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RC-CLASS AND LC-CLASS ON FIXED POINT THEOREMS FOR α-CARISTI TYPE CONTRACTION MAPPINGS

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Abstract. In this paper, we introduce the notion of $(\alpha, \mathcal{H}_{LC}, f_{RC})$-Caristi type contraction mappings and prove fixed point theorem by using this notion on complete metric space. To illustrate our result, we construct an example.

1 Introduction

Caristi [9] proved that if a self mapping T on a complete metric space (X, d) satisfies the condition:

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \ \forall \ x \in X \tag{1.1}$$

where $\phi : X \to [0, \infty)$ is a lower semicontinuous function, then T has a fixed point. The mapping T satisfying the condition (1.1) is known as Caristi mapping. Kirk [15] showed that if Caristi mapping for (X, d) has a fixed point, then (X, d) is complete and viceversa. Semat *et al*. [19] introduced the notion of α-admissible and α-ψ-contractive type mappings. These notions were extended by several authors, see for example, [1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 20]. Recently, Ali [1] introduced the notion of α-Caristi type contraction mapping and proved a fixed point theorem on complete metric space. On the other hand Ansari [2] introduced the family of functions known as RC-class and LC-class to generalize some existing fixed point theorems. In this paper we introduce a new Caristi type contraction condition by combining the above ideas. Note that, we denote by $CL(X)$ the space of all nonempty closed subsets of X. For $x \in X$ and $A \in CL(X)$, $d(x, A) = \inf\{d(x, a) : a \in A\}$. A function $H : CL(X) \times CL(X) \to [0, \infty]$ defined by

$$H(A, B) = \begin{cases} \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\} & \text{if exists} \\ \infty & \text{otherwise} \end{cases}$$

is a generalized Hausdorff metric space induced by metric d.

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Mohammadi *et al.* [18] and Asl *et al.* [8] extended the notion of α-admissible mapping from singlevalued to multivalued mapping in the following way:

Definition 1. *[18] Let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CL(X)$ is α-admissible if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for each $z \in Ty$.*

Definition 2. *[8] Let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CL(X)$ is α_*-admissible mapping if for each $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(u, v) : u \in Tx$ and $v \in Ty\}$.*

Minak and Altun [17] showed that every α_*-admissible mapping is α-admissible, but converse is not true in general, and gave the following example.

Example 3. *Let $X = [-1, 1]$. Define $T : X \to CL(X)$ by*

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 0 \\ \{-x\} & \text{if } x \notin \{-1, 0\} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

This T is α-admissible but not α_*-admissible.

Kutbi and Sintunavarat [16] introduced the notion of α-continuous multivalued mapping which is more general than continuous multivalued mappings.

Definition 4. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to CL(X)$ is said to be an α-continuous, if for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$, that is, $\lim_{n\to\infty} H(Tx_n, Tx) = 0$.*

Definition 5. *[2] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The function f is said to be a RC-class if f is continuous and satisfies*

$$\begin{aligned} f(s, t) &> 0 \Longrightarrow s > t; \\ f(t, t) &= 0; \\ s &\leq t \Longrightarrow f(e, s) \geq f(e, t) \text{ for each } e \in \mathbb{R}; \\ t &\leq e \leq s \Longrightarrow f(s, e) + f(e, t) \leq f(s, t); \\ \exists\, g &: \mathbb{R} \to \mathbb{R},\ f(g(s), g(t)) \geq 0 \Longrightarrow s \leq t, \end{aligned}$$

where $s, t, e \in \mathbb{R}$.

In the following, we can see some examples for RC-class functions.

Example 6. *For $n \in \mathbb{N}$ and $a > 1$,*

$$\begin{array}{ccc} f(s,t) = s - t & & g(t) = -t \\ f(s,t) = \frac{s-t}{1+t} & , & g(t) = \frac{1}{t} - 1 \\ f(s,t) = s^{2n+1} - t^{2n+1} & , & g(t) = -t \\ f(s,t) = a^s - a^t & , & g(t) = -t \\ f(s,t) = a^s - a^t + t - s & , & g(t) = -t \\ f(s,t) = e^{s^{2n+1} - t^{2n+1}} - 1 & , & g(t) = -t \\ f(s,t) = e^{s-t} - 1 & , & g(t) = -t. \end{array}$$

Definition 7. *[2] We say that $\mathcal{H}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a LC-class if h is continuous and satisfies the following conditions*

$$\begin{aligned} &\mathcal{H}(t) > 0 \text{ if and only if } t > 0; \\ &\mathcal{H}(0) = 0; \\ &\mathcal{H}(s+t) \leq \mathcal{H}(s) + \mathcal{H}(t); \end{aligned}$$

and

$$x \leq y \Longrightarrow \mathcal{H}(x) \leq \mathcal{H}(y).$$

Example 8. *For $a > 1, m > 0$ and $n \in \mathbb{N}$*

$$\begin{aligned} \mathcal{H}(t) &= 1 - a^{-t} \\ \mathcal{H}(t) &= mt \\ \mathcal{H}(t) &= m\sqrt[n]{t} \\ \mathcal{H}(t) &= \log_a 1 + t \\ \mathcal{H}(t) &= \log_a 1 + \sqrt[n]{t}, \end{aligned}$$

are some examples for $LC-$class.

