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DIFFERENT VERSIONS OF THE IMPRIMITIVITY THEOREM

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Abstract. In this paper we present different versions of the imprimitivity theorem hoping that this might become a support for the ones who are interested in the subject. We start with Mackey's theorem [26] and its projective version [29]. Then we remind Mackey's fundamental imprimitivity theorem in the bundle context [14]. Section 5 is dedicated to the imprimitivity theorem for systems of G-covariance [6]. In Section 6 and 7 we refer to the imprimitivity theorem in the context of C^*-algebras [39] and to the symmetric imprimitivity theorem [36], [42], [11].

1 Introduction

The importance of induced representations was recognized and emphasized by George Mackey [26], who first proved the imprimitivity theorem and used that to analyze the representation theory of some important classes of groups (which include the Heisenberg group and semi-direct products where the normal summand is abelian). Mackey's imprimitivity theorem gives a way of identifying those representations of a locally compact group G which are induced from a given closed subgroup H. This theorem has played a fundamental role in the development of the representation theory of locally compact groups and has found applications in other fields of mathematics as well as in quantum mechanics. In [29], Mackey also proved the imprimitivity theorem for projective representations. During the years it has been extended to mathematical different structures from groups (for example [14]) and new versions have appeared, some of them avoiding Mackey's separability assumptions.

Fell [14] showed that a locally compact group extension H, N (that is, a locally compact group H together with a closed normal subgroup N of H) can be regarded as a special case of a more general object, called a homogeneous Banach $*$-algebraic

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bundle. In other words, the generalization of a group extension H, N to a homogeneous Banach $*$-algebraic bundle consists in letting H and N become Banach $*$-algebras, while their quotient $G = H/N$ remains a group. His purpose was to classify the $*$-representations of the Banach $*$-algebra H in terms of the $*$-representations of N and the projective representations of subgroups of G. By a Banach $*$-algebraic bundle over a locally compact group G with unit e, we mean a Hausdorff space B together with an open surjection $\pi\colon B \to G$ such that each fiber $B_x = \pi^{-1}(x)$ $(x \in G)$ has the structure of a Banach space and a binary operation "$\cdot$" on B and a unary operation "$*$" on B which are equivariant under π with the multiplication and inverse in G, i.e. $\pi(s \cdot t) = \pi(s)\pi(t)$, $\pi(s^*) = (\pi(s))^{-1}$, $s, t \in B$ and satisfying the laws in a Banach $*$-algebra, i.e. $r \cdot (s \cdot t) = (r \cdot s) \cdot t$, $(r \cdot s)^* = s^* \cdot r^*$, $\|r \cdot s\| \leq \|r\|\|s\|$ (see Definition 13). The equivariance conditions show that B_e is closed under "$\cdot$" and "$*$", which is in fact a Banach $*$-algebra, called the unit fiber subalgebra of B. If λ is Haar measure on G, the operations "$\cdot$" and "$*$" induce a natural Banach $*$-algebra structure on the cross-sectional space $L = L_1(B, \lambda)$, consisting of all measurable functions $f\colon G \to B$ such that $\pi \circ f = 1_G$ and such that $\|f\| = \int_G \|f(g)\| d\lambda(g) < \infty$, called the cross-sectional algebra of B, π. Thus, one may think of a Banach $*$-algebraic bundle over G as a Banach $*$-algebra L together with a distinguished continuous direct sum decomposition of L (as a Banach space), the decomposition being based on the group G and the operations "$\cdot$" and "$*$" of L being equivariant with the multiplication and inverse in G. This latter view of a Banach $*$-algebraic bundle is clearly analogous to the concept of systems of imprimitivity for representations of G. In Theorem 23 we remind Mackey's fundamental imprimitivity theorem in the bundle context.

Cattaneo [6] proved a generalization of the imprimitivity theorem by admitting subrepresentations of induced representations. The imprimitivity theorem is still valid provided transitive systems of imprimitivity are replaced by transitive systems of covariance, i.e. provided positive-operator-valued measures take the place of projection-valued measures. In particular, he showed that a strongly continuous unitary representation of a second countable locally compact group G on a separable (complex) Hilbert space is unitarily equivalent to a representation induced from a closed subgroup of G if and only if there is an associated transitive system of covariance. Then he extended the theorem to projective representations.

Ørsted presented in [32] an elementary proof of Mackey's imprimitivity theorem, not involving any measure theory beyond Fubini's theorem for continuous functions. For projective systems of imprimitivity a similar proof was given, but for unitary systems, in the general case of non-unimodular groups.

S.T. Ali [1] proved a generalization of Mackey's imprimitivity theorem in the special case where projection-valued measure is replaced by a commutative positive-operator-valued measure to the system of covariance.

When Rieffel [39] viewed inducing in the context of C^*-algebras, the imprimitivity theorem emerged as a Morita equivalence between the group C^*-algebra $C^*(H)$

and the transformation group C^*-algebra $C^*(G, G/H)$. The theorem states in fact that the unitary representation U is induced precisely when it is part of a covariant representation (π, U) of the dynamical system $(C_0(G/H), G)$. In other words, Rieffel proved the theorem by showing that the crossed product $C_0(G/H) \bowtie G$ is Morita equivalent to the group algebra $C^*(H)$. This reformulation has found many generalizations, both to other situations involving transformation groups and to crossed products of non-commutative C^*-algebras. Of particular interest has been the realization that the imprimitivity theorem has a symmetric version: if K is another closed subgroup of G, then H acts naturally on the left of the right coset space G/K, K acts on the right of $H \setminus G$ and the transformation group C^*-algebras $C^*(H, G/K)$ and $C^*(K, H \setminus G)$ are Morita equivalent [40]. The symmetric imprimitivity theorem of Green and Rieffel involves commuting free and proper actions of two groups, G and H, on a space X and asserts that $C_0(G \setminus X) \bowtie H$ is Morita equivalent to $C_0(X/H) \bowtie G$; one recovers Mackey's theorem by taking $H \subset G$ and $X = G$. The extensions of Rieffel's imprimitivity theorem to cover actions of G on a non-commutative C^*-algebra A is due to Green [18]; it asserts that the crossed product $C^*(H, A)$ is Morita equivalent to $C^*(G, C_0(G/H, A))$ where G acts diagonally on $C_0(G/H, A) \cong C_0(G/H) \otimes A$. As Rieffel observed in [41], it is not clear to what extent the symmetric imprimitivity theorem works for actions on non-commutative algebras. Raeburn [36] formulated such a theorem and investigated some of its consequences.

In several projects it was shown that imprimitivity thorems and other Morita equivalences are equivariant, in the sense that the bimodules implementing the equivalences between crossed products carry actions or coactions compatible with those on the crossed products (see [10]). In [11], Echterhoff and Raeburn proved an equivariant version of Raeburn's symmetric imprimitivity theorem ([36]) for the case when two subgroups act on opposite sides of a locally compact group.

We point out some recent papers without presenting the results, but only summarize them. Suppose that (X, G) is a second countable locally compact transformation group and that $S_G(X)$ denotes the set of Morita equivalences classes of separable dynamical systems (G, A, α), where A is a $C_0(X)$-algebra and α is compatible with the given G-action on X. Huef, Raeburn and Williams proved ([21, Theorem 3.1]) that if G and H act freely and properly on the left and right of a space X, then $S_G(X/H)$ and $S_H(G \setminus X)$ are isomorphic as semigroups and if the isomorphism maps the class of (G, A, α) to the class of (H, B, β), then $A \bowtie_\alpha G$ is Morita equivalent to $B \bowtie_\beta H$. In [33] the authors proved an analogue of the symmetric imprimitivity theorem of [36] concerning commuting free and proper actions of two different groups. In fact, they proved two symmetric imprimitivity theorems, one for reduced crossed products ([33, Theorem 1.9]) and one for full crossed products (Theorem 2.1, [33]). Pask and Raeburn also showed how comparing the two versions of the imprimitivity theorem can lead to amenability results ([33, Corollary 3.1]). Huef and Raeburn identified the representations which induce to regular representation under the Morita equivalence of the symmetric imprimitivity theorem ([22, Theorem 1, Corollary 6]) and obtained a direct proof of the theorem of Quigg and Spielberg ([22, Corollary 3]) that in [35] proved that the symmetric imprimitivity theorem has analogues for reduced crossed products. The results in [23] showed that the different proofs of the symmetric imprimitivity theorem for actions on graph algebras yield isomorphic equivalences and this gives a new information about the amenability of actions on graph algebras.

2 Mackey's imprimitivity theorem

Let M be a separable locally compact space and let G be a separable locally compact group. Let $x, s \longrightarrow (x)s$ denote a map of $M \times G$ onto M which is continuous and is such that for fixed $s, x \longrightarrow (x)s$ is a homeomorphism and such that the resulting map of G into the group of homeomorphisms of M is a homomorphism.

Let $P(E \longrightarrow P_E)$ be a σ homomorphism of the σ Boolean algebra of projections in a separable Hilbert space H such that P_M is the identity I.

Let $U(s \longrightarrow U_s)$ be a representation of G in H, that is a weakly (and hence strongly) continuous homomorphism of G into the group of unitary operators in H.

Definition 1. *([26]) If $U_s P_E U_s^{-1} = P_{(E)s^{-1}}$ for all E and s and if P_E takes on values other than 0 and I, we say that U is* ***imprimitive*** *and that P is a* ***system of imprimitivity for*** *U.*

We call M ***the base of*** *P.*

Definition 2. *([26]) P is a* ***transitive system of imprimitivity for*** *U if for each $x, y \in M$ there is $s \in G$ for which $(x)s = y$.*

In general we define a pair to be a unitary representation for the group G together with a particular system of imprimitivity for this representation.

Definition 3. *([26]) If U, P and U', P' are two pairs with the same base M we say that they are* ***unitary equivalent*** *if there is a unitary transformation V from the space of U and P to the space of U' and P' such that $V^{-1}U'_s V = U_s$ and $V^{-1}P'_E V = P_E$ for all s and E.*

Theorem 4. *(Theorem 2, [26]) Let G be a separable locally compact group and let G_0 be a closed subgroup of G. Let U' and P' be any pair based on G/G_0. Let μ be any quasi invariant measure in G/G_0. Then there is a representation L of G_0 such that U', P' is unitarily equivalent to the pair generated by L and μ. If L and L' are representations of G_0 and μ and μ' are quasi invariant measures in G/G_0 then the pair generated by L' and μ' is unitary equivalent to the pair generated by L and μ if and only if L and L' are unitary equivalent representations of G_0.*

3 Mackey's imprimitivity theorem for projective representations

Definition 5. *([29]) Let G be a separable locally compact group. A* ***projective representation*** *L of G is a map $x \longrightarrow L_x$ of G into the group of all unitary transformations of a separable Hilbert space H onto itself such that :*

a) $L_e = I$, where e is the identity of G and I is the identity operator;

b) $L_{xy} = \sigma(x, y)L_xL_y$ for all $x, y \in G$, where $\sigma(x, y)$ is a constant;

c) the function $x \longrightarrow (L_x(\phi), \psi)$ is a Borel function on G for each $\phi, \psi \in H$.

The function $\sigma : x, y \longrightarrow \sigma(x, y)$ is uniquely determined by L and it is called the **multiplier** of L.

By a **σ-representation** of G we mean a projective representation whose multiplier is σ.

The multiplier σ of the projective representation L has the following properties :

1. $\sigma(e, x) = \sigma(x, e) = 1$ and $|\sigma(x, y)| = 1$ for all $x, y \in G$;

2. $\sigma(xy, z)\sigma(x, y) = \sigma(x, yz)\sigma(y, z)$ for all $x, y, z \in G$;

3. σ is a Borel function on $G \times G$.

Any function from $G \times G$ to the complex numbers which has these three properties is called a **multiplier** for G.

If σ is a multiplier for G we define a group G^σ whose elements are pairs (λ, x), where λ is a complex number of modulus one and $x \in G$ and in which the multiplication is given by $(\lambda, x)(\mu, y) = (\lambda\mu/\sigma(x, y), xy)$. In G^σ the identity element is $(1, e)$ and the inverse of (λ, x) is $(\sigma(x, x^{-1})/\lambda, x^{-1})$. Let $\mathbf{T}$ denote the compact group of all complex numbers of modulus one. $\mathbf{T}$ and G, as separable locally compact groups, have natural Borel structures which are standard (in the sense described in [28]). The direct product of these defines a standard Borel structure in G^σ with respect to which $(x, y) \longrightarrow xy^{-1}$ is a Borel function. Thus G^σ is a standard Borel group (in the sense of [28]). Moreover, the direct product of Haar measure in $\mathbf{T}$ with a right invariant Haar measure in G is a right invariant measure in G^σ. Thus it can be applied [28, Theorem 7.1] and it results that G^σ admits a unique locally compact topology under which it is a separable locally compact group whose associated Borel structure is that just described. We suppose G^σ equipped with this topology.

