HEREDITY IN FUNDAMENTAL LEFT COMPLEMENTED ALGEBRAS

Marina Haralampidou and Konstantinos Tzironis

Abstract. In the present paper, we introduce the notion of a fundamental complemented linear space, through continuous projections. This notion is hereditary. Relative to this, we prove that if a certain topological algebra is fundamental, then a concrete subspace is fundamental too. For a fundamental complemented linear space, we define the notion of continuity of the complementor. In some cases, we employ a generalized notion of complementation, that of (left) precomplementation. In our main result, the continuity of the complementor for a certain fundamental complemented (topological) algebra is inherited to the induced vector complementor of the underlying linear space of a certain right ideal. Weakly fundamental algebras are also considered in the context of locally convex ones.

1 Introduction and Preliminaries

In 1970, F.A. Alexander dealt with representation theorems in the context of Banach complemented algebras [1]. For this, and among others, continuity of the complementor is assumed. A respective representation theory, in the non-normed case, is faced in [11]. An appropriate context to work in this theory, is that of fundamental (pre)complemented algebras, studied in [6] and [18]. Here, the genetic property has to do with the existence of certain continuous linear maps (Definition 1). Fundamentality is also considered in the context of topological linear spaces (Definition 17). For certain topological algebras being also fundamental, the latter property is inherited to appropriate subspaces (Theorem 19); this information is used to obtain our main result (Theorem 20). Besides, an issue relative to the representation theory of non-normed complemented topological algebras is again the continuity of the complementor, which is faced via the "fundamental property". The latter continuity is inherited to the induced vector complementor of the underlying linear space of a certain right ideal, say, R [ibid.]. In fact, the continuity of a complementor turns

out to be a key property of the aforementioned representation theory, and this was the motivation for the present work. Under conditions, a complemented algebra E admits a faithful continuous representation T in an inner product space. The question that arises then is, when every complete subalgebra F of the topological algebra of all continuous linear operators on R as in Theorem 20, that contains the image $T(E)$, and certain projections, relative to closed subspaces of R, gains "complementation" and "fundamentality". Indeed, this is true in the context of certain fundamental complemented algebras [10]. A further application of our main Theorem 20 assures a kind of the continuity of the complementor on F [ibid.]. A quick reference to weakly fundamental algebras is given in Section 2.

All algebras, employed below, are taken over the field $\mathbb{C}$ of complexes. A *topological algebra* E is an algebra which is a topological vector space and the ring multiplication is separately continuous (see e.g., [13]). If, in particular, the topology is defined by a family $(p_\alpha)_{\alpha \in A}$ of seminorms (resp. submultiplicative seminorms), then E is named a *locally convex* (in particular, *locally m-convex*) *algebra*. We use the notation $(E, (p_\alpha)_{\alpha \in A})$. We also employ the notation $\overline{S}$ for the (topological) closure of a subset S of a topological algebra E.

We denote by $\mathcal{A}_l(S)$ (resp. $\mathcal{A}_r(S)$) the left (right) annihilator of a (non empty) subset S of an algebra E, being a left (resp. right) ideal of E. If S is a left (resp. right) ideal of E, then the ideals $\mathcal{A}_l(S)$ and $\mathcal{A}_r(S)$ are two-sided. If $\mathcal{A}_l(E) = \{0\}$ (resp. $\mathcal{A}_r(E) = \{0\}$) we say that E is a *left* (resp. *right*) *preannihilator algebra*. For a right preannihilator algebra, it is also used the term *proper algebra*. E is named *preannihilator* if it is both left and right preannihilator.

For a topological algebra E, $\mathcal{L}_l(E) \equiv \mathcal{L}_l$ (resp. $\mathcal{L}_r(E) \equiv \mathcal{L}_r, \mathcal{L}(E) \equiv \mathcal{L}$) stands for the set of all closed left (right, two-sided) ideals of E. If for $I \in \mathcal{L}$ the relation $I^2 = \{0\}$ implies $I = \{0\}$, then E is called *topologically semiprime*, while E is *topologically simple*, if it has no proper closed two-sided ideals.

The next notion was introduced in [6, p. 3723, Definition 2.1].

A topological algebra E is called *left complemented*, if there exists a mapping $\perp : \mathcal{L}_l \longrightarrow \mathcal{L}_l : I \mapsto I^\perp$, such that

$$\text{if } I \in \mathcal{L}_l, \text{ then } E = I \oplus I^\perp. \tag{1.1}$$

$I^\perp$ is called a *complement* of I.

$$\text{If } I, J \in \mathcal{L}_l, I \subseteq J, \text{ then } J^\perp \subseteq I^\perp. \tag{1.2}$$

$$\text{If } I \in \mathcal{L}_l, \text{ then } (I^\perp)^\perp = I. \tag{1.3}$$

$\perp$, as before, is called a *left complementor* on E. In what follows, we denote by $(E, \perp)$ a left complemented algebra with a left complementor $\perp$.

A *right complemented algebra* is defined analogously and we talk about a *right complementor*. A left and right complemented algebra is simply called a *complemented algebra*.

A topological algebra E is named *left precomplemented*, if for every $I \in \mathcal{L}_l$, there exists $I' \in \mathcal{L}_l$ such that $E = I \oplus I'$. Similarly, a *right precomplemented algebra* is defined. A left and right precomplemented algebra is called a *precomplemented algebra* (see [6, p. 3725, Definition 2.7]).

