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GREENLEES-MAY DUALITY IN A NUTSHELL

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Abstract. This expository article delves deep into Greenlees-May Duality which is widely thought of as a far-reaching generalization of Grothendieck's Local Duality. Despite its focal role in the theory of derive local homology and cohomology, in the literature this theorem did not get the treatment it deserves, as indeed its proof is a tangled web in a series of scattered papers. By carefully scrutinizing the requisite tools, we present a clear-cut well-documented proof of this theorem for the sake of reference.

1 Introduction

Throughout this note, all rings are assumed to be commutative and noetherian with identity.

The Riemann-Roch Theorem is a ground-breaking result in mathematics, which is especially important in the realms of complex analysis and algebraic geometry. Quite unexpectedly, it establishes a formula for the computation of the dimension of the space of meromorphic functions with prescribed zeroes and allowed poles. It relates the complex analysis of a connected compact Riemann surface with the surface's purely topological genus, in a way that can be carried over into purely algebraic settings. Initially proved as Riemann's inequality by Riemann in 1857, the theorem reached its definitive form for Riemann surfaces after the work of Riemann's student Gustav Roch in 1865. It was later generalized to algebraic curves, to higher-dimensional varieties and beyond.

Serre Duality is a duality theory, generalizing the Riemann-Roch Theorem in some sense, which shows that the cohomology group in degree i of a non-singular projective algebraic variety of dimension n is the dual space of the cohomology group in degree $n-i$. Grothendieck vastly generalized this result to his Coherent Duality Theories. In the present article we are, however, mainly interested in the algebraic side of the theory, i.e. the algebraic counterpart to Serre Duality, the so-called Local Duality.

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In his algebraic geometry seminars of 1961-2, Grothendieck founded the theory of local cohomology as an indispensable tool in both algebraic geometry and commutative algebra. Given an ideal $\mathfrak{a}$ of a ring R, the local cohomology functor $H^i_{\mathfrak{a}}(-)$ is defined as the ith right derived functor of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^n, -)$. Among a myriad of exceptional results, he proved the so-called Local Duality Theorem:

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a local ring (i.e. $\mathfrak{m}$ is the only maximal ideal of R and $k = R/\mathfrak{m}$) with a dualizing module ω_R, and M a finitely generated R-module. Let $(-)^\vee := \operatorname{Hom}_R(-, E_R(k))$, where $E_R(k)$ is the injective envelope of k. Then*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}_R^{\dim(R)-i}(M, \omega_R)^\vee$$

for every $i \geq 0$.

The dual theory to local cohomology, i.e. local homology, was initiated by Matlis [12] in 1974, and its study was continued by Simon in [17] and [18]. Given an ideal $\mathfrak{a}$ of R, the local homology functor $H_i^{\mathfrak{a}}(-)$ is defined as the ith left derived functor of the $\mathfrak{a}$-adic completion functor $\Lambda^{\mathfrak{a}}(-) \cong \varprojlim(R/\mathfrak{a}^n \otimes_R -)$.

The existence of a dualizing module in Theorem 1 is rather restrictive as it forces R to be Cohen-Macaulay. To proceed further and generalize Theorem 1, Greenlees and May [7, Propositions 3.1 and 3.8], established a spectral sequence

$$E^2_{p,q} = \operatorname{Ext}_R^{-p}\left(H^q_{\mathfrak{a}}(R), M\right) \underset{p}{\Rightarrow} H^{\mathfrak{a}}_{p+q}(M) \tag{1.1}$$

for any R-module M. One can also settle the dual spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^R\left(H^q_{\mathfrak{a}}(R), M\right) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M) \tag{1.2}$$

for any R-module M.

