

## ON TEA, DONUTS AND NON-COMMUTATIVE GEOMETRY

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**Abstract.** As many will agree, it feels good to complement a cup of tea by a donut or two. This sweet relationship is also a guiding principle of non-commutative geometry known as Serre Theorem. We explain the algebra behind this theorem and prove that elliptic curves are complementary to the so-called non-commutative tori.

### 1 Introduction

*“...in divinity opposites are always reconciled.”* — Walter M. Miller Jr.

An algebraic curve  $C$  is the set of points on the affine plane whose coordinates are zeros of a polynomial in two variables with real or complex coefficients, like the one shown in Figure 1.

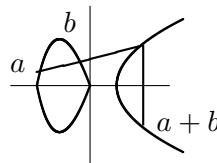


Figure 1: Affine cubic  $y^2 = x(x - 1)(x + 1)$  with addition law.

The mathematical theory of algebraic curves emerged from the concept of an analytic surface created by Georg Friedrich Bernhard Riemann (1826-1866). These are called Riemann surfaces and are nothing but algebraic curves over the field  $\mathbb{C}$ . The proper mathematical language – algebraic geometry – is the result of an inspiration and the hard work of Julius Wilhelm Richard Dedekind (1831-1916),

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Heinrich Martin Weber (1842-1913), David Hilbert (1862-1943), Wolfgang Krull (1899-1971), Oscar Zariski (1899-1986), Alexander Grothendieck (1928-2014) and Jean-Pierre Serre (born 1926) among others.

The  $\mathbb{C}$ -valued polynomial functions defined on the curve  $C$  can be added and multiplied pointwise. This makes the totality of such functions into a ring, say,  $A$ . Since multiplication of complex numbers is commutative,  $A$  is a *commutative ring*. As an algebraic object, it is *dual* to the geometric object  $C$ . In particular,  $C$  can be recovered up to isomorphism from  $A$  and vice versa. In simple terms, this is the duality between systems of algebraic equations and their solutions.

Instead of the ring  $A$  itself, it is often useful to work with *modules* over  $A$ ; the latter is a powerful tool of modern algebra synonymous with representations of the ring  $A$ . Moreover, according to Maurice Auslander (1926-1994), one should study morphisms between modules rather than modules themselves [1]. Here we are talking about the *category of modules*, i.e. the collection of all modules and all morphisms between them. (In other words, instead of the individual modules over  $A$  we shall study their “sociology”, i.e. relationships between modules over  $A$ .) We shall write  $\mathbf{Mod}(A)$  to denote a category of finitely generated graded modules over  $A$  factored by a category of modules of finite length. In geometric terms, the rôle of modules is played by the so-called *sheaves* on the curve  $C$ . Likewise, we are looking at the category  $\mathbf{Coh}(C)$  of *coherent* sheaves on  $C$ , i.e. a class of sheaves having particularly nice properties related to the geometry of  $C$ . Now the duality between curve  $C$  and algebra  $A$  is a special case of the famous *Serre Theorem* [6] saying that the two categories are equivalent:

$$\mathbf{Coh}(C) \cong \mathbf{Mod}(A). \quad (1.1)$$

The careful reader may notice that  $\mathbf{Mod}(A)$  is well-defined for all rings – commutative or *not*. For instance, consider the non-commutative ring  $M_2(A)$  of two-by-two matrices over  $A$ . It is a trivia to verify that  $\mathbf{Mod}(M_2(A)) \cong \mathbf{Mod}(A)$  are equivalent categories, yet the ring  $M_2(A)$  is not isomorphic to  $A$ . (Notice that  $A$  is the center of the ring  $M_2(A)$ .) Rings whose module categories are equivalent are said to be *Morita equivalent*.