An idempotent element $0 \neq e = e^2$ is called *minimal*, if the algebra eEe is a division one. On the other hand, if e is a minimal (idempotent) element, then Ee is a minimal left ideal and eE is a minimal right ideal of E (see [15, p. 45, Lemma 2.1.8]).

If E is a left precomplemented algebra and $I, I' \in \mathcal{L}_l$ with $E = I \oplus I'$, then there exists a linear map $T : E \to E$ with $T^2 = T$ (projection) such that $\operatorname{Im} T = I$ and $\ker T = I'$. Indeed, for $x \in E$, there are unique $y \in I$ and $z \in I'$ with $x = y + z$. Then the mapping $T(x) = y$ is well defined, linear and unique with the aforementioned properties.

For the sake of completeness, we refer some notions, which were introduced in [18, Definitions 2.5, 2.8, 2.9 and 2.10].

Definition 1. A left precomplemented algebra E is called *fundamental* if, for any $I \in \mathcal{L}_l$, and a (pre)complement of I, say $I' \in \mathcal{L}_l$ (viz. $E = I \oplus I'$) there is a continuous linear mapping $T = T(I, I') : E \to E$ such that $T^2 = T$, $\operatorname{Im} T = I$ and $\ker T = I'$.

A *fundamental right precomplemented algebra* is defined analogously. A fundamental left and right precomplemented algebra is simply called a *fundamental precomplemented algebra*.

A left complemented algebra is named *fundamental,* if it is fundamental as left precomplemented. Similarly, for a right complemented algebra, and for a complemented one.

In the locally convex case, if E is the topological direct sum of I and I' (in the sense of [16, p. 90]), then E is named a *weakly fundamental algebra.*

EXAMPLE. *Every left precomplemented Banach algebra is fundamental* (see [18, Proposition 2.6]).

Definition 2. Let $(E, \perp)$ be a fundamental left complemented algebra. A net $(I_\delta)_{\delta \in \Delta}$ of minimal closed left ideals (of E) is $\perp$-*convergent to* $I_0 \in \mathcal{L}_l$, if

$$T_\delta \equiv T_\delta(I_\delta, I_\delta^\perp) \underset{\text{uniformly}}{\longrightarrow} T_0(I_0, I_0^\perp)$$

on any minimal right ideal of E.

Definition 3. An element x in a topological algebra E is said to be *axially closed* if the left ideal Ex is minimal closed.

****

Surveys in Mathematics and its Applications **11** (2016), 93 – 106
http://www.utgjiu.ro/math/sma

In particular, a subset of E is named *axially closed* if each of its elements is axially closed.

Examples 4. (1) Any primitive idempotent element x in a precomplemented algebra E is axially closed (see [4, p. 964, Theorem 2.1]).

Recall that x is *primitive* if it can not be expressed as the sum of two orthogonal idempotents; namely, of some non-zero idempotents $y, z \in E$ with $yz = zy = 0$.

(2) Any primitive idempotent element in a topologically semiprime algebra, in which, moreover, every left ideal contains a minimal left ideal, (in short (D_l)-algebra) is axially closed (see [5, p. 154, Theorem 3.9]).

(3) Any non-unital commutative semisimple topological algebra E with discrete space of maximal regular ideals is a (D_l)-algebra (see [9, p. 148, Examples 3.8, (1)]). So, since E is also topologically semiprime, all its primitive idempotents are axially closed.

(4) Any semisimple finite-dimensional topological algebra is a (D_l)-algebra [ibid. Examples 3.8, (2)], and thus its primitive idempotents are axially closed (see also (3)).

For more examples of (D_l)-algebras see [ibid.].

Definition 5. Let $(E, \perp)$ be a fundamental left complemented algebra. The mapping $\perp$ is called *continuous* whenever for each convergent, axially closed net $(a_\delta)_{\delta \in \Delta}$ with $a_\delta \xrightarrow{\delta} a_0 \in E$, $a_0 \neq 0$, and such that $Ea_0 \in \mathcal{L}_l$, the net $(Ea_\delta)_{\delta \in \Delta}$ is $\perp$-convergent in Ea_0. Namely,

$$T_\delta \equiv T_\delta(Ea_\delta, Ea_\delta^\perp) \xrightarrow[\text{uniformly}]{} T_0(Ea_0, Ea_0^\perp)$$

on any minimal right ideal of E.

In the sequel, all results also hold by interchanging "left" by "right".

Lemma 6. *Let E be a topologically simple algebra, and $\{0\} \neq I$ (resp. $\{0\} \neq R$) a closed left (right) ideal of E. Then $\mathcal{A}_l(I) = \{0\}$ (resp. $\mathcal{A}_r(R) = \{0\}$).*

Proof. Since $\mathcal{A}_l(I)$ is a closed two-sided ideal, either $\mathcal{A}_l(I) = E$ or $\mathcal{A}_l(I) = \{0\}$. If the first case holds, $EI = \{0\}$. In particular, $I^2 = \{0\}$ and by [5, p. 149, Theorem 2.1], $I = \{0\}$, a contradiction. $\square$

2 Weakly fundamental (pre)complemented algebras

In this section, we get realizations of the notion "weakly fundamental" for locally convex algebras (see Definition 1). The same results are obviously applied in locally convex spaces when the notion of (pre)complemented linear spaces is considered in

the fashion of Definition 15. In what follows, "topological direct sum" is taken in the sense of [16, p. 90].