For each σ-representation L of G let $L^0_{\lambda,x} = \lambda L_x$ and denote by L^0 the map $(\lambda, x) \longrightarrow L^0_{\lambda,x}$. By [29, Theorem 2.1], for each σ-representation L of G, the map L^0 is an ordinary representation of G^σ.

Let H be a closed subgroup of G. If σ is a multiplier for G, then the restriction of σ to H is a multiplier for H and we may speak of the σ-representations of H as well as of G. In particular, the restriction to H of a σ-representation of G is a σ-representation of H. In [26] it is discussed a process for going from ordinary representation L of H to certain ordinary representation U^L of G, called **induced representation**. This process can be generalized for σ-representations as well. Let θ denote the identity map of H^σ into G^σ. The range of θ is the inverse image of the closed subgroup H under the canonical homomorphism of G^σ on G. Hence this range is a close subgroup of G^σ and is locally compact. Since θ is both an algebraic isomorphism and a Borel isomorphism, it follows from Theorem 7.1, [28] that it is a homeomorphism. Let L be an arbitrary σ-representation of H. Then L^0 is an ordinary representation of H^σ which may be regarded as an ordinary representation of the closed subgroup $\theta(H^\sigma)$ of G^σ. As described in [27] it can be formed U^{L_0} and from [27, Theorem 12.1] and [29, Theorem 2.1] it follows that U^{L_0} is of the form V^0 for a uniquely determined σ-representation V of G. Actually U^{L_0} is only defined up to an equivalence. V is called **the σ-representation of G induced by the σ-representation L of H** and it is denoted by U^L.

Definition 6. *([29]) Let S be a metrically standard Borel space. A **projection valued measure** on S is a map P, $E \longrightarrow P_E$, of the Borel subsets of S into the projections on a separable Hilbert space $H(P)$ such that $P_{E\cap F} = P_E P_F$, $P_S = I$, $P_0 = 0$ and $P_E = \sum_{j=1}^{\infty} P_{E_j}$, when $E = \bigcup_{j=1}^{\infty} E_j$ and the E_j are disjoint.*

Definition 7. *([29]) Let L be a σ-representation of a separable locally compact group G. A **system of imprimitivity for** L is a pair consisting of a projection valued measure P with $H(P) = H(L)$ and an anti homomorphism h of G into the group of all Borel automorphisms of the domain S of P such that:*

a) if $[x]y$ denotes the action of $h(y)$ on x then $y, x \longrightarrow [x]y$ is a Borel function;

b) $L_y P_E L_y^{-1} = P_{[E]y^{-1}}$ for all $y \in G$ and all Borel sets $E \subseteq S$.

*We call S the **base** of the system of imprimitivity.*

Definition 8. *([29]) Let P, h and P', h' be systems of imprimitivity for the same σ-representation L. We say that P, h and P', h' are **strongly equivalent** if there is a Borel isomorphism φ of the base S of P onto the base S' of P' such that $P'_{\varphi[E]} = P_E$ for all E and $h'(y) = \varphi h(y)\varphi^{-1}$ for all $y \in G$.*

Theorem 9. *(Theorem 6.6, [29]) Let G be a separable locally compact group, let H be a close subgroup of G and let σ be a multiplier for G. Let V be a σ-representation of G and let P' be a projection valued measure based on G/H such that P', h is a* *system of imprimitivity for V. Then there is a σ-representation L of H such that the pair P', is equivalent to the pair P, U^L, where P, h is the canonical system of imprimitivity for U^L based on G/H. If L_1 and L_2 are two σ-representations of H and P_1, h and P_2, h are the corresponding canonical systems of imprimitivity then the pairs P_1, U^{L_1} and P_2, U^{L_2} are equivalent if and only if L_1 and L_2 are equivalent σ-representations of H.*

4 Mackey's imprimitivity theorem in the bundle context

Definition 10. *([14]) Let G be a fixed (Hausdorff) topological group with unit e. A **bundle** $\mathcal{B}$ over G is a pair $\langle B, \pi\rangle$, where B is a Hausdorff topological space and π is a continuous open map of B onto G.*

*G is called the **base space** and π the **bundle projection** of $\mathcal{B}$. For each $x \in G$, $\pi^{-1}(x)$ is the **fiber over** x and is denoted by B_x.*

Definition 11. *([14]) Let X and B be two Hausdorff topological spaces and let π be a continuous open map of B onto X. A **cross-sectional function for** $\mathcal{B}$ is a map $\gamma\colon X \to B$ such that $\pi \circ \gamma$ is the identity map on X. A continuous cross-sectional function is called a **cross-section** of $\mathcal{B}$.*

We denote by $L(\mathcal{B})$ the linear space of all cross-sections f of $\mathcal{B}$ which have compact support (that is, $f(x) = 0_x$ for all x outside some compact subset K of X).

Definition 12. *([14]) A **Banach bundle** $\mathcal{B}$ over G is a bundle $\langle B, \pi\rangle$ over G together with operations and a norm making each fiber B_x ($x \in G$) into a complex Banach space and satisfying the following conditions:*

i) $s \longrightarrow \|s\|$ *is continuous on B to $\mathbb{R}$;*

ii) *the operation $+$ is continuous on $\{\langle s, t\rangle \in B \times B |\ \pi(s) = \pi(t)\}$ to B;*

iii) *for each complex number λ, the map $s \longrightarrow \lambda \cdot s$ is continuous on B to B;*

iv) *if $x \in G$ and $(s_i)_i$ is a net of elements of B such that $\|s_i\| \longrightarrow 0$ and $\pi(s_i) \longrightarrow x$ in G, then $s_i \longrightarrow 0$ in B, where 0 is the zero element of the Banach space $B_x = \pi^{-1}(x)$.*

Definition 13. *([14]) A **Banach $*$-algebraic bundle $\mathcal{B}$ over** G is a Banach bundle $\langle B, \pi\rangle$ over G together with a binary operation "$\cdot$" on $B \times B$ to B and a unary operation "$*$" on B satisfying :*

i) $\pi(s \cdot t) = \pi(s)\pi(t)$ *for all $s, t \in B$ (equivalently, $B_x \cdot B_y \subset B_{xy}$ for all $x, y \in G$);*

ii) *for each* $x, y \in G$*, the product* $\langle s, t\rangle \longrightarrow s \cdot t$ *is bilinear on* $B_x \times B_y$ *to* B_{xy}*;*

iii) $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ *for all* $r, s, t \in B$*;*

iv) $\|s \cdot t\| \leq \|s\|\|t\|$ *for all* $s, t \in B$*;*

v) *"·" is continuous on* $B \times B$ *to* B*;*

vi) $\pi(s^*) = (\pi(s))^{-1}$ *for all* $s \in B$ *(equivalently,* $(B_x)^* \subset B_{x^{-1}}$ *for all* $x \in G$*);*

vii) *for each* $x \in G$*, the map* $s \longrightarrow s^*$ *is conjugate-linear on* B_x *to* $B_{x^{-1}}$*;*

viii) $(s \cdot t)^* = t^* \cdot s^*$ *for all* $s, t \in B$*;*

ix) $s^{**} = s$ *for all* $s \in B$*;*

x) $\|s^*\| = \|s\|$ *for all* $s \in B$*;*

xi) $s \longrightarrow s^*$ *is continuous on* B *to* B*.*

Remark 14. *([14]) Let* $\mathcal{B} = \langle B, \pi, \cdot, ^* \rangle$ *be a Banach* $*$*-algebraic bundle. If* H *is a topological subgroup of* G*, the reduction of* $\mathcal{B}$ *to* H *is a Banach* $*$*-algebraic bundle over* H *(with the restrictions of the norm and the operations of* $\mathcal{B}$*).*

Definition 15. *([14]) Let* $\mathcal{B} = \langle B, \pi, \cdot, ^* \rangle$ *be a Banach* $*$*-algebraic bundle over the topological group* G *(with unit* e*). If* $x \in G$*, a map* $\lambda \colon B \to B$ *is* ***of left order*** x *(respectively* ***of right order*** x *if* $\lambda(B_y) \subset B_{xy}$ *(respectively,* $\lambda(B_y) \subset B_{yx}$*) for all* $y \in G$*. We say that* λ *is* ***bounded*** *if there is a non-negative constant* k *such that* $\|\lambda(s)\| \leq k\|s\|$ *for all* $s \in B$*. If* λ *is of some left or right order* x*, it is said to be* ***quasi-linear*** *if, for each* $y \in G$*,* $\lambda|B_y$ *is linear (on* B_y *to* B_{xy} *or* B_{yx}*).*

Definition 16. *([14]) A* ***multiplier of*** $\mathcal{B}$ ***of order*** x *is a pair* $u = \langle \lambda, \mu\rangle$*, where* λ *and* μ *are continuous bounded quasi-linear maps of* B *into* B*,* λ *is of left order* x*,* μ *is of right order* x *and*

(i) $s \cdot \lambda(t) = \mu(s) \cdot t$

(ii) $\lambda(s \cdot t) = \lambda(s) \cdot t$

(iii) $\mu(s \cdot t) = s \cdot \mu(t)$ *for all* $s, t \in B$*.*

If $x \in G$, we denote by $M_x(\mathcal{B})$ the set of all multipliers of $\mathcal{B}$ of order x.

Definition 17. *([14]) A* $*$***-representation*** *of a Banach* $*$*-algebraic bundle* $\mathcal{B}$ *over* G *on a Hilbert space* X *is a map* T *assigning to each* $s \in B$ *a bounded linear operator* T_s *on* X *such that :*

i) $s \longrightarrow T_s$ *is linear on each fiber* B_x $(x \in G)$*;*

ii) $T_{st} = T_s T_t$ *for all* $s, t \in B$*;*

iii) $T_{s^*} = (T_s)^*$ *for all* $s \in B$*;*

iv) the map $s \longrightarrow T_s$ *is continuous with respect to the topology of* B *and the strong operator topology.*

X *is called the* ***space of*** T *and is denoted by* $X(T)$*.*

Definition 18. *([14]) A* $*$*-representation* T *of* $\mathcal{B}$ *is* ***non-degenerate*** *if the union of the ranges of* T_s $(s \in B)$ *spans a dense linear subspace of* $X(T)$*, or, equivalently, if* $0 \neq \xi \in X(T)$ *implies that* $T_s\xi \neq 0$ *for* $s \in B$*.*

Definition 19. *([14]) Let* X *be a Hilbert space and let* M *be a locally compact Hausdorff space. A* ***Borel*** X***-projection-valued measure on*** M *is a map* P *assigning to each Borel subset* W *of* M *a projection* $P(W)$ *on* X *such that :*

a) $P(M) = I_X$ *(=identity operator on* X*);*

b) if $W_1, W_2, \ldots$ *is a sequence of pairwise disjoint Borel subsets of* M*, then* $P(W_n)$ $(n = 1, 2, \ldots)$ *are pairwise orthogonal and*

$$P(\bigcup_{n=1}^{\infty}) = \sum_{n=1}^{\infty} P(W_n).$$

P *is* ***regular*** *if, for every Borel set* W*,*

$$P(W) = \sup\left\{P(C)|\ C \text{ is a compact subset of } W\right\}.$$

Definition 20. *([14]) Let* M *be a locally compact Hausdorff space on which the locally compact group* G *acts continuously to the left as a group of transformations* $\langle x, m\rangle \longrightarrow xm$*. A* ***system of imprimitivity for*** $\mathcal{B}$ ***over*** M *is a pair* $\langle T, P\rangle$*, where* T *is a non-degenerate* $*$*-representation of* $\mathcal{B}$ *and* P *is a regular Borel* $X(T)$*-projection-valued measure on* M *satisfying*

$$T_s P(W) = P(\pi(s)W)T_s$$

for all $s \in B$ *and all Borel subsets* W *of* M*.*

Definition 21. *([14]) If* $\mathcal{T} = \langle T, P\rangle$ *and* $\mathcal{T}' = \langle T', P'\rangle$ *are two systems of imprimitivity over the same* M*, a bounded linear operator* $F\colon X(T) \to X(T')$ *is* $\mathcal{T}, \mathcal{T}'$ ***interwining*** *if* $FT_s = T'_s F$ *and* $FP(W) = P'(W)F$ *for all* $s \in B$ *and all Borel subsets* W *of* M*. If* F *is isometric and onto* $X(T')$*, then* $\mathcal{T}$ *and* $\mathcal{T}'$ *are* ***equivalent****.*

Definition 22. *([14]) The non-degenerate $*$-representation T^0 of $L(M,\mathcal{B})$ defined by*

$$T^0_f\xi = \int_G \left[\int_M dPmT_{f(m,x)}\right]\xi d\lambda x \ \ (f\in L(M,\mathcal{B}))$$

*is called the **integrated form** of the system of imprimitivity $\langle T,P\rangle$.*

Theorem 23. *([14, Theorem 15.1]) Let K be a closed subgroup of G and M the G-transformation space G/K. Let $\langle T,P\rangle$ be a system of imprimitivity for $\mathcal{B}$ over M. Then there is a non-degenerate $*$-representation S of $\mathcal{B}_K$, unique to within unitary equivalence, such that $\langle T,P\rangle$ is equivalent to the system of imprimitivity attached to U^S, the induced representation (see [14, Section 11]).*

Proof. We assume that there is a cyclic vector ξ for $\langle T,P\rangle$. Let T^0 be the integrated form of $\langle T,P\rangle$.

