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C-CLASS FUNCTIONS ON COMMON FIXED POINTS FOR MAPPINGS SATISFYING LINEAR CONTRACTIVE CONDITION

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Abstract. The purpose of this paper is to prove a common fixed point theorem, by using the concept of weakly subsequential continuity and compatibility of type (E) for two pairs of self mappings satisfying a linear contractive condition in metric spaces, we give two examples to illustrate our results.

1 Introduction

Since the famous Banach principle, the metric fixed point theory was known some improvement, especially the contractive condition was generalized in various type, as contraction of Kannan [16], Ciric contraction [10], Chaettjea type contraction [9] and Zamferscu contraction [27]. But the most important those of Meir-Keeler [17], Geraghty [11], Boyd et Wong[7](contraction nonlinear) and Branciari [8](integral type contraction), recently Rhoades [22]extended the concept of weak contraction due to Alber and Guerre-Delabriere [3] in the setting of metric spaces, which was generalized to generalized weak contraction. In 2008, Suzuki[26] introduced a new kind of contractions, which called Suzuki-contraction, more recently he first author [2] introduced $\mathcal{C}$-class functions, he gave some new results, which generalized and improved some ones.

Jungck[12]introduced and used the concept of commuting mappings to establish a common fixed point theorem for two self mappings in metric spaces, Sessa[23] generalized it to the weakly commuting mappings. Jungck[13] introduced the concept of compatibility mappings in metric space, it is weaker than the last notions. After that many authors introduced various type of compatibility, compatibility of type (A),of type (B),of type (C) and of type (P) for two self mappings f and g on metric space (X, d) respectively in [14], [19],[21] and [20] as follows: the pair(S, T)

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is compatible of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0,$$

S and T are compatible of type (B) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, S^2x_n)\Big] \text{ and}$$

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, gz) + \lim_{n\to\infty} d(Tz, T^2x_n)\Big],$$

they are compatible of type (C) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, T^2x_n) + \lim_{n\to\infty} d(Sz, T^2x_n)]$$

and

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, T^2x_n) + \lim_{n\to\infty} d(Tz, S^2x_n)],$$

and said to be compatible of type(P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0,$$

whenever in the all above definitions, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.
Aamri and Moutawakil [1] defined two self maps S and T on a metric space (X, d) are said to be satisfy property (E,A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some z in X.

2 Preliminaries

Pant[18] introduced the notion of reciprocal continuity as follows:

Definition 1. *Self maps S and T of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n\to\infty} STx_n = St$ and $\lim_{n\to\infty} TSx_n = Tt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

In 2009, Bouhadjera and Godet Thobie [6] introduced the concept of subcompatibility and subsequential continuity as follows:
Two self-mappings S and T on a metric space (X, d) are said to be subcompatible, if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ and } \lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

for some $t \in X$

Definition 2. *[6] The pair (S, T) is called to be subsequentially continuous, if there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Motivated by Defintion2, the second author [5] gave the following definition:

Definition 3. *[5] Let S and T to be two self mappings of a metric space (X, d), the pair (S, T) is said to be weakly subsequentially continuous (shortly wsc), if there exists a sequence $\{x_n\}$, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Notice that subsequentially continuous or, reciprocally continuous maps are weakly subsequentially continuous, but the converse may be not.

Example 4. *Let $X = [0, 2]$ and d is the euclidian metric, we define S, T as follows:*

$$Sx = \begin{cases} 1 + x, & 0 \le x \le 1 \\ \frac{x+1}{2}, & 1 < x \le 2 \end{cases}, \quad Tx = \begin{cases} 1 - x, & 0 \le x \le 1 \\ x - 2, & 1 < x \le 2 \end{cases}$$

Clearly that S and T are discontinuous at 1.
We consider a sequence $\{x_n\}$, which defined for each $n \ge 1$ by: $x_n = \dfrac{1}{n}$, clearly that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$, also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S(2 - \frac{1}{n}) = 2 = S(1),$$

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(2 + \frac{1}{n}) = 2 \ne T(1),$$

then (S, T) is S-subsequentially continuous, so it is wsc .
On other hand, let $\{y_n\}$ be a sequence which defined or each $n \ge 1$ by: $y_n = 1 + \dfrac{1}{n}$, we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = 1,$$

but

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(1+\frac{2}{n}) = 1 \neq S(1),$$
$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(4+\frac{1}{n}) = 1 \neq T(1),$$

then S and T are never reciprocally continuous.