In 1982 Evgeny Konstantinovich Sklyanin (born 1955) was busy with a difficult problem of quantum physics [8], when he discovered a remarkable non-commutative ring  $S(\alpha, \beta, \gamma)$  with the following property, see Section 4. If one calculates the right hand side of (1.1) for the  $S(\alpha, \beta, \gamma)$ , then it will be equivalent to the left hand side of (1.1) calculated for an *elliptic curve*; by such we understand a subset  $\mathcal{E}$  of complex projective plane  $\mathbb{C}P^2$  given by the *Legendre* cubic:

$$y^2z = x(x-z)(x-\lambda z), \quad \lambda \in \mathbb{C} - \{0; 1\}. \quad (1.2)$$

Moreover, the ring  $S(\alpha, \beta, \gamma)$  gives rise to an automorphism  $\sigma : \mathcal{E} \rightarrow \mathcal{E}$  [8]. Sklyanin’s example hints, that at least some parts of algebraic geometry can be recast in terms of non-commutative algebra; but what such a generalization is good for?

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Long time ago, Yuri Ivanovich Manin (born 1937) and his student Andrei Borisovich Vervekin (born 1964) suggested that in the future all algebraic geometry will be non-commutative; the classical case corresponds to an “abelianization” of the latter [10]. No strict description of such a process for rings exists, but for groups one can think of the abelianized fundamental group of a knot or a manifold; the latter is still a valuable topological invariant – the first homology – yet it cannot distinguish between knots or manifolds nearly as good as the fundamental group does. This simple observation points to advantages of non-commutative geometry.

In this note we demonstrate an equivalence between the category of elliptic curves and the category of so-called *non-commutative tori*; the latter is closely related (but not identical) to the category of Sklyanin algebras, see Section 4. We explore an application to the rank problem for elliptic curves, see Section 5.

## 2 Elliptic curves on breakfast

An imaginative reader may argue that the affine cubic in Figure 1 reminds a part of a tea pot; what will be the donut in this case? (The situation is sketched in Figure 2 by the artistically impaired author.) We show in Section 4 that the algebraic dual of  $\mathcal{E}$  (the “donut”) corresponds to a *non-commutative torus* to be defined below. But first, let us recall some standard facts.

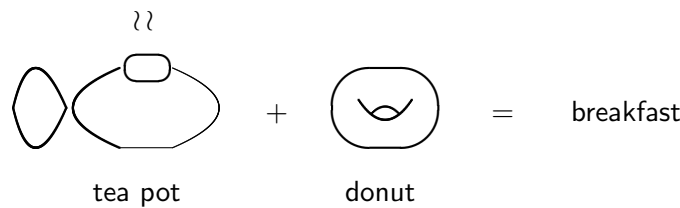


Figure 2: Mathematical breakfast.

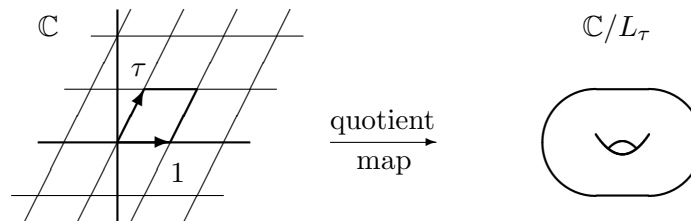
An *elliptic curve*  $\mathcal{E}$  is a subset of  $\mathbb{C}P^2$  given by equation (1.2). There exist two more equivalent definitions of  $\mathcal{E}$ .

(1) The curve  $\mathcal{E}$  can be defined as the intersection of two quadric surfaces in complex projective space  $\mathbb{C}P^3$ :

$$\begin{cases} u^2 + v^2 + w^2 + z^2 = 0, \\ Av^2 + Bw^2 + z^2 = 0, \end{cases} \tag{2.1}$$

where  $A$  and  $B$  are some complex constants and  $(u, v, w, z) \in \mathbb{C}P^3$ . (Unlike the Legendre form (1.2), this representation of  $\mathcal{E}$  is *not* unique.) The system of equations (2.1) is called the *Jacobi form* of  $\mathcal{E}$ .

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Figure 3: Complex torus  $\mathbb{C}/L_\tau$ .

(2) The analytic form of  $\mathcal{E}$  is given by a *complex torus*, i.e. a surface of genus 1 whose local charts are patched together by a complex analytic map. The latter is simply the quotient space  $\mathbb{C}/L_\tau$ , where  $L_\tau := \mathbb{Z} + \mathbb{Z}\tau$  is a lattice and  $\tau \in \mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$  is called the complex modulus, see Figure 3.