We start with the following useful result.

Theorem 7. *Let E be a locally convex algebra. Consider the assertions:*

(1) For every $I \in \mathcal{L}_l$, there exists a continuous linear mapping $T : E \to E$ with $T^2 = T$, $\operatorname{Im} T = I$ and $E = I \oplus \ker T$.

(2) E is a left precomplemented algebra.

Then $(1) \Rightarrow (2)$. Besides, $(2) \Rightarrow (1)$, if in particular,

$$E \text{ is the topological direct sum of } I \text{ and } I' \text{ (: the complement of } I \text{ in } E). \quad (2.1)$$

Proof. It is enough to show that $(2) \Rightarrow (1)$. Consider $I \in \mathcal{L}_l$ and its complement $I' \in \mathcal{L}_l$. Since both of them are locally convex spaces, the assumption that $E = I \oplus I'$ is the topological direct sum of I and I' is meaningful. Thus, the projections $P_I : E \to I$ and $P_{I'} : E \to I'$ are continuous (see [ibid. p. 90, Proposition 21 and p. 95, Proposition 29]). The comments preceding Definition 1 complete the proof. $\square$

Concerning the previous result, we note that E need not be the topological direct sum of I and I' when they carry the induced topologies (see the comments before Proposition 29 in [16, p. 95]). This reveals the necessity of the assumption (2.1).

Corollary 8. *Every left complemented locally convex algebra, satisfying (2.1), is weakly fundamental.*

In the next, by a *locally C^*-algebra* we mean an involutive complete locally (m)-convex algebra $(E, (p_\alpha)_{\alpha \in A})$, such that $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in E$ and $\alpha \in A$. We also remind that a topological algebra E is said to be an *annihilator algebra,* if it is preannihilator with $\mathcal{A}_r(I) \neq \{0\}$ for every $I \in \mathcal{L}_l$, $I \neq E$ and $\mathcal{A}_l(J) \neq \{0\}$ for every $J \in \mathcal{L}_r$, $J \neq E$ (see [5]).

Theorem 9. *Every annihilator locally C^*-algebra, satisfying (2.1), is weakly fundamental.*

Proof. Let E be a topological algebra as in the statement. Then, by [8, p. 226, Theorem 3.1], E is left complemented. The assertion now follows from Corollary 8. $\square$

The next example is, under (2.1), a realization of the previous theorem.

Example 10. [6, p. 3724, Example 2.4]. Let X be a discrete (completely regular, k-space). Consider the locally m-convex algebra $\mathcal{C}_c(X)$ of all $\mathbb{C}$-valued continuous functions on X in the topology of compact convergence, defined by the family of seminorms $(p_K)_K$, where K runs over all compact subsets of X, where

$$p_K(f) = \sup_{x \in K} |f(x)|, f \in \mathcal{C}_c(X)$$

(cf., for instance [13, p. 19, Example 3.1]). $\mathcal{C}_c(X)$ is complete, since X is a k-space (see [12, p. 230 and p. 231, Theorem 12]), and it is a *locally C^*-algebra* under the involution given by $f^*(x) = \overline{f(x)}$, $f \in \mathcal{C}_c(X)$, $x \in X$. Since a locally C^*-algebra is *semisimple*, $\mathcal{C}_c(X)$ is topologically semiprime and hence *preannihilator* (cf. [5, p. 150, Lemma 2.3]). Besides, by [9, p. 144, Examples 2.2, (4)], if I is a proper closed ideal in $\mathcal{C}_c(X)$, there exists some closed maximal ideal M with $I \subset M$. Hence, $\mathcal{A}(M) \subseteq \mathcal{A}(I)$, where $\mathcal{A} \equiv \mathcal{A}_l = \mathcal{A}_r$. Since X is a completely regular space,

$$M = M_x = \{f \in \mathcal{C}_c(X) : f(x) = 0\} \text{ for every } x \in X$$

(cf. [13, p. 223, Corollary 1.3; see also its proof]). On the other hand, the closed subset $\{x\}^c$ corresponds to the proper closed ideal of $\mathcal{C}_c(X)$ given by

$$I_{\{x\}^c} = \{g \in \mathcal{C}_c(X) : g(y) = 0 \text{ for every } y \in \{x\}^c\}.$$

(Of course, $\{x\}^c$ is a non empty proper subset of X and hence $I_{\{x\}^c}$ is a non trivial proper ideal of $\mathcal{C}_c(X)$; see [ibid., p. 221, Lemma 1.5, and p. 222, Remark 1.2]). Obviously, $I_{\{x\}^c} \subseteq \mathcal{A}(M_x)$ and hence $\mathcal{A}(I) \neq \{0\}$. Thus, $\mathcal{C}_c(X)$ *is an annihilator algebra*.

3   Complemented subspaces in (left) complemented algebras

In this section, we pave the way to succeed vector complementors topically, in the sense that minimal closed one-sided ideals in certain left complemented algebras to be complemented as topological vector spaces (Theorem 14). Continuity of the vector complementor in question, is faced in the context of Theorem 20. So, we start with the next result, that generalizes Lemma 10 in [17, p. 658].