Example 5. *Let $X = [0,\infty)$ endowed with the euclidian metric. Define S, T as follows:*

$$Sx = \begin{cases} \frac{x+2}{2}, & 0 \leq x < 2 \\ 1, & x = 2 \\ 0, & x > 2 \end{cases} \qquad Tx = \begin{cases} 4-x, & 0 \leq x \leq 2 \\ x+1, & x > 2 \end{cases}$$

Clearly, S et T are discontinuous at 2. Consider a sequence $\{x_n\}$ in X satisfying $0 \leq x_n < 2$ and $\lim_{n\to\infty} x_n = 2$.
we have:

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 2$$

. Moreover

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(1+\frac{x_n}{2})$$
$$= \lim_{n\to\infty} (3-\frac{x_n}{2}) = T(2) = 2$$

. Hence (S,T) is wsc.
On other hand, if there exists a sequence $\{y_n\}$ satisfying

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = l,$$

so must be $y_n \in [0,2)$, then we have:

$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(1+\frac{y_n}{2})$$
$$= \lim_{n\to\infty} 3-\frac{y_n}{2} = 3-\frac{l}{2},$$

$T(l) = 3-\frac{l}{2}$ if and only if $l = 2$.

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(4-y_n) = 0,$$

$S(l) = 0$ if and only if $l > 2$. Consequently, (S,T) is never sequentially continuous.

Definition 6. *[5] The pair (S,T) is said to be S-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$.*

Definition 7. *[5] The pair (S,T) is said to be T-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} TSx_n = Tt$.*

Singh and Mahendra Singh [24] introduced the notion of compatibility of type (E), and gave some properties about this type as follows:

Definition 8. *[24] Self maps S and T on a metric space (X,d), are said to be compatible of type (E), if $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$ and $\lim_{n\to\infty} S^2x_n = \lim_{n\to+\infty} STx_n = Tt$, whenever $\{x_n\}$ is a sequence in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

Remark 9. *If $St = Tt$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not true. Generally compatibility of type (E) implies compatibility of type (B).*

Definition 10. *[25] Two self maps S and T of a metric space (X,d) are S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = Tt$, for some $t \in X$.*
The pair (S,T) is said to be T-compatible of type (E), if $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$, for some $t \in X$.

Notice that if S and T are compatible of type (E), then they are S-compatible and T-compatible of type (E), but the converse is not true.

Example 11. *Let $X = [0,\infty)$ endowed with the euclidian metric, we define S,T as follows:*

$$Sx = \begin{cases} 2, & 0 \leq x \leq 2 \\ x+1, & x > 2 \end{cases} \qquad Tx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 0, & x > 2 \end{cases}$$

Consider the sequence $\{x_n\}$ which defined by: $x_n = 2 - \dfrac{1}{n}$, for all $n \geq 1$.

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 2,$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STxn) = 2 = T(2)$$

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSxn) = 2 = S(2)$$

then (S,T) is compatible of type (E).

In 2014 the concept of C-class functions A. H. Ansari [2]. By using this concept we can generalize many fixed point theorems in the literature.

Definition 12. *Let $F : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous mapping, it is called a C-class function if it satisfies the following conditions:*

(F_1) : *$F(s,t) \leq s$, for all $(s,t) \in \mathbb{R}_+^2$.*

(F_2) : *$F(s,t) = s$ implies that $s = 0$, or $t = 0$,for all $(s,t) \in \mathbb{R}_+^2$.*

Note for some F we have that $F(0,0) = 0$.

We denote C-class functions as $\mathcal{C}$.