There exists a one-to-one correspondence  $\mathcal{E} \rightarrow \mathbb{C}/L_\tau$  given by a meromorphic  $\wp$ -function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L_\tau - \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right). \quad (2.2)$$

The lattice  $L_\tau$  is connected to the so-called *Weierstrass normal form* of  $\mathcal{E}$  in  $\mathbb{C}P^2$ :

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3, \quad (2.3)$$

where  $g_2 = 60 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^4}$  and  $g_3 = 140 \sum_{\omega \in L_\tau - \{0\}} \frac{1}{\omega^6}$ . In turn, the Weierstrass normal form (2.3) is linked to the Legendre cubic (1.2) by the formulas  $g_2 = \frac{3\sqrt{4}}{(\lambda^2 - \lambda + 1)}$  and  $g_3 = \frac{1}{27}(\lambda + 1)(2\lambda^2 - 5\lambda + 2)$ . Thus, it is possible to recover  $\mathcal{E}$  – up to an isomorphism – from  $L_\tau$ ; we shall denote by  $\mathcal{E}_\tau$  the elliptic curve corresponding to the complex torus of modulus  $\tau$ .

**Remark 1.** *It is easy to see, that each automorphism  $\sigma : \mathcal{E}_\tau \rightarrow \mathcal{E}_\tau$  is given by a shift of lattice  $L_\tau$  in the complex plane  $\mathbb{C}$  and vice versa; thus points  $p \in \mathbb{C}/L_\tau$  are bijective with the automorphisms of  $\mathcal{E}_\tau$ . In particular, points of finite order on  $\mathcal{E}_\tau$  correspond to finite order automorphisms of  $\mathcal{E}_\tau$ .*

Notice that  $L_\tau$  can be written in a new basis  $\{\omega_1, \omega_2\}$ , where  $\omega_2 = a\tau + b$  and  $\omega_1 = c\tau + d$  for some  $a, b, c, d \in \mathbb{Z}$ , such that  $ad - bc = 1$ . Normalization  $\tau := \frac{\omega_2}{\omega_1}$  of basis  $\{\omega_1, \omega_2\}$  to a standard basis  $\{1, \tau'\}$  implies that  $\mathcal{E}_\tau$  and  $\mathcal{E}_{\tau'}$  are isomorphic, if and only if:

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \quad (2.4)$$

An elliptic curve  $\mathcal{E}_\tau$  is said to have *complex multiplication* (CM) if there is a complex number  $\alpha \notin \mathbb{Z}$  such that  $\alpha L_\tau \subseteq L_\tau$ . Applying multiplication by  $\alpha$  to

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$1 \in L_\tau$ , we have  $\alpha \in L_\tau$ , i.e.,  $\alpha = a + b\tau$  for some  $a, b \in \mathbb{Z}$ . Applying multiplication by  $\alpha$  to  $\tau \in L_\tau$ , we have  $(a + b\tau)\tau = c + d\tau$  for some  $c, d \in \mathbb{Z}$ . Since  $\tau$  is not a real number and the ratio  $(c + d\tau)(a + b\tau)^{-1}$  is uniquely determined,  $c + d\tau$  cannot be a real multiple of  $a + b\tau$ . Thus  $a + b\tau$  and  $c + d\tau$ , viewed as vectors of the real vectors space  $\mathbb{C}$ , are linearly independent, and therefore  $ad - bc \neq 0$ . Rewriting the above equality as a quadratic equation,  $b\tau^2 + (a - d)\tau - c = 0$ , we have  $\tau = \frac{(d-a) \pm \sqrt{-D}}{2b}$ , where  $D = -(a - d)^2 - 4ac$ . Since  $\tau$  is not a real number, we must have  $D > 0$ , which tells us that  $\tau$  (and hence  $\alpha = a + b\tau$ ) belongs to the imaginary quadratic field  $\mathbb{Q}(\sqrt{-D})$ . In particular, not every elliptic curve admits complex multiplication. Conversely, if  $\tau \in \mathbb{Q}(\sqrt{-D})$  then  $\mathcal{E}_\tau$  has CM; thus one gets a necessary and sufficient condition for complex multiplication.

No surprise, that complex multiplication is used to construct extensions of the imaginary quadratic fields with abelian Galois group, thus providing a powerful link between complex analysis and number theory [7]. Perhaps that is why David Hilbert counted complex multiplication as not only the most beautiful part of mathematics but also of entire science.