It is by and large more palatable to have isomorphisms rather than spectral sequences. But the problem is that the category of R-modules $\mathcal{M}(R)$ is not rich enough to allow for such isomorphisms. We need to enlarge this category to the category of R-complexes $\mathcal{C}(R)$, and even enhance it further, to the derived category $\mathcal{D}(R)$. This is a standard context in which the sought isomorphisms do indeed exist. As a matter of fact, the spectral sequence (1.1) turns into the isomorphism

$$\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R), X\right) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(X), \tag{1.3}$$

and the spectral sequence (1.2) turns into the isomorphism

$$\mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R X \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(X) \tag{1.4}$$

in $\mathcal{D}(R)$ for any R-complex X. Patching the two isomorphisms (1.3) and (1.4) together, we obtain the celebrated Greenlees-May Duality Theorem:

Theorem 2. *Let $\mathfrak{a}$ be an ideal of R, and $X, Y \in \mathcal{D}(R)$. Then there is a natural isomorphism*

$$\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X), Y) \simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Y))$$

in $\mathcal{D}(R)$.

This was first proved by Alonso Tarrío, Jeremías López and Lipman in [2]. Theorem 2 is a far-reaching generalization of Theorem 1 and indeed extends it to its full generality. This theorem also demonstrates perfectly some sort of adjointness between derived local cohomology and homology.

Despite its incontrovertible impact on the theory of derived local homology and cohomology, we regretfully notice that there is no comprehensive and accessible treatment of the Greenlees-May Duality in the literature. There are some papers that touch on the subject, each from a different perspective, but none of them present a clear-cut and thorough proof that is fairly readable for non-experts; see for example [7], [2], [13], and [15]. In order to remedy this defect, our approach is to present this theorem starting from scratch, providing all prerequisites in a self-contained text. Our aim is to present a well-documented and rigorous proof, accessible to non-specialists. Our proof mixes standard arguments with new ones; however, in any case, all details are fully worked out. Our final goal is to explain the highly non-trivial fact that Greenlees-May Duality generalizes Local Duality in simple and traceable steps.

2 Module Prerequisites

In this section, we embark on providing the requisite tools on modules which are needed in Section 4.

First we recall the notion of a δ-functor which will be used as a powerful tool to establish natural isomorphisms.

Definition 3. *Let R and S be two rings. Then:*

(i) A homological covariant δ-functor is a sequence $(\mathcal{F}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$ of additive covariant functors with the property that every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules gives rise to a long exact sequence

$$\cdots \to \mathcal{F}_2(M'') \xrightarrow{\delta_2} \mathcal{F}_1(M') \to \mathcal{F}_1(M) \to \mathcal{F}_1(M'') \xrightarrow{\delta_1} \mathcal{F}_0(M') \to \mathcal{F}_0(M) \to \mathcal{F}_0(M'') \to 0$$

of S-modules, such that the connecting morphisms δ_i are natural in the sense *that any commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows induces a commutative diagram

$$\begin{array}{ccccccccccccccc} \cdots \to & \mathcal{F}_2(M'') & \xrightarrow{\delta_2} & \mathcal{F}_1(M') & \to & \mathcal{F}_1(M) & \to & \mathcal{F}_1(M'') & \xrightarrow{\delta_1} & \mathcal{F}_0(M') & \to & \mathcal{F}_0(M) & \to & \mathcal{F}_0(M'') & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \mathcal{F}_2(N'') & \xrightarrow{\Delta_2} & \mathcal{F}_1(N') & \to & \mathcal{F}_1(N) & \to & \mathcal{F}_1(N'') & \xrightarrow{\Delta_1} & \mathcal{F}_0(N') & \to & \mathcal{F}_0(N) & \to & \mathcal{F}_0(N'') & \to 0 \end{array}$$

of S-modules with exact rows.