Proposition 11. *Let E be a topologically simple left precomplemented algebra and $I \in \mathcal{L}_l(E)$, $\{0\} \neq R \in \mathcal{L}_r(E)$. Put $S = I \cap R$. If, for some $x \in E$, $rx \in S$ for every $r \in R$, then $x \in I$. Namely $\{x \in E : Rx \subseteq S\} \subseteq I$.*

Proof. By hypothesis, there exists $I' \in \mathcal{L}_l(E)$ such that $E = I \oplus I'$. Thus, $x = x_1 + x_2$ with $x_1 \in I$ and $x_2 \in I'$. For every $r \in R$, $rx = rx_1 + rx_2$. Since $rx \in S \subseteq I$ and $rx_1 \in I$, we get $rx_2 \in I$. Therefore, $rx_2 \in I \cap I'$ and hence $x_2 \in \mathcal{A}_r(R)$. But, $\mathcal{A}_r(R) = \{0\}$ (see Lemma 6). Thus, $x_2 = 0$ and $x \in I$. $\square$

Corollary 12. *Let E be a topologically simple left precomplemented algebra, $I_1, I_2 \in \mathcal{L}_l(E)$, $\{0\} \neq R \in \mathcal{L}_r(E)$. Put $S_1 = I_1 \cap R$ and $S_2 = I_2 \cap R$. Then*
   (a) $S_1 \subseteq S_2$ if and only if $I_1 \subseteq I_2$.
   (b) $S_1 = S_2$ if and only if $I_1 = I_2$.

Proof. (a) If $S_1 \subseteq S_2$ and $x \in I_1$, then $rx \in R$ and $rx \in I_1$ for all $r \in R$. Therefore, $rx \in S_1$ and thus $rx \in S_2$. Thus, by Proposition 11, $x \in I_2$, and hence, $I_1 \subseteq I_2$.

(b) Apply (a). □

Proposition 13. *Let E be a Hausdorff topological algebra and e an idempotent element of E. Consider the closed right ideal $R = eE$. Then, for every closed left ideal I of E, $I \cap R = \overline{RI} = RI$.*

If, in particular, E is a topologically simple left precomplemented locally m-convex algebra and e is minimal, then for any closed subspace S of R,

$$\overline{ES} = \{a \in E : Ra \subseteq S\}.$$

(Here, ES stands for the left ideal of E, generated by S).

Proof. Obviously $RI \subseteq I \cap R$, and thus $\overline{RI} \subseteq I \cap R$. Now, for $x \in I \cap R$, $x = ex$ and since $e \in R$, we get $x \in RI$. Namely, $I \cap R \subseteq RI$, which in connection with $RI \subseteq \overline{RI} \subseteq I \cap R$ yields the first part of the assertion.

Now, since $S \subseteq R = eE = \{ex : x \in E\}$, there exists F with $F \subseteq E$ so that $S = \{ex : x \in F\}$. Moreover, $eS = \{ex : x \in F\} = S$. Besides, $S = eS \subseteq \overline{ES}$ and $S \subseteq R$, that yield

$$S \subseteq \overline{ES} \cap R. \tag{3.1}$$

If $z \in \overline{ES} \cap R$, then there exists a net $(z_\delta)_{\delta \in \Delta} \subseteq ES$ so that $z = \lim_\delta z_\delta$. Since $z \in R$, we get $z = ez = e\lim_\delta(ez_\delta)$. But $ez_\delta \in ES \cap R$. Therefore, $z \in \overline{ES \cap R}$ and thus $\overline{ES} \cap R \subseteq \overline{ES \cap R}$. Since R is closed, we also get $\overline{ES \cap R} \subseteq \overline{ES} \cap R$, and hence

$$\overline{ES} \cap R = \overline{ES \cap R}. \tag{3.2}$$

Since R is a right ideal and ES is a left ideal of E, we get $RES \subseteq ES \cap R$. Now, for $x \in ES \cap R$, $x \in ES$ and $x = ex$, since $x \in R$. Therefore, $x = ex \in RES$ and thus $ES \cap R \subseteq RES$. The previous argumentation yields $RES = ES \cap R$. Thus, since ES is a left ideal of E, and as we saw above, $eS = S$, we have $ES \cap R = RES = eEES \subseteq eES = eEeS \cong \mathbb{C}S \cong S$ (see also [13, p. 62, Corollary 5.1]). Namely, $ES \cap R \subseteq S$ and $\overline{ES \cap R} \subseteq S$. The latter, in connection with (3.1) and (3.2), yields $\overline{ES} \cap R = S$. Applying $I \cap R = RI$ for $I = \overline{ES}$, we get $\overline{ES} \cap R = R\overline{ES}$ and thus $S = R\overline{ES}$, from which $\overline{ES} = \{x \in E : Rx \subseteq S\}$. Indeed, put $A = \{x \in E : Rx \subseteq S\}$. If $x \in \overline{ES}$, then $Rx \subseteq R\overline{ES} = S$. Therefore $x \in A$ and hence $\overline{ES} \subseteq A$. Besides, for $x \in A$, $Rx \subseteq S = R\overline{ES} = \overline{ES} \cap R$. So, by Proposition 11, $x \in \overline{ES}$ and thus $A \subseteq \overline{ES}$, as asserted, and this completes the proof. □

Now, we state one of the main results concerning the existence of complementors in closed linear subspaces.

