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SOME COMMON FIXED POINT THEOREMS USING IMPLICIT RELATION IN 2-BANACH SPACES

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Abstract. In this article, we study the existence and uniqueness of a common fixed point of family of self mappings satisfying implicit relation on a 2-Banach space. We also prove well-posedness of a common fixed point problem.

1 Introduction

The theory of 2-Banach space was introduced by Gähler [6, 7] who have proved few basic fixed point results in such spaces. Subsequently, several authors including Iseki [8], Rhoades [15] and Whites [21] studied various aspects of the fixed point theory and proved fixed point theorems in 2-metric spaces and 2-Banach spaces. These aspects have been motivated by concepts already known for ordinary metric spaces. Recently, the study about fixed point theory for expansive and non expansive mappings is deeply explored and has extended too many other directions (see, eg.,[1, 2, 5, 10, 11, 12, 13, 16, 18]). Veerapandi and Anil Kumar [20] investigated the properties of fixed points of sequence of mappings under contraction condition in Hilbert spaces. In [17, 19], Saluja obtained fixed point for two self mappings using implicit relation.

Motivated and inspired by the above work, in this paper, we investigate the existence and uniqueness of the common fixed point for a family of self mappings under implicit relation in 2-Banach spaces which generalizes the results of Saluja [17]. Further, we study the well-posedness of the common fixed point problem of a pair of self mappings in 2-Banach space setting.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are required in the sequel. $\mathbb{R}$ denotes the set of all real numbers throughout this paper.

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Definition 1. *Let X be a real linear space and $\|\cdot,\cdot\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions :*

(i) $\|x, y\| = 0$ *if and only if x and y are linearly independent;*

(ii) $\|x, y\| = \|y, x\|$, *for all $x, y \in X$;*

(iii) $\|x, ay\| = |a|\|x, y\|$, *for all $x, y \in X$ and $a \in \mathbb{R}$;*

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, *for all $x, y, z \in X$;*

Then $\|\cdot,\cdot\|$ is called a 2 - norm and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and $a \in \mathbb{R}$.

Definition 2. *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$, for all $y \in X$.*

Definition 3. *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to converge to a point $x \in X$ if $\lim_{n\to\infty} \|x_n - x, y\| = 0$, for all $y \in X$.*

Definition 4. *A linear 2-normed space $(X, \|\cdot,\cdot\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.*

Definition 5. *A sequence $\{x_n\}$ in a 2-Banach space X is said to be asymptotically T- regular if $\lim_{n\to\infty} \|x_n - Tx_n, y\| = 0$, for all $y \in X$.*

Definition 6. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mappings on X. Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.*

Definition 7. ***(Implicit Relation)*** *Let Φ be the class of real valued continuous functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the first argument and satisfying the following condition: for $x, y > 0$,*

(i) $x \leq \phi\left(y, \frac{x+y}{2}, \frac{x+y}{2}\right)$

or

(ii) $x \leq \phi(x, 0, x)$

there exists a real number $0 < h < 1$ such that $x \leq hy$.

Example 8. *Let $\phi(r, s, t) = r - \alpha \min(s, t) + (2 + \alpha)t$, where $\alpha > 0$.*

Example 9. *Let* $\phi(r,s,t) = r^2 - ar\max(s,t) - bs,$ *where* $a > 0, b > 0.$

Example 10. *Let* $\phi(r,s,t) = r + c\max(s,t),$ *where* $c \geq 0.$

Recently, Saluja [17] proved a result in 2-Banach space for two self mappings as follows:

Theorem 11. *Let X be a 2-Banach space (with dim $X \geq 2$) and let S and T be two continuous self mappings of X such that for all $x, y, u \in X$ satisfying the condition:*

$$\begin{aligned} \|Sx - Ty, u\| \leq \phi\Big(& \|x - y, u\|, \frac{\|x - Sx, u\| + \|y - Ty, u\|}{2}, \\ & \frac{\|x - Ty, u\| + \|y - Sx, u\|}{2}\Big), \end{aligned} \tag{2.1}$$

then S and T have a unique common fixed point in X.

Now we are going to generalize and extend Theorem 11 for a family of self mappings.

3 Common Fixed Point Theorems

In this section, we first extend the work of Saluja [17] to a case of pair of mappings S^p and T^q where p and q are some positive integers satisfying the condition (2.1).