For each pair of elements ϕ,ψ of $L(\mathcal{B})$ we define $\alpha=\alpha[\phi,\psi]$ of $L(M,\mathcal{B})$ as follows :

$$\alpha(yK,x)=\int_K \psi^*(yk)\phi(k^{-1}y^{-1}x)d\nu k. \tag{4.1}$$

Notice that $\alpha=\alpha[\phi,\psi]$ depends linearly on ϕ and conjugate-linearly on ψ.

We define a conjugate-bilinear form $(\cdot,\cdot)_0$ on $L(\mathcal{B})\times L(\mathcal{B})$ as follows :

$$(\phi,\psi)_0=(T^0_{\alpha[\phi,\psi]}\xi,\xi) \tag{4.2}$$

We define a representation Q of $M(\mathcal{B}_K)$ on $L(\mathcal{B})$ ($M(\mathcal{B}_K)$ is identified as usual with $M_K(\mathcal{B})=\bigcup\limits_{x\in K} M_x(\mathcal{B})$). For $t\in M(\mathcal{B}_K)$, $\phi\in L(\mathcal{B})$, put

$$(Q_t\phi)(x)=(\delta(\pi(t)))^{\frac{1}{2}}(\Delta(\pi(t)))^{-\frac{1}{2}}t\cdot\phi(\pi(t)^{-1}t) \ \ (x\in G) \tag{4.3}$$

Clearly, $Q_t\phi\in L(\mathcal{B})$, Q_t is linear in t on each fiber and

$$Q_{st}=Q_sQ_t \ \ (s,t\in M(\mathcal{B}_{\mathcal{K}})). \tag{4.4}$$

We claim that

$$(Q_t\phi,\psi)_0=(\phi,Q_{t^*}\psi)_0 \ \ (t\in M(\mathcal{B}_K),\phi,\psi\in L(\mathcal{B})). \tag{4.5}$$

Indeed, it is sufficient to show

$$\alpha[Q_t\phi,\psi]=\alpha[\phi,Q_{t^*}\psi]. \tag{4.6}$$

If $y,x\in G$, we have

$$\alpha[Q_t\phi,\psi](yK,x)=$$

$$= (\delta(\pi(t)))^{\frac{1}{2}}(\Delta(\pi(t)))^{-\frac{1}{2}}\int_K \Delta(k^{-1}y^{-1})(\psi(k^{-1}y^{-1}))^* t\phi(\pi(t)^{-1}k^{-1}y^{-1}x)d\nu k$$

$$= (\Delta(\pi(t)))^{\frac{1}{2}}(\delta(\pi(t)))^{-\frac{1}{2}}\int_K \Delta(k^{-1}y^{-1})(\psi(\pi(t)k^{-1}y^{-1}))^* t\phi(k^{-1}y^{-1}x)d\nu k$$

$$= \int_K \Delta(k^{-1}y^{-1})((Q_{t^*}\psi)(k^{-1}y^{-1}))^*\phi(k^{-1}y^{-1}x)d\nu k$$

$$= \int_K (Q_{t^*}\psi)^*(yk)\phi(k^{-1}y^{-1}x)d\nu k$$

$$= \alpha[\phi, Q_{t^*}\psi](yK, x).$$

So (4.6) is proved and hence (4.5) also.

Next we claim that if $\phi_i \longrightarrow \phi$ and $\psi_i \longrightarrow \psi$ in $L(\mathcal{B})$ uniformly on G with uniformly bounded compact supports, then

$$(\phi_i, \psi_i)_0 \longrightarrow (\phi, \psi). \tag{4.7}$$

Indeed, one verifies that $\alpha[\phi_i, \psi_i] \longrightarrow \alpha[\phi, \psi]$ uniformly on $M \times G$ with uniformly bounded compact supports; hence $(T^0_{\alpha[\phi_i,\psi_i]}\xi, \xi) \longrightarrow (T^0_{\alpha[\phi,\psi]}\xi, \xi)$.

For each $f \in L(M, \mathcal{B})$ we define a map $\widehat{f}\colon U \to L(\mathcal{B})$ as follows :

$$\widehat{f}(u)(x) = u^{-1}f(\pi(u)K, \pi(u)x) \ \ (u \in U, x \in G). \tag{4.8}$$

It is clear that $\widehat{f}(u)$ belongs to $L(\mathcal{B})$ and that $u \longrightarrow \widehat{f}(u)$ is continuous in the sense that, if $u_i \longrightarrow u$ in U, $\widehat{f}(u_i) \longrightarrow \widehat{f}(u)$ uniformly on G with uniformly bounded compact support. It follows that this and (4.7) that, if $f, g \in L(M, \mathcal{B})$, the function $u \longrightarrow (\widehat{f}(u), \widehat{g}(u))_0$ is continuous on U. We also observe that

$$\widehat{f}(ut) = (\delta(\pi(t)))^{\frac{1}{2}}(\Delta(\pi(t)))^{-\frac{1}{2}}Q_{t^{-1}}(\widehat{f}(u)) \ \ (u \in U, t \in U_K) \tag{4.9}$$

From (4.5) and (4.9) it follows that

$$(\widehat{f}(ut), \widehat{f}(ut))_0 = \delta(\pi(t))(\Delta(\pi(t)))^{-1}(\widehat{f}(u), \widehat{f}(u))_0 \ \ (u \in U, t \in U_K). \tag{4.10}$$

We prove that for each $f \in L(M, \mathcal{B})$,

$$(T^0_f\xi, T^0_f\xi) = \int_M (\rho(\pi(u)))^{-1}(\widehat{f}(u), \widehat{f}(u))_0 d\mu_\rho(\pi(u)K) \tag{4.11}$$

(By (4.10), the integrant in (4.11) depends only on $\pi(u)K$; it has compact support in M and we have seen that it is continuous), where ρ is a G, K rho-function, i.e. a non-negative-valued continuous function ρ on G satisfying $\rho(xk) = \delta(k)(\Delta(k))^{-1}\rho(x)$ for all $x \in G$, $k \in K$ and ρ gives rise to a unique regular Borel measure μ_ρ on M satisfying $\int_G \rho(x)f(x)d\lambda x = \int_M d\mu_\rho(xK)\int_K f(xk)d\nu k$, for all $f \in L(G)$.

Fix an element $f \in L(M,\mathcal{B})$. Since $E = \{\pi(u)K \mid \widehat{f}(u) \neq 0\}$ has compact closure in M, we can choose an element $\sigma \in L(G)$ such that $\int_K \sigma(xk)d\nu k = 1$ whenever $xK \in E$. Then

$$\int_M (\rho(\pi(u)))^{-1}(\widehat{f}(u),\widehat{f}(u))_0 d\mu_\rho(\pi(u)K) = \int_G \sigma(\pi(u))(\widehat{f}(u),\widehat{f}(u))_0 d\lambda(\pi(u)) \tag{4.12}$$

To evaluate the right side of (4.12), we observe first by (4.6) and (4.9) that $\alpha[\widehat{f}(u),\widehat{f}(u)]$ depends only on $\pi(u)$. Therefore, if $u \in U$ and $\pi(u) = x$, we denote $\alpha[\widehat{f}(u),\widehat{f}(u)] = \alpha_x$. From the openness of π and from the continuity of $u \longmapsto \widehat{f}(u)$ and $\langle\phi,\psi\rangle \longmapsto \alpha[\phi,\psi]$, we deduce that $x \longmapsto \alpha_x$ is continuous in the sense that, if $x_i \longmapsto x$ in G, then $\alpha_{x_i} \longmapsto \alpha_x$ uniformly with uniformly bounded compact support. It follows that

$$\beta = \int_G \sigma(x)\alpha_x d\lambda x \tag{4.13}$$

exists as a Bochner integral in $L(M,\mathcal{B})$ (with the supremum norm over a large compact set). Thus, since $(\widehat{f}(u),\widehat{f}(u))_0 = (T^0_{\alpha_{\pi(u)}}\xi,\xi)$, it follows from (4.12) and (4.13) that

$$\int_M (\rho(\pi(u)))^{-1}(\widehat{f}(u),\widehat{f}(u))_0 d\mu_\rho(\pi(u)K) = (T^0_\beta\xi,\xi) \tag{4.14}$$

Consequently, since T^0 ia a $*$-representation of $L(M,\mathcal{B})$, we see that (4.11) will be proved if we can show that

$$\beta = f^*f \tag{4.15}$$

To prove (4.15) we evaluate each side of (4.13) at the arbitrary point $\langle yK,z\rangle \in M\times G$, getting by Proposition 2.5, [14]

$$\beta(yK,z) = \int_G \sigma(x)\alpha_x(yK,z)d\lambda x. \tag{4.16}$$

But, if $u \in U$, $\pi(u) = x$, then

$$\alpha_x(yK,z) = \int_K (\widehat{f}(u))^*(yk)(\widehat{f}(u))(k^{-1}y^{-1}z)d\nu k$$

$$= \Delta(y^{-1})\int_K \Delta(k)(\delta(k))^{-1}(\widehat{f}(u)(ky^{-1}))^*\widehat{f}(u)(ky^{-1}z)d\nu k$$

$$= \Delta(y^{-1})\int_K \Delta(k)(\delta(k))^{-1}(f(xK,xky^{-1}))^*f(xK,xky^{-1}z)d\nu k$$

So by (4.16), we have

$$\beta(yK,z) = \Delta(y^{-1})\int_G\int_K \Delta(k)(\delta(k))^{-1}\sigma(x)(f(xK,xky^{-1}))^*f(xK,xky^{-1}z)d\nu k d\lambda x \tag{4.17}$$

Now the integrand on the right side of (4.17) is a continuous function with compact support $K \times G$ to B_z. So we may use Fubini's Theorem to interchange the order of integration, replace x by xk^{-1} and interchange back again, getting

$$\beta(yK, z) = \Delta(y^{-1}) \int_G \int_K (\delta(k))^{-1} \sigma(xk^{-1})(f(xK, xy^{-1}))^* f(xK, xy^{-1}z) d\nu k d\lambda x$$

$$= \Delta(y^{-1}) \int_G \int_K \sigma(xk)(f(xK, xy^{-1}))^* f(xK, xy^{-1}z) d\nu k d\lambda x$$

$$= \Delta(y^{-1}) \int_G (f(xK, xy^{-1}))^* f(xK, xy^{-1}z) d\lambda x$$

$$= \int_G (f(xyK, x))^* f(xyK, xz) d\lambda x$$

$$= \int_G \Delta(x^{-1})(f(x^{-1}yK, x^{-1}))^* f(x^{-1}yK, x^{-1}z) d\lambda x = (f^* f)(yK, z).$$

By the arbitrariness of yK and z, this implies (4.15). So (4.11) is proved.

It follows from (4.11) that

$$(\phi, \phi)_0 \geq 0 \ \ (\phi \in L(\mathcal{B})) \tag{4.18}$$

Indeed, if τ is an arbitrary element of $L(M)$, we may replace f in (4.11) by

$$g : \langle m, x\rangle \longmapsto \tau(m) f(m, x),$$

getting

$$\int_M |\tau(\pi(u)K)|^2 (\rho(\pi(u)))^{-1} (\widehat{f}(u), \widehat{f}(u))_0 d\mu_\rho(\pi(u)K) = (T_g^0 \xi, T_g^0 \xi) \geq 0 \tag{4.19}$$

Since $u \longmapsto (\widehat{f}(u), \widehat{f}(u))_0$ is continuous, the arbitrariness of τ implies by (4.19) that $(\widehat{f}(u), \widehat{f}(u))_0 \geq 0$ for all $u \in U$; this holds for all $f \in L(M, \mathcal{B})$. So (4.18) is established.

We'll write $\|\phi\|_0$ for $(\phi, \phi)_0^{\frac{1}{2}}$ $(\phi \in L(\mathcal{B}))$.

We prove that for all $\phi \in L(\mathcal{B})$ and $t \in \mathcal{B}_K$ we have

$$\|Q_t \phi\|_0 \leq \|t\| \|\phi\|_0 \tag{4.20}$$

Indeed, from (4.18) and (4.6) we obtain

$$\|Q_t\phi\|_0 = (Q_{t^*t}\phi, \phi)_0 \leq \|Q_{t^*t}\phi\|_0 \|\phi\|_0 = (Q_{(t^*t)^2}\phi, \phi)_0^{\frac{1}{2}} \|\phi\|_0 \leq$$

$$\|Q_{(t^*t)^2}\phi\|_0^{\frac{1}{2}} \|\phi\|_0^{\frac{3}{2}} = (Q_{(t^*t)^4}\phi, \phi)^{\frac{1}{4}} \|\phi\|_0^{\frac{3}{2}} = \ldots = \|Q_{(t^*t)^{2^n}}\phi\|_0^{2^{-n}} \|\phi\|_0^{2-2^{-n}} \tag{4.21}$$

for each positive integer n. We estimate now the right side of (4.21).