(ii) A cohomological covariant δ-functor is a sequence $\left(\mathcal{F}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$ of additive covariant functors with the property that every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules gives rise to a long exact sequence

$$0 \to \mathcal{F}^0(M') \to \mathcal{F}^0(M) \to \mathcal{F}^0(M'') \xrightarrow{\delta^0} \mathcal{F}^1(M') \to \mathcal{F}^1(M) \to \mathcal{F}^1(M'') \xrightarrow{\delta^1} \mathcal{F}^2(M') \to \cdots$$

of S-modules, such that the connecting morphisms δ^i are natural in the sense that any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

of R-modules with exact rows induces a commutative diagram

$$\begin{array}{ccccccccccccccc} 0 \to & \mathcal{F}^0(M') & \to & \mathcal{F}^0(M) & \to & \mathcal{F}^0(M'') & \xrightarrow{\delta^0} & \mathcal{F}^1(M') & \to & \mathcal{F}^1(M) & \to & \mathcal{F}^1(M'') & \xrightarrow{\delta^1} & \mathcal{F}^2(M') & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{F}^0(N') & \to & \mathcal{F}^0(N) & \to & \mathcal{F}^0(N'') & \xrightarrow{\Delta^0} & \mathcal{F}^1(N') & \to & \mathcal{F}^1(N) & \to & \mathcal{F}^1(N'') & \xrightarrow{\Delta^1} & \mathcal{F}^2(N') & \to \cdots \end{array}$$

of S-modules with exact rows.

Example 4. *Let R and S be two rings, and $\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(S)$ an additive covariant functor. Then the sequence $(L_i\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$ of left derived functors of $\mathcal{F}$ is a homological covariant δ-functor, and the sequence $\left(R^i\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$ of right derived functors of $\mathcal{F}$ is a cohomological covariant δ-functor.*

Definition 5. *Let R and S be two rings. Then:*

(i) A morphism

$$\tau : (\mathcal{F}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0} \to (\mathcal{G}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$$

of homological covariant δ-functors is a sequence $\tau = (\tau_i : \mathcal{F}_i \to \mathcal{G}_i)_{i\geq 0}$ of natural transformations of functors, such that any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules induces a commutative diagram

$$\begin{array}{ccccccccccccccc}
\cdots \to & \mathcal{F}_2(M'') & \xrightarrow{\delta_2} & \mathcal{F}_1(M') & \to & \mathcal{F}_1(M) & \to & \mathcal{F}_1(M'') & \xrightarrow{\delta_1} & \mathcal{F}_0(M') & \to & \mathcal{F}_0(M) & \to & \mathcal{F}_0(M'') & \to 0 \\
 & \downarrow{\scriptstyle \tau_2(M'')} & & \downarrow{\scriptstyle \tau_1(M')} & & \downarrow{\scriptstyle \tau_1(M)} & & \downarrow{\scriptstyle \tau_1(M'')} & & \downarrow{\scriptstyle \tau_0(M')} & & \downarrow{\scriptstyle \tau_0(M)} & & \downarrow{\scriptstyle \tau_0(M'')} & \\
\cdots \to & \mathcal{G}_2(M'') & \xrightarrow{\Delta_2} & \mathcal{G}_1(M') & \to & \mathcal{G}_1(M) & \to & \mathcal{G}_1(M'') & \xrightarrow{\Delta_1} & \mathcal{G}_0(M') & \to & \mathcal{G}_0(M) & \to & \mathcal{G}_0(M'') & \to 0
\end{array}$$

of S-modules with exact rows. If in particular, τ_i is an isomorphism for every $i \geq 0$, then τ is called an isomorphism of δ-functors.

(ii) A morphism

$$\tau : \left(\mathcal{F}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0} \to \left(\mathcal{G}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$$

