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NORMED ALGEBRAS AND THE GEOMETRIC SERIES TEST

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Abstract. The purpose of this article is to survey a class of normed algebras that share many central features of Banach algebras, save for completeness. The likeness of these algebras to Banach algebras derives from the fact that the geometric series test is valid, whereas the lack of completeness points to the failure of the absolute convergence test for series in the algebra. Our main result is a compendium of conditions that are all equivalent to the validity of the geometric series test for commutative unital normed algebras. Several examples in the final section showcase some incomplete normed algebras for which the geometric series test is valid, and still others for which it is not.

1 Introduction

A useful and oft-invoked fact from functional analysis is that a normed linear space is complete if and only if the absolute convergence test for series is valid (Theorem 2.8 of [3] or Lemma 3.20 of [20]). The theorem migrates over to the realm of normed algebras unadulterated. Specifically, a normed algebra $(A, \|\cdot\|)$ is a Banach algebra if and only if the series $\sum a_n$ of elements of A is convergent whenever the series $\sum \|a_n\|$ of real numbers is. In a similar vein, but with emphasis on the algebra multiplication, it was recently confirmed in [11] that a normed algebra with identity is complete precisely when the algebra version of Dedekind's classical series test [2] holds in the sense that the series $\sum a_n b_n$ is convergent in A whenever (a_n) and (b_n) are both sequences in A such that $\sum a_n$ converges and (b_n) is of bounded variation.

Certain series tests are thus able to characterize completeness for normed algebras, but the *geometric series test* is not one of them. Perhaps this comes as no surprise. After all, the geometric series test may be viewed as a very special case of the absolute convergence test; it states that, in the normed algebra $(A, \|\cdot\|)$, the geometric series $\sum a^n$ converges whenever the element $a \in A$ satisfies $\|a\| < 1$. In the setting of operator theory and integral equations, it is common to refer to geometric series as

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Neumann series, after the German mathematician Carl Gottfried Neumann (1832-1925). While the geometric series test holds for Banach algebras, completeness is assuredly not necessary. As will be discussed later, among the algebras in which it is valid are certain algebras of differentiable functions that cannot be endowed with a complete algebra norm. Thus, the class of normed algebras for which the geometric series test holds is strictly larger than the class of Banach algebras, but still there are normed algebras in which the test fails.

For simplicity, we concentrate exclusively on commutative algebras and, except for a brief foray at the end of the second section, algebras that are unital. It turns out that many features of commutative unital Banach algebras are in place for commutative unital algebras in which the geometric series test is valid. For example, as will be seen, the fundamental theorem of Banach algebras concerning compactness of spectra and the Gel'fand representation theorem both hold when the geometric series test does. Even more astonishing is the fact that the validity of the geometric series test is equivalent to many familiar hallmarks of Banach algebra theory. Among these are the Beurling–Gel'fand formula for the spectral radius, the openness of the set of invertible elements, the fact that maximal ideals are always closed, the upper semi-continuity of the spectrum as a set-valued mapping, and many others. Our goal is to compile a list of such conditions.

In his review [18] of the textbook [4] of Bonsall and Duncan, Rickart writes:

> It is remarkable that in a series of papers, published between 1939 and 1944 by Gel'fand and his collaborators, virtually all of the main lines along which the theory of Banach algebras would develop for a period of 25 or 30 years were already laid down.

Like Bonsall and Duncan (page 4 of [4]), we recognize the lasting influence and the distinguished pioneering work of Israel M. Gel'fand (1913-2009) in this field even when the completeness requirement for the algebras is here weakened.

We were led to the present study from the arena of totally ordered fields, where a robust program to characterize completeness and the attendant Archimedean property has been blossoming. In this context, the validity of the absolute convergence test for series, or of the classical series tests of Dedekind or Dirichlet, characterizes the Cauchy completeness of an ordered field [12], whereas the geometric series test is too weak to do so. Rather, as shown in [10], the geometric series test holds in an ordered field precisely when the field has the *Archimedean property,* meaning that the canonically-embedded copy of the natural numbers is not bounded above. A surprising fact about ordered fields with the Archimedean property is that they are exactly those that are isomorphic to subfields of $\mathbb{R}$. For this and further information and references about the *real analysis in reverse* program, see [17]. The present article may be viewed as a modest contribution to what may be called *Banach algebras in reverse.*

The idea of looking at geometric series in incomplete normed algebras seems to have been initiated by Fuster and Marquina in their influential paper [6]. Palmer later expounded the connection between the geometric series test and spectral theory. Proposition 3.3 of [15] and Proposition 2.2.7 of [16] forge the link between the geometric series test in an algebra and norms on the algebra that are spectral in the sense that they dominate the spectral radius.

The present survey inventories an eclectic list of twenty-two conditions that are equivalent to the validity of the geometric series test in the simple environment of a commutative unital algebra for which an algebra norm is already in place. These conditions, all familiar from the general theory of Banach algebras, are provided in Theorems 1 and 2. The list is not intended to be exhaustive. We hope our adherence to the classical setting offers a resource, a primer, and a portal through which readers who are conversant with the fundamental notions of normed algebras might pass en route to tackling the encyclopedic treatment of spectral algebras – algebras on which some spectral pseudo-norm can be defined – in the comprehensive treatise [16].

The third section of this article is devoted to detailing some examples of algebras in which the geometric series test is not valid, and some natural examples of incomplete algebras in which it is. In this final section, we also confront the limitations of the present study by exposing certain aspects of Banach algebra theory that do not carry over to the larger class of algebras in which the geometric series test is known to hold.