Theorem 14. *Let $(E, \perp)$ be a Hausdorff topologically simple left complemented locally m-convex algebra and e a minimal element in E. Consider the (minimal closed right) ideal $R = eE$ of E. If $\mathcal{V}_R$ stands for the set of all closed linear subspaces of R, then there exists a map $p : \mathcal{V}_R \to \mathcal{V}_R, S \mapsto S^p$, such that*

$$\text{If }\ S \in \mathcal{V}_R, \ \text{ then }\ R = S \oplus S^p. \tag{3.3}$$

$$\text{If }\ S_1, S_2 \in \mathcal{V}_R \ \text{ with }\ S_1 \subseteq S_2, \ \text{ then }\ S_2^p \subseteq S_1^p. \tag{3.4}$$

$$\text{If }\ S \in \mathcal{V}_R, \ \text{ then }\ (S^p)^p = S. \tag{3.5}$$

Proof. Let I be a closed left ideal in E. Then $E = I \oplus I^\perp$. For $x \in R$, there are unique $x_1 \in I$, $x_2 \in I^\perp$ with $x = x_1 + x_2$. Since $x = ex$, $x = ex_1 + ex_2$, while $ex_1 \in I$, $ex_2 \in I^\perp$. The uniqueness implies $x_1 = ex_1$, $x_2 = ex_2$. Namely, $x_1, x_2 \in R$ that yields $x_1 \in R \cap I$ and $x_2 \in R \cap I^\perp$. Put $S = R \cap I$ and $S^p = R \cap I^\perp$. Obviously, S satisfies (3.3). Besides, $\mathcal{V} = \{S_I = R \cap I : I \in \mathcal{L}_l(E)\} \subseteq \mathcal{V}_R$ and thus the map $p : \mathcal{V} \to \mathcal{V} : S_I \mapsto S_I^p = R \cap I^\perp$ is well defined. Claim that $\mathcal{V} = \mathcal{V}_R$. To this end, let S be a proper closed subspace of R. Denote by ES the left ideal of E generated by S (see also Proposition 13). Consider $K = \overline{ES} \in \mathcal{L}_l(E)$. Since $S = eS$ (e, idempotent; see the proof of Proposition 13), we have $S = eS \subseteq ES \subseteq K$, and thus $S \subseteq R \cap K$. If $z \in R \cap K$, there exists a net $(z_\delta)_{\delta \in \Delta}$ in ES with $z = \lim_\delta z_\delta$ and $z \in R$. Hence $z = ez = e\lim_\delta z_\delta = \lim_\delta (ez_\delta)$ and $e(ES) \subseteq eEeS \cong \mathbb{C}S \cong S$. Here, we also used the fact that E is a locally convex algebra with continuous quasi-inversion, and the ideal eE is minimal (see [13, p. 52, Lemma 3.1] and [5, p. 155, Theorem 3.11]). Thus $z \in S$. Therefore $S = R \cap K$ with K a closed left ideal of E. The previous argumentation completes the assertion. Thus, for any $S \in \mathcal{V}_R$, $S = R \cap \overline{ES}$ and $S^p = R \cap (\overline{ES})^\perp$. Moreover, $S^p = R \cap \overline{ES^p}$. Applying Corollary 12, (b), we get $\overline{ES^p} = (\overline{ES})^\perp$. Now, $(S^p)^p = R \cap (K^\perp)^\perp = R \cap K = S$. Moreover, if $K_1, K_2 \in \mathcal{L}_l(E)$ and if $S_1 = R \cap K_1$, $S_2 = R \cap K_2$, $S_1^p = R \cap K_1^\perp$, $S_2^p = R \cap K_2^\perp$, then $S_1 \subseteq S_2$ if and only if $K_1 \subseteq K_2$ (see Corollary 12, (a)). Thus $K_2^\perp \subseteq K_1^\perp$ if and only if $R \cap K_2^\perp \subseteq R \cap K_1^\perp$ if and only if $S_2^p \subseteq S_1^p$ and the proof is complete. $\qquad\square$

Based on Theorem 14, we set the next.

Definition 15. Let X be a topological linear space and $\mathcal{V}_X$ the family of its closed linear subspaces. X is named a *complemented linear space*, if there exists a mapping $p : \mathcal{V}_X \to \mathcal{V}_X : S \mapsto S^p$ such that

$$\text{If } S \in \mathcal{V}_X, \quad \text{then } X = S \oplus S^p \ (S^p \text{ is called the } \textit{complement} \text{ of } S). \tag{3.6}$$

$$\text{If } S_1, S_2 \in \mathcal{V}_X \ \text{ with } S_1 \subseteq S_2, \quad \text{then } S_2^p \subseteq S_1^p. \tag{3.7}$$

$$\text{If } S \in \mathcal{V}_X, \quad \text{then } (S^p)^p = S. \tag{3.8}$$

p is called a *vector complementor* on X.

Besides, the topological linear space X is named *precomplemented* if, for $S \in \mathcal{V}_X$, there exists $S' \in \mathcal{V}_X$, so that $X = S \oplus S'$.

In the sequel, (X, p) will stand for a complemented linear space X with respect to a vector complementor p.

EXAMPLE. *Any Hilbert space is complemented*, in the sense of Definition 15 (see, e.g., [2, p. 201, Theorem 15.1.1 and p. 202, Corollary 15.1.1]). In that case, the vector complementor is defined via orthogonality.