2 Main Results

We begin this section with the following definition.

Definition 9. *Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ and $\phi : X \to [0,\infty)$ be two mappings, further, f is a RC-class and $\mathcal{H}$ is a LC-class function. A mapping $T : X \to CL(X)$ is said to be an $(\alpha, \mathcal{H}_{LC}, f_{RC})$-Caristi type contraction if for each $x \in X$ and $u \in Tx$ with $\alpha(x,u) \geq 1$ there exists $v \in Tu$ such that*

$$\mathcal{H}(d(u,v)) \leq f(\phi(x), \phi(u)). \tag{2.1}$$

Remark 10. *If we take* $\mathcal{H}(t) = t$ *and* $f(s,t) = s - t$*, then above definition reduces to the Definition 2.1 [1].*

Theorem 11. *Let* (X, d) *be a complete metric space and let* $T : X \to CL(X)$ *be an* $(\alpha, \mathcal{H}_{LC}, f_{RC})$*-Caristi type contraction mapping. Assume that the following conditions hold:*

(i) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$*;*

(ii) T *is* α*-admissible;*

(iii) T *is* α*-continuous.*

Then T *has a fixed point.*

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. By Definition 9, for $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$ there exists $x_2 \in Tx_1$ such that

$$\mathcal{H}(d(x_1, x_2) \leq f(\phi(x_0), \phi(x_1)).$$

As T is α-admissible, then $\alpha(x_0, x_1) \geq 1$ implies $\alpha(x_1, x_2) \geq 1$. Again, by Definition 9, for $x_1 \in X$ and $x_2 \in Tx_1$ with $\alpha(x_1, x_2) \geq 1$ there exists $x_3 \in Tx_2$ such that

$$\mathcal{H}(d(x_2, x_3)) \leq f(\phi(x_1), \phi(x_2)).$$

Continuing in the same way, we get a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$ and

$$0 \leq \mathcal{H}(d(x_n, x_{n+1})) \leq f(\phi(x_{n-1}), \phi(x_n)) \text{ for each } n \in \mathbb{N}. \tag{2.2}$$

By using the properties of $\mathcal{H}$, f and above inequality, we conclude that the sequence $\{\phi(x_{n-1})\}$ is a nonincreasing sequence, there exists $r \geq 0$ such that $\phi(x_n) \to r$. Now consider $n, p \in \mathbb{N}$, by using the triangular inequality and subadditivity of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}(d(x_n, x_{n+p})) &\leq \mathcal{H}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + d(x_{n+p-1}, x_{n+p})) \\
&\leq \mathcal{H}(d(x_n, x_{n+1})) + \mathcal{H}(d(x_{n+1}, x_{n+2})) + \mathcal{H}(d(x_{n+2}, x_{n+3})) \\
&\quad + \cdots + \mathcal{H}(d(x_{n+p-1}, x_{n+p})) \\
&\leq f(\phi(x_{n-1}), \phi(x_n)) + f(\phi(x_n), \phi(x_{n+1})) + f(\phi(x_{n+1}), \phi(x_{n+2})) \\
&\quad + \cdots + f(\phi(x_{n+p-2}), \phi(x_{n+p-1})) \\
&= f(\phi(x_{n-1}), \phi(x_{n+p-1})).
\end{aligned} \tag{2.3}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X, since $\phi \to r$. By completeness of X, we have $x^* \in X$ such that $x_n \to x^*$. By hypothesis (iii), we have $\lim_{n\to\infty} H(Tx_n, Tx^*) = 0$. By using the triangular inequality, we have

$$\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\
&\leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have $d(x^*, Tx^*) = 0$. This implies that $x^* \in Tx^*$. ☐

Example 12. *Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x-y|$. Define $T : X \to CL(X)$ by*

$$Tx = \begin{cases} [0, x] & \text{if } x \geq 0 \\ \{-e^x\} & \text{if } x < 0, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi : X \to [0, \infty)$ by

$$\phi(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Take $H(x) = \frac{x}{2}$ and $f(x, y) = x - y$ for each $x, y \in X$. Then, for each $x \in X$ and $u \in Tx$ with $\alpha(x, u) = 1$, there exists $v \in Tu$ such that

$$\mathcal{H}(d(u, v)) \leq f(\phi(x), \phi(u)).$$

Therefore, T is $(\alpha, \mathcal{H}_{LC}, f_{RC})$-Caristi type contraction mapping. For $x_0 = 3$ we have $x_1 = 3/2 \in Tx_0$ such that $\alpha(x_0, x_1) = 1$. Clearly, T is α-admissible. Let $\{x_n\}$ is any sequence in X such that $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) = 1$ for each $n \in \mathbb{N}$, then by definition of α, it clear that $x_n \geq 0$ for each $n \in \mathbb{N}$. Since $x_n \to x^*$, then $x^* \geq 0$. Thus, $Tx_n = [0, x_n]$ and $Tx^* = [0, x^*]$. Therefore, $\lim_{n\to\infty} H(Tx_n, Tx^*) = 0$. This shows that T is α-continuous. Thus, by Theorem 11, T has a fixed point.