2 Main results

We first review some basic definitions and standard notation. Throughout we focus on a normed commutative complex algebra $(A, \|\cdot\|)$ with multiplicative identity element e having $\|e\| = 1$. We do not, of course, assume that A is complete. It is well known that A is included, as a dense subalgebra, in a Banach algebra $\tilde{A}$, called the *completion of* A. The set of *invertible elements* of A is denoted by $\mathrm{Inv}(A)$, and, for $a \in A$, the *spectrum of* a is the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda e - a \notin \mathrm{Inv}(A)\}.$$

It is an important fact at the heart of the celebrated Gel'fand–Mazur theorem (Theorem 4.19 of [1]) that the spectrum $\sigma(a)$ is non-empty for all $a \in A$. In the literature, this fact is sometimes only stated for the case of a Banach algebra, but it easily extends from this special case to our setting, since $\sigma(a)$ contains the spectrum of a with respect to the completion $\tilde{A}$; we refer to Corollary 4.18 of [1] and Theorem 5.7 of [4] for classical treatments and Theorem 1 of [21] for a recent alternative proof. While, in general, $\sigma(a)$ need not be closed or bounded, $\sigma(a)$ is closed for all $a \in A$ provided that every element of A has bounded spectrum; see Proposition 2.7 of [15] for a short proof. The *spectral radius* of $a \in A$ is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \in \mathbb{R} \cup \{\infty\}.$$

More may be said if A is a Banach algebra. For example, Allan refers to the fact that, in this case, the spectrum $\sigma(a)$ of any element $a \in A$ is a non-empty and compact subset of the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$ as the *fundamental theorem of Banach algebras* (Theorem 4.17 of [1]), and an analytic description of the spectral radius is provided by the *Beurling–Gel'fand spectral radius formula*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf\{\|a^n\|^{1/n} : n \in \mathbb{N}\}$$

(Theorem 4.23 of [1]). As indicated in the introduction, we will see that both results hold in the wider class of normed algebras for which the geometric series test is valid, and that each of them actually characterizes this class.

As usual, the *character space* Φ_A of the algebra A consists of all non-zero multiplicative linear functionals on A, that is, all algebra homomorphisms from A onto the field $\mathbb{C}$ of complex numbers. On the present level of generality, characters may well be discontinuous.

If A is a subalgebra of an algebra B with the same identity element, then A is said to be an *inverse-closed* subalgebra of B provided that, whenever $a \in A$ has a multiplicative inverse a^{-1} in B, then a^{-1} in fact belongs to A. The term *full* subalgebra is also used in the literature in place of inverse-closed subalgebra.

Without further ado, here is the first main result of the article. The characterizations provided in Theorem 1 cover not only a number of natural variations of the geometric series test, but also the Beurling–Gel'fand spectral radius formula and important properties of the invertible elements. We mention that unital topological algebras for which the invertible elements form an open set are known as Q-algebras. More generally, an arbitrary topological algebra is called a *Q-algebra* if the subset of quasi-invertible elements is open. The investigation of such algebras dates back to Kaplansky [13] even in the context of topological rings; see also Dales [5] and Mallios [14] for further information. Theorem 1 confirms, in particular, that if the geometric series test holds for some algebra norm on A, then it does for all equivalent algebra norms.

Theorem 1. *In a normed commutative complex algebra $(A, \|\cdot\|)$ with identity e, the following statements are equivalent:*

(1) *the geometric series test holds for A;*
(2) *if $a \in A$ satisfies $\|e - a\| < 1$, then $a \in Inv(A)$;*
(3) *if $a \in A$ and $b \in Inv(A)$ satisfy $\|b - a\| < 1/\|b^{-1}\|$, then $a \in Inv(A)$;*
(4) *$Inv(A)$ is an open subset of A;*
(5) *$Inv(A)$ has a non-empty interior;*
(6) *if $x, y_n \in A$ for all $n \in \mathbb{N}$ such that $xy_n \to e$, then $x \in Inv(A)$ and $y_n \to x^{-1}$;*

(7) *A is an inverse-closed subalgebra of $\tilde{A}$;*
(8) $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf\{\|a^n\|^{1/n} : n \in \mathbb{N}\}$ *for all* $a \in A$;
(9) $r(a) \le \|a\|$ *for all* $a \in A$;
(10) *there is a number* $c > 0$ *such that* $r(a) \le c\,\|a\|$ *for all* $a \in A$;
(11) *if* $a \in A$ *satisfies* $\|a\| < 1$, *then* $e - a^n \in Inv(A)$ *for some* $n \in \mathbb{N}$;
(12) *if* $a \in A$ *satisfies* $\|a\| < 1$, *then* $e - a^n \in Inv(A)$ *for all* $n \in \mathbb{N}$;
(13) *if* $a \in A$ *satisfies* $\|a\| < 1$, *then* $e + a + a^2 + \cdots + a^n \in Inv(A)$ *for all* $n \in \mathbb{N}$;
(14) *if* $a \in A$ *satisfies* $\|a^k\| < 1$ *for some* $k \in \mathbb{N}$, *then* $e - a \in Inv(A)$ *and* $(e-a)^{-1} = \sum_{n=0}^{\infty} a^n$;
(15) *if* $a \in A$ *has the property that* $\sum_{n=0}^{\infty} \|a^n\|$ *converges, then* $\sum_{n=0}^{\infty} a^n$ *converges.*