Theorem 12. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, S and T be two continuous self mappings of X such that*

$$\begin{aligned} \|S^p x - T^q y, u\| \leq \phi\Big(& \|x - y, u\|, \frac{\|x - S^p x, u\| + \|y - T^q y, u\|}{2} \\ & + \frac{\|x - T^q y, u\| + \|y - S^p x, u\|}{2}\Big), \end{aligned}$$

for all $x, y, u \in X$ where p and q are some positive integers. Then S and T have a unique common fixed point.

Proof. Since S^p and T^q satisfy the conditions of Theorem 11, S^p and T^q have a unique common fixed point. Let v be the common fixed point. Now

$$\begin{aligned} S^p v = v \quad &\Rightarrow \quad S(S^p v) = Sv, \\ S^p(Sv) \quad &= \quad Sv. \end{aligned}$$

If $Sv = x_0$ then $S^p(x_0) = x_0$. So, Sv is a fixed point of S^p. Similarly, $T^q(Tv) = Tv$. Now, we have

$$\begin{aligned}\|v-Tv,u\| &= \|S^pv-T^q(Tv),u\|\\ &\leq \phi\Big(\|v-Tv,u\|, \frac{\|v-S^pv,u\|+\|Tv-T^q(Tv),u\|}{2},\\ &\quad \frac{\|v-T^q(Tv),u\|+\|Tv-S^pv,u\|}{2}\Big)\\ &= \phi\Big(\|v-Tv,u\|,0,\|v-Tv,u\|\Big).\end{aligned}$$

Hence, by definition 7 (ii), we obtain

$$\|v-Tv,u\| \leq 0.$$

Thus $v=Tv$ for all $u\in X$. Similarly, $v=Sv$.

For uniqueness of v, let $w\neq v$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p and T^q which implies $w=v$. Hence S and T have a unique common fixed point. □

Hence we have proved that if x_0 is a unique common fixed point of S^p and T^q for some positive integers p and q then x_0 is a unique common fixed point of S and T. Next we generalize Theorem 11 to the case of family of mappings satisfying the condition (2.1).

Theorem 13. *Let $(X,\|\cdot,\cdot\|)$ be a 2-Banach space and $\{F_\alpha\}$ be a family of continuous self mappings on X satisfying*

$$\begin{aligned}\|F_\alpha x-F_\beta y,u\| &\leq \phi\Big(\|x-y,u\|, \frac{\|x-F_\alpha x,u\|+\|y-F_\beta y,u\|}{2},\\ &\quad \frac{\|x-F_\beta y,u\|+\|y-F_\alpha x,u\|}{2}\Big),\end{aligned} \tag{3.1}$$

for $\alpha,\beta\in\Lambda$ with $\alpha\neq\beta$ and $x,y,u\in X$. Then there exists a unique $v\in X$ satisfying $F_\alpha v=v$ for all $\alpha\in\Lambda$.

Proof. For $x_0\in X$, we define a sequence $\{x_n\}$ as follows:

$$x_{2n+1}=F_\alpha x_{2n}, x_{2n+2}=F_\beta x_{2n+1}, n=0,1,2,\ldots$$

Now, for all $u\in X$, from (3.1), we have

$$\begin{aligned}
\|x_{2n+1}-x_{2n},u\| &= \|F_\alpha x_{2n}-F_\beta x_{2n-1},u\| \\
&\leq \phi\Bigg(\|x_{2n}-x_{2n-1},u\|,\frac{\|x_{2n}-F_\alpha x_{2n},u\|+\|x_{2n-1}-F_\beta x_{2n-1},u\|}{2}, \\
&\quad \frac{\|x_{2n}-F_\beta x_{2n-1},u\|+\|x_{2n-1}-F_\alpha x_{2n},u\|}{2}\Bigg) \\
&\leq \phi\Bigg(\|x_{2n}-x_{2n-1},u\|,\frac{\|x_{2n}-x_{2n+1},u\|+\|x_{2n-1}-x_{2n},u\|}{2}, \\
&\quad \frac{\|x_{2n}-x_{2n},u\|+\|x_{2n-1}-x_{2n+1},u\|}{2}\Bigg) \\
&\leq \phi\Bigg(\|x_{2n}-x_{2n-1},u\|,\frac{\|x_{2n}-x_{2n+1},u\|+\|x_{2n-1}-x_{2n},u\|}{2}, \\
&\quad \frac{\|x_{2n-1}-x_{2n},u\|+\|x_{2n}-x_{2n+1},u\|}{2}\Bigg).
\end{aligned}$$