If $s \in \mathcal{B}_K$, we have $\|Q_s\phi\|_0^2 = (T^0_\alpha \xi, \xi)$, where $\alpha = \alpha[Q_s\phi, Q_s\phi]$; that is, for $x, y \in G$,

$$\alpha(yK, x) = \int_K (Q_s\phi)^*(yK)(Q_s\phi)(k^{-1}y^{-1}x)d\nu k =$$

$$\int_K (\Delta(k)(\delta(k))^{-1}\Delta(y^{-1})((Q_s\phi)(ky^{-1}))^*(Q_s\phi)(ky^{-1}x)d\nu k =$$

$$\Delta(y^{-1})\int_K \Delta(k)(\delta(k))^{-1}(\phi(ky^{-1}))^* s^* s\phi(ky^{-1}x)d\nu k. \tag{4.22}$$

We define the numerical function γ on $M \times G$ as follows :

$$\gamma(yK, x) = \Delta(y^{-1})\int_K \Delta(k)(\delta(k))^{-1}\|\phi(ky^{-1})\|\|\phi(ky^{-1}x)\|d\nu k \ \ (x, y \in G)$$

It is easy to see that the definition is legitimate and that γ is continuous with compact support on $M \times G$. Comparing γ with (4.22) we see that

$$\|\alpha(yK, x)\| \leq \|s^*s\|\gamma(yK, x) \ \ (x, y \in G).$$

Therefore the $L(M, \mathcal{B})$- norm of α satisfies

$$\|\alpha\| \leq k\|s^*s\| \tag{4.23}$$

where $k = \int_G \sup_{m\in M} \gamma(m, x)d\lambda x$. Here k depends only on ϕ. By (4.23), $\|T^0_\alpha\| \leq k\|s^*s\|$, so that $\|Q_s\phi\|_0 = (T^0_\alpha\xi, \xi)^{\frac{1}{2}} \leq k^{\frac{1}{2}}\|\xi\|\|s^*s\|^{\frac{1}{2}}$. Applying this to (4.21), with $s = (t^*t)^{2^n}$, we get for each positive integer n

$$\|Q_t\phi\|_0^2 \leq k^{2^{-n-1}}\|\xi\|^{2^{-n}}\|(t^*t)^{2^{n+1}}\|^{2^{-n-1}}\|\phi\|_0^{2-2^{-n}} \leq$$

$$k^{2^{-n-1}}\|\xi\|^{2^{-n}}\|t\|^2\|\phi\|_0^{2-2^{-n}} \tag{4.24}$$

Letting $n \longrightarrow \infty$ in (4.24) we obtain (4.20). So the claim is proved.

Now, having established in (4.18) that the form $(,)_0$ is positive, we define N to be the linear subspace $\{\phi \in L(\mathcal{B}) \mid (\phi, \phi)_0 = 0\}$ of $L(\mathcal{B})$ and Y to be the pre-Hilbert space $L(\mathcal{B})/N$ with the inner product $(\kappa(\phi), \kappa(\psi))_0 = (\phi, \psi)_0$ $(\phi, \psi \in L(\mathcal{B})$, $\kappa\colon L(\mathcal{B}) \to Y)$ being the quotient map. Let Y_c be the Hilbert space completion of Y and $\| \ \|_0$ the norm in Y_c. Note that, by (4.7), κ is continuous with respect to the inductive limit topology of $L(\mathcal{B})$.

In virtute of (4.20), each $t \in \mathcal{B}_K$ gives rise to a continuous linear operator S_t on Y_c satisfying $\|S_t\|_0 \leq \|t\|$ ($\| \ \|_0$ being here the operator norm on Y_c) and $S_t \circ \kappa = \kappa \circ Q_t$ $(t \in \mathcal{B}_K)$. In view of (4.24) and (4.15), the same relations hold for S. If $\phi, \psi \in L(\mathcal{B})$ and $t \in \mathcal{B}_K$, we have $(S_t\kappa(\phi), \kappa(\psi))_0 = (Q_t\phi, \psi)_0$; so the continuity of $(S_t\kappa(\phi), \kappa(\psi))_0$ in t follows from (4.7). Thus we have established all the conditions for S to be a $*$-representation of $\mathcal{B}_K$ on Y_c.

We prove that S is non-degenerate. Let F be the linear span of $Q_t\phi$ ($t \in A$, $\phi \in L(\mathcal{B})$). Evidently F is closed under multiplication by continuous complex functions on G. Using an approximate unit in $\mathcal{B}$, we see that for each $x \in G$, the set $\{\psi(x) \mid \psi \in F\}$ is a dense subspace of $\mathcal{B}_x$. So by the proof of Proposition 2.2, [14], any function $\psi \in L(\mathcal{B})$ can be uniformly approached by functions $\{\psi_\alpha\}$ in F having uniformly bounded compact support. Applying κ we see that $\kappa(\psi_\alpha) \longrightarrow \kappa(\psi)$ in Y_c. But $\kappa(\psi_\alpha)$ is in the linear span of the ranges of S_t. So S is non-degenerate.

We have constructed a non-degenerate $*$-representation S of $\mathcal{B}_K$. We show that $\langle T, P\rangle$ is equivalent to the system of imprimitivity $\langle U^S, P'\rangle$ attached to the induced representation U^S of $\mathcal{B}$.

For each $f \in L(M,\mathcal{B})$, let $\widetilde{f}\colon U \to Y$ be given by $\widetilde{f}(u) = \kappa(\widehat{f}(u))$ ($u \in U$). By the continuity of κ and of $\widehat{f}$, $\widetilde{f}$ is continuous. Applying κ to (4.9) we find that

$$\widetilde{f}(ut) = (\delta(\pi(t)))^{\frac{1}{2}}(\Delta(\pi(t)))^{-\frac{1}{2}}S_{t^{-1}}(\widetilde{f}(u)) \ \ (u \in U, t \in U_K).$$

Since $\widetilde{f}$, like $\widehat{f}$, has compact support in M, it follows that $\widetilde{f} \in X(U^S)$. So $f \longmapsto \widetilde{f}$ is a linear map of $L(M,\mathcal{B})$ into $X(U^S)$. We notice that the right side of (4.11) is just $\|\widetilde{f}\|^2$ (norm in $X(U^S)$). So (4.11) asserts that

$$\|\widetilde{f}\|^2 = \|T^0_f\xi\| \tag{4.25}$$

for all $f \in L(M,\mathcal{B})$. Since T^0 is non-degenerate, (4.25) shows that there is a (unique) linear isometry ι of X (=$X(T)$) into $X(U^S)$ satisfying

$$\iota(T^0_f\xi) = \widetilde{f} \ \text{ for all } \ f \in L(M,\mathcal{B}) \tag{4.26}$$

We show that ι intertwines $\langle T, P\rangle$ and $\langle U^S, P'\rangle$. Let $s = av$ ($a \in A, v \in U$) and let $\eta = T^0_f\xi$ ($f \in L(M,\mathcal{B})$). Defining sf by $(sf)(m,x) = s\cdot f(\pi(s)^{-1}m, \pi(s)^{-1}x)$, $\langle m, x\rangle \in M \times G$ and taking into account that $T_sT^0_f = T^0_{sf}, P_\psi T^0_f = T^0_{\psi f}$, we find

$$\iota(T_s\eta) = (sf)\tilde{} \tag{4.27}$$

Now, if $u \in U, y \in G, x = \pi(s)$, we have

$$\begin{aligned}
[Q_{u^{-1}au}\widehat{f}(v^{-1}u)](y) &= u^{-1}au(\widehat{f}(v^{-1}u))(y) \\
&= u^{-1}auu^{-1}vf(\pi(v^{-1}u)K, \pi(v^{-1}u)y) \\
&= u^{-1}sf(x^{-1}\pi(u)K, x^{-1}\pi(u)y) = u^{-1}(sf)(\pi(u)K, \pi(u)y) \\
&= (sf)\widehat{}(u)(y).
\end{aligned}$$

So $Q_{u^{-1}au}(\widehat{f}(v^{-1}u)) = (sf)\widehat{}(u)$. Applying κ to both sides of this we get

$$(sf)\tilde{}(u) = S_{u^{-1}au}(\widetilde{f}(v^{-1}u)). \tag{4.28}$$

By relation (21), § 11, [14], $S_{u^{-1}au}(\widetilde{f}(v^{-1}u)) = (U^S_s\widetilde{f})(u)$. So by (4.28), $(sf)^{\sim} = U^S_s\widetilde{f}$. Combining this with (4.26) and (4.27), we have (by the denseness of the η in X)

$$\iota \circ \circ T_s = U^S_s \circ \iota \ \ (s \in B). \tag{4.29}$$

A similar calculation shows that

$$\iota \circ \circ P_\phi = P'_\phi \circ \iota \ \ (\phi \in L(M)). \tag{4.30}$$

But (4.29) and (4.30) together assert that ι intertwines $\langle T, P\rangle$ and $\langle U^S, P'\rangle$.

To prove that $\langle T, P\rangle$ and $\langle U^S, P'\rangle$ are equivalent under ι it remains only to show that ι is onto $X(U^S)$. Since ι is an isometry, it is in fact sufficient to show that range(ι) is total in $X(U^S)$.

By Proposition 11.2, [14], $\{F_{\phi,\kappa(\psi)} \mid \phi, \psi \in L(\mathcal{B})\}$ is total in $X(U^S)$. We show that, if $\phi, \psi \in L(\mathcal{B})$, $F_{\phi,\kappa(\psi)}$ belongs to range(ι). For this purpose we define $f = \alpha[\psi, \phi^*]$, i.e.

$$f(yK, z) = \int_K \phi(yk)\psi(k^{-1}y^{-1}z)d\nu k \ \ (y, z \in G).$$

Thus $f \in L(M, \mathcal{B})$ and

$$\widehat{f}(u)(x) = u^{-1}f(\pi(u)K, \pi(u)x) = \int_K u^{-1}\phi(\pi(u)k)\psi(k^{-1}x)d\nu k. \tag{4.31}$$

On the other hand, consider the Bochner integral in $L(\mathcal{B})$ (with the supremum norm over a large compact set)

$$\zeta(u) = \int_K (\Delta(k))^{\frac{1}{2}}(\delta((k))^{-\frac{1}{2}}(Q_{u^{-1}\phi(\pi(u)k)}\psi)d\nu k \ \ (u \in U). \tag{4.32}$$

Applying κ to both sides of (4.32) we obtain (by the continuity of κ and Proposition 2.5, [14])

$$\kappa(\zeta(u)) = \int_K (\Delta(k))^{\frac{1}{2}}(\delta(k))^{-\frac{1}{2}}S_{u^{-1}\phi(\pi(u)k)}(\kappa(\psi))d\nu k = F_{\phi,\kappa(\psi)}(u) \tag{4.33}$$

Again, applying to both sides of (4.32) the continuous functional of evaluation at a point x of G, we have by (4.33) :

$$\zeta(u)(x) = \int_K (\Delta(k))^{\frac{1}{2}}(\delta(k))^{-\frac{1}{2}}(Q_{u^{-1}\phi(\pi(u)k)}(\psi))(x)d\nu k =$$

$$\int_K u^{-1}\phi(\pi(u)k)\psi(k^{-1}x)d\nu k = \widehat{f}(u)(x). \tag{4.34}$$

Combining (4.33), (4.34) and (4.26), we get $F_{\phi,\kappa(\psi)} = \widetilde{f} \in \text{range}(\iota)$. Thus the range$(\iota)$ contains all $F_{\phi,\kappa(\psi)}$ $(\phi, \psi \in L(\mathcal{B}))$; and hence coincides with $X(U^S)$. Consequently, $\langle T, P\rangle \cong \langle U^S, P'\rangle$. Thus we have proved the existence part of Theorem for those $\langle T, P\rangle$ which have cyclic vectors. But an arbitrary $\langle T, P\rangle$ is a direct sum $\bigoplus\limits_i \langle T^{(i)}, P^{(i)}\rangle$ of systems of imprimitivity which have cyclic vectors. By what is already proved, for each i there is a non-degenerate $*$-representation $S^{(i)}$ of $\mathcal{B}_K$ such that the system of imprimitivity attached to $U^{S^{(i)}}$ is equivalent $\langle T^{(i)}, P^{(i)}\rangle$. By the Remark preceding Proposition 13.1, [14], the system of imprimitivity attached to U^S, where $S = \bigoplus\limits_i S^{(i)}$ is equivalent to $\langle T, P\rangle$. Thus the existence of S has been completely proved. □

5 Imprimitivity theorem for systems of G-covariance

For each topological space X we denote by $\mathcal{B}_X$ the Borel structure (i.e. σ-field) generated by the closed sets of X. Every Hilbert space $\mathcal{H}$ considered is understood to be a complex one and $\mathcal{L}(\mathcal{H})$ is the complex vector space of all continuous linear operators in $\mathcal{H}$. We denote the characteristic function of a set A by ψ_A.