Example 13. *[2]The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:*

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;

(7) $F(s,t) = s \log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;

(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$,and is continuous, $F(s,t) = s \Rightarrow s = 0$;

(10) $F(s,t) = s - \frac{t}{k+t}$, $F(s,t) = s \Rightarrow t = 0$;

(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$,here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$,here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$;

(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$.(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;

(14) $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$,here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,

(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;

(17) $F(s,t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function , $F(s,t) = s \Rightarrow s = 0$;

(18) $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, where Γ is the Euler Gamma function.

Let Ψ be a set of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and strictly increasing.

(ψ_2) $\psi(t) = 0$ if and only of $t = 0$.

Let Φ set of all continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\phi(t) = 0$ if and only if $t = 0$.

3 Main results

Theorem 14. *Let (X, d) be a metric space, A, B, S are four self mappings a on X such for all $x, y \in X$ we have:*

$$\psi(d(Sx, Ty)) \leq F(\psi(M(x, y)), \phi(M(x, y))), \tag{3.1}$$

where $\varphi \in \Phi_{u},$, $\psi \in \Psi$, $F \in \mathbb{C}$ and

$$M(x,y) = \frac{1}{a+b+c+2h}(ad(Ax, By)+bd(Ax, Sx)+cd(By, Ty)+h[(d(Ax, Ty)+d(By, Sx)])$$

that $a, b, c, h \geq 0$ and $a + h > 0$ .If the two pairs (A, S) and (B, T) are weakly subsequentially continuous (wsc) and compatible of type (E), then A, B, S and T have a unique common fixed point in X.

Proof. Since (A, S) is wsc, there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim\limits_{n\to\infty} ASx_n = Az$, $\lim\limits_{n\to\infty} SAx_n = Sz$ again (A, S) is compatible of type (E) implies that

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A^2x_n = Sz$$

and

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = Az,$$

consequently we obtain $Az = Sz$ and z is a coincidence point for A and S.

Similarly for B and T, since (B, T) is wsc (suppose that it is B-subsequentially continuous) there exists a sequence $\{y_n\}$ such

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$ and

$$\lim_{n\to\infty} BTy_n = Bt,$$

again$\{B, T\}$ is compatible of type (E), we get

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} B^2y_n = Tt$$

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = Bt,$$

so we have $Bt = Tt$.
We claim $Az = Bt$, by using 3.1 we get:

$$\begin{aligned}
& \psi(d(Az, Bt)) = \psi(d(Sz, Tt)) \\
\leq\ & F(\psi(\frac{1}{a+b+c+2h}(ad(Az, Bt) + bd(Az, Sz) + cd(Bt, Tt) + h[(d(Az, Tt) + d(Bt, Sz)])), \\
& \phi(\frac{1}{a+b+c+2h}(ad(Az, Bt) + bd(Az, Sz) + cd(Bt, Tt) + h[(d(Az, Tt) + d(Bt, Sz)]))) \\
=\ & F(\psi(\frac{a+2h}{a+b+c+2h}d(Az, Bt)), \phi(\frac{a+2h}{a+b+c+2h}d(Az, Bt))) \\
\leq\ & \psi(d(Az, Bt))
\end{aligned}$$

So, $\psi(\frac{a+2h}{a+b+c+2h}d(Az, Bt)) = 0$, or $phi(\frac{a+2h}{a+b+c+2h}d(Az, Bt)) = 0$, thus $d(Az, Bt) = 0$, which implies that $Az = Bt$.

Now we will prove $z = Az$, if not by using(3.1) we get:

$$\begin{aligned}
\psi(d(Sx_n, Tt)) &\leq F(\psi(M(x_n, t)), \phi(M(x_n, t))) \\
&\leq \psi(M(x_n, t)),
\end{aligned}$$

where

$$\begin{aligned}
& M(x_n, t) = \\
& \frac{1}{a+b+c+2h}(ad(Ax_n, Bt) + bd(Ax_n, Sx_n) + cd(Bt, Tt) + h[(d(Ax_n, Tt) + d(Bt, Sx_n)]) \\
& \to \frac{a+2h}{a+b+c+2h}d(z, t)
\end{aligned}$$

letting $n \to \infty$ and since F, ψ, ϕ are continuous we get:

$$\begin{aligned}
\psi(d(z, Az)) &= F(\psi(\frac{a+2h}{a+b+c+2h}d(z, Az)), \phi(\frac{a+2h}{a+b+c+2h}d(z, Az))) \\
&\leq \psi(d(z, Az)),
\end{aligned}$$

So, $\psi(\frac{a+2h}{a+b+c+2h}d(z, Az)) = 0$, or $\phi(\frac{a+2h}{a+b+c+2h}d(z, Az)) = 0$, thus $d(z, Az) = 0$, which implies that $z = Az$. Hence $z = Az = Sz$.