### 3 A non-commutative donut

The product of complex numbers  $a$  and  $b$  is commutative, i.e.  $ab = ba$ . Such a property cannot be taken for granted, since more than often  $ab \neq ba$ . (For instance, putting on a sock  $a$  then a rollerblade shoe  $b$  feels better than the other way around.) Below we consider a non-commutative algebra related to elliptic curves.

A  $C^*$ -algebra  $\mathcal{A}$  is an algebra over  $\mathbb{C}$  with a norm  $a \mapsto \|a\|$  and an involution  $a \mapsto a^*$ ,  $a \in \mathcal{A}$ , such that  $\mathcal{A}$  is complete with respect to the norm, and such that  $\|ab\| \leq \|a\| \|b\|$  and  $\|a^*a\| = \|a\|^2$  for every  $a, b \in \mathcal{A}$ . We assume that our algebra  $\mathcal{A}$  has a unit. Such algebras provide an axiomatic way to describe rings of bounded linear operators acting on a Hilbert space. These rings appear in the works of John von Neumann (1903-1957) and his former student Francis Joseph Murray (1911-1996) on quantum physics. Axioms for  $C^*$ -algebras were introduced by Israel Moiseevich Gelfand (1913-2009). The usage of  $C^*$ -algebras in non-commutative geometry was pioneered by Alain Connes (born 1947).

A  $C^*$ -algebra is called *universal* if for any other  $C^*$ -algebra with the same number of generators satisfying the same relations, there is a unique unital  $C^*$ -morphism from the former to the latter. (For  $C^*$ -algebras, the universal problem does not always have a solution, but it does have a solution for a class of  $C^*$ -algebras which we introduce below.) An element  $u \in \mathcal{A}$  is called a *unitary*, if  $u^{-1} = u^*$ .

**Definition 2.** ([5]) *By a non-commutative torus one understands the universal  $C^*$ -algebra  $\mathcal{A}_\theta$  generated by unitaries  $u$  and  $v$  satisfying the commutation relation  $vu = e^{2\pi i\theta}uv$  for a real constant  $\theta$ .*

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**Remark 3.** If one denotes  $x_1 = u, x_2 = u^*, x_3 = v, x_4 = v^*$  and if  $e$  is the unit, then the relations for  $\mathcal{A}_\theta$  take the form:

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta} x_1x_3, \\ x_1x_2 &= e = x_2x_1, \\ x_3x_4 &= e = x_4x_3, \end{cases} \tag{3.1}$$

with the involution  $x_1^* = x_2$  and  $x_3^* = x_4$ .

Different  $\mathcal{A}_\theta$  may have equivalent module categories. It was shown by Marc Aristide Rieffel (born 1937) and others, that  $\mathcal{A}_\theta$  and  $\mathcal{A}_{\theta'}$  are Morita equivalent if and only if:

$$\theta' = \frac{a\theta + b}{c\theta + d} \text{ for some matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{3.2}$$

Denote by  $\Lambda_\theta$  a *pseudo-lattice*, i.e. the abelian subgroup  $\mathbb{Z} + \mathbb{Z}\theta$  of  $\mathbb{R}$ . The  $\mathcal{A}_\theta$  is said to have *real multiplication* (RM), if there exists a real number  $\alpha \notin \mathbb{Z}$  such that  $\alpha\Lambda_\theta \subseteq \Lambda_\theta$ . The above inclusion implies, that  $\theta, \alpha \in \mathbb{Q}(\sqrt{D})$  for some integer  $D > 0$ . Indeed, from  $\alpha\Lambda_\theta \subseteq \Lambda_\theta$  one gets  $\alpha = a + b\theta$  and  $\alpha\theta = c + d\theta$  for some  $a, b, c, d \in \mathbb{Z}$ , such that  $ad - bc \neq 0$ . Eliminating  $\alpha$  in the above two equations, one obtains a quadratic equation  $b\theta^2 + (a - d)\theta - c = 0$  with discriminant  $D$ . Because  $\theta$  is a real root, one gets  $D > 0$ . Since  $\alpha = a + b\theta$ , one has an inclusion  $\alpha \in \mathbb{Q}(\sqrt{D})$ ; whence the name.