Proof. (1) $\implies$ (2) If $a \in A$ satisfies $\|e - a\| < 1$, then the hypothesis ensures that the sequence of partial sums

$$s_n = e + (e-a) + (e-a)^2 + \cdots + (e-a)^n$$

converges to an element $s \in A$ as $n \to \infty$. The usual telescoping behavior gives

$$as_n = (e - (e-a))\, s_n = s_n - (e-a)s_n = e - (e-a)^{n+1}$$

for all $n \in \mathbb{N}$. Because $\|(e-a)^n\| \le \|e-a\|^n < 1$ for all $n \in \mathbb{N}$, we see that the sequence of elements $(e-a)^n$ converges to 0 as $n \to \infty$. It follows that $as_n \to e$ as $n \to \infty$. On the other hand, $as_n \to as$ as $n \to \infty$. Thus $as = e$, and hence $a \in \mathrm{Inv}(A)$.

(2) $\implies$ (3) Since $\left\|e - b^{-1}a\right\| = \left\|b^{-1}(b-a)\right\| \le \left\|b^{-1}\right\| \|b-a\| < 1$, condition (2) yields $b^{-1}a \in \mathrm{Inv}(A)$. Hence a is invertible.

(3) $\implies$ (4) $\implies$ (5) is immediate.

(5) $\implies$ (6) Let a be an interior point of $\mathrm{Inv}(A)$, and choose an open neighborhood U of a such that $U \subseteq \mathrm{Inv}(A)$. Then, given $x, y_n \in A$ for which $xy_n \to e$ as $n \to \infty$, we obtain $xy_n a \to a$ as $n \to \infty$ and therefore $xy_n a \in U$ for all sufficiently large $n \in \mathbb{N}$. For such n we have $xy_n a(xy_n a)^{-1} = e$, which reveals that x is invertible. Finally, from $xy_n \to e$ as $n \to \infty$ we obtain $y_n \to x^{-1}$ as $n \to \infty$, as desired.

(6) $\implies$ (7) Let x be an element of A that has an inverse in its completion $\tilde{A}$; specifically, there exists an element $y \in \tilde{A}$ such that $xy = e$. Since A is dense in $\tilde{A}$, there is a sequence (y_n) of elements in A that converges to y as $n \to \infty$. Thus the sequence (xy_n) converges to $xy = e$, from which it follows by the hypothesis that $x \in \mathrm{Inv}(A)$.

(7) $\implies$ (8) Condition (7) ensures that, for arbitrary $a \in A$ and $\lambda \in \mathbb{C}$, the element $\lambda e - a$ is invertible in A precisely when it is invertible in $\tilde{A}$. This shows that the spectrum and the spectral radius of a are the same with respect to A and $\tilde{A}$. Hence (8) follows from the classical spectral radius formula for the Banach algebra $\tilde{A}$ (Theorem 4.23 of [1]).

(8) $\implies$ (9) $\implies$ (10) is obvious.

(10) $\implies$ (11) Given $a \in A$ with $\|a\| < 1$, let $n \in \mathbb{N}$ be so large that $\|a^n\| \leq \|a\|^n < 1/c$. For arbitrary $\lambda \in \sigma(a^n)$, condition (10) ensures that $|\lambda| \leq r(a^n) \leq c\|a^n\| < 1$ and hence $\lambda \neq 1$. Thus $1 \notin \sigma(a^n)$, so that $e - a^n \in \mathrm{Inv}(A)$.

(11) $\implies$ (12) Suppose that $a \in A$ satisfies $\|a\| < 1$, and let $n \in \mathbb{N}$. Then $\|a^n\| \leq \|a\|^n < 1$, so the hypothesis ensures the existence of a number $k \in \mathbb{N}$ such that $e - (a^n)^k = e - a^{kn} \in \mathrm{Inv}(A)$. By the familiar telescoping argument, it follows that

$$(e - a^n)(e + a^n + a^{2n} + \cdots + a^{(k-1)n})(e - a^{kn})^{-1} = (e - a^{kn})(e - a^{kn})^{-1} = e,$$

exposing the fact that $e - a^n \in \mathrm{Inv}(A)$.

(12) $\implies$ (13) If $a \in A$ satisfies $\|a\| < 1$, then we know that $e - a^n \in \mathrm{Inv}(A)$ for arbitrary $n \in \mathbb{N}$. Thus, again by telescoping,

$$e + a + a^2 + \cdots + a^n = (e - a^{n+1})(e - a)^{-1} \in \mathrm{Inv}(A),$$

since the product of two invertible elements is invertible.

(13) $\implies$ (14) Suppose that $a \in A$ satisfies $\|a^k\| < 1$ for some $k \in \mathbb{N}$. Then condition (13) applies to $-a^k$ and shows, in particular, that $e - a^k$ is invertible. Telescoping then entails that

$$(e - a)(e + a + a^2 + \cdots + a^{k-1})(e - a^k)^{-1} = (e - a^k)(e - a^k)^{-1} = e,$$

which confirms that $e - a$ is invertible. Moreover, choose any $\delta > 0$ for which $\left\|a^k\right\|^{1/k} < \delta < 1$, and observe that

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \inf\{\|a^n\|^{1/n} : n \in \mathbb{N}\} \leq \left\|a^k\right\|^{1/k},$$

by the Remark following Theorem 4.23 of [1] and also by Proposition 2.8 of [4]. Thus $\|a^n\| < \delta^n$ for almost all $n \in \mathbb{N}$ and therefore $a^n \to 0$ as $n \to \infty$. Again by telescoping, we conclude that

$$e + a + a^2 + \cdots + a^n = (e - a)^{-1}(e - a^{n+1}) \to (e - a)^{-1}$$

as $n \to \infty$, as desired.