of cohomological covariant δ-functors is a sequence $\tau = \left(\tau^i : \mathcal{F}^i \to \mathcal{G}^i\right)_{i\geq 0}$ of natural transformations of functors, such that any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules induces a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 \to & \mathcal{F}^0(M') & \to & \mathcal{F}^0(M) & \to & \mathcal{F}^0(M'') & \xrightarrow{\delta^0} & \mathcal{F}^1(M') & \to & \mathcal{F}^1(M) & \to & \mathcal{F}^1(M'') & \xrightarrow{\delta^1} & \mathcal{F}^2(M') & \to \cdots \\
 & \downarrow{\scriptstyle \tau^0(M')} & & \downarrow{\scriptstyle \tau^0(M)} & & \downarrow{\scriptstyle \tau^0(M'')} & & \downarrow{\scriptstyle \tau^1(M')} & & \downarrow{\scriptstyle \tau^1(M)} & & \downarrow{\scriptstyle \tau^1(M'')} & & \downarrow{\scriptstyle \tau^2(M')} & \\
0 \to & \mathcal{G}^0(M') & \to & \mathcal{G}^0(M) & \to & \mathcal{G}^0(M'') & \xrightarrow{\Delta^0} & \mathcal{G}^1(M') & \to & \mathcal{G}^1(M) & \to & \mathcal{G}^1(M'') & \xrightarrow{\Delta^1} & \mathcal{G}^2(M') & \to \cdots
\end{array}$$

of S-modules with exact rows. If in particular, τ^i is an isomorphism for every $i \geq 0$, then τ is called an isomorphism of δ-functors.

The following remarkable theorem due to Grothendieck provides hands-on conditions that ascertain the existence of isomorphisms between δ-functors.

Theorem 6. *Let R and S be two rings. Then the following assertions hold:*

(i) Assume that $(\mathcal{F}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$ and $(\mathcal{G}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$ are two homological covariant δ-functors such that $\mathcal{F}_i(F) = 0 = \mathcal{G}_i(F)$ for every free R-module F and every $i \geq 1$. If there is a natural transformation $\eta : \mathcal{F}_0 \to \mathcal{G}_0$ of functors which is an isomorphism on free R-modules, then there is a unique isomorphism $\tau : (\mathcal{F}_i)_{i\geq 0} \to (\mathcal{G}_i)_{i\geq 0}$ of δ-functors such that $\tau_0 = \eta$.

(ii) Assume that $\left(\mathcal{F}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$ and $\left(\mathcal{G}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$ are two cohomological covariant δ-functors such that $\mathcal{F}^i(I) = 0 = \mathcal{G}^i(I)$ for every injective R-module I and every $i \geq 1$. If there is a natural transformation $\eta : \mathcal{F}^0 \to \mathcal{G}^0$ of functors which is an isomorphism on injective R-modules, then there is a unique isomorphism $\tau : (\mathcal{F}^i)_{i\geq 0} \to (\mathcal{G}^i)_{i\geq 0}$ of δ-functors such that $\tau^0 = \eta$.

Proof. The proof is standard and can be found in almost every book on homological algebra. For example, see [14, Corollaries 6.34 and 6.49]. One should note that the above version is somewhat stronger than what is normally recorded in the books. However, the same proof can be modified in a suitable way to imply the above version. □

The following corollary sets forth a special case of Theorem 6 which frequently occurs in practice.

Corollary 7. *Let R and S be two rings. Then the following assertions hold:*

(i) Assume that $\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(S)$ is an additive covariant functor, and $(\mathcal{F}_i : \mathcal{M}(R) \to \mathcal{M}(S))_{i\geq 0}$ is a homological covariant δ-functor such that $\mathcal{F}_i(F) = 0$ for every free R-module F and every $i \geq 1$. If there is a natural transformation $\eta : L_0\mathcal{F} \to \mathcal{F}_0$ of functors which is an isomorphism on free R-modules, then there is a unique isomorphism $\tau : (L_i\mathcal{F})_{i\geq 0} \to (\mathcal{F}_i)_{i\geq 0}$ of δ-functors such that $\tau_0 = \eta$.

(ii) Assume that $\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(S)$ is an additive covariant functor, and $\left(\mathcal{F}^i : \mathcal{M}(R) \to \mathcal{M}(S)\right)_{i\geq 0}$ is a cohomological covariant δ-functor such that $\mathcal{F}^i(I) = 0$ for every injective R-module I and every $i \geq 1$. If there is a natural transformation $\eta : \mathcal{F}^0 \to R^0\mathcal{F}$ of functors which is an isomorphism on injective R-modules, then there is a unique isomorphism $\tau : (\mathcal{F}^i)_{i\geq 0} \to (R^i\mathcal{F})_{i\geq 0}$ of δ-functors such that $\tau^0 = \eta$.