Hence by definition 7 (i), we have

$$\|x_{2n+1}-x_{2n},u\| \leq h\|x_{2n}-x_{2n-1},u\| \;\text{ where }\; 0<h<1.$$

Proceeding in the similar way, we obtain

$$\|x_{2n+1}-x_{2n},u\| \leq h^{2n}\|x_1-x_0,u\|, \quad n=1,2,3,\ldots.$$

Also for $n>m$, we have

$$\begin{aligned}
\|x_n-x_m,u\| &\leq \|x_n-x_{n-1},u\|+\|x_{n-1}-x_{n-2},u\|+\cdots\cdots+\|x_{m+1}-x_m,u\| \\
&\leq (h^{n-1}+h^{n-2}+\cdots\cdots+h^m)\|x_1-x_0,u\| \\
&\leq \frac{h^m}{1-h}\|x_1-x_0,u\|.
\end{aligned}$$

Note that $\frac{h^m}{1-h}\to 0$ as $m\to\infty$, since $0<h<1$. Thus $\|x_n-x_m,u\|\to 0$ as $m,n\to\infty$, which shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point $z\in X$ such that $x_n\to z$ as $n\to\infty$. By the continuity of F_α and F_β, it is clear that $F_\alpha z=F_\beta z=z$. Therefore, z is a common fixed point of F_α for all $\alpha\in\Lambda$.

In order to prove the uniqueness, lets take another common fixed point, say v of

F_α and F_β where $v \neq z$. Then

$$\begin{aligned} \|v - z, u\| &= \|F_\alpha v - F_\beta z, u\| \\ &\leq \phi\Big(\|v - z, u\|, \frac{\|v - F_\alpha v, u\| + \|z - F_\beta z, u\|}{2}, \\ &\quad \frac{\|v - F_\beta z, u\| + \|z - F_\alpha v, u\|}{2}\Big) \\ &\leq \phi\Big(\|v - z, u\|, 0, \|v - z, u\|\Big). \end{aligned}$$

Now, by definition 7 (ii), we get

$$\|v - z, u\| \leq h\|v - z, u\| \text{ where } 0 < h < 1.$$

Thus $v = z$ which implies that z is a unique common fixed point of F_α for all $\alpha \in \Lambda$. □

Theorem 14. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $\{F_n\}$ be a sequence of self mappings on X such that $\{F_n\}$ converging pointwise to a self mapping F and*

$$\|F_n x - F_n y, u\| \leq \phi\Big(\|x - y, u\|, \frac{\|x - F_n x, u\| + \|y - F_n y, u\|}{2}, \frac{\|x - F_n y, u\| + \|y - F_n x, u\|}{2}\Big),$$

for all $x, y, u \in X$. If $\{F_n\}$ has a fixed point v_n and F has a fixed point v, then the sequence $\{v_n\}$ converges to v.

Proof. Note that $F_n v_n = v_n$ and $Fv = v$. Now consider

$$\begin{aligned} \|v - v_n, u\| &= \|Fv - F_n v_n, u\| \\ &\leq \|Fv - F_n v, u\| + \|F_n v - F_n v_n, u\| \\ &\leq \|Fv - F_n v, u\| + \phi\Big(\|v - v_n, u\|, \frac{\|v - F_n v, u\| + \|v_n - F_n v_n, u\|}{2}, \\ &\quad \frac{\|v - F_n v_n, u\| + \|v_n - F_n v, u\|}{2}\Big). \end{aligned}$$

By the fact that $F_n v \to Fv$ as $n \to \infty$, we get

$$\|v - v_n, u\| \leq \phi\Big(\|v - v_n, u\|, 0, \|v - v_n, u\|\Big).$$

Hence, by Implicit Relation 7 (ii), we obtain

$$\|v - v_n, u\| \leq 0,$$

which implies that $v_n \to v$ as $n \to \infty$. □

Theorem 15. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T : X \to X$ such that*

$$\|Tx-Ty,u\| \leq \phi\Big(\|x-y,u\|, \frac{\|x-Tx,u\|+\|y-Ty,u\|}{2}, \frac{\|x-Ty,u\|+\|y-Tx,u\|}{2}\Big),$$

for all $x, y, u \in X$. Then T has a unique fixed point $v \in X$ and $\{z_n\}$ is asymptotically T- regular if and only if T is continuous at $v \in X$.

