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# ON ONE INTERESTING INEQUALITY

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**Abstract**. In this paper, we give a classification of points under which the generalization of Cîrtoaje's inequality or the reverse inequality are valid.

## 1 Introduction

Nowadays, inequalities with power-exponential functions are intensively studied. The power-exponential functions have many useful applications in mathematical analysis and in other theories like statistics, biology, optimization, ordinary differential equations, probability,.... The history and the literature review of some interesting inequalities with power-exponential functions can be found for example in [2]. Some other interesting problems concerning inequalities of power-exponential functions can be found for example in [6]. Cîrtoaje, in the paper [1], has posted the following interesting conjecture on the inequalities with power-exponential functions. We note that the inequality is similar to the reverse arithmetic-geometric mean inequality with the rearrangement of its terms.

**Conjecture 1.** *If $a, b \in (0; 1]$ and $r \in [0; e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}. \tag{1.1}$$

The conjecture was proved by Matejíčka [3]. Matejíčka also proved (1.1) under other conditions in the papers [4, 5]. For example, it was proved that (1) is valid for $a, b, r \in (0; e]$. In the paper [5], it was also showed that the certain generalization of Cîrtoaje's inequality fulfils an interesting property with some applications. The one of this applications is a classification of solution points of Cîrtoaje's inequality (CI), which we make in this paper.

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*Dedicated to my grandmother Žofia Čuchorová.*

## 2 Notations and preliminaries.

For the convenience of the reader, we provide a summary of the mathematical notations and definitions used in this paper (see also [5]). Put

$$F(r) = \ln n + \frac{r}{n}\left(\sum_{i=1}^{n} x_i \ln x_i\right) - \ln\left(e^{rx_1 \ln x_n} + \sum_{i=1}^{n-1} e^{rx_{i+1} \ln x_i}\right). \tag{2.1}$$

The function $F(r)$ is defined on $R_+^n$ where $n \in \mathbf{N}$, $r \geq 0$, $R_+^n = \{(x_1, ..., x_n),\ x_i > 0,\ i = 1, ..., n\}$. We note that $F(r) \geq 0$ is equivalent to the following generalization of Cîrtoaje's inequality (again CI)

$$n\sqrt[n]{\prod_{i=1}^{n} x_i^{rx_i}} \geq x_n^{rx_1} + \sum_{i=1}^{n-1} x_i^{rx_{i+1}}. \tag{2.2}$$

The inequality (2.2) was published for first time as a conjecture in the paper [2]. In the paper [4] it was shown that (3) for $n = 3$ does not valid on $M = \{(x_1, x_2, x_3),\ 1 \geq x_i > 0,\ i = 1, 2, 3\}$ for $r \in [0; e]$.

The reverse inequality to the generalization of Cîrtoaje's inequality

$$n\sqrt[n]{\prod_{i=1}^{n} x_i^{rx_i}} < x_n^{rx_1} + \sum_{i=1}^{n-1} x_i^{rx_{i+1}} \tag{2.3}$$

we denote by RCI.

The function

$$g(x_1, ..., x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \log(x_i) - m_x, \tag{2.4}$$

$$\text{where} \quad m_x = \max_{1 \leq m \leq n} \{x_{m+1} \log(x_m)\}, \quad x_1 = x_{n+1}. \tag{2.5}$$

we will call characteristic function of CI.

Put

$$\begin{aligned}
O_\varepsilon(A) =& \{X \in R^n;\ | X - A | < \varepsilon, \varepsilon > 0\},\\
S^n =& \{(x_1, ..., x_n) \in R_+^n;\ x_i = x_j,\ i, j = 1, ..., n\},\\
M_+^n =& \{(x_1, ..., x_n) \in R_+^n;\ g(x_1, ..., x_n) > 0\},\\
M_0^n =& \{(x_1, ..., x_n) \in R_+^n;\ g(x_1, ..., x_n) = 0\},\\
M_-^n =& \{(x_1, ..., x_n) \in R_+^n;\ g(x_1, ..., x_n) < 0\}.
\end{aligned}$$

Denote $r_A$ the positive root of $F(r) = 0$ (if the root exists) for $A \in R_+^n$-$S^n$. From the results of [5] we get that for each $A \in R_+^n$ there is a finite limit

$$g(A) = \lim_{r \to +\infty} F'(r) = \frac{1}{n}\sum_{i=1}^{n} x_i \log(x_i) - m_x.$$

## 3 Main Results

We prove two interesting results, which follow from results of the paper [5]. In the paper [5] it was showed that $F(r)$ is a concave function for each $A \in R^n_+$-$S^n$ and $F(0) = 0$, $F'(0) > 0$ again for each $A \in R^n_+$-$S^n$. The results have significant consequences.