Proof. (i): We note that $(L_i\mathcal{F})(F) = 0$ for every $i \geq 1$ and every free R-module F. Now the result follows from Theorem 6 (i).

(ii): We note that $(R^i\mathcal{F})(I)=0$ for every $i\geq 1$ and every injective R-module I. Now the result follows from Theorem 6 (ii). □

We next recall the Koszul complex and the Koszul homology briefly. The Koszul complex $K^R(a)$ on an element $a\in R$ is the R-complex

$$K^R(a) := \operatorname{Cone}(R \xrightarrow{a} R),$$

and the Koszul complex $K^R(\underline{a})$ on a sequence of elements $\underline{a}=a_1,\ldots,a_n\in R$ is the R-complex

$$K^R(\underline{a}) := K^R(a_1)\otimes_R\cdots\otimes_R K^R(a_n).$$

It is easy to see that $K^R(\underline{a})$ is a complex of finitely generated free R-modules concentrated in degrees $n,\ldots,0$. Given any R-module M, there is an isomorphism of R-complexes

$$K^R(\underline{a})\otimes_R M \cong \Sigma^n \operatorname{Hom}_R\left(K^R(\underline{a}),M\right),$$

which is sometimes referred to as the self-duality property of the Koszul complex. Accordingly, we feel free to define the Koszul homology of the sequence $\underline{a}$ with coefficients in M, by setting

$$H_i(\underline{a};M) := H_i\left(K^R(\underline{a})\otimes_R M\right)\cong H_{i-n}\left(\operatorname{Hom}_R\left(K^R(\underline{a}),M\right)\right)$$

for every $i\geq 0$.

One can form both direct and inverse systems of Koszul complexes and Koszul homologies as explicated in the next remark.

Remark 8. *We have:*

(i) Given an element $a\in R$, we define a morphism $\varphi_a^{k,l}: K^R(a^k)\to K^R(a^l)$ of R-complexes for every $k\leq l$ as follows:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "a^k"] \arrow[d, equal] & R \arrow[r] \arrow[d, "a^{l-k}"] & 0 \\ 0 \arrow[r] & R \arrow[r, "a^l"] & R \arrow[r] & 0 \end{tikzcd}$$

It is easily seen that $\left\{K^R(a^k),\varphi_a^{k,l}\right\}_{k\in\mathbb{N}}$ is a direct system of R-complexes. Given elements $\underline{a}=a_1,\ldots,a_n\in R$, we let $\underline{a}^k=a_1^k,\ldots,a_n^k$ for every $k\geq 1$. Now

$$K^R(\underline{a}^k)=K^R(a_1^k)\otimes_R\cdots\otimes_R K^R(a_n^k),$$

and we let

$$\varphi^{k,l} := \varphi_{a_1}^{k,l}\otimes_R\cdots\otimes_R\varphi_{a_n}^{k,l}.$$

It follows that $\left\{K^R(\underline{a}^k), \varphi^{k,l}\right\}_{k\in\mathbb{N}}$ *is a direct system of* R*-complexes. It is also clear that* $\left\{H_i\left(\underline{a}^k; M\right), H_i\left(\varphi^{k,l} \otimes_R M\right)\right\}_{k\in\mathbb{N}}$ *is a direct system of* R*-modules for every* $i \in \mathbb{Z}$*.*

(ii) Given an element $a \in R$*, we define a morphism* $\psi_a^{k,l} : K^R(a^k) \to K^R(a^l)$ *of* R*-complexes for every* $k \geq l$ *as follows:*

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "a^k"] \arrow[d, "a^{k-l}"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r, "a^l"] & R \arrow[r] & 0
\end{tikzcd}$$