Proof. Let $v \in X$ and $z_n \to v$ as $n \to \infty$. Now

$$\|Tz_n - Tv, u\| \quad \leq \quad \phi\Big(\|z_n - v, u\|, \frac{\|z_n - Tz_n, u\| + \|v - Tv, u\|}{2}, \frac{\|z_n - Tv, u\| + \|v - Tz_n, u\|}{2}\Big).$$

Since T has a fixed point and $\{z_n\}$ is asymptotically T- regular, we get

$$\|Tz_n - Tv, u\| \quad \leq \quad \phi\Big(\|Tz_n - Tv, u\|, 0, \|Tz_n - Tv, u\|\Big).$$

By Implicit relation 7 (ii), there exists $0 < h < 1$ such that

$$\|Tz_n - Tv, u\| \leq h\|Tz_n - Tv, u\|,$$

this shows that

$$Tz_n \to Tv \text{ as } n \to \infty.$$

Hence T is continuous at $v \in X$. Conversely, assume that T is continuous at $v \in X$. Note that

$$z_n \to v \;\Rightarrow\; Tz_n \to Tv \text{ as } n \to \infty,$$

which implies that

$$\|z_n - Tz_n, u\| \to \|v - Tv, u\| = 0,$$

since T has a fixed point. This completes the proof. □

Remark 16. Theorem 13 *extends and generalizes the study of* Saluja [17] *to a family of continuous self mappings using implicit relation in Banach space. In* Theorem 14, *convergence of sequence of self mappings to another self mapping implies convergence of corresponding sequence of fixed points. Note that, continuity of mappings is not necessary in* Theorem 14.

4 Well Posedness

The notion of well - posedness of a fixed point problem has generated much interest to several mathematicians, for example [3, 4, 9, 14]. Here, we study well-posedness of a common fixed point problem of mappings in Theorem 11.

Definition 17. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and f be a self mapping. The fixed point problem of f is said to be well-posed if*

(i) *f has a unique fixed point $x_0 \in X$*

(ii) *for any sequence $\{x_n\} \subset X$ and $\lim_{n\to\infty} \|x_n - fx_n, u\| = 0$ we have $\lim_{n\to\infty} \|x_n - x_0, u\| = 0$.*

Let $CFP(T, f, X)$ denote a common fixed point problem of self mappings T and f on X and $CF(T, f)$ denote the set of all common fixed points of T and f.

Definition 18. *$CFP(T, f, X)$ is called well-posed if $CF(T, f)$ is singleton and for any sequence $\{x_n\}$ in X with*

$$\tilde{x} \in CF(T, f) \text{ and } \lim_{n\to\infty} \|x_n - fx_n, u\| = \lim_{n\to\infty} \|x_n - Tx_n, u\| = 0$$

implies $\tilde{x} = \lim_{n\to\infty} x_n$.

Theorem 19. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T, f be self mappings on X as in* Theorem 11. *Then the common fixed point problem of f and T is well posed.*

Proof. From Theorem 11, the mappings f and T have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n\to\infty} \|fx_n - x_n, u\| = \lim_{n\to\infty} \|Tx_n - x_n, u\| = 0$. Without loss of generality, assume that $v \neq x_n$ for any non-negative integer n. Using (2.1) and $fv = Tv = v$, we get

$$\begin{aligned}
\|v - x_n, u\| &\leq \|Tv - Tx_n, u\| + \|Tx_n - x_n, u\| \\
&= \|fv - Tx_n, u\| + \|Tx_n - x_n, u\| \\
&\leq \|Tx_n - x_n, u\| + \phi\Big(\|v - x_n, u\|, \frac{\|v - fv, u\| + \|x_n - Tx_n, u\|}{2}, \\
&\qquad \frac{\|v - Tx_n, u\| + \|x_n - fv, u\|}{2}\Big) \\
&= \phi(\|v - x_n, u\|, 0, \|v - x_n, u\|).
\end{aligned}$$

Hence by Implicit relation 7 (ii), we obtain $\|v - x_n, u\| \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 20. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T be a self mapping on X such that*

$$\|Tx - Ty, u\| \leq \phi\Big(\|x - y, u\|, \frac{\|x - Tx, u\| + \|y - Ty, u\|}{2}, \frac{\|x - Ty, u\| + \|y - Tx, u\|}{2}\Big),$$

for all $x, y, u \in X$. Then the fixed point problem of T is well posed.