**Proposition 2.** *CI is valid locally for all $X \in R^n_+ - S^n$.*

*Proof.* Proof follows from the following Lemma 3. □

**Lemma 3.**

- *Let $A \in M^n_+$. Then there is $O_\varepsilon(A) \subset R^n_+$ such that CI is valid for all $r \geq 0$ on $O_\varepsilon(A)$,*
- *Let $A \in M^n_0$. Then CI is valid for all $r \geq 0$ in $A$,*
- *Let $A \in M^n_-$. Then there is $O_\varepsilon(A) \subset R^n_+$, $0 < p \leq q < \infty$ such that CI is valid for all $0 \leq r \leq p$ on $O_\varepsilon(A)$ and RCI is valid for all $r > q$ on $O_\varepsilon(A)$.*

*Proof.* If $A \in M^n_+$ then we have $g(A) > 0$. From continuity of $g$ we get there is $O_\varepsilon(A) \subset R^n_+$ such that $g(X) > 0$ on $O_\varepsilon(A)$. If $g(X) > 0$ then from $F(0) = 0$, $F'(0) > 0$, $F''(r) < 0$ and $\lim\limits_{r \to +\infty} F'(r) = g(X) > 0$ (see [5]) we obtain that CI is valid in $X$ for all $r \geq 0$.

If $A \in M^n_-$ then we have $g(A) < 0$. From $F(0) = 0$, $F'(0) > 0$, we have $F(A, r_{00}) > 0$ for some $r_{00} > 0$. From continuity of $F$ we obtain $F(X, r_0) > 0$ for some $r_0 > 0$ and $X \in O_1(A)$ and from $\lim\limits_{r \to +\infty} F'(r) = g(A) < 0$ we have $F(A, s_0) < 0$ for some $s_0 \geq r_0$. It implies $F(X, r) > 0$ for $X \in O_1(A)$ and $0 \leq r \leq r_0$. We also have there is $O_2(A) \subset R^n_+$ such that $F(X, s_0) < 0$ for $X \in O_2(A)$. So $F(X, r) < 0$ for $X \in O_2(A)$ and for $r > s_0$. Put $O_\varepsilon(A) = O_1(A) \cap O_2(A)$. The proof is complete. □

From results of the paper [5] we can obtain even more information about points where CI and RCI is valid.

For example it is easy to show that:

- There is no $A \in R^n_+$ such that RCI is valid in $A$ for all $r > 0$.
- If $M \subset M^n_-$ is a compact set, then there is $0 < p \leq q < \infty$ such that CI is valid for all $0 \leq r \leq p$ on $M$ and RCI is valid for all $r > q$ on $M$.

A suitable choices of points from $R_+^n$ give that $M_+^n$, $M_-^n$, $M_0^n \neq \varnothing$ for $n \geq 2$. Indeed, $M_0^n \neq \varnothing$ is evident.

Put $x_1 = 1, ... x_{n-1} = 1$, $x_n = e$. Then

$$g(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \log(x_i) - m_x = \frac{e}{n} - 1.$$

It implies $g(X) > 0$ for $n = 2$ and $g(X) < 0$ for $n \geq 3$.

Put $x_1 = 1, ... x_{n-1} = 1$, $x_n = e^n$. Then

$$g(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \log(x_i) - m_x = e^n - n.$$

It implies $g(X) > 0$ for $n \geq 2$.

Put $n = 2$, $x_1 = e^2$, $x_2 = e^3$. Then

$$g(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \log(x_i) - m_x = e^2 - \frac{e^3}{2} < 0.$$

It implies $M_+^n$, $M_-^n \neq \varnothing$ for $n \geq 2$.

**Example 4.** *Using Matlab for fitting of the curves which are solution of the characteristic equation $g(X) = 0$ for n=2 we obtain the following figure 1 of points of solution of CI and RCI. In the figure 1 we denote by $CI + RCI$ points where CI and also RCI are locally valid. By $CI$ we denote points where CI is valid for all $r > 0$.*

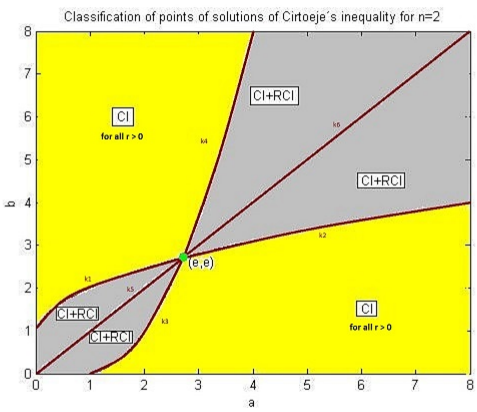


Figure 1:

**Remark 5.** *Our method can be used for the analysis of other suitable power-exponential inequalities.*

**Remark 6.** *Let $n \geq 2$. We note that CI is not valid globally for any $r > 0$.*

Indeed, let $n \geq 2$ is a natural number, and $a$ real number such that $a > 4n^2$. Put $x_1 = x_2 = ...x_{n-1} = a$, $x_n = 2a$, $r = 1/a$. Easy to see that

$$H(X) = n \sqrt[n]{\prod_{i=1}^{n} x_i^{rx_i}} \geq x_n^{rx_1} - \sum_{i=1}^{n-1} x_i^{rx_{i+1}} < 0.$$

It follows from

$$H(X) = na\sqrt[n]{4a} - na - a^2 < na2\sqrt{a} - a^2 < a\sqrt{a}(2n - \sqrt{a}) < 0.$$

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**Competing Interest.** The author declares that he has no competing interests.

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