In the context of Theorem 14, the mapping

$$s : \mathcal{L}_l(E) \to \mathcal{V}_R : I \mapsto s(I) := I \cap R$$

is well defined, and in view of Corollary 12, (b), it is $1 - 1$. Besides, by the proof of Theorem 14, $s(\mathcal{L}_l(E)) = \mathcal{V}_R$. Moreover,

$$j : \mathcal{V}_R \to \mathcal{L}_l(E) : S \mapsto j(S) := \overline{ES}$$

is a well defined map, and since $\overline{ES} \cap R = S$ (see also the proof of Theorem 14), is the inverse of s. Finally, we consider the map

$$p : \mathcal{V}_R \to \mathcal{V}_R : S \mapsto S^p := s((j(S))^\perp) = (s \circ \perp \circ j)(S). \qquad (3.9)$$

In that case, according to Theorem 14, R, as a linear space, is complemented with an (induced) vector complementor p as in (3.9).

Definition 15 is realized in the next result.

Theorem 16. *Let $(E, \perp)$ be a Hausdorff topologically simple left complemented locally m-convex algebra and $e \in E$ minimal. Consider the (minimal closed right) ideal $R = eE$ of E. Then R, as linear space, is complemented with a vector complementor p as in* (3.9).

Proof. Immediate from Theorem 14, Definition 15 and the comments that follow. $\qquad \square$

4 Continuity of complementors in linear spaces. Fundamental linear subspaces

Heredity of the "fundamentality" from a certain fundamental left complemented algebra to concrete complemented subspaces is a key result in this section (Theorem 19). Based on this, we state the main result, which concerns conditions under which the continuity of the complementor of a certain fundamental left complemented

algebra induces continuity of the vector complementor for appropriate closed ideals in the topological algebra concerned (Theorem 20).

Let A be a subset of a linear space X. Consider the smallest subspace of X that derived from A. Namely, take the intersection of all linear subspaces of X containing A; we denote it by $[A]$. If $A = \{x\}, x \in X$, we simply write $[x]$.

If (X, p) is a complemented linear space and $S \in \mathcal{V}_X$, then there exists a unique linear map $T : X \to X$ with $T^2 = T$ (projection) such that $\operatorname{Im} T = S$ and $\ker T = S^p$ (see [18, Remark 2.4]). We use the symbol $T = T(S, S^p)$. If, in particular, T is continuous, we set the next.

Definition 17. A complemented (topological) linear space (X, p) is called *fundamental* if, for any closed linear subspace S of X, there is a continuous linear mapping $T : X \to X$ such that $T^2 = T$, $\operatorname{Im} T = S$ and $\ker T = S^p$.

EXAMPLES.

(1) According to example after Definition 15, *any Hilbert space is fundamental.* See also [3, p. 198, Theorem 14.9, p. 199, Corollary 14.10 and Remark 14.12].

Hence

(2) *Every complemented Banach space X is fundamental*, since as it is known, for any closed linear subspace of X, there is defined a projection as in Definition 17. See also the Example after Definition 1, and [14], as well.

(3) *Every precomplemented locally convex space is, under the respective property* (2.1), *weakly fundamental.* This is based on the fact that *any locally convex space X is precomplemented if and only if for any $S \in \mathcal{V}_X$ there exists a continuous linear mapping $T : X \to X$ with $T^2 = T$, $\operatorname{Im} T = S$ and $X = S \oplus \ker T$.* See also the proof of Theorem 7 and the comments at the beginning of Section 2.

Provided that a finite dimensional linear subspace of a Hausdorff topological vector space is closed, we give the next.

Definition 18. Let (X, p) be a Hausdorff fundamental complemented linear space. The mapping p is said to be *continuous* if for every net $(x_\delta)_{\delta \in \Delta}$ of elements of X with $\lim_\delta x_\delta = x_0 \in X$ and $x_0 \neq 0$, the net $(T_\delta([x_\delta], [x_\delta]^p))_{\delta \in \Delta}$ converges uniformly to $T_0([x_0], [x_0]^p)$.

The *fundamental property* is inherited to certain linear subspaces of a concrete topological algebra as the following result shows.

Theorem 19. *Let $(E, \perp)$ be a Hausdorff topologically simple left complemented locally m-convex algebra and $e \in E$ a minimal element. Consider the (minimal closed right) ideal $R = eE$ of E. If E is fundamental, then the complemented linear subspace (R, p) is fundamental too, where p is the vector complementor as in (3.9).*

Proof. Let S be a closed subspace of R. Consider $I_S = ES$ and $I = \overline{ES}$. Since E is fundamental, there exists a continuous linear map $T : E \to E$ with $T^2 = T$, Im $T = I$ and ker $T = I^\perp$. In view of the proof of Theorem 14, $S = R \cap I$ and $S^p = R \cap I^\perp$. Consider the mapping $T_R = T|_R : R \to R$. For $x \in R$, there are unique $x_1 \in I$, $x_2 \in I^\perp$ with $x = x_1 + x_2$, whence $x = ex = ex_1 + ex_2$. Since $ex_1 \in I$ and $ex_2 \in I^\perp$, the uniqueness, in the analysis of x, yields $x_1 = ex_1$ and $x_2 = ex_2$. Thus, $T_R(x) = T(x_1 + x_2) = x_1 = ex_1 \in R$ and hence T_R is well defined. Since T is continuous and R closed, T_R is continuous, as well. Obviously, T_R is linear and $T_R^2 = T_R$. The preceding argumentation also shows that $T_R(R) \subseteq S$. Now, for $x \in S$, $T_R(x) = x$ and thus $T_R(R) = S$, namely Im $T_R = S$. Similarly, ker $T_R = S^p$, completing thus, the fundamentality of R. $\qquad\square$