References

[1] M. Abbas, H. Aydi, *On common fixed point of generalized contractive mappings in metric spaces*, Surveys in Mathematics and its Applications, **7** (2012) 39-47. MR2981738.

[2] M. A. Ahmed, *A common fixed point theorem for expansive mappings in 2-metric spaces and its application*, Chaos, Solitons and Fractals, **42**(5) (2009) 2914-2920. MR2560004. Zbl 1198.54068.

[3] M. Akkouchi, *Well-posedness of the fixed point problem for certain asymptotically regular mappings*, Annalas Mathematicae Silesianae, **23** (2009) 43-52. MR2741844 . Zbl 1234.54049.

[4] M. Akkouchi, V. Popa, *Well-posedness of the fixed point problem for mappings satisfying an implicit relations*, Demonstratio Mathematica, **43**(4) (2010) 923-929. MR2761650 . Zbl 1238.54018.

[5] Lj. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd), **12**(26) (1971) 19-26. MR0309092 (46#8203). Zbl 0234.54029.

[6] S. Gähler, *2-metric Raume and ihre topologische strucktur*, Math. Nachr., **26** (1963) 115-148. MR0162224 (28#5423). Zbl 0117.16003.

[7] S. Gähler, *Uber die unifromisieberkeit 2-metrischer Raume*, Math. Nachr., **28** (1965) 235-244. MR0178452 (31#2709). Zbl 0142.39804.

[8] K. Iseki, *Fixed point theorems in 2-metric space*, Math. Seminar. Notes, Kobe Univ., **3** (1975) 133-136. MR0405395 (53#9189).

[9] B. K. Lahiri, *Well-posedness and certain classes of operators*, Demonstratio Mathematica, **38** (2005) 169-176. MR2123731 (2005k:47121). Zbl 1159.47305.

[10] S. N. Lal, A. K. Singh, *An analogue of Banach's contraction principle for 2-metric spaces*, Bull. Austral. Math. Soc., **18**(1) (1978) 137-143. MR0645161 (58#31029). Zbl 0385.54028.

[11] S. V. R. Naidu, J. R. Prasad, *Fixed point theorems in 2-metric spaces*, Indian J. Pure. Appl. Math., **17**(8) (1986) 974-993. MR0856334 (87i:54096).

[12] W. G. Park, *Approximate additive mapping in 2-Banach spaces and related topics*, J. Math. Anal. Appl., **376**(1) (2011) 193-202. MR2745399. Zbl 1213.39028.

[13] M. Pitchaimani, D. Ramesh kumar, *Common and coincidence fixed point theorems for asymptotically regular mappings in 2-Banach Space*, Nonlinear Func. Anal. Appl., (Accepted).

[14] S. Reich, A. T. Zaslawski, *Well-posedness of fixed point problems*, Far East J. Math. sci, Special volume, part **III** (2001) 393-401. MR1888108 (2003d:54058).

[15] B. E. Rhoades, *Contraction type mappings on a 2-metric space*, Math. Nachr., **91** (1979) 151-155. MR0563606 (81b:54047). Zbl 0424.54038.

[16] M. Saha, D. Dey, *Fixed point of expansive mappings in a 2-Banach space*, Int. J. Math. Sci. Eng. Appl., **4**(4) (2010) 355-362. MR2768749.

[17] G. S. Saluja, *On common fixed point theorems under implicit relation in 2-Banach spaces*, Nonlinear Func. Anal. Appl., **20**(2) (2015) 215-221. Zbl 1321.47121.

[18] G. S. Saluja, *Existence results of unique fixed point in 2-Banach spaces*, Int. J. Math. Combin., **1** (2014) 13-18. Zbl 1325.47130.

[19] G. S. Saluja, *On common fixed point theorems satisfying implicit relation in 2-Banach spaces*, Bull. Kerala Math. Assoc., **12**(1) (2015) 77-85.

[20] T. Veerapandi, S. Anil Kumar, *Common fixed point theorems of a sequence of mappings on Hilbert space*, Bull. Cal. Math. Soc., **91(4)** (1999) 299-308. MR1748540 (2000k:47089). Zbl 0956.54023.

[21] A. White, *2-Banach spaces*, Math. Nachr., **42** (1969) 43-60. MR0257716(41#2365). Zbl 0185.20003.

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