Next we shall prove $z = t$, if not by using (3.1) we get:

$$\begin{aligned}
\psi(d(Sx_n, Ty_n) &\leq F(\psi(M(x_n, y_n)), \phi(M(x_n, y_n))) \\
&\leq \psi(M(x_n, y_n)),
\end{aligned}$$

where

$$\begin{aligned}
& M(x_n, y_n) = \\
& \frac{1}{a+b+c+2h}(ad(Ax_n, By_n) + bd(Ax_n, Sx_n) + cd(By_n, Ty_n) + h[(d(Ax_n, Ty_n) + d(By_n, Sx_n)]) \\
& \to \frac{a+2h}{a+b+c+2h}d(z, t)
\end{aligned}$$

letting $n \to \infty$ we get:

$$\psi(d(z,t)) \leq F(\psi(\frac{a+2h}{a+b+c+2h}d(z,t)), \phi(\frac{a+2h}{a+b+c+2h}d(z,t)))$$

So, $\psi(\frac{a+2h}{a+b+c+2h}d(z,t)) = 0$, or $\phi(\frac{a+2h}{a+b+c+2h}d(z,t)) = 0$, thus $d(z,t) = 0$, we obtain $z = t$, then z is a fixed point for A, B, S and T.

For the uniqueness suppose that there is another fixed point w and using (3.1) we get:

$$\begin{aligned}\psi(d(z,w)) &= \psi(d(Sz,Tw)) \leq F(\psi(M(z,w), \phi(M(z,w)))),\\ &= F(\psi(\frac{a+2h}{a+b+c+2h}d(z,w), \phi(\frac{a+2h}{a+b+c+2h}d(z,w)))),\end{aligned}$$

So, $\psi(\frac{a+2h}{a+b+c+2h}d(z,w)) = 0$, or $phi(\frac{a+2h}{a+b+c+2h}d(z,w)) = 0$, thus $d(z,w) = 0$, which implies that $z = w$. □

If $A = B$ and $S = T$, we obtain the following corollary:

Corollary 15. *Let (X,d) be a metric space and let $S, A : X \to X$ two self mappings such for all $x, y \in X$ we have:*

$$\psi((d(Sx,Sy)) \leq F(\psi(N(x,y)), \phi(N(x,y)),$$

where $N(x,y) = \frac{1}{a+b+c+2h}(ad(Ax,Ay) + bd(Ax,Sx) + cd(Ay,Sy) + h[d(Ax,Sy) + d(Ay,Sx)])$ and where $F \in \mathcal{C}$, assume that the pair (A,S) is wsc A-subsequentially continuous and A-compatible of type (E), then A and S have a unique common fixed point in X.

If we take $F(s,t) = s - t$ in Theorem 14, we obtain the following corollary:

Corollary 16. *For four self mappings A, B, S and T on metric space (X,d) such for all $x, y \in X$ we have:*

$$\psi(d(Sx,Ty))) \leq \psi(M(x,y)) - \phi(M(x,y).$$

If the two pairs (A,S) and (B,T) are wsc and compatible of type (E), then A, B, S and T have a unique common fixed point.

Corollary 16 is Theorem 1 in [5].