(14) $\implies$ (15) $\implies$ (1) is immediate. $\square$

Some of the equivalences of the preceding result have a long history and remain valid in more general situations. For instance, in Lemma 2.1 of [23], Yood established that each of the conditions (8) and (9) characterizes the class of normed Q-algebras which need not be unital or commutative. Moreover, the equivalence of conditions (4) and (5) has a natural extension well beyond the framework of our unital and commutative normed algebras. Specifically, by Lemma 6.4 of [14], an arbitrary topological algebra is a Q-algebra precisely when the subset of quasi-invertible elements has a non-empty interior, and this happens if and only if the set of quasi-invertible elements is a neighborhood of zero. Furthermore, Tsertos [22] proved that an arbitrary topological algebra A is a Q-algebra exactly when there exists a balanced neighborhood U of zero in A such that $r(a) \leq g_U(a)$ for all $a \in A$, where g_U stands for the gauge functional of U.

Our list of conditions characterizing algebras in which the geometric series test holds resumes in Theorem 2. During this interlude, we review some more terminology. An element a in a commutative normed algebra A is a *topological divisor of zero* if there exists a sequence (u_n) of unit vectors in A for which $au_n \to 0$ as $n \to \infty$, and $a \in A$ is *permanently singular* if a fails to be invertible in any normed algebra that contains A as a subalgebra.

Also recall that, for arbitrary non-empty compact subsets K and L of $\mathbb{C}$, the *Hausdorff distance* of K and L is defined to be

$$\Delta(K, L) = \max\left\{\sup\left\{\operatorname{dist}(z, L) : z \in K\right\}, \sup\left\{\operatorname{dist}(w, K) : w \in L\right\}\right\},$$

where, as usual, $\operatorname{dist}(z, L) = \inf\left\{|z - w| : w \in L\right\}$. It is well known and easily seen that Δ is a metric on the collection of all non-empty compact subsets of $\mathbb{C}$.

Theorem 2. *In a normed commutative complex algebra $(A, \|\cdot\|)$ with identity e, the following statements are equivalent:*

(1) *the geometric series test holds for A;*

(2) *if (a_n) is a sequence in $Inv(A)$ that converges to some $a \in \partial Inv(A)$, then $\|a_n^{-1}\| \to \infty$;*

(3) *every $a \in \partial\, Inv(A)$ is a topological divisor of zero;*

(4) *every $a \in \partial\, Inv(A)$ is permanently singular;*

(5) *every maximal ideal of A is closed;*

(6) *every $\varphi \in \Phi_A$ is continuous with $\|\varphi\| = 1$, and $\sigma(a) = \{\varphi(a) : \varphi \in \Phi_A\}$ for all $a \in A$;*

(7) *the mapping that assigns to each $a \in A$ its spectrum $\sigma(a)$ is upper semi-continuous, i.e., for every $a \in A$ and every open neighborhood V of $\sigma(a)$ in $\mathbb{C}$, there exists some open neighborhood U of a in A such that $\sigma(x) \subseteq V$ for all $x \in U$;*

(8) *$\sigma(a)$ is compact and non-empty for each $a \in A$, and the mapping that assigns to each $a \in A$ its spectrum $\sigma(a)$ is uniformly continuous with respect to the Hausdorff metric; in fact $\Delta(\sigma(a), \sigma(b)) \leq \|a - b\|$ for all $a, b \in A$;*

(9) *the fundamental theorem of Banach algebra theory holds for A, i.e., $\sigma(a)$ is a non-empty compact subset of the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$ for all $a \in A$.*

Proof. (1) $\Longrightarrow$ (2) Consider a sequence of elements $a_n \in \mathrm{Inv}(A)$ that converges to some limit $a \in \partial\mathrm{Inv}(A)$. Since we know from Theorem 1 that $\mathrm{Inv}(A)$ is open, we conclude that a fails to be invertible. Condition (3) of Theorem 1 thus implies that

$\|a_n - a\| \geq 1/\left\|a_n^{-1}\right\|$ for all $n \in \mathbb{N}$. Because $a_n \to a$ as $n \to \infty$, we conclude that $\left\|a_n^{-1}\right\| \to \infty$ as $n \to \infty$.

(2) $\implies$ (3) Given an arbitrary $a \in \partial\,\mathrm{Inv}(A)$, we choose elements $a_n \in \mathrm{Inv}(A)$ such that $a_n \to a$ as $n \to \infty$ and define the unit vectors $u_n = a_n^{-1}/\left\|a_n^{-1}\right\|$ for all $n \in \mathbb{N}$. Then condition (2) entails that $au_n = (a - a_n)u_n + e/\left\|a_n^{-1}\right\| \to 0$ as $n \to \infty$, as desired.

(3) $\implies$ (4) It is an elementary general fact that topological divisors of zero in any unital normed algebra are permanently singular (Proposition 2.16 of [4]).

(4) $\implies$ (1) Condition (4) implies that every boundary point of $\mathrm{Inv}(A)$ fails to be invertible. This shows that $\mathrm{Inv}(A)$ is open and hence implies, by Theorem 1, that (1) holds.

