5(w(t))\big], \ t \in I. \end{aligned}$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s + 2\Omega_6(s) + \Omega_4(s) + \Omega_5(s)} \le \int_0^T \widehat{M}(s)ds < \int_{c_0}^\infty \frac{ds}{s + 2\Omega_6(s) + \Omega_4(s) + \Omega_5(s)}$$

This inequality implies that there is a constant K such that $w(t) \le K$, $t \in I$. Then

$$|x(t)| \le \mu(t) \le u(t), \ \ t \in I$$
$$|x'(t)| \le v(t) \le z(t), \ \ t \in I$$

and hence

$$\|x^*\| \le \|x\|_r + \|x\|_1 \le K.$$

In the second step we rewrite the initial value problem (1.4)-(1.7) as an integral operator and will prove that this operator is completely continuous.

Define the operator $Q : B \to B$, $B = \Omega$ by

$$(Qx)(t) = \begin{cases} \phi(t), \qquad t \in [-r, 0] \\ \phi(0) + [\eta - g(0, \phi)]t + \int_0^t g(s, y_s)ds + \int_0^t \int_0^s F\Big(\tau, x_\tau, x'(\tau), \\ \int_0^\tau a(s,\sigma)W(\sigma, x_\sigma, x'(\sigma))d\sigma, \int_0^T b(s,\sigma)H(\tau, x_\sigma, x'(\sigma))d\tau\Big)d\tau ds \\ + \sum_{0<t_k<t}[I_k(x(t_k^-)) + (T - t_k)J_k(x(t_k^-))], \quad t \in I. \end{cases}$$

Let a bounded sequence $\{z_v\}$ in B, i.e $\|z_v\| \le b$, for all v. Then the sequence $\{z_v(t)\}$, $t \in I$ is bounded in B. Using the fact that

$$\left|F\Big(t, z_{vt}, z_v'(s), \int_0^t a(t,s)W(s, z_{vs}, z_v'(s))ds, \int_0^T b(t,s)H(s, z_{vs}, z_v'(s))ds\Big)\right| \le M_1$$

where

$$M_1 = M^*\Omega_6\big[2b + M^*\Omega_4(2b)T + M^*\Omega_5(2b)T\big], \ \ M^* = \max\{\|\phi\|, \max\{\widetilde{M}(t), t \in I\}\}$$

we can easily prove that there exists a constant

$$M = \max\Big\{\|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + c_1 bT + M_1\frac{T^2}{2} + \sum_{k=1}^{n}(c_k + (T - t_k)d_k)|z(t_k^-)|, |\eta| + (1+T)c_1\|\phi\| + (2+T)c_2 + M_1 T\Big\}$$

such that

$$\|Qz_v\|_r \le M \quad \text{and} \quad \|(Qz_v)'\|_1 \le M$$

which means that $\{Qz_v\}$ and $\{(Qz_v)'\}$ are uniformly bounded.

Moreover rewriting the operator Qx for $t \in I$ as $(Qx)(t) = (Q_1x)(t) + (Q_2x)(t)$ where

$$\begin{aligned}(Q_1x)(t) =&\phi(0) + [\eta - g(0,\phi)]t + \int_0^t g(s, y_s)ds \\ &+ \sum_{0<t_k<t} [I_k(x(t_k^-)) + (T - t_k)J_k(x(t_k^-))], \\ (Q_2x)(t) =& \int_0^t \int_0^s F\Big(\tau, x_\tau, x'(\tau), \int_0^\tau a(s,\sigma)W(\sigma, x_\sigma)d\sigma, \\ &\int_0^T b(s,\sigma)H(\tau, x_\sigma)d\tau\Big)d\tau ds,\ t \in I,\end{aligned}$$

it is easy to see that the sequences $\{Q_1z_v\}$, $\{(Q_1z_v)'\}$ and $\{Q_2z_v\}$, $\{(Q_2z_v)'\}$ are equicontinuous.

Thus, by the Arzela-Ascoli theorem the operator Q is completely continuous.

Finally, the set $\Phi(Q) = \{x \in B : x = \lambda Qx,\ \lambda \in (0,1)\}$ is bounded, as we proved in the first part. Hence by *Lemma 1*, the operator Q has a fixed point in B. Then it is clear that the initial value problem (1.4)-(1.7) has at least one solution.

Hence the proof of the theorem is complete. □

5 Application

In this section, we apply some of the results established in this paper.