Definition 24. *([6]) Let X be a topological space and let $\mathcal{H}$ be a Hilbert space. A (weak)* ***Borel positive-operator-valued measure*** *on X acting in $\mathcal{H}$ is a map $P\colon \mathcal{B}_X \to \mathcal{L}(\mathcal{H})$ such that*

i) P is positive, i.e. $P(\emptyset) = 0$ and $P(B) \geq 0$ for all $B \in \mathcal{B}_X$;

ii) P is (weakly) countably additive, i.e. if $(B_i)_{i\in\mathbb{N}}$ is a sequence of mutually disjoint elements of $\mathcal{B}_X$, then $P(\bigcup\limits_{i=0}^{\infty} B_i) = w - \sum\limits_{i=0}^{\infty} P(B_i)$, where $w - \sum$ means that the series $(P(B_i))$ converges in the weak operator topology on $\mathcal{L}(\mathcal{H})$.

If $P(X) = I_{\mathcal{H}}$, then P is said to be ***normalized****.*

If in addition P satisfies iii) $P(B)P(B') = P(B \cap B')$ for all $B, B' \in \mathcal{B}_X$, then P is a ***Borel projection-valued measure****.*

Definition 25. *([6]) If G be a topological group. A topological space $X \neq \emptyset$ is a* ***topological (left) G-space*** *if G operates continuously on (the left of) X, i.e. if there is a continuous map $(g, x) \longrightarrow g(x)$ of the topological product space $G \times X$ into X such that for each $x \in X$ we have $1(x) = x$ and $(gg')(x) = g(g'(x))$ for all $g, g' \in G$.*

If H is a subgroup of G, we denote by G/H the topological homogeneous space of left cosets of H in G, which is a topological G-space in a canonical way.

Definition 26. *([6]) Let G be a topological group, let X be a topological G-space, let U be a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}$ and let P be a normalized Borel positive-operator-valued measure on X acting in $\mathcal{H}$. We say that that P is G-**covariant** and that the ordered pair (U,P) is a **system of** G-**covariance** in $\mathcal{H}$ based on X if U,P satisfy*

$$U_g P(B) U_g^{-1} = P(g(B))$$

*for all $g \in G$ and all $B \in \mathcal{B}_X$. The system (U,P) is called **transitive** if so is the G-space X.*

Remark 27. *([6]) If P is a Borel projection-valued measure, then (U,P) is a Mackey's system of imprimitivity for G based on X and acting on $\mathcal{H}$.*

Definition 28. *([6]) Two systems of G-covariance, (U,P) in $\mathcal{H}$ and (U',P') in $\mathcal{H}'$, both based on X are **unitarily equivalent** if there is a unitary map V of $\mathcal{H}$ onto $\mathcal{H}'$ such that*

$$VU_g = U'_g V \ \text{ for all } \ g \in G$$

and

$$VP(B) = P'(B)V \ \text{ for all } \ B \in \mathcal{B}_X.$$

Proposition 29. *(Proposition 1, [6]) Let G be a second countable locally compact group, let X be a countably generated Borel G-space and let $\mathcal{H}, \mathcal{H}'$ be separable Hilbert spaces. If (U,M) is a system of G-covariance in $\mathcal{H}$ based on X there are a separable Hilbert space $\mathcal{H}_e$, an isometric map W of $\mathcal{H}$ into $\mathcal{H}_e$ and a system of imprimitivity (U_e,P) for G based on X and acting in $\mathcal{H}_e$ satisfying*

$$WU(g) = U_e(g)W \ \text{ for all } \ g \in G \tag{5.1}$$

$$WM(B) = P(B)W \ \text{ for all } \ B \in \mathcal{B}_X \tag{5.2}$$

and such that the set

$$\mathcal{M} = \left\{ P(B)W\psi | \ B \in \mathcal{B}_X \ \text{ and } \ \psi \in \mathcal{H} \right\}$$

is total in $\mathcal{H}_e$.

The mapping W is surjective if and only if (U,M) is a system of imprimitivity.

Let (U',M') be a system of G-covariance in $\mathcal{H}'$ based on X and unitarily equivalent to (U,M). If there are $\mathcal{H}'_e, W', P', U'_e, \mathcal{M}'$ mutually satisfying the same relations as, respectively, $\mathcal{H}_e, W, P, U_e, \mathcal{M}$ when $\mathcal{H}', U', M'$ replace $\mathcal{H}, U, M$, then the systems of imprimitivity (U_e,P) and (U'_e,P') are unitarily equivalent.

Proof. By a theorem of Neumark [31], there are a Hilbert space $\mathcal{H}_e$, an isometric map W of $\mathcal{H}$ into $\mathcal{H}_e$ and a normalized Borel positive-valued-measure P on X acting in $\mathcal{H}_e$ such that

$$WM(B) = P(B)W \ \text{ for all } \ B \in \mathcal{B}_X.$$

Let $\mathcal{E}_{\mathcal{H}}(\mathcal{B}_X)$ be the complex vector space of all step functions based on $\mathcal{B}_X$ taking values in $\mathcal{H}$. Define a positive Hermitian sesquilinear form $\langle\ \cdot\ \rangle$ on $\mathcal{E}_{\mathcal{H}}(\mathcal{B}_X)$ by

$$\langle \sum_i \psi_i \phi_{B_i} | \sum_j \psi_j \phi_{B_j} \rangle = \sum_{i,j} (M(B_i \cap B_j)\psi_i|\psi_j), \tag{5.3}$$

where the sums are finite and $(\cdot|\cdot)$ is the inner multiplication on $\mathcal{H}$. The positivity of $\langle\cdot|\cdot\rangle$ is a consequence of the positivity of M. Let $\mathcal{J}$ be the subspace of all $f \in \mathcal{E}_{\mathcal{H}}(\mathcal{B}_X)$ such that $\langle f|f\rangle = 0$; then $\mathcal{H}_e$ is the completion of the quotient space $\mathcal{E}_{\mathcal{H}}(\mathcal{B}_X)/\mathcal{J}$ equipped with the extended quotient form which we denote by $(\cdot|\cdot)_e$. The map W is defined by

$$W\psi = [f_\psi],$$

where $f_\psi \in \mathcal{E}_{\mathcal{H}}(\mathcal{B}_X)$ is the constant map with the value ψ and $[f_\psi]$ denotes the equivalence class of f_ψ modulo $\mathcal{J}$; the positive valued-measure P is given by

$$P(B)\left[\sum_i \psi_i \phi_{B_i}\right] = \left[\sum_i \psi_i \phi_{B \cap B_i}\right] \quad (\psi_i \in \mathcal{H}, B_i \in \mathcal{B}_X)$$

and extension by continuity. We remark that, for each $B \in \mathcal{B}_X)$, we have $M(B) = W^* P(B) W$ and $W^*W = \mathrm{Id}_{\mathcal{H}}$, where W^* is the adjoint of W. The set $\mathcal{M} = \{P(B)W\psi | B \in \mathcal{B}_X \text{ and } \psi \in \mathcal{H}\}$ is total in $\mathcal{H}_e$; it follows that P is weakly countably additive because the subset $\{P(B)|B \in \mathcal{B}_X\}$ of $\mathcal{L}(\mathcal{H}_e)$ is norm-bounded and for each sequence (B_i) of mutually disjoint elements of $\mathcal{B}_X$ and each pair $P(B)W\psi$, $P(B')W\psi'$ of elements of $\mathcal{M}$, we have

$$(P(\bigcup_{i=0}^{\infty} B_i)P(B)W\psi|P(B')W\psi')_e = (M(\bigcup_{i=0}^{\infty}(B \cap B' \cap B_i))\psi|\psi') =$$

$$\sum_{i=0}^{\infty}(M(B \cap B' \cap B_i)\psi|\psi') = \sum_{i=0}^{\infty}(P(B_i)P(B)W\psi|P(B')W\psi')_e$$

Since $\mathcal{B}_X$ is countably generated and $\mathcal{H}$ is separable, $\mathcal{H}_e$ is separable. Let $\{B_i\}_{i\in\mathbb{N}}$ be a clan of elements of $\mathcal{B}_X)$ generating $\mathcal{B}_X)$ and let $\{\psi_i\}_{i\in\mathbb{N}}$ be a dense subset of elements of $\mathcal{H}$; the set $\mathcal{M}_0 = \{P(B_k)\psi_l | B_k \in \{B_i\} \text{ and } \psi_l \in \{\psi_i\}\}$ is dense in $\mathcal{M}$. In fact, for each $P(B)W\psi \in \mathcal{M}$ and an arbitrary positive real number ε, we can choose $B_k \in \{B_i\}$ such that $(P(B\Delta B_k)W\psi|W\psi)_e^{\frac{1}{2}} < \frac{\varepsilon}{2}$ ([20], § 13, Theorem D) and $\psi_l \in \{\psi_i\}$ such that $\|\psi - \psi_l\| < \frac{\varepsilon}{2}$; then we have

$$\|P(B)W\psi - P(B_k)W\psi_l\|_e \leq \|(P(B) - P(B_k))W\psi\|_e + \|P(B_k)W(\psi - \psi_l)\|_e \leq$$

$$(P(B\Delta B_k)W\psi|W\psi)_e^{\frac{1}{2}} + \|\psi - \psi_l\| < \varepsilon.$$

For each $g \in G$, let $U_e(g)$ be the unitary operator in $\mathcal{H}_e$ defined in $\mathcal{M}$ by

$$U_e(g)P(B)W\psi = P(g.B)WU(g)\psi \tag{5.4}$$

and extended to $\mathcal{H}_e$ by linearity and continuity. This definition makes sense since

$$(U_e(g)P(B)W\psi|U_e(g)P(B')W\psi')_e = (M(g.(B\cap B'))U(g)\psi|U(g)\psi') =$$

$$(M(B\cap B')\psi|\psi') = (P(B)W\psi|P(B')W\psi')_e$$

for all $P(B)W\psi, P(B')W\psi'$ in $\mathcal{M}$. Let $\mathcal{L}_s(\mathcal{H}_e)_1$ be the closed unit ball of $\mathcal{L}(\mathcal{H}_e)$ equipped with the strong operator topology. The map $g \longmapsto (P(g.B), WU(g)\psi)$ of G into the topological product space $\mathcal{L}_s(\mathcal{H}_e)_1 \times \mathcal{H}_e$ is Borel for all $B \in \mathcal{B}_X$ and all $\psi \in \mathcal{H}$ by Lemma 3 (Remark 3), [6] and because the Borel structure of $\mathcal{L}_s(\mathcal{H}_e)_1 \times \mathcal{H}_e$ coincides with the product Borel structure. In addition, the map $(A,\psi) \longmapsto A\psi$ of $\mathcal{L}_s(\mathcal{H}_e)_1 \times \mathcal{H}_e$ into $\mathcal{H}_e$ is continuous. From this, Lemma 1, [6] and the equicontinuity of the unitary group $\mathbf{U}(\mathcal{H}_e)$, we can conclude ([4], Chapter III, § 3, Proposition 5) that the homomorphism $g \longmapsto U_e(g)$ of G into $\mathbf{U}(\mathcal{H}_e)$ equipped with the strong operator topology is identical on $\mathbf{U}(\mathcal{H}_e)$ with the weak operator one and makes $\mathbf{U}(\mathcal{H}_e)$ into a Polish group ([8], Lemma 4). Finally, we get that (U_e, P) is a system of imprimitivity for G based on X and acting in $\mathcal{H}_e$.