Theorem 20. *Let $(E, \perp)$ be a Hausdorff topologically simple fundamental left complemented locally m-convex algebra with continuous complementor. Let e be a minimal element in E, such that the respective (minimal closed right) ideal $R = eE$ has no nilpotent elements of order 2. Then the induced vector complementor p on $\mathcal{V}_R$ (the set of all closed subspaces of R) is continuous.*

Proof. Let $(x_\delta)_{\delta \in \Delta}$ be a net of nonzero elements in R with $\lim_\delta x_\delta = x_0 \in R, x_0 \neq 0$ and $S_\delta = [x_\delta], \delta \in \Delta$, $S_0 = [x_0]$, closed subspaces of R (see also [16, p. 38, Theorem 5]). Consider the ideals $I_\delta = ES_\delta$ and $I_0 = ES_0$ of E (cf. Proposition 13). Then, for any $\delta \in \Delta$, we have $ES_\delta = E\{\lambda x_\delta : \lambda \in \mathbb{C}\} = \{\lambda x_\delta + ax_\delta : \lambda \in \mathbb{C}, a \in E\}$. Since $ex_\delta = x_\delta$, $ES_\delta = \{xx_\delta : x \in E\}$. Thus, $ES_\delta = Ex_\delta = Eex_\delta$. Similarly, $ES_0 = Ex_0 = Eex_0$. Take now $x \in R$ with $x \neq 0$. Since $x \in R$, $x = ex$. Claim that $xe \neq 0$. Otherwise, $xex = 0$ or $x^2 = 0$ and by hypothesis for R, $x = 0$, a contradiction. The ideals $Eex_\delta, \delta \in \Delta$ and Eex_0 are minimal and closed. Since, $x_\delta, x_0 \in R, \delta \in \Delta$, we have $x_\delta = ex_\delta, \delta \in \Delta$ and $x_0 = ex_0$. So, we prove more generally, that the ideals $Eex = Ex, x \in R, x \neq 0$, are minimal and closed. We first prove the closedness of Ex. Indeed take $a \in \overline{Ex}$, then there is a net $(a_\delta x)_{\delta \in \Delta}$, in Ex with $a_\delta x \underset{\delta}{\to} a$ or yet $a_\delta xe \underset{\delta}{\to} ae$. So, since $x = ex$, we also have $a_\delta exe \underset{\delta}{\to} ae$. Since $exe = xe \neq 0$ and the algebra eEe is a division one (by hypothesis, e is a minimal element), we get $a_\delta(exe)(exe)^{-1} \underset{\delta}{\to} ae(exe)^{-1}$. Thus, $a_\delta e \underset{\delta}{\to} ae(exe)^{-1}$. Therefore $a_\delta ex \underset{\delta}{\to} ae(exe)^{-1}x$ and $a_\delta x \underset{\delta}{\to} ae(exe)^{-1}x \in Ex$. So, Ex is closed. We prove now that the closed ideal Ex is a minimal one. So, let L be a nonzero closed left ideal of E with $L \subseteq Eex$. Now, E, as topologically simple, is obviously, topologically semiprime, so we get $L^2 \neq \{0\}$ (see also, [5, p. 149, Theorem 2.1]). Therefore, there are $yex, zex \in L$ such that $yexzex \neq 0$, and so $exze \neq 0$. So, there is $c \in eEe$ with $cexze = e$. Since $Ezex \subseteq L$, we get $Eex = Ecexzex \subseteq Ezex \subseteq L \subseteq Eex$. Thus, $L = Eex$ that yields the minimality of $Ex(= Eex)$.

According to the preceding argumentation, the net $(x_\delta)_{\delta \in \Delta}$ is axially closed in E (see Definition 3), and the ideal Ex_0 is closed, as well. Since the complementor

****

Surveys in Mathematics and its Applications **11** (2016), 93 – 106
http://www.utgjiu.ro/math/sma

$\perp$ is continuous, we have (Definition 5)

$$T_\delta \equiv T_\delta(Ex_\delta, Ex_\delta^\perp) \underset{\text{uniformly},\delta}{\longrightarrow} T_0(Ex_0, Ex_0^\perp)$$

on any minimal right ideal of E, thus and in R. For any $\delta \in \Delta$, we consider the restrictions $T_\delta' = T_\delta|_R$ and $T_0' = T_0|_R$. For $S = [x]$, $0 \neq x \in R$, we have $Ex = E[x]$ (see also at the beginning of the proof). Since Ex is closed, we also have $Ex = E[x] = \overline{E[x]} = \overline{ES}$ and $\overline{ES} \cap R = S$ (see the proof of Theorem 14). The linear space (R, p) is fundamental since E is fundamental (see Theorem 19). Moreover, $T_\delta(Ex_\delta, Ex_\delta^\perp)|_R = T_\delta'(Ex_\delta \cap R, Ex_\delta^\perp \cap R) = T_\delta'([x_\delta], [x_\delta]^p)$ for all $\delta \in \Delta$ and $T_0(Ex_0, Ex_0^\perp)|_R = T_0'(Ex_0 \cap R, Ex_0^\perp \cap R) = T_0'([x_0], [x_0]^p)$. Hence, $T_\delta'([x_\delta], [x_\delta]^p) \underset{\text{uniformly},\delta}{\longrightarrow} T_0'([x_0], [x_0]^p)$. Therefore p is continuous (see also Definition 18). $\qquad\square$