Example 13. *Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x-y|$. Define $T : X \to CL(X)$ by*

$$Tx = \begin{cases} [0, \frac{x}{x+1}] & \text{if } x \geq 0 \\ \{-x^2\} & \text{if } x < 0, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi : X \to [0, \infty)$ by

$$\phi(x) = \begin{cases} \frac{x}{2} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Take $H(x) = \frac{x}{4}$ *and* $f(x, y) = x - y$ *for each* $x, y \in X$. *Then, for each* $x \in X$ *and* $u \in Tx$ *with* $\alpha(x, u) = 1$, *there exists* $v \in Tu$ *such that*

$$\mathcal{H}(d(u, v)) \leq f(\phi(x), \phi(u)).$$

Therefore, T *is* $(\alpha, \mathcal{H}_{LC}, f_{RC})$*-Caristi type contraction mapping. It is easy to see that the rest of the conditions of Theorem 11 hold. Thus,* T *has a fixed point. Note that Theorem 2.1 of [1] is not applicable here, to see consider* $x = \frac{1}{3}$ *and* $u = \frac{1}{4} \in Tx$.

Definition 14. *Let* (X, d) *be a metric space,* $\alpha : X \times X \to [0, \infty)$ *and* $\phi : X \to [0, \infty)$ *be two mappings, further,* f *is a* RC*-class and* $\mathcal{H}$ *is a* LC*-class function. A mapping* $T : X \to CL(X)$ *is said to be an* $(\alpha_T, \mathcal{H}_{LC}, f_{RC})$*-Caristi type contraction if for each* $x \in X$ *and* $u \in Tx$ *there exists* $v \in Tu$ *such that*

$$\mathcal{H}(d(u, v)) \leq f(\phi(x), \phi(u)) \text{ whenever } \alpha(u, v) \geq 1. \tag{2.4}$$

Theorem 15. *Let* (X, d) *be a complete metric space and let* $T : X \to CL(X)$ *be an* $(\alpha_T, \mathcal{H}_{LC}, f_{RC})$*-Caristi type contraction mapping. Assume that the following conditions hold:*

(i) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$;

(ii) T *is* α_**-admissible;*

(iii) T *is* α*-continuous.*

Then T *has a fixed point.*

Proof. The proof of this theorem can be obtained on the same lines as the proof of last theorem is done. □

3 Consequence

In this section we list some fixed point theorems which can be obtained from our results:

Theorem 16. *Let* $(X, d, \preceq)$ *be a complete ordered metric space and let* $T : X \to CL(X)$ *be a mapping such that for each* $x \in X$ *and* $u \in Tx$ *with* $x \preceq u$ *there exists* $v \in Tu$ *satisfying*

$$\mathcal{H}(d(u, v)) \leq f(\phi(x), \phi(u))$$

where $\phi : X \to [0, \infty)$ *be a function. Assume that the following conditions hold:*

(i) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $x_0 \preceq x_1$;

(ii) for each $x \in X$ *and* $y \in Tx$ *with* $x \preceq y$, *we have* $y \preceq z$ *for each* $z \in Ty$;

(iii) *T is ordered-continuous, that is, for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$.*

Then T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

Then by the hypothesis of this theorem, it is easy to see that all conditions of Theorem 11 hold. Thus, T has a fixed point. □

In following result, we assume that (X, d) is a metric space and $G = (V(G), E(G))$ is a directed graph such that the set of its vertices $V(G)$ coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $E(G) \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Further assume that G has no parallel edges.

Theorem 17. *Let (X, d) be a complete metric space endowed with the graph G and let $T : X \to CL(X)$ be a mapping such that for each $x \in X$ and $u \in Tx$ with $(x, u) \in E(G)$ there exists $v \in Tu$ satisfying*

$$\mathcal{H}(d(u, v)) \leq f(\phi(x), \phi(u))$$

where $\phi : X \to [0, \infty)$ be a function. Assume that the following conditions hold:

(i) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*

(ii) *for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for each $z \in Ty$;*

(iii) *T is G-continuous, that is, for each sequence $\{x_n\}$ in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$, we have $Tx_n \to Tx$.*

Then T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Then by the hypothesis of this theorem, it is easy to see that all the conditions of Theorem 11 hold. Thus, T has a fixed point. □

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