### 4 Bon appetit!

The reader noticed already something unusual: transformations (2.4) and (3.2) are given by the same formulas! This observation hints at a possibility of equivalence of the corresponding categories as shown on the diagram below.

$$\begin{array}{ccc} \mathcal{E}_\tau & \longrightarrow & \mathcal{E}_{\tau' = \frac{a\tau+b}{c\tau+d}} \\ F \downarrow & & \downarrow F \\ \mathcal{A}_\theta & \longrightarrow & \mathcal{A}_{\theta' = \frac{a\theta+b}{c\theta+d}} \end{array}$$

**Example 4.** If  $D > 1$  is a square-free integer, then  $F$  maps  $\mathcal{E}_\tau$  with CM by  $\sqrt{-D}$  to  $\mathcal{A}_\theta$  with RM by  $\sqrt{D}$ ; in other words,  $F(\mathcal{E}_{\sqrt{-D}}) = \mathcal{A}_{\sqrt{D}}$ . Note however, that no explicit formula for the function  $\theta = \theta(\tau)$  exists in general.

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The *category of elliptic curves* consists of all  $\mathcal{E}_\tau$  and all algebraic (or holomorphic) maps between  $\mathcal{E}_\tau$ . Likewise, the *category of non-commutative tori* consists of all  $\mathcal{A}_\theta$  and all  $C^*$ -algebra homomorphisms between  $\mathcal{A}_\theta \otimes \mathcal{K}$ , where  $\mathcal{K}$  is the  $C^*$ -algebra of all compact operators on an infinite dimensional separable Hilbert space. (It was proved by Marc Aristide Rieffel, that  $\mathcal{A}_\theta \otimes \mathcal{K}$  and  $\mathcal{A}_{\theta'} \otimes \mathcal{K}$  are isomorphic if and only if  $\mathcal{A}_\theta$  and  $\mathcal{A}_{\theta'}$  are *Morita equivalent*.) The following result tells us, that indeed we have an equivalence of two categories.

**Theorem 5.** *There exists a covariant functor,  $F$ , from the category of elliptic curves to the category of noncommutative tori, such that  $F(\mathcal{E}_\tau)$  and  $F(\mathcal{E}_{\tau'})$  are isomorphic if and only if  $\mathcal{E}_\tau$  and  $\mathcal{E}_{\tau'}$  are related by an algebraic map.*

*Proof.* A *Sklyanin algebra*  $S(\alpha, \beta, \gamma)$  is a  $\mathbb{C}$ -algebra on four generators  $x_1, \dots, x_4$  and six quadratic relations:

$$\begin{cases} x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3), \\ x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3, \\ x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4), \\ x_1x_3 + x_3x_1 = x_4x_2 - x_2x_4, \\ x_1x_4 - x_4x_1 = \gamma(x_2x_3 + x_3x_2), \\ x_1x_4 + x_4x_1 = x_2x_3 - x_3x_2, \end{cases} \tag{4.1}$$

where  $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$ , see [9], p. 260. It was proved that the center of  $S(\alpha, \beta, \gamma)$  gives rise to a family of elliptic curves  $\mathcal{E}_\tau$  in the Jacobi form:

$$\begin{cases} u^2 + v^2 + w^2 + z^2 = 0, \\ \frac{1-\alpha}{1+\beta}v^2 + \frac{1+\alpha}{1-\gamma}w^2 + z^2 = 0, \end{cases} \tag{4.2}$$

and an automorphism  $\sigma : \mathcal{E}_\tau \rightarrow \mathcal{E}_\tau$  [9]. As explained in Section 1,  $\mathbf{Coh}(\mathcal{E}_\tau) \cong \mathbf{Mod}(S(\alpha, \beta, \gamma))$  for each admissible values of  $\alpha, \beta$  and  $\gamma$  even though  $S(\alpha, \beta, \gamma)$  is a non-commutative ring.