(1) $\implies$ (5) From Theorem 1 we know that condition (1) means precisely that the open ball U of radius 1 centered at e is included in $\mathrm{Inv}(A)$. Consequently, if M is a maximal ideal in A, then $M \cap U = \emptyset$ and hence $\overline{M} \cap U = \emptyset$. In particular, we have $\overline{M} \neq A$ and therefore $M = \overline{M}$ by maximality. Thus (5) holds.

(5) $\implies$ (6) It is an established result from functional analysis (Theorem 3.2 of [3]) that the continuity of any linear functional φ on A is equivalent to $\ker\varphi$ being a closed subset of A. Now, for every $\varphi \in \Phi_A$, it is easily seen that $\ker\varphi$ is an ideal of codimension 1 in A, hence a maximal ideal of A, and therefore closed by hypothesis. Thus φ is continuous and consequently extends to a character of the same norm on the completion $\tilde{A}$. But it is well known that characters on unital Banach algebras have norm 1 (Theorem 4.43 of [1]). Thus $\|\varphi\| = 1$, as desired.

Moreover, if an element $a \in A$ is invertible, then it is obvious that $\varphi(a) \neq 0$ for all $\varphi \in \Phi_A$. Conversely, if a fails to be invertible, then the ideal aA does not contain e and hence is included in some maximal ideal M of A. Since, by hypothesis, M is closed, it follows that the quotient algebra A/M is a normed division algebra over $\mathbb{C}$ and thus, by the Gel'fand–Mazur theorem, isomorphic to $\mathbb{C}$ (Theorem 4.19 of [1]). We may therefore view the quotient mapping from A onto A/M as a character and thus obtain some $\varphi \in \Phi_A$ for which $\varphi(a) = 0$. Consequently, a fails to be invertible precisely when $\varphi(a) = 0$ for some $\varphi \in \Phi_A$. The stipulated formula for $\sigma(a)$ is now immediate.

(6) $\implies$ (1) For arbitrary $a \in A$ and $\lambda \in \sigma(a)$, there exists some $\varphi \in \Phi_A$ for which $\lambda = \varphi(a)$ and therefore $|\lambda| = |\varphi(a)| \leq \|\varphi\|\,\|a\| = \|a\|$. Thus $r(a) \leq \|a\|$ for all $a \in A$, which, by Theorem 1, is equivalent to condition (1).

(1) $\implies$ (8) Again by Theorem 1, condition (1) ensures that $\mathrm{Inv}(A)$ is open and that $\sigma(a)$ is bounded for every $a \in A$. Moreover, $\sigma(a)$ is closed and therefore compact, since $\mathrm{Inv}(A)$ and hence $\mathbb{C} \setminus \sigma(a)$ are open for all $a \in A$. Finally, since we already know that conditions (1) and (6) of the present result are equivalent, for arbitrary $a, b \in A$ the formula for the spectrum provided in (6) yields

$$\sigma(a) = \sigma(b + a - b) = \{\varphi(b) + \varphi(a - b) : \varphi \in \Phi_A\} \subseteq \sigma(b) + \sigma(a - b).$$

Because $\sigma(a-b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|a-b\|\}$ by condition (9) of Theorem 1, we conclude that

$$\sigma(a) \subseteq \sigma(b) + \{\lambda \in \mathbb{C} : |\lambda| \leq \|a-b\|\} .$$

Consequently, for each $\lambda \in \sigma(a)$, there exists some $\mu \in \sigma(b)$ such that $|\lambda - \mu| \leq \|a-b\|$. This entails that $\operatorname{dist}(\lambda, \sigma(b)) \leq \|a-b\|$ for all $\lambda \in \sigma(a)$ and hence, by symmetry, also that $\operatorname{dist}(\mu, \sigma(a)) \leq \|a-b\|$ for all $\mu \in \sigma(b)$. Thus $\Delta(\sigma(a), \sigma(b)) \leq \|a-b\|$, as claimed.

(8) $\Longrightarrow$ (7) If the set-valued spectrum mapping is continuous at the element $a \in A$ with respect to the Hausdorff metric, then, in particular, we have

$$\sup\{\operatorname{dist}(z, \sigma(a)) : z \in \sigma(x)\} \to 0$$

as $x \to a$. It is straightforward to see that this condition is equivalent to the upper semi-continuity of the spectrum mapping at a.

(7) $\Longrightarrow$ (1) Evidently, the upper semi-continuity of the set-valued spectrum mapping at the element $0 \in A$ means precisely that, for each $\varepsilon > 0$, there exists some $\delta > 0$ with the property that $|\mu| < \varepsilon$ for all $\mu \in \sigma(x)$ and $x \in A$ with $\|x\| < \delta$. Take any $\delta > 0$ such that the preceding condition holds for the choice $\varepsilon = 1$, and consider a non-zero element $a \in A$ and an arbitrary real number t with $0 < t < 1$. Then the element $x = t\delta a / \|a\| \in A$ satisfies $\|x\| < \delta$. Moreover, for every $\lambda \in \sigma(a)$, we infer from $\lambda e - a \notin \operatorname{Inv}(A)$ that $(t\delta\lambda / \|a\|)e - x \notin \operatorname{Inv}(A)$ and hence $t\delta\lambda / \|a\| \in \sigma(x)$. From the choice of δ and $\varepsilon = 1$ we conclude that $t\delta\,|\lambda| / \|a\| < 1$ whenever $0 < t < 1$ and hence $\delta\,|\lambda| / \|a\| \leq 1$. Thus $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq (1/\delta)\,\|a\|\}$ for all $a \in A$, which shows that condition (10) of Theorem 1 holds with the choice $c = 1/\delta$. Consequently, (8) implies (1) by Theorem 1.