It is easily seen that $\left\{K^R(a^k), \varphi_a^{k,l}\right\}_{k\in\mathbb{N}}$ *is an inverse system of* R*-complexes. Given elements* $\underline{a} = a_1, ..., a_n \in R$*, we let* $\underline{a}^k = a_1^k, ..., a_n^k$ *for every* $k \geq 1$*. Now*

$$K^R(\underline{a}^k) = K^R(a_1^k) \otimes_R \cdots \otimes_R K^R(a_n^k),$$

and we let

$$\psi^{k,l} := \psi_{a_1}^{k,l} \otimes_R \cdots \otimes_R \psi_{a_n}^{k,l}.$$

It follows that $\left\{K^R(\underline{a}^k), \psi^{k,l}\right\}_{k\in\mathbb{N}}$ *is an inverse system of* R*-complexes. It is also clear that* $\left\{H_i\left(\underline{a}^k; M\right), H_i\left(\psi^{k,l} \otimes_R M\right)\right\}_{k\in\mathbb{N}}$ *is an inverse system of* R*-modules for every* $i \in \mathbb{Z}$*.*

Recall that an inverse system $\{M_\alpha, \varphi_{\alpha,\beta}\}_{\alpha\in\mathbb{N}}$ of R-modules is said to satisfy the trivial Mittag-Leffler condition if for every $\beta \in \mathbb{N}$, there is an $\alpha \geq \beta$ such that $\varphi_{\alpha\beta} = 0$. Besides, the inverse system $\{M_\alpha, \varphi_{\alpha,\beta}\}_{\alpha\in\mathbb{N}}$ of R-modules is said to satisfy the Mittag-Leffler condition if for every $\beta \in \mathbb{N}$, there is an $\alpha_0 \geq \beta$ such that $\operatorname{im}\varphi_{\alpha\beta} = \operatorname{im}\varphi_{\alpha_0\beta}$ for every $\alpha \geq \alpha_0 \geq \beta$. It is straightforward to verify that the trivial Mittag-Leffler condition implies the Mittag-Leffler condition.

The following lemma reveals a significant feature of Koszul homology and lies at the heart of the proof of Greenlees-May Duality. The idea of the proof is taken from [15].

Lemma 9. *Let* $\underline{a} = a_1, ..., a_n \in R$*, and* $\underline{a}^k = a_1^k, ..., a_n^k$ *for every* $k \geq 1$*. Then the inverse system* $\left\{H_i\left(\underline{a}^k; R\right)\right\}_{k\in\mathbb{N}}$ *satisfies the trivial Mittag-Leffler condition for every* $i \geq 1$*.*

Proof. Let $a \in R$ and M a finitely generated R-module. The transition maps of the inverse system $\left\{K^R(a^k) \otimes_R M\right\}_{k\in\mathbb{N}}$ can be identified with the following morphisms of R-complexes for every $k \geq l$:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "a^k"] \arrow[d, "a^{k-l}"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "a^l"] & M \arrow[r] & 0 \end{tikzcd}$$

Since $H_1\left(a^k; M\right) = \left(0 :_M a^k\right)$, the transition maps of the inverse system

$$\left\{H_1\left(a^k; M\right)\right\}_{k\in\mathbb{N}}$$

can be identified with the R-homomorphisms

$$\left(0 :_M a^k\right) \xrightarrow{a^{k-l}} \left(0 :_M a^l\right)$$

for every $k \geq l$. Fix $l \in \mathbb{N}$. Since R is noetherian and M is finitely generated, the ascending chain

$$(0 :_M a) \subseteq \left(0 :_M a^2\right) \subseteq \cdots$$

of submodules of M stabilizes, i.e. there is an integer $t \geq 1$ such that

$$\left(0 :_M a^t\right) = \left(0 :_M a^{t+1}\right) = \cdots .$$

Set $k := t + l$. Then the transition map $\left(0 :_M a^k\right) \xrightarrow{a^{k-l