**Remark 6.** *The algebra  $S(\alpha, \beta, \gamma)$  depends on two variables, say,  $\alpha$  and  $\beta$ . It is not hard to see that  $\alpha$  corresponds to  $\tau$  and  $\beta$  corresponds to  $\sigma$ , since the automorphism  $\sigma$  is given by the shift  $0 \mapsto p$ , where  $p$  is a point of  $\mathcal{E}_\tau$ .*

Let  $\mathcal{E}_\tau$  be any elliptic curve, but choose  $\sigma : \mathcal{E}_\tau \rightarrow \mathcal{E}_\tau$  to be of order 4. Following ([2], Remark 1), in this case system (4.1) can be brought to the *skew-symmetric* form:

$$\begin{cases} x_3x_1 = \mu e^{2\pi i\theta} x_1x_3, \\ x_4x_2 = \frac{1}{\mu} e^{2\pi i\theta} x_2x_4, \\ x_4x_1 = \mu e^{-2\pi i\theta} x_1x_4, \\ x_3x_2 = \frac{1}{\mu} e^{-2\pi i\theta} x_2x_3, \\ x_2x_1 = x_1x_2, \\ x_4x_3 = x_3x_4, \end{cases} \tag{4.3}$$

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where  $\theta = \text{Arg}(q)$  and  $\mu = |q|$  for some complex number  $q \in \mathbb{C} - \{0\}$ . (In general, any Sklyanin algebra on  $n$  generators can be brought to the skew-symmetric form if and only if  $\sigma^n = \text{Id}$  [2], Remark 1.) We shall denote by  $S(q)$  the Sklyanin algebra given by relations (4.3).

The reader can verify that relations (4.3) are invariant under the involution  $x_1^* = x_2$  and  $x_3^* = x_4$ . This involution turns  $S(q)$  into an algebra with involution.

To continue our proof, we need the following result.

**Lemma 7.** *The system of relations (3.1) for any free  $\mathbb{C}$ -algebra on generators  $x_1, \dots, x_4$  is equivalent to the following system of relations:*

$$\begin{cases} x_3x_1 &= e^{2\pi i\theta} x_1x_3, \\ x_4x_2 &= e^{2\pi i\theta} x_2x_4, \\ x_4x_1 &= e^{-2\pi i\theta} x_1x_4, \\ x_3x_2 &= e^{-2\pi i\theta} x_2x_3, \\ x_2x_1 &= x_1x_2 = e, \\ x_4x_3 &= x_3x_4 = e. \end{cases} \quad (4.4)$$

*Proof.* Notice that the first and the two last equations of (3.1) and (4.4) coincide; we shall proceed stepwise for the rest of the equations.

Let us prove that relations (3.1) imply  $x_1x_4 = e^{2\pi i\theta} x_4x_1$ . It follows from  $x_1x_2 = e$  and  $x_3x_4 = e$  that  $x_1x_2x_3x_4 = e$ . Since  $x_1x_2 = x_2x_1$  we can bring the last equation to the form  $x_2x_1x_3x_4 = e$  and multiply both sides by  $e^{2\pi i\theta}$ ; thus one gets  $x_2(e^{2\pi i\theta} x_1x_3)x_4 = e^{2\pi i\theta}$ . But  $e^{2\pi i\theta} x_1x_3 = x_3x_1$  and our main equation takes the form  $x_2x_3x_1x_4 = e^{2\pi i\theta}$ . Multiplying both sides on the left by  $x_1$  we have  $x_1x_2x_3x_1x_4 = e^{2\pi i\theta} x_1$ ; since  $x_1x_2 = e$  one has  $x_3x_1x_4 = e^{2\pi i\theta} x_1$ . Again, one can multiply both sides on the left by  $x_4$  and thus get  $x_4x_3x_1x_4 = e^{2\pi i\theta} x_4x_1$ ; since  $x_4x_3 = e$ , one gets the required identity  $x_1x_4 = e^{2\pi i\theta} x_4x_1$ .