If we take $\psi(t) = t$ in Corolllary16, and putting $\varphi(t) = (I - \phi)(t)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an upper semi continuous function such $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t$ for all $t > 0$, we obtain the following corollary:

Corollary 17. *Let (X,d) be a space metric and let A, B, S and T self mappings on X such for all $x, y \in X$ we have:*

$$d(Sx,Ty))) \leq \varphi(M(x,y)),$$

where $M(x,y) = \frac{1}{a+b+c+2h}(ad(Ax,By) + bd(Ax,Sx) + cd(By,Ty) + h[(d(Ax,Ty) + d(By,Sx)])$. If the following conditions are satisfied:

1. (A,S) *is* A*-subsequentially continuous and* A*-compatible of type (E),*

2. (B,T) *is* B*-subsequentially continuous and* B*-compatible of type (E),*

then A,B,S *and* T *have a unique common fixed point.*

Corollary 17 generalizes and improves theorem 1 in [28].
If we take $\psi(t)=t$ in Theorem 14, we obtain the following corollary:

Corollary 18. *Let* (X,d) *be a space metric and let* A,B,S *and* T *self mappings on* X *such for all* $x,y\in X$ *we have:*

$$d(Sx,Ty)))\leq F(M(x,y),\phi(M(x,y)),$$

where $\phi\in\Phi$.
If the following conditions are satisfied:

1. (A,S) *is* S*-subsequentially continuous and* S*-compatible of type (E),*

2. (B,T) *is* T*-subsequentially continuous and* T*-compatible of type (E),*

then A,B,S *and* T *have a unique common fixed point.*

Corollary 18 generalizes Corollary 1 in [5].
If we take $F(s,t)=ks$ in Corollary 18, we obtain the following corollary:

Corollary 19. *For four self mappings* A,B,S *and* T *on metric space* (X,d) *such for all* $x,y\in X$ *we have:*

$$d(Sx,Ty))\leq kM(x,y)),$$

where $M(x,y)=\frac{1}{a+b+c+2h}(ad(Ax,By)+bd(Ax,Sx)+cd(By,Ty)+h[(d(Ax,Ty)+d(By,Sx)])$, *and* $0\leq k<1$ *if the four mappings satisfying:*

1. (A,S) *is* A*-subsequentially continuous and* A*-compatible of type (E),*

2. (B,T) *is* B*-subsequentially continuous and* B*-compatible of type (E),*

then A,B,S *and* T *have a unique common fixed point.*

If we $F(s,t)=\frac{s}{(1+t)^r}$, we obtain the following corollary:

Corollary 20. *Let* (X,d) *be a space metric and let* A,B,S *and* T *self mappings on* X *such for all* $x,y\in X$ *we have:*

$$\psi(d(Sx,Ty)))\leq\frac{\psi(M(x,y)}{(1+\phi(M(x,y))^r},$$

where $r>0$, $\psi\in\Psi$ *and* $\phi\in\Phi$.
If the following conditions are satisfied:

1. *$\{A,S\}$ is S-subsequentially continuous and S-compatible of type (E),*
2. *$\{B,T\}$ is T-subsequentially continuous and T-compatible of type (E),*

then A,B,S and T have a unique common fixed point.

Now we can obtain the same result in Theorem 14, by using the subsequential continuity with compatibility of type (E) as follows:

Theorem 21. *Let (X,d) be a space metric and let A,B,S and T be four self mappings on X such for all $x,y\in X$ we have:*

$$d(Sx,Ty))\leq F(\psi(M(x,y),\phi(M(x,y)),$$

where $M=\frac{1}{a+b+c+2h}(ad(Ax,By)+bd(Ax,Sx)+cd(By,Ty)+h[d(Ax,Ty)+d(By,Sx)])$, if the following conditions hold:

1. *(A,S) is wsc and compatible of type (E),*
2. *(B,T) is wsc and compatible of type (E),*

then A,B,S and T have a unique common fixed point.