(8) $\Longrightarrow$ (9) Since the identity $r(a) = \Delta(\sigma(a), \{0\})$ holds for all $a \in A$, the choice $b = 0$ in the estimate provided in (8) leads to $r(a) = \Delta(\sigma(a), \{0\}) \leq \|a - 0\| = \|a\|$ for arbitrary $a \in A$.

(9) $\Longrightarrow$ (1) Since (9) entails that $r(a) \leq \|a\|$ for all $a \in A$, the desired implication follows from Theorem 1. □

We point out that Theorem 2 exposes an error in Exercise 4.5 of [1] in which the reader is asked to prove that, in an arbitrary normed algebra A with identity, every element in the boundary of $\operatorname{Inv}(A)$ is a topological divisor of zero. As we now know, this property actually characterizes the special class of normed algebras under consideration.

While many of the preceding characterizations may be extended to the case of non-commutative normed algebras, we note that, for a non-commutative Banach algebra, the spectrum need not be continuous with respect to the Hausdorff metric, as shown by the famous example, due to Kaplansky, of a sequence of nilpotent Hilbert space operators that converges to an operator whose spectrum contains a non-trivial disc centered at the origin (Example 2.2.16 of [16]).

As a consequence of the preceding result, we obtain that all the essential features of the famous *Gel'fand representation theorem* (Theorem 4.59 of [1] or Theorem 3.1.20 of [19]) remain valid in the setting of a normed commutative complex algebra A with identity for which the geometric series test holds. Specifically, since we know from Theorem 2 that $\|\varphi\| = 1$ for all $\varphi \in \Phi_A$, it follows from the Banach–Alaoglu theorem that Φ_A is compact in the weak-star topology inherited from the topological dual space A^*. Moreover, the mapping that assigns to every element $a \in A$ its Gel'fand transform $\hat{a}$ given by $\hat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \Phi_A$ is a unital continuous algebra homomorphism from A into the Banach algebra $C(\Phi_A)$ of all continuous complex-valued function on Φ_A, where $C(\Phi_A)$ is, as usual, endowed with the supremum norm $\|\cdot\|_\infty$. Finally, by condition (6) of Theorem 2, an element $a \in A$ is invertible in A precisely when $\hat{a}$ is invertible in $C(\Phi_A)$, the spectrum $\sigma(a)$ coincides with the range of $\hat{a}$, and the identity $r(a) = \|\hat{a}\|_\infty$ holds for all $a \in A$.

In the setting of arbitrary topological algebras, a natural counterpart of condition (5) of Theorem 2 has been of interest for quite some time. According to Haralampidou [8], a topological algebra is named a *Q'-algebra* provided that all of its maximal regular left or right ideals are closed. By Theorem 6.1 of [14], every Q-algebra is a Q'-algebra, while, by Example 3.6.6 of [9], the converse is not true in general.

We close this section with another application of Theorem 1.

Corollary 3. *If the geometric series test holds in a non-unital commutative normed algebra, then the test also holds in its unitization.*

Proof. Let $(A, \|\cdot\|)$ be a non-unital commutative normed algebra over $\mathbb{C}$ for which the geometric series test holds, and let $A \oplus \mathbb{C}e$ denote the unitization of A, endowed with the canonical algebra operations and the norm given by $\|a + \lambda e\|_e = \|a\| + |\lambda|$ for all $a \in A$ and $\lambda \in \mathbb{C}$. To see that condition (2) of Theorem 1 is fulfilled in $A \oplus \mathbb{C}e$, suppose that the element $a + \lambda e \in A \oplus \mathbb{C}e$ satisfies $\|e - (a + \lambda e)\|_e < 1$ and thus $\|a\| + |1 - \lambda| < 1$. It follows that $\lambda \neq 0$. Also, $\|a\| + |1 - \lambda| < 1 \leq |\lambda| + |1 - \lambda|$ and hence $\|a\| < |\lambda|$. Because the geometric series test holds in A, the sequence of partial sums

$$s_n = (-a/\lambda) + (-a/\lambda)^2 + \cdots + (-a/\lambda)^n$$

converges to an element $s \in A$ as $n \to \infty$. Moreover, the sequence of elements

$$\begin{aligned}
(a/\lambda)s_n + (a/\lambda) + s_n &= (a/\lambda)\sum_{k=1}^{n}(-a/\lambda)^k + \sum_{k=2}^{n}(-a/\lambda)^k \\
&= -\sum_{k=1}^{n}(-a/\lambda)^{k+1} + \sum_{k=2}^{n}(-a/\lambda)^k \\
&= -\sum_{k=2}^{n+1}(-a/\lambda)^k + \sum_{k=2}^{n}(-a/\lambda)^k \\
&= -(-a/\lambda)^{n+1}
\end{aligned}$$

converges to 0 as $n \to \infty$. This enables us to conclude that $(a/\lambda)s + (a/\lambda) + s = 0$ and therefore

$$(a + \lambda e)\left(\frac{1}{\lambda}s + \frac{1}{\lambda}e\right) = \left(\frac{a}{\lambda}s + \frac{a}{\lambda} + s\right) + e = e.$$

This reveals that $a + \lambda e \in \mathrm{Inv}(A \oplus \mathbb{C}e)$, to complete the proof by Theorem 1. $\square$