The proof of the remaining relations  $x_4x_2 = e^{2\pi i\theta} x_2x_4$  and  $x_3x_2 = e^{-2\pi i\theta} x_2x_3$  is similar and is left to the reader. Lemma 7 is proved.  $\square$

Returning to the proof of Theorem 5, we see that relations (4.3) are equivalent to (4.4) plus the following relations:

$$x_1x_2 = x_3x_4 = \frac{1}{\mu} e. \quad (4.5)$$

(The reader is encouraged to verify the equivalence.) We shall call (4.5) the *scaling of the unit* relations and denote by  $I_\mu$  the two-sided ideal in  $S(q)$  generated by such relations. It is easy to see that the scaling of the unit relations are invariant under the involution  $x_1^* = x_2$  and  $x_3^* = x_4$ . Thus involution on algebra  $S(q)$  extends to

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an involution on the quotient algebra  $S(q)/I_\mu$ . In view of the above, one gets an isomorphism of rings with involution:

$$\mathcal{A}_\theta \cong S(q)/I_\mu. \quad (4.6)$$

Theorem 5 is proved.  $\square$

## 5 Know your rank

Why is Theorem 5 useful? Roughly, it says that instead of the elliptic curves  $\mathcal{E}_\tau$ , one can study the corresponding non-commutative tori  $\mathcal{A}_\theta$ ; in particular, geometry of  $\mathcal{E}_\tau$  can be recast in terms of  $\mathcal{A}_\theta$ . It is interesting, for instance, to compare isomorphism invariants of  $\mathcal{E}_\tau$  with invariants of the algebra  $\mathcal{A}_\theta$ . Below we relate one such invariant, the rank of  $\mathcal{E}_\tau$ , to the so-called arithmetic complexity of  $\mathcal{A}_\theta$ .

Recall that if  $a, b \in \mathcal{E}_\tau$  are two points, then their sum is defined as shown in Figure 1. (The addition is simpler on the complex torus  $\mathbb{C}/L_\tau$ , where the sum is just the image of sum of complex numbers  $a, b \in \mathbb{C}$  under projection  $\mathbb{C} \rightarrow \mathbb{C}/L_\tau$ .) Since each point has an additive inverse and there is a zero point, the set of all points of  $\mathcal{E}_\tau$  has the structure of an *abelian group*. If  $\mathcal{E}_\tau$  is defined over  $\mathbb{Q}$  (or a finite extension of  $\mathbb{Q}$ ), then the Mordell-Weil theorem says that the subgroup of points with rational coordinates is a *finitely generated* abelian group. The number of generators in this group is called the *rank* of  $\mathcal{E}_\tau$  and is denoted by  $rk(\mathcal{E}_\tau)$ . Little is known about ranks in general, except for the famous *Birch and Swinnerton-Dyer Conjecture* comparing  $rk(\mathcal{E}_\tau)$  to the order at zero of the Hasse-Weil  $L$ -function attached to  $\mathcal{E}_\tau$ .

Let  $\mathcal{E}_\tau$  be an elliptic curve with CM by  $\sqrt{-D}$  for some square-free positive integer  $D$ . In this case  $rk(\mathcal{E}_\tau)$  is finite, because  $\mathcal{E}_\tau$  can be defined over  $\mathbb{Q}$  or a finite extension of  $\mathbb{Q}$ ; moreover, the integer  $rk(\mathcal{E}_\tau)$  is an isomorphism invariant of  $\mathcal{E}_\tau$  modulo a finite number of twists. The ranks of CM elliptic curves were calculated by Benedict Hyman Gross (born 1950) in his doctoral thesis [3]; they are reproduced in the table below for some primes  $D \equiv 3 \pmod{4}$ .

Let  $\mathcal{A}_\theta$  be a non-commutative torus with RM by  $\sqrt{D}$ , see Example 4 for motivation. By an *arithmetic complexity*  $c(\mathcal{A}_\theta)$  of  $\mathcal{A}_\theta$  one understands the number of independent entries in the period  $(a_1, \dots, a_n)$  of continued fraction of  $\sqrt{D}$ . Roughly speaking, the  $c(\mathcal{A}_\theta)$  is equal to the Krull dimension of an irreducible component of the affine variety defined by a diophantine equation obtained from the continued fraction of  $\sqrt{D}$ ; we refer the interested reader to [4, Section 6.2.1] for the details. It follows from the standard properties of continued fractions and formula (3.2), that  $c(\mathcal{A}_\theta)$  is an invariant of the Morita equivalence of  $\mathcal{A}_\theta$ . The values of  $c(\mathcal{A}_\theta)$  for primes  $D \equiv 3 \pmod{4}$  are given in the table below.