Given the system of G-covariance (U', M'), suppose that we have Hilbert space $\mathcal{H}'_e$ an isometric map W' of $\mathcal{H}'$ into $\mathcal{H}'_e$, a system of imprimitivity (U'_e, P') for G based on X and acting in $\mathcal{H}'_e$ satisfying $W'^*P'(B)W = M'(B)$ for all $B \in \mathcal{B}_X$, $W'^*U'_e(g)W' = U'(g)$ for all $g \in G$ and suppose that the set $\mathcal{M}' = \{P'(B)W'\psi | B \in \mathcal{B}_X \text{ and } \psi \in \mathcal{H}'\}$ is total in $\mathcal{H}'_e$. If Z is a unitary map of $\mathcal{H}$ onto H' establishing the equivalence of (U, M) to (U', M'), then the map $P(B)W\psi \longmapsto P'(B)W'Z\psi$ of $\mathcal{M}$ onto $\mathcal{M}'$' extends by linearity and continuity to a unitary map of $\mathcal{H}_e$ onto $\mathcal{H}'_e$ making (U_e, P) and (U'_e, P') unitarily equivalent. □

Let G be a locally compact group and let H be a closed subgroup of G. We denote by $Ind_H^G U$ the (strongly continuous unitary) representation of G induced from H by a strongly continuous representation U of H on a Hilbert space $\mathcal{H}$. In what follows, whenever G is second countable and $\mathcal{H}$ separable, we assume that $Ind_H^G U$ is realized on $L^2(G/H, \mu)$, the Hilbert space of all equivalence classes of μ-square integrable maps of G/H into $\mathcal{H}$, where μ is a G-quasi-invariant measure on G/H. Moreover, we denote by $P_{\mathcal{H}}$ the standard Borel projection-valued measure on G/H acting in $L^2_{\mathcal{H}}(G/H, \mu)$ defined by $P_{\mathcal{H}}(B)f = \psi_B f \ \ (f \in L^2_{\mathcal{H}}(G/H, \mu))$

Theorem 30. *(Proposition 2, [6]) Let G be a second countable locally compact group, let H be a closed subgroup of G, let μ be a G-quasi-invariant measure on G/H and let $\mathcal{H}, \mathcal{H}'$ be separable Hilbert spaces. If (U, M) is a system of G-covariance in $\mathcal{H}$ based on G/H, there are a strongly continuous unitary representation $\gamma(U)$ of*

H on a separable Hilbert space $\mathcal{K}$ and an isometric map V of $\mathcal{H}$ into $L^2_{\mathcal{K}}(G/H,\mu)$ satisfying

$$VU_g = Ind_H^G\gamma(U)_g V \text{ for all } g \in G \tag{5.5}$$

$$VM(B) = P_{\mathcal{K}}(B)V \text{ for all } B \in \mathcal{B}_X \tag{5.6}$$

and such that the set

$$\{P_{\mathcal{K}}(B)V\xi| \ B \in \mathcal{B}_{G/H} \text{ and } \xi \in \mathcal{H}\}$$

is total in $L^2_{\mathcal{K}}(G/H,\mu)$.

The map V is surjective if and only if (U,M) is a system of imprimitivity.

If (U',M') is a system of G-covariance in $\mathcal{H}'$ based on G/H and unitarily equivalent to (U,M) and if $\mathcal{K}'$ is the carrier space of $\gamma(U')$, then the systems of imprimitivity $(Ind_H^G\gamma(U),P_{\mathcal{K}})$ and $(Ind_H^G\gamma(U'),P_{\mathcal{K}'})$ are unitarily equivalent.

Proof. Applying Mackey's imprimitivity theorem to the system of imprimitivity (U_e,P) constructed in Proposition 29 with $X = G/H$; so we get $\gamma(U)$ and a unitary map W_e of H_e onto $L^2_{\mathcal{K}}(G/H,\mu)$ making $(Ind_H^G\gamma(U),P_{\mathcal{K}})$ unitarily equivalent to (U_e,P) and such that (5.5), (5.6) are satisfied with $V = W_eW$. Obviously, V is onto $L^2_{\mathcal{K}}(G/H,\mu)$ if and only if W is onto H_e, i.e. if and only if (U,M) is a system of imprimtivity. □

Remark 31. *Equation (5.5) expresses the unitary equivalence of U to a subrepresentation of $Ind_H^G\gamma(U)$; conversely, if an isometric map V of $\mathcal{H}$ into $L^2_{\mathcal{K}}(G/H,\mu)$ establishes such an equivalence, then (5.5) is satisfied and we have $V^*V = I_{\mathcal{H}}$. Moreover, if M is defined by (5.6), i.e. by $M(B) = V^*P_{\mathcal{K}}(B)V$, then (U,M) is a system of G-covariance in $\mathcal{H}$ based on G/H.*

We present now Ali's generalization of Mackey's imprimitivity theorem in the special case where the positive-operator-valued measure associated to the system of covariance is commutative ([1]).

Let X be a metrizable locally compact topological space, let G be a metrizable locally compact topological group, let $\mathcal{H}$ be a separable Hilbert space and let P be a normalized positive-operator-valued measure as in Definition 24. We assume that P is commutative, i.e. for all $B_1, B_2 \in \mathcal{B}_X$, $P(B_1)P(B_2) = P(B_2)P(B_1)$. Let U be a strongly continuous unitary representation of G on $\mathcal{H}$. The pair (U,P) forms a **commutative** system of covariance if, for all $g \in G$ and $B \in \mathcal{B}_X$, $U_gP(B)U_g^* = P(g(B))$.

Let $\mathcal{A}(P)$ be the commutative von Neumann algebra generated by the operators $P(E)$ for all $E \in \mathcal{B}_X$ and denote $\mathcal{M}_I(X;\mathcal{A}(P))$ the set of all positive-operator-valued measures b defined on $\mathcal{B}_X$ such that $b(E) \in \mathcal{A}(P)$ for all $E \in \mathcal{B}_X$ and which satisfy the normalization condition $b(x) = I_{\mathcal{H}}$. $\mathcal{M}_I(X;\mathcal{A}(P))$ has a natural topology under which it is compact and convex. Furthermore, the set of its extreme points $\mathcal{E}$ is a G_δ and consists of all the positive-valued measures in it.

Proposition 32. *(Proposition 1, [1]) Let U be a strongly continuous unitary representation of the metrizable locally compact group G on the separable Hilbert space $\mathcal{H}$ and let X be a metrizable locally compact homogeneous G-space. Then there is a normalized positive-operator-valued measure P on $\mathcal{B}_X$ for which (U,P) is a transitive commutative system of covariance if and only if U is a representation which is induced from a subgroup H of G and there is a probability measure ν on $\mathcal{B}_X$ which is invariant under H. Furthermore, given P, ν is uniquely fixed and conversely.*

Proof. Let U be induced from the unitary representation V of H acting on the Hilbert space $\mathcal{H}_0$ and let $Y = G/H$. Corresponding to H, let $g = k_g h_g$ be the Mackey decomposition for any element $g \in G$ such that $k_g \in G, h_g \in H$. The coset representative $k_g \in G/H$ is to be chosen in such a way that $k_e = e$, the neutral element of G. Let $\beta\colon Y \to G$ be the Borel section for which, for all $y \in Y$, $\beta(y) = k_{\beta(y)}$. Then, following [26] we write $\mathcal{H}$ in the form $\mathcal{H} = \mathcal{H}_0 \otimes L^2(Y,\lambda)$ and

$$(U_g\phi)(y) = B(g,y)\phi(g^{-1}(y)) \tag{5.7}$$

for all $\phi \in \mathcal{H}$, where the multiplier $B(g,y)$ is given by

$$B(g,y) = [\xi(g,y)]^{\frac{1}{2}} V(h_{g^{-1}\beta(y)})^{-1},$$

where ξ is the usual Radon-Nikodym derivate for the quasi-invariant measure λ with

$$\int_Y u(g(y))d\lambda(y) = \int_Y u(y)\xi(g,y)d\lambda(y).$$

Suppose that ν is a probability measure on $\mathcal{B}_X$ which is invariant H, i.e. $\nu(h(E)) = \nu(E)$ for all $E \in \mathcal{B}_X$ and $h \in H$. For each $x \in X$ and $E \in \mathcal{B}_X$ consider the operator $P_x(E)$ on $\mathcal{H}$,

$$(T_x(E)\phi)(y) = \chi_E(\beta(y)(x))\phi(y), \tag{5.8}$$

where χ_E is the characteristic function of the set E. It is straightforward to check that for fixed x, $E \longmapsto T_x(E)$ is a positive-valued measure on $\mathcal{H}$. Moreover, for fixed E, the function $x \longmapsto \chi_E(\beta(y)(x))$ is measurable. Hence consider

$$P(E) = \int_X T_x(E)d\nu(x) \tag{5.9}$$

the integral being defined strongly. It is easy to verify that $E \longmapsto P(E)$ is a normalized commutative positive-operator-valued measure. Furthermore, (U,P) is a system of covariance. Indeed, by (5.7) and (5.8), for all $\phi \in \mathcal{H}$,

$$\begin{aligned}
(U_g P(E) U_g^* \phi)(y) &= \int_X d\nu(x)\chi_E(\beta(g^{-1}(y))(x))\phi(y) \\
&= \int_X d\nu(x)\chi_E(\beta(g^{-1}\beta(y)h^{-1}_{g^{-1}\beta(y)}(x))\phi(y) \\
&= \int_X d\nu(x)\chi_{(g^{-1}\beta(y))^{-1}(E)}(h^{-1}_{g^{-1}\beta(y)}(x))\phi(y).
\end{aligned}$$

Hence, since ν is invariant under $h \in H$,

$$(U_g P(E) U_g^* \phi)(y) = \int_X d\nu(x) \chi_{g(E)}(\beta(y)(x)) \phi(y)$$

so that

$$U_g P(E) U_g^* = P(g(E)).$$

From Theorem, Section 2, [1], it follows that the positive-operator-valued measure P in (5.9) is uniquely determined by ν.

Suppose that (U, P) is a commutative transitive system of covariance. Then, since the commutative von Neumann algebra $\mathcal{A}(P)$ is invariant under the action of G and the spectrum Y of $\mathcal{A}(P)$ is a transitive G-space, it follows from Takesaki's extension of Mackey's imprimitivity theorem to invariant von Neumann algebras, that U is induced from a unitary representation of H . □

6 Imprimitivity theorems in the context of C^*-algebras

Definition 33. *([39]) If A is a $*$-normed algebra ([37]) then a **Hermitian left A-module** is a Hilbert space W on which A acts by means of a norm continuous non-degenerate $*$-representation by bounded operators (2.2. [9]), action denoted by aw for $a \in A$, $w \in W$. If this action of A on W is by means of an antirepresentation, then we speak about a **Hermitian right A-module**.*

Let G be a locally compact group. We denote by $L(G)$ the group algebra of G, that is the $*$-normed algebra of all complex-valued functions on G which are integrable with respect to left Haar measure on G with convolution as multiplication and with the usual involution. If R is a strongly continuous unitary representation of G on a Hilbert space W, then a $*$-representation, also denoted by R, of $L(G)$ can be defined by $R_f w = \int_G f(x) R_x w dx$, for all $f \in L(G)$, $w \in W$. Then we can define on $L(G)$ a C^*-algebra norm by

$$\|f\|_{C^*(G)} = \sup \left\{ \|R_f\| \ : \ R \text{ is a unitary representation of } G \right\}.$$

The C^*-algebra obtained by completing $L(G)$ with respect to this norm is called **the group C^*-algebra of G** denoted by $C^*(G)$ (see 13.9.1, [9]).

Definition 34. *([39]) If A is a C^*-algebra and if B is a C^*-subalgebra of A, then a **conditional expectation** from A to B is a continuous positive projection of A onto B which satisfies the conditional expectation property*

$$P(ab) = bP(a) \ \text{ and } \ P(ab) = P(a)b$$

for $b \in B$ and $a \in A$.

Let the pre C^*-algebra $C = L(A)/J$, where $L(A)$ consists of operators on the C^*-algebra A which provide bounded operators on every Hermitian A-module $A \otimes_B V$ which is induced from the subalgebra B via the conditional expectation P (see Lemma 1.7, Theorem 1.8, [39]) and J is the ideal of operators of norm zero in $L(A)$.

For each $a \in A$ let L_a denote the operator of left multiplication on A by a. By Proposition 3.2, [39], $L_a \in L(A)$. Let E denote the linear span of the set of elements of $L(A)$ of the form $L_a P L_c$ for $a, c \in A$. By Proposition 3.5, [39], E is a two-sided ideal in $L(A)$ and every element of E has an adjoint in E. Moreover, the image of E in C is a two-sided $*$-ideal of C and, in particular, is a pre-C^*-algebra, also denoted by E, called **the imprimitivity algebra of the conditional expectation** P.

Definition 35. *([39]) Let B be pre-C^*-algebra. A **right B-rigged space** is a right B-module X (in the algebraic sense) which is a pre-B-Hilbert space (with compatible multiplication by complex numbers on B and X) with preinner product conjugate linear in the first variable such that $\langle x, yb\rangle_B = \langle x, y\rangle_B b$ for all $x, y \in X$ and $b \in B$, which implies that $\langle xb, y\rangle_B = b^*\langle x, y\rangle_B$ and such that the range of $\langle , \rangle_B$ generates a dense subalgebra of B. **Left B-rigged spaces** are defined similarly except that it is required that B acts on the left of X, that the preinner product be conjugate linear in the second variable and that $\langle bx, y\rangle_B = b\langle x, y\rangle_B$.*

It can be defined a seminorm on a B-rigged space by setting $\|x\|_B = \|\langle x, x\rangle_B\|^{\frac{1}{2}}$.