Proof. It is similar as in proof of Theorem 14. □

Example 22. *Let $X=[0,2]$ and d is the euclidian metric, we define A,B,S and T by*

$$Ax=\begin{cases}\frac{x+1}{2}, & 0\leq x\leq 1\\ \frac{3}{4}, & 1<x\leq 2\end{cases}\qquad Bx=\begin{cases}2-x, & 0\leq x\leq 1\\ 2, & 1<x\leq 2\end{cases}$$

$$Sx=\begin{cases}1, & 0\leq x\leq 2\\ \frac{1}{4}, & 1<x\leq 2\end{cases}\qquad Tx=\begin{cases}\frac{1}{3}x, & 0\leq x<1\\ 1, & x=1\\ \frac{1}{2}, & 1<x\leq 2\end{cases}$$

We consider a sequence $\{x_n\}$ which defined for each $n\geq 1$ by: $x_n=1-\frac{1}{n}$, clearly that $\lim_{n\to\infty}Ax_n=1$ and $\lim_{n\to\infty}Sx_n=1$, also we have:

$$\lim_{n\to\infty}ASx_n=\lim_{n\to\infty}A(1-\frac{1}{2n})=A(1)$$
$$=S(1)=1,$$

then (A,S) is A-subsequentially continuous and A-compatible of type (E), on the other hand consider a sequence defined by: $y_n=1-e^{-n}$, for all $n>1$.It is clear that $\lim_{n\to\infty}By_n=\lim_{n\to\infty}Tx_n=1$, and $\lim_{n\to\infty}BTy_n=B(1)=T(1)=1$, this yields that (B,T) is B-subsequentially continuous and B-compatible of type (E). For the contractive condition, we have the following cases:

1. *For $x, y \in [0,1]$, we have*

$$d(Sx,Ty) = \frac{1}{3}|y-3| \leq \frac{2}{3}|y-\frac{3}{2}| = \frac{1}{2}d(By,Ty).$$

2. *For $x \in [0,1]$ and $y \in (1,2]$, we have*

$$d(Sx,Ty) = \frac{1}{2} \leq \frac{3}{4} = \frac{1}{2}d(By,Ty)$$

3. *For $x \in (1,2]$ and $y \in [0,1]$, we have*

$$d(Sx,Ty) = \frac{1}{3}|y-\frac{3}{4}| \leq \frac{1}{4} = \frac{1}{2}d(Ax,Sx)$$

4. *For $x, y \in (1,2]$, we have*

$$d(Sx,Ty) = \frac{1}{4} \leq \frac{5}{8} = \frac{1}{2}d(Ax,By)$$

Consequently all hypotheses of Corollary 17 with $k = \frac{2}{3}$ satisfy, therefore 1 is the unique common fixed for A, B, S and T.

Example 23. *Let $X = [0,4]$ and d is the euclidian metric, we define A, B, S and T by*

$$Ax = Bx \begin{cases} x, & 0 \leq x \leq 2 \\ 3, & 2 < x \leq 4 \end{cases} \qquad Sx = Tx = \begin{cases} 4-x, & 0 \leq x \leq 2 \\ 3, & 2 < x \leq 4 \end{cases}$$

We consider a sequence $\{x_n\}$ which defined for each $n \geq 1$ by:
$x_n = 2 - \frac{1}{n}$, clearly that $\lim\limits_{n\to\infty} Ax_n = 2$ and $\lim\limits_{n\to\infty} Sx_n = 2$, we have also:

$$\begin{aligned} \lim_{n\to\infty} ASx_n &= \lim_{n\to\infty} A(2-\frac{1}{2n}) = A(2) \\ &= S(2) = 2, \end{aligned}$$

then (A, S) is A-subsequentially continuous and A-compatible of type (E).
For the contractive condition, we have the following cases:

1. *For $x, y \in [0,2]$, we have*

$$d(Sx,Ty) = \frac{1}{2}|x-y| \leq \frac{2}{3}|x-y| = \frac{2}{3}d(Ax,By)$$

2. For $x \in [0,2]$ and $y \in (2,4]$, we have

$$d(Sx,Ty) = \frac{1}{2}x \leq 2 = \frac{2}{3}d(By,Ty)$$

3. For $x \in (2,4]$ and $y \in [0,2]$, we have

$$d(Sx,Ty) = \frac{1}{2}y) \leq \frac{4}{3n} = \frac{2}{3}d(Ax,Sx)$$

4. For $x, y \in (2,4]$, we have $d(Sx,Ty) = 0$ and so obviously is satisfied.

Consequently all hypotheses of Corollary 19 with $k = \frac{2}{3}$ satisfy, therefore $\frac{1}{2}$ is the unique common fixed for A, B, S and T.

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