3 Examples

3.1. The familiar set $C([a, b])$ of all continuous complex-valued functions on the compact interval $[a, b]$ is rendered a commutative unital algebra when endowed with pointwise algebraic operations. The multiplicative identity element is, of course, the constant function 1, so a function $f \in C([a, b])$ belongs to $\mathrm{Inv}(C([a, b]))$ precisely when $0 \notin \mathrm{range}(f)$, which ensures that $\sigma(f) = \mathrm{range}(f)$. It is well known that $C([a, b])$ is a Banach algebra with respect to the supremum norm $\|\cdot\|_\infty$, and so is, for arbitrary $n \in \mathbb{N}$, the algebra $C^n([a, b])$ of all n-times continuously differentiable complex-valued functions on $[a, b]$ when endowed with the norm $\|\cdot\|_n$ given by

$$\|f\|_n = \sum_{k=0}^{n} \frac{1}{k!} \left\| f^{(k)} \right\|_\infty \qquad \text{for all } f \in C^n([a, b]).$$

Clearly, $C^n([a, b])$ fails to be complete with respect to $\|\cdot\|_\infty$ or $\|\cdot\|_m$ whenever $m < n$. The geometric series test is valid, however, when $C^n([a, b])$ is equipped with *any* algebra norm that dominates the supremum norm. This follows, for example, from condition (9) of Theorem 1, since $\sigma(f) = \mathrm{range}(f)$ for any $f \in C^n([a, b])$.

3.2. Even worse, the subalgebra $C^\infty([a, b])$ of $C([a, b])$ comprised of all infinitely differentiable functions supports a non-zero derivation, namely the operator of differentiation. Hence, as alluded to in the introduction, a classical result due to Johnson ensures that $C^\infty([a, b])$ can never be endowed with a complete algebra norm (Corollary 18.22 of [4]). But, as above, condition (9) of Theorem 1 holds with respect to *any* algebra norm on $C^\infty([a, b])$ that dominates the supremum norm. In particular, the algebra $(C^\infty([a, b]), \|\cdot\|_\infty)$ is a member of the class under consideration, as is $(C^\infty([a, b]), \|\cdot\|_n)$ for arbitrary $n \in \mathbb{N}$. The norms $\|\cdot\|_\infty$ and $\|\cdot\|_n$ are not compatible, however, exposing the fact that semi-simple normed algebras for which the geometric series test holds need not carry a unique algebra norm topology. This represents a point of departure from Banach algebra theory, where a classical result due to Shilov ensures the uniqueness of the complete algebra norm topology for a semi-simple commutative Banach algebra (Corollary 2.3.4 of [5]).

In the same vein, we note that the identity mapping from $(C^1([a, b]), \|\cdot\|_\infty)$ onto $(C^1([a, b]), \|\cdot\|_1)$ provides an example of a discontinuous homomorphism with closed graph from a semi-simple normed algebra for which the geometric series test holds onto a semi-simple Banach algebra. Thus, despite the continuity of all multiplicative linear functionals on normed algebras for which the geometric series test holds, there is no hope for any deeper automatic continuity results in this context.

3.3. For a different class of examples, let Ω be a locally compact Hausdorff space, and let $C_*(\Omega)$ consist of all continuous functions $f : \Omega \to \mathbb{C}$ that are constant outside of some compact subset of Ω depending on f. With respect to the usual pointwise operations, $C_*(\Omega)$ is a unital subalgebra of the commutative algebra $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω. Clearly, $C_*(\Omega)$ may be viewed as the unitization of the classical algebra $C_{00}(\Omega)$ of all continuous complex-valued functions on Ω with compact support. With respect to the supremum norm, $C_b(\Omega)$ is a Banach algebra, while $C_*(\Omega)$ fails to be complete, for instance, when Ω is an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and, of course, also in many other cases. However, the geometric series test always holds for $C_*(\Omega)$, since it turns out that condition (7) of Theorem 1 is satisfied. In fact, $C_*(\Omega)$ is inverse-closed in the Banach algebra $C_b(\Omega)$, since, if a continuous function $f : \Omega \to \mathbb{C}$ is constant outside of some compact set and invertible in $C_b(\Omega)$, then f never vanishes on Ω and $1/f$ is constant outside of the same compact set. Alternatively, it is straightforward to see that the geometric series test holds in $C_{00}(\Omega)$ and therefore, by Corollary 3, also in $C_*(\Omega)$.

Of course, similar examples exist in the context of algebras of differentiable functions. On the other hand, specializing to the case $\Omega = \mathbb{N}$, we conclude that the geometric series test holds for the incomplete normed algebra of all eventually constant sequences of complex numbers with respect to the supremum norm. The same result may be established for this sequence algebra with respect to the bounded variation norm, but we leave the details to the interested reader.

3.4. We next turn to the commutative unital algebra $\mathrm{Pol}(\mathbb{C})$ of all complex polynomials, considered here as complex-valued functions on $\mathbb{C}$ with the usual pointwise operations. Since $\mathrm{Pol}(\mathbb{C})$ as a vector space has a countable Hamel basis, a well known application of the Baire category theorem ensures that $\mathrm{Pol}(\mathbb{C})$ cannot be complete with respect to any vector space norm (see Remark (2) of Section I.1 of [4]). Even more may be said about algebra norms on $\mathrm{Pol}(\mathbb{C})$. Indeed, since only non-zero constant polynomials have multiplicative inverses in $\mathrm{Pol}(\mathbb{C})$, the spectrum of any non-constant polynomial in this algebra is the entire complex plane. Hence both of our main results, Theorem 1 and 2, immediately reveal that the geometric series test does not hold for any algebra norm on $\mathrm{Pol}(\mathbb{C})$. This has an interesting consequence regarding the automatic continuity of characters on this algebra.