Definition 36. *([39]) Let A and B be pre-C^*-algebras. A **left pre-Hermitian B-rigged A-module** is a right B-rigged space X which is a left A-module by means of a continuous $*$-homomorphism of A into $L(X)$ which is non-degenerate in the sense that AX is dense in X with respect to the B-seminorm on X. If the B-preinner product on X is definite and if X is complete with respect to the B-norm, then we call X a **Hermitian B-rigged A-module**. Right pre-Hermitian left B-rigged A-modules are defined similarly. If $B = \mathbb{C}$, we say simply **pre-Hermitian A-module**.*

Given A and B two pre-C^*-algebras and X a pre-Hermitian B-rigged A-module, by Theorem 5.1, [39], for any Hermitian B-module V we define a preinner product on $X \otimes_B V$ whose value on elementary tensors is given by $\langle x \otimes v, x' \otimes v'\rangle = \langle\langle x', x\rangle_B v, v'\rangle$ and the action of $L(X)$ on $X \otimes_B V$ which is defined on elementary tensors by $T(x \otimes v) = (Tx) \otimes v$ for $T \in L(X)$, $x \in X$, $v \in V$ is an action by bounded operators of norm no greater that $\|T\|$ with respect to this preinner product and any adjoint of T acts as an adjoint of T on $X \otimes_B V$. In this way we obtain a continuous non-degenerate $*$-representation of the quotient C^*-algebra, $C = L(X)/J$, on the corresponding Hilbert space ${}^A V$. When restricted to A, this representation is still non-degenerate, so that ${}^A V$ becomes a Hermitian A-module, called the **Hermitian A-module obtained by inducing V from B to A via X.**

We define now an analog of the imprimitivity algebra E. We assume that X a B-rigged space; $L(X)$ is the natural analog for B-rigged spaces of the algebra of all bounded operators on an ordinary Hilbert space. In view of this, it is natural to look for the analog of the two-sided ideal of compact operators. Now the ideal of compact operators is generated by the operators of rank one. In the present setting we define an analog of these operators. For any $x, y \in X$ we let $T_{(x,y)}$ be the operator on X defined by $T_{(x,y)}z = x\langle y, z\rangle_B$ for all $z \in X$. By Proposition 6.3, [39], E, the linear span of the set of operators in $L(X)$ of the form $T_{(x,y)}$, $x, y \in X$, is a two-sided ideal in $L(X)$, called the **imprimitivity algebra of the B-rigged space X**.

Theorem 37. *(Theorem 6.29, [39]) Let A and B be pre-C^*-algebras and let X be a Hermitian B-rigged A-module. Let E be the imprimitivity algebra of the B-rigged space X. Then a Hermitian A-module W is unitarily equivalent to a Hermitian A-module induced from a Hermitian B-module via X if and only if W can be made into a Hermitian E-module such that*

$$a(ex) = (ae)x \tag{6.1}$$

for all $a \in A$, $e \in E$, $x \in X$, where ae is the product of a and e as elements of $L(X)$ (this product is an element of E).

Proof. If W is induced from a Hermitian B-module, then it follows from Theorem 5.1, [39] that W is also a Hermitian E-module satisfying (6.1).

Suppose, conversely, that W can be made into a Hermitian E-module satisfying (6.1). Then by Theorem 6.23, [39] there is a Hermitian B-module V such that as E-modules ${}^EV$ is unitarily equivalent to W. Let S be a unitary E-isomorphism of ${}^EV$ onto W. By Theorem 5.1, [39], ${}^EV$ is an A-module such that $a(eu) = (ae)u$ for $a \in A, e \in E, u \in^E V$. Since the action of A on W is assumed to satisfy the same relation, we have

$$S(a(eu)) = S((ae)u) = (ae)S(u) = aS(eu)$$

for all $a \in A, e \in E$ and $u \in^E V$. But the linear span of the elements of the form eu in ${}^EV$ is dense in ${}^EV$ and so S is a unitary A-isomorphism as well. $\square$

Rieffel showed how Mackey's imprimitivity theorem for induced representations of locally compact groups can be derived from the imprimitivity theorem for induced representations of C^*-algebras (Theorem 37).

Let G be a locally compact group and let H be a closed subgroup of G. $C_c(G)$ denotes the algebra of the continuous complex-valued functions on G of compact support. Let A and B denote the pre-C^*-algebras $C_c(G)$ and $C_c(H)$ respectively, with A viewed as a pre-Hermitian B-rigged A-module. Let $C(G/H)$ denote the C^*-algebra of bounded continuous complex-valued functions on G/H with pointwise multiplication and supremum norm $\|\cdot\|_\infty$ and let $C_\infty(G/H)$ denote its C^*-subalgebra of functions vanishing at infinity. Whenever convenient we tacitly identify elements of $C(G/H)$ with the corresponding bounded continuous functions on G which are constant on the cosets of H. To facilitate this, our notation will not distinguish between points of G and points of G/H. According to Blattner's formulation [3] of Mackey's imprimitivity theorem, a system of imprimitivity based on G/H for a unitary G-module W is a representation of $C_\infty(G/H)$ on W such that $x(Fw) = (xF)(xw)$ for all $x \in G, F \in C_\infty(G/H)$ and $w \in W$, where $(xF)(y) = F(x^{-1}y)$ for all $y \in G$.

Theorem 38. *(Theorem 7.18, [39]) Let G be a locally compact group, let H be a closed subgroup of G and let W be a unitary G-module. Then W is unitarily equivalent to a unitary G-module induced from a unitary H-module if and only if W can be made into a Hermitian $C_\infty(G/H)$-module such that*

$$x(Fw) = (xF)(xw) \tag{6.2}$$

for all $x \in G, F \in C_\infty(G/H)$ and $w \in W$. (A representation of $C_\infty(G/H)$ on W satisfying the relation (6.2) is called a ***system of imprimitivity for*** W ***based on*** *G/H).*

Proof. The necessity of the conditions follows from Proposition 7.3, [39].

Suppose that W is a unitary G-module which satisfies the conditions of the theorem. We make W a Hermitian E-module. For any $\Phi \in E$ and $w \in W$, view Φ as an element of $C_c(G, C_\infty(G/H))$ and define Φw by

$$\Phi w = \int_G \Phi(y)(yw)dy \tag{6.3}$$

Following the proof of Proposition 7.6, [39], it is easily seen that W becomes an E-module giving a $*$-representation of E by bounded operators and in fact that

$$\|\Phi w\| \leq \|w\| \int_G \|\Phi(y)\|_\infty dy.$$

In particular, this last fact shows that the representation is continuous for the inductive limit topology in the sense of Theorem 7.16, [39]. Finally, a standard argument shows that the representation is nondegenerate ([3], [13], [16]). Applying Theorem 7.16, [39], we conclude that W is a Hermitian E-module.

Now, if $f \in A, \Phi \in E$ and $w \in W$ then, by using 7.14, 7.19, 7.20, [39], it is easily calculated that

$$f(\Phi w) = (f \star \Phi)w.$$

An application of Theorem 37 concludes the proof. □

7 The symmetric imprimitivity theorem

There are a number of Morita equivalences that play a fundamental role in the study of the representation theory of crossed products. These equivalences go by the name of imprimitivity theorems as the original motivation and statements can be traced back through Rieffel's work ([39]) and from there to Mackey's systems of imprimitivity [26], which we presented before. Most of these are subsumed by the Raeburn's symmetric imprimitivity theorem [36] which we reproduce here.

Definition 39. *([36], [42]) Let X be a (left) G-space and le $x \in X$. The **orbit** through x is the set $G \cdot x = \{s \cdot x \in X | \ s \in G$ and $x \in X\}$. The **stability group** at x is $G_x = \{s \in G | \ s \cdot x = x\}$. The G-action is called **free** if $G_x = \{e\}$ for all $x \in X$. The set of orbits is denoted by $G \setminus X$.*

Definition 40. *([36], [42]) If X and Y are locally compact Hausdorff spaces, then a continuous map $f \colon X \to Y$ induces a map $f^* \colon C^b(Y) \to C^b(X)$ via $f^*(\varphi)(y) = \varphi(f(y))$. If $f^{-1}(K)$ is compact in X when K is compact in Y, we call f a **proper map**.*

Definition 41. *([36], [42]) A locally compact G-space X is called **proper** if the map $(s, x) \longrightarrow (s \cdot x, x)$ is a proper map from $G \times X$ to $X \times X$.*

*If X is a proper G-space we say that G **acts properly on** X.*

Let X be a G-space and let (G, A, α) be a C^*-dynamical system, i.e. a triple consisting of a locally compact group G, a C^*-algebra A and a continuous homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$. If $f \colon X \to A$ is a continuous function such that

$$f(s \cdot x) = \alpha_s(f(x)) \tag{7.1}$$

for all $x \in X$ and $s \in G$, then $x \longrightarrow \|f(x)\|$ is constant on G-orbits and gives a well-defined function on $G \setminus X$. The **induced algebra** is

$$\mathrm{Ind}_G^X(A, \alpha) = \{f \in C^b(X, G) | \ f \text{ satisfies (7.1) and } G{\cdot}x \longrightarrow \|f(x)\| \text{ is in } C_0(G \backslash X)\}.$$

Since $\mathrm{Ind}_G^X(A, \alpha)$ is a closed $*$-subalgebra of $C^b(X, A)$, it is a C^*-algebra with respect to the supremum norm. When the context is clear, we'll shorten $\mathrm{Ind}_G^X(A, \alpha)$ to $\mathrm{Ind}_G^X \alpha$ or $\mathrm{Ind}\ \alpha$.

Example 42. *([42]) Let H be a closed subgroup of a locally compact group G and let (H, D, β) be a dynamical system. Then G is a right H-space with orbit space the set of left cosets G/H. Then*

$$Ind_H^G(D, \beta) = \{f \in C^b(G, D) | \ f(sh) = \beta_h^{-1}(f(s)) \ \text{for } s \in G, \ h \in H$$
$$\text{and } sH \longrightarrow \|f(s)\| \text{ is in } C_0(G/H)\}$$

$Ind_H^G(D,\beta)$ is nontrivial if β can't be lifted to an action of G. If $\beta = \alpha|_H$ for a dynamical system (G,A,α), then $\varphi\colon C^b(G,D) \to C^b(G,D)$ given by $\varphi(f)(s) = \alpha_s(f(s))$ defines an isomorphism of $Ind_H^G(D,\beta)$ onto $C_0(G/H,D)$. The algebra $Ind_H^G\beta$ is special to the imprimitivity theory.

The setup requires two commuting free and proper actions of locally compact groups K and H on a locally compact space X. It is convenient to have one group, K in this case, acts on the left and the other, H, on the right. Then the fact that the actions commute simply amounts to the condition

$$t\cdot(p\cdot h) = (t\cdot p)\cdot s \text{ for all } t\in K, p\in X \text{ and } s\in H.$$

In addition, we suppose that there are commuting strongly continuous actions α and β of K and H, respectively on a C^*-algebra A.

By [42, Lemma 3.57], there are dynamical systems

$$(\mathrm{Ind}_H^X(A,\alpha), K, \sigma) \text{ and } (\mathrm{Ind}_K^X(A,\alpha), H, \tau),$$

where

$$\sigma\colon K \to \mathrm{Aut}(\mathrm{Ind}_H^X(A,\beta)) \text{ and } \tau\colon H \to \mathrm{Aut}(\mathrm{Ind}_K^X(A,\alpha))$$

are strongly continuous actions given by

$$\sigma_t(f)(p) = \alpha_t(f(t^{-1}\cdot p)) \text{ and } \tau_s(f)(p) = \beta_s(f(p\cdot s)). \tag{7.2}$$

The actions σ and τ are often called **diagonal actions**, because, for example, σ is the restriction to Ind β of the canonical extension of lt $\otimes\,\alpha$ on $C_0(X)\otimes A$ to $C^b(X,A)$.

The symmetric imprimitivity theorem states that the crossed products Ind $\beta \rtimes_\sigma K$ and Ind $\alpha \rtimes_\tau H$ are Morita equivalent.

We denote by $C_{cc}(Y, \mathrm{Ind}_c\alpha)$ the collection of functions $f \in C_c(Y\times X, A)$ such that

(a) $f(y, s\cdot x) = \alpha_s(f(y,x))$ for $y\in Y, x\in X$ and $s\in G$;

(b) there are compact sets $C_1 \subset Y$ and $C_2 \subset G\setminus X$ such that $f(y,x) = 0$ if the element $(y, G\cdot x)$ is not in $C_1\times C_2$, where Y is a locally compact space, X is a locally compact G-space and (G,A,α) is a dynamical system.

By Lemma 3.53, [42] we can define $E_0 = C_{cc}(K, \mathrm{Ind}_c\beta) \subset$ Ind $\beta \rtimes_\sigma K$ and $B_0 = C_{cc}(H, \mathrm{Ind}_c\alpha) \subset$ Ind $\alpha \rtimes_\tau H$. Therefore E_0 consists of A-valued functions on $K\times X$ and B_0 of A-valued functions on $H\times X$ which are viewed as dense $*$-subalgebras of the corresponding crossed products as in [42, Lemma 3.53].

Definition 43. *([12]) Let A and B be C^*-algebras. A* ***right-Hilbert*** *A-B* ***bimodule*** *is a Hilbert B-module F which is also a non-degenerate left A-module (i.e. $AF = F$) satisfying*

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

$$\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B$$

for all $a \in A$, $x, y \in F$ and $b \in B$.