Example 6. *First, we consider the partial first order differential equation of the form*

$$\begin{aligned}w_t(u,t) - g_1(t, w(x, t-r)) =&P\Big(t, w(u,t), \int_0^t k_1\big(t, w(x, t-r)\big)ds, \\ &\int_0^b h_1\big(t, w(x,t-r)\big)ds\Big), \quad (5.1)\end{aligned}$$

$$w(0,t) = w(\pi, t) = 0, \quad 0 \le t \le b, \quad (5.2)$$

$$w(u,t) = \phi(x,t), \quad 0 \le u \le \pi, \quad (5.3)$$

$$w(t_k^+, y) - w(t_k^-, y) = I_k(w(t_k^-, y)),\ k = 1, 2, ..., m, \quad (5.4)$$

where $P : [0,b] \times D \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is a function and* $g_1, k_1, h_1 : [0,b] \times D \to \mathbb{R}^n$ *are continuous functions. We assume that the functions* P, g_1, k_1 *and* h_1 *in (5.1)-(5.4) satisfy the following conditions.*

(1) *There exist constants* c_1 *and* c_2 *such that* $|g(t,\phi)| \le c_1|\phi| + c_2$, *for* $t \in [0,b]$, $\phi \in D$.

(**2**) *There exists a nonnegative function* p_1 *defined on* $[0,b]$ *such that*

$$\Big|\int_0^t k_1(t,x)ds\Big| \leq p_1(t)|x|$$

for $t \in [0,b]$ *and* $x \in D$.

(**3**) *There exists a nonnegative function* q_1 *defined on* $[0,b]$ *such that*

$$\Big|\int_0^b h_1(t,x)ds\Big| \leq q_1(t)|x|$$

for $t \in [0,b]$ *and* $x \in D$.

(**4**) *There exists nonnegative real valued continuous function* l_1 *defined on* $[0,b]$ *and a positive continuous increasing function* K_1 *defined on* $\mathbb{R}_+$ *such that*

$$|P(t,x,y,z)| \leq l_1(t)K_1(|x|+|y|+|z|)$$

for $t,x,y,z \in [0,b] \times D \times \mathbb{R}^n \times \mathbb{R}^n$.

(**5**) *There exist constants* c_k *such that* $|I_k(x)| \leq c_k|x|$, $k = 1,...,m$ *for each* $x \in \mathbb{R}^n$.

Let us take $X = L^2[0,\pi]$. *Suppose that*

$$\int_0^T \widehat{m}(s)ds < \int_c^{+\infty} \frac{ds}{\Omega_3(s)+\Omega_1(s)+\Omega_2(s)}$$

where

$$\widehat{m}(t) = \max\Big\{\frac{1}{(1-c_1)}p(t),\ L_1m_1(t),\ L_2m_2(t)\Big\} \text{ and}$$
$$c = \frac{1}{(1-c_1)}\Big\{(1+c_1)\|\phi\| + 2c_2 + \sum_{k=1}^n c_k|x(t_k^-)|\Big\}$$

is satisfied. Define the functions $f : [0,b]\times X\times X\times X \to X$, $a,b : [0,b]\times[0,b] \to X$, $g,w,h : [0,b]\times X \to X$, $I_k \in (X,X)$ *as follows*

$$f(t,x,y,z)(u) = P(t,x(u,t),y(u,t),z(u,t)),$$
$$k(t,x)(u) = k_1(t,x(u,t)) \quad \text{and}$$
$$h(t,x)(u) = h_1(t,x(u,t))$$

for $t \in [0,b], x,y,z \in X$ *and* $0 \leq u \leq \pi$.

With these choices of the functions, the equations (5.1)-(5.4) can be modelled abstractly as nonlinear mixed Volterra-Fredholm integrodifferential equation with nonlocal condition in Banach space X:

$$\frac{d}{dt}[x(t) - g(t, x_t)] = f\Big(t, x_t, \int_0^t a(t,s)w(s,x_s)ds, \int_0^T b(t,s)h(s,x_s)ds\Big),$$
$$t \in I = [0,T] \setminus \{t_1, ..., t_m\}, \qquad (5.5)$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, ..., m, \qquad (5.6)$$
$$x_0 = \phi \qquad (5.7)$$

Since all the hypotheses of the Theorem 3 are satisfied, the Theorem 3 can be applied to guarantee the solution of the nonlinear mixed Volterra-Fredholm type neutral impulsive integrodifferential equation.

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