For this we first observe that every $\lambda \in \mathbb{C}$ induces a multiplicative linear functional φ_λ on $\mathrm{Pol}(\mathbb{C})$, namely the point evaluation at λ given by $\varphi_\lambda(p) = p(\lambda)$ for arbitrary $p \in \mathrm{Pol}(\mathbb{C})$. Conversely, it is easily verified that every $\varphi \in \Phi_{\mathrm{Pol}(\mathbb{C})}$ is of the form $\varphi = \varphi_\lambda$ with the choice $\lambda = \varphi(z)$. Thus $\Phi_{\mathrm{Pol}(\mathbb{C})}$ may be identified with $\mathbb{C}$, and we obtain the identity $\sigma(p) = \{\varphi(p) : \varphi \in \mathrm{Pol}(\mathbb{C})\}$ for all $p \in \mathrm{Pol}(\mathbb{C})$. Consequently, condition (6) of Theorem 2 leads to the conclusion that, for every algebra norm on $\mathrm{Pol}(\mathbb{C})$, there exists at least one discontinuous point evaluation.

To exemplify, we first endow $\mathrm{Pol}(\mathbb{C})$ with the norm $\|\cdot\|_1$ defined by

$$\|p\|_1 = |a_0| + |a_1| + |a_2| + \cdots + |a_n|\,,$$

where $p \in \mathrm{Pol}(\mathbb{C})$ is given by $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ for all $z \in \mathbb{C}$. It is a routine exercise to check that $\|\cdot\|_1$ does, indeed, provide an algebra norm for $\mathrm{Pol}(\mathbb{C})$; in fact, $(\mathrm{Pol}(\mathbb{C}), \|\cdot\|_1)$ may be construed as a dense subalgebra of the classical *discrete semi-group algebra* $\ell^1(\mathbb{N}_0)$; see Example 4.4 of [1] or Example 1.23 of [4]. For arbitrary $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, we obtain $|\varphi_\lambda(p)| = |p(\lambda)| \leq \|p\|_1$ for all $p \in \mathrm{Pol}(\mathbb{C})$, while, in the case $|\lambda| > 1$, we have $|\varphi_\lambda(z^n)| = |\lambda|^n \to \infty$ as $n \to \infty$. This shows that the continuous characters for $(\mathrm{Pol}(\mathbb{C}), \|\cdot\|_1)$ are precisely the point evaluations induced by points of the closed unit disc in $\mathbb{C}$.

We may also equip $\mathrm{Pol}(\mathbb{C})$ with the supremum norm $\|\cdot\|_K$ corresponding to an arbitrary infinite compact subset K of $\mathbb{C}$. Then we know that the geometric series test fails in $\mathrm{Pol}(\mathbb{C})$ with respect to this norm. Moreover, the continuous characters for $\|\cdot\|_K$ on $\mathrm{Pol}(\mathbb{C})$ are precisely the point evaluations corresponding to the points of the polynomially convex hull $\widehat{K}$ of K, i.e., the complement of the unbounded component of $\mathbb{C} \setminus K$, since the corresponding result holds for the completion of $\mathrm{Pol}(\mathbb{C})$ with respect to $\|\cdot\|_K$; see Proposition 4.3.12 of [5]. The reader is invited to check how the other conditions of Theorems 1 and 2 are violated in this context.

3.5. Finally, the set $(\mathrm{Rat}(K), \|\cdot\|_K)$ of all rational functions with poles outside the infinite compact subset K of the complex plane, endowed with pointwise operations and the supremum norm over K, is another example of an incomplete normed algebra, but one for which the geometric series test holds.

Indeed, because evaluations at the points of K provide characters on $\mathrm{Rat}(K)$, it is clear that the algebra $\mathrm{Rat}(K)$ is semi-simple, while the operator of complex differentiation is a non-zero derivation on $\mathrm{Rat}(K)$. Thus, as above, a theorem due to Johnson ensures that $\mathrm{Rat}(K)$ cannot be equipped with *any* complete algebra norm; see Theorem 18.21 of [4]. In particular, $(\mathrm{Rat}(K), \|\cdot\|_K)$ fails to be complete. To illustrate how much bigger its completion may be, we point to the Hartogs–Rosenthal theorem which confirms that $\mathrm{Rat}(K)$ is dense in $C(K)$ with respect to $\|\cdot\|_K$ whenever K has zero planar measure (Corollary II.8.4 of [7]).

On the other hand, if $f \in \mathrm{Rat}(K)$ satisfies $\|f\|_K < 1$, then the geometric series $1 + f + f^2 + \cdots$ converges uniformly on K to the rational function $1/(1-f)$. The algebra $(\mathrm{Rat}(K), \|\cdot\|_K)$ is thus an incomplete extension of $(\mathrm{Pol}(K), \|\cdot\|_K)$ in which the geometric series test holds. From Theorem 2 we know that all characters on $\mathrm{Rat}(K)$ are continuous. It follows that these characters are precisely the point evaluations induced by points of K, since, by Proposition 4.3.12 of [5], this is true for the characters on the completion of $\mathrm{Rat}(K)$ with respect to $\|\cdot\|_K$.

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