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$D \equiv 3 \pmod{4}$	$rk(\mathcal{E}_\tau)$	continued fraction of $\sqrt{D}$	$c(\mathcal{A}_\theta)$
3	1	$[1, \overline{1, 2}]$	2
7	0	$[2, \overline{1, 1, 1, 4}]$	1
11	1	$[3, \overline{3, 6}]$	2
19	1	$[4, \overline{2, 1, 3, 1, 2, 8}]$	2
23	0	$[4, \overline{1, 3, 1, 8}]$	1
31	0	$[5, \overline{1, 1, 3, 5, 3, 1, 1, 10}]$	1
43	1	$[6, \overline{1, 1, 3, 1, 5, 1, 3, 1, 1, 12}]$	2
47	0	$[6, \overline{1, 5, 1, 12}]$	1
59	1	$[7, \overline{1, 2, 7, 2, 1, 14}]$	2
67	1	$[8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}]$	2
71	0	$[8, \overline{2, 2, 1, 7, 1, 2, 2, 16}]$	1
79	0	$[8, \overline{1, 7, 1, 16}]$	1
83	1	$[9, \overline{9, 18}]$	2

The reader may notice a correlation between  $rk(\mathcal{E}_\tau)$  and  $c(\mathcal{A}_\theta)$ , which can be generalized as follows.

**Theorem 8.** ([4]) *If  $D \equiv 3 \pmod{4}$  is a prime number, then*

$$rk(\mathcal{E}_\tau) = c(\mathcal{A}_\theta) - 1. \quad (5.1)$$

Perhaps (5.1) is a reconciliation formula for the “opposites”  $\mathcal{E}_\tau$  and  $\mathcal{A}_\theta$  meant by the science fiction writer Walter M. Miller Jr. (1923-1996). The reader is to judge!

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## References

- [1] M. Auslander, *A functorial approach to representation theory*, in: Representations of algebras (Puebla, 1980), pp. 105-179, Lecture Notes in Math. **944**, Springer, Berlin-New York, 1982. [MR0672116\(83m:16027\)](#). [Zbl 0486.16027](#).
- [2] B. L. Feigin and A. V. Odesskii, *Sklyanin’s elliptic algebras*, Functional Anal. Appl. **23** (1989), 207-214. [MR1026987\(91e:16037\)](#). [Zbl 0713.17009](#).

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<http://www.utgjiu.ro/math/sma>

- [3] B. H. Gross, *Arithmetic on Elliptic Curves with Complex Multiplication*, Lecture Notes Math. **776** (1980), Springer. [MR0563921\(81f:10041\)](#). [Zbl 0433.14032](#).
- [4] I. V. Nikolaev, *Noncommutative Geometry*, De Gruyter Studies in Mathematics **66**, De Gruyter, Berlin, 2017. [Zbl 06790762](#).
- [5] M. A. Rieffel, *Non-commutative tori – a case study of non-commutative differentiable manifolds*, Contemp. Math. **105** (1990), 191-211. [MR1047281\(91d:58012\)](#). [Zbl 0713.46046](#).
- [6] J. P. Serre, *Fasceaux algébriques cohérents*, Ann. of Math. **61** (1955), 197-278. [MR0068874\(16,953c\)](#). [Zbl 0067.16201](#).
- [7] J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992. [MR1171452\(93g:11003\)](#). [Zbl 0752.14034](#).
- [8] E. K. Sklyanin, *Some algebraic structures connected to the Yang–Baxter equations*, Functional Anal. Appl. **16** (1982), 27-34. [MR0684124\(84c:82004\)](#). [Zbl 0513.58028](#).
- [9] S. P. Smith and J. T. Stafford, *Regularity of the four dimensional Sklyanin algebra*, Compositio Math. **83** (1992), 259-289. [MR1175941\(93h:16037\)](#). [Zbl 0758.16001](#).
- [10] A. B. Verevkin, *On a noncommutative analogue of the category of coherent sheaves on a projective scheme*, in Algebra and analysis (Tomsk, 1989), 41-53; Amer. Math. Soc. Transl. Ser. 2, **151**, Amer. Math. Soc., Providence, RI, 1992. [MR1191171\(93j:14002\)](#). [Zbl 0920.14001](#).

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