Definition 44. *([38], [12]) Let A and B be C^*-algebras. A* ***partial*** *A-B* ***imprimitivity bimodule*** *is a complex vector space F which is a right Hilbert B-module and a left Hilbert A-module such that*

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \ \text{ and } \ {}_A\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_B$$

for all $a \in A, b \in B$ and $x, y, z \in F$. If ${}_A\overline{\langle F, F \rangle} = A$ (i.e. F is full as a Hilbert A-module), we say that F is a ***right-partial imprimitivity bimodule****. If $\overline{\langle F, F \rangle}_B = B$, F is* ***left-partial imprimitivity bimodule****. If both ${}_A\langle , \rangle$ and $\langle , \rangle_B$ are full, then F is called an A-B* ***imprimitivity bimodule****. A and B are called* ***Morita equivalent*** *if there is at least one A-B imprimitivity bimodule.*

Theorem 45. *([36], [42], [12]) (****Raeburn's Symmetric Imprimitivity Theorem****) Suppose that we have free and proper actions of locally compact groups K and H on the left and right, respectively, of a locally compact space X and commuting strongly continuous actions α and β of K and H, respectively, on a C^*-algebra A. Let E_0 and B_0 be viewed as dense $*$-subalgebras of Ind $\beta \bowtie_\sigma K$ and Ind $\alpha \bowtie_\tau H$, respectively, and define $Z_0 = C_c(X, A)$. If $c \in E_0, b \in B_0$ and $f, g \in Z_0$, then define*

$$c \cdot f(p) = \int_K c(t,p)\alpha_t(t^{-1} \cdot p))\Delta_K(t)^{\frac{1}{2}} d\mu_K(t)$$

$$f \cdot b(p) = \int_H \beta_s^{-1}(f(p \cdot s^{-1})b(s, p \cdot s^{-1}))\Delta_H(s)^{-\frac{1}{2}} d\mu_H(s)$$

$${}_{E_0}\langle f, g \rangle(t,p) = \Delta_K(t)^{-\frac{1}{2}} \int_H \beta_s(f(p \cdot s)\alpha_t(g(t^{-1} \cdot p \cdot s)^*))d\mu_H(s)$$

$$\langle f, g \rangle_{B_0}(s,p) = \Delta_H(s)^{-\frac{1}{2}} \int_K \alpha_t(f(t^{-1} \cdot p)^* \beta_s(g(t^{-1} \cdot p \cdot s)))d\mu_K(t),$$

where Δ_K and Δ_H are the modular functions of K, respectively H. Then the completion $Z = Z_H^K$ is a Ind $\beta \bowtie_\sigma K$-Ind $\alpha \bowtie_\tau H$-imprimitivity bimodule and Ind $\beta \bowtie_\sigma K$ is Morita equivalent to Ind $\alpha \bowtie_\tau H$.

A similar symmetric imprimitivity theorem has been deduced independently by Kasparov ([24, Theorem 3.15]) which gave a Morita equivalence for the reduced crossed products.

There is a number of special cases of the symmetric theorem that pre-date Raeburn's theorem. One result due to Green is the case where A is the one-dimensional algebra $\mathbb{C}$ of the complex number. This follows from Theorem 45 together with the observation that $\mathrm{Ind}_H^X(C_0(X), \mathrm{rt})$ and $\mathrm{Ind}_K^X(C_0(X), \mathrm{lt})$ are identified with $C_0(X/H)$ and $C_0(K \setminus X)$, respectively.

Corollary 46. *([42], [12])(**Green's Symmetric Imprimitivity Theorem**) Suppose that K and H are locally compact groups acting freely and properly on the right and left, respectively, of a locally compact space X. If the actions commute then $C_0(X/H) \rtimes_{lt} K$ and $C_0(K \setminus X) \rtimes_{rt} H$ are Morita equivalent via an imprimitivity bimodule Z which is the completion of $Z_0 = C_c(X)$ equipped with actions and inner products given by*

$$c \cdot f(p) = \int_K c(t, p \cdot H) f(t^{-1} \cdot p) \Delta_K(t)^{\frac{1}{2}} d\mu_K(t)$$

$$f \cdot b(p) = \int_H f(p \cdot s^{-1}) b(s, K \cdot p \cdot s^{-1}) \Delta_H(s)^{-\frac{1}{2}} d\mu_H(s)$$

$${}_{E_0}\langle f, g\rangle(t, p \cdot H) = \Delta_K(t)^{-\frac{1}{2}} \int_H f(p \cdot s) \overline{g(t^{-1} \cdot p \cdot s)} d\mu_H(s)$$

$$\langle f, g\rangle_{B_0}(s, K \cdot p) = \Delta_H(s)^{-\frac{1}{2}} \int_K \overline{f(t^{-1} \cdot p)} g(t^{-1} \cdot p \cdot s) d\mu_K(t)$$

for all $c \in C_c(K \times X/H)$, $b \in C_c(H \times K \setminus P)$ and $f, g \in C_c(X)$.

Corollary 47. *([42], [12]) Suppose that H is a closed subgroup of a locally compact group G and that (H, D, β) is a dynamical system. Let $\sigma \colon G \to Aut(Ind_H^G(B, \beta))$ be defined by $\sigma_r(f)(t) = f(r^{-1}t)$. View $E_0 = C_c(G, Ind_c\beta)$ and $B_0 = C_c(H, D)$ as dense subalgebras of $Ind_H^G(D, \beta) \rtimes_\tau G$ and $D \rtimes_\beta H$, respectively. Let $Z_0 = C_c(G, D)$. If $c \in E_0$, $f, g \in Z_0$ and $b \in B_0$, then define*

$$c \cdot f(r) = \int_G c(t, r) f(t^{-1} \cdot r) \Delta_G(t)^{\frac{1}{2}} d\mu_G(t)$$

$$f \cdot b(r) = \int_H \beta_s^{-1}(f(rs^{-1}) b(s)) \Delta_H(s)^{-\frac{1}{2}} d\mu_H(s)$$

$${}_{E_0}\langle f, g\rangle(t, r) = \Delta_G(t)^{-\frac{1}{2}} \int_G \beta_s(f(rs) g(t^{-1}rs)^*) d\mu_H(s)$$

$$\langle f, g\rangle_{B_0}(s) = \Delta_H(s)^{-\frac{1}{2}} \int_G f(t^{-1})^* \beta_s(g(t^{-1}s)) d\mu_G(t).$$

Then the completion Z of Z_0 is a $Ind_H^G(D, \beta) \rtimes_\sigma G$-$D \rtimes_\beta H$-imprimitivity bimodule and $Ind_H^G(D, \beta) \rtimes_\sigma G$ is Morita equivalent to $D \rtimes_\beta H$.

The next result is a really corollary of the symmetric imprimitivity theorem.

Theorem 48. *([42], [12]) (**Green's imprimitivity theorem**) Suppose that* (G, A, α) *is a dynamical system, that* H *is a closed subgroup of* G *and that* σ *is the diagonal action of* G *on* $C_0(G/H, A)$ *defined in (2). Let* $\gamma(s) = \Delta_G(s)^{\frac{1}{2}}\Delta_H(s)^{-\frac{1}{2}}$*. View* $E_0 = C_c(G \times G/H, A)$ *and* $B_0 = C_c(H, A)$ *as* $*$*-subalgebras of* $C_0(G/H, A) \rtimes_\sigma G$ *and* $A \rtimes_{\alpha|_H} H$*, respectively. Let* $Y_0 = C_c(G, A)$*. If* $c \in E_0, f, g \in Y_0$ *and* $b \in B_0$*, then define*

$$c \cdot f(s) = \int_G c(r, sH)\alpha_r(f(r^{-1}s))d\mu(r)$$

$$f \cdot b(s) = \int_H f(sh)\alpha_{sh}(b(h^{-1}))\gamma(h)d\nu(h)$$

$$\langle f, g\rangle_{B_0}(h) = \gamma(h)\int_G \alpha_s^{-1}(f(s)^* g(sh))d\mu(s).$$

$${}_{E_0}\langle f, g\rangle(r, sH) = \int_H f(sh)\alpha_r(g(r^{-1}sh)^*)\Delta(r^{-1}sh)d\nu(h).$$

Then the completion $Y = Y_H^G$ *of* Y_0 *is a* $C_0(G/H, A) \rtimes_\sigma G$*-*$A \rtimes_{\alpha|_H} H$*-imprimitivity bimodule and* $C_0(G/H, A) \rtimes_\sigma G$ *is Morita equivalent to* $A \rtimes_{\alpha|_H} H$*.*

If G is a locally compact group, we denote by λ^G the left regular representation of G, by $C_r^*(G)$ the reduced group C^*-algebra of G which is the norm closure of $\lambda^G(L^1(G))$, by $M(A\otimes C_r^*(G))$ the multiplier algebra of all adjointable operators from $A \otimes C_r^*(G)$ to itself and by δ_G the usual comultiplication on $C_r^*(G)$, $\delta_G\colon C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$, which is the integrated form of the representation $s \longrightarrow \lambda(s) \otimes \lambda(s)$.

A **coaction** of a locally compact group G on a C^*-algebra A is an injective non-degenerate homomorphism

$$\delta\colon A \to M(A \otimes C_r^*(G))$$

satisfying

$$(\delta \otimes \mathrm{id}_G) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta \ \text{ and } \ \delta(A)(1 \otimes C_r^*(G)) \subset A \otimes C_r^*(G).$$

If $\alpha\colon G \to \mathrm{Aut}(\alpha)$ is an action, we denote the canonical embeddings in the crossed product by $i_A\colon A \to M(A \rtimes_\alpha G)$ and $i_G\colon G \to UM(A \rtimes_\alpha G)$. We write $u_G\colon G \to UM(C^*(G))$ for the canonical embedding of G in its group algebra. Composing the integrated form $(i_A \otimes 1) \rtimes (i_G \otimes u_G)$ with the regular representations of $A \rtimes_\alpha G$ and $C^*(G)$ gives a homomorphism which factors through a coaction $\widehat{\alpha}\colon A \rtimes_{\alpha,r} G \to M((A \rtimes_\alpha G) \otimes C_r^*(G))$, called the **dual coaction** of G on $A \rtimes_{\alpha,r} G$ ([34]).

If H is a closed subgroup of G, a theorem of Herz implies that the integrated forms of $\lambda^G|_H$ and λ^H have the same kernel in $C^*(H)$. Thus $\lambda^G|_H$ factors through an injective and non-degenerate homomorphism $C_H\colon C_r^*(H) \to M(C_r^*(H))$. By [34, Example 2.4], for normal H, if $\varepsilon\colon A \to M(A \otimes C_r^*(H))$ is a coaction of H, then $\mathrm{Inf}_H^G \varepsilon = (\mathrm{id}_A \otimes C_H \circ \varepsilon)\colon A \to M(A \otimes C_r^*(G))$ is a coaction of G on A, called the **inflation** of ε to G.

Echterhoff and Raeburn inflated the dual coactions on the crossed products in the symmetric imprimitivity theorem and the next theorem states these inflated systems are Morita equivalent in the sense of [5], [10].

Theorem 49. *([11, Theorem 2]) Let K and H be closed subgroups of a locally compact group G. Then the systems*

$$(Ind_H^G(D,\beta) \rtimes_{\tau,r} K, Inf_K^G(\widehat{\tau})) \text{ and } (Ind_K^G(D,\alpha) \rtimes_{\sigma,r} H, Inf_H^G(\widehat{\sigma}))$$

are Morita equivalent, where τ and σ are the diagonal actions.

Remark 50. *([11]) In Theorem 49 the coactions are inflated up to the group G: K and H could both lie in a smaller subgroup L, but the inflated coactions of L need not be Morita equivalent. For example, suppose $K = \{e\}$, so that we can take $L = H$. If further $D = \mathbb{C}$, then $Ind_K^G \mathbb{C} = C_0(G)$, $Ind_H^G \mathbb{C} = C_0(G/H)$ and the symmetric imprimitivity theorem states that $C_0(G) \rtimes_\sigma H$ is Morita equivalent to $C_0(G/H)$. Since $K = \{e\}$, the inflated coaction of H on $C_0(G/H)$ is trivial and an H-equivariant version of the theorem would imply that $(C_0(G) \rtimes_\sigma H) \rtimes_{\widehat{\sigma}} H$ is Morita equivalent to $C_0(G/H) \otimes C_0(H)$; by the Rieffel correspondence, this equivalence would induce a homeomorphism on spectra. But the spectrum G of $(C_0(G) \rtimes_\sigma H) \rtimes_{\widehat{\sigma}} H \cong C_0(G) \otimes \mathcal{K}(L^2(H))$ need not be homeomorphic to the spectrum $G/H \times \times H$ of $C_0(G/H) \otimes C_0(H)$ for example if $G = \mathbb{R}$ and $H = \mathbb{Z}$.*

Remark 51. *([11]) The symmetric imprimitivity theorem of [36] concerns a locally compact space X which carries commuting free and proper actions of two groups K and H. Remark 50 shows why we do not expect an equivariant version in this generality.*

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