Remark.- The hypothesis "the ideal $R = eE$ has no nilpotent elements of order 2", of the previous theorem, is fulfilled e.g. when E is preannihilator and satisfies *Le Page condition*, namely $Ex = Ex^2$ for all $x \in E$ (see [7]). Indeed, if $x \in R$ with $x^2 = 0$, then $Ex = \{0\}$ and thus $x = 0$ (see [5, p. 150, Lemma 2.3]).

Corollary 21. *Let E be a topological algebra as in Theorem 20. Let e be a minimal element in E which is not a right divisor of zero. Then the induced vector complementor p on $\mathcal{V}_R$ is continuous.*

Proof. Claim that the ideal $R = eE$ has no nilpotent elements of order 2. Otherwise, take $x \in E$ with $ex \neq 0$ and $(ex)^2 = 0$, namely $exex = 0$. Since $ex \neq 0$ and e is not a right divisor of zero, we get $exe \neq 0$. As eEe is a division algebra, we get $ex = (exe)^{-1}(exe)x = (exe)^{-1}(exex) = 0$, a contradiction. Theorem 20 completes the proof. $\qquad\square$

References

[1] F. E. Alexander, *Representation theorems for complemented algebras*, Trans. Amer. Math. Soc., **148**(1970), 385–398. MR0275159 (43 #916). Zbl 0196.14901.

[2] Ph. Blanchard, E. Brüning, *Mathematical Methods in Physics. Distributions, Hilbert Space Operators, and Variational Methods*, Birkhäuser, 2003. MR1936762 (2004c:46001). Zbl Zbl 0941.46001.

[3] J. R. Giles, *Introduction to the Analysis of Normed Linear Spaces*, Cambridge University Press, 2000. MR1745287 (2000m:46001). Zbl Zbl 0941.46001.

[4] M. Haralampidou, *Structure theorems for complemented topological algebras*, Boll. U.M.I. **7**(1993), 961–971. MR1255657 (94k:46091). Zbl 0892.46053.

[5] M. Haralampidou, *Annihilator topological algebras*, Portug. Math. **51**(1994), 147–162. MR1281963 (95f:46076). Zbl 0806.46051.

[6] M. Haralampidou, *On complementing topological algebras*, J. Math. Sci., **96**(1999), No 6, 3722–3734. MR1724441 (2000j:46085). Zbl 0953.46024.

[7] M. Haralampidou, *Classification of locally m-convex algebras through Le Page condition*, Comment. Math. Prace Mat. **44**(2004), 255–269. MR2118012 (2005k: 46111). Zbl 1086.46035.

[8] M. Haralampidou, *Dual complementors in topological algebras*, Banach Center Publications, Institute of Math., Polish Academy of Sci. **67**(2005), 219–233. MR2143927 (2006e:46053). Zbl 1090.46036.

[9] M. Haralampidou, *On the Krull property in topological algebras*, Comment. Math. XLVI, **2**(2006), 141–162. MR2287681 (2007i:46045). Zbl 1180.46035.

[10] M. Haralampidou and K. Tzironis, *An application of the Kakutani-Mackey like theorem in the representation of complemented topological algebras* (in preparation)

[11] M. Haralampidou and K. Tzironis, *Representation theorems for fundamental complemented algebras* (in preparation)

[12] J. L. Kelley, *General Topology*, Springer-Verlag, New York, 1955. MR0070144 (16,1136c).

[13] A. Mallios, *Topological algebras. Selected topics*, North-Holland, Amsterdam, 1986. MR0857807 (87m:46099). Zbl 0597.46046.

[14] M. S. Moslehian, *A survey on the complemented subspace problem*, Trends in Math. **9**(2006), no. 1, 91–98.

[15] C. E. Rickart, *General theory of Banach algebras*, R.E. Krieger, Huntington, N.Y., 1974. MR0115101 (22 #5903). Zbl 095.09702.

[16] A. P. Robertson and W. Robertson, *Topological Vector Spaces*, Cambridge Univ. Press, 1964. MR0162118 (28 #5318). Zbl 0123.30202.

[17] B. J. Tomiuk, *Structure theory of complemented Banach algebras*, Can. J. Math. **14**(1962), 651–659. MR0143060 (26 #626).

[18] K. Tzironis, *On continuity of complementors in topological algebras*, Conference on Topological Algebras and their Applications (under publication, Proceedings of ICTAA 2014 - "De Gruyter Proceedings in Mathematics").

Marina Haralampidou
Department of Mathematics, University of Athens,
Panepistimioupolis, Athens 15784, Greece.
e-mail: mharalam@math.uoa.gr

Konstantinos Tzironis
Department of Mathematics, University of Athens,
Panepistimioupolis, Athens 15784, Greece.
e-mail: tzirk@math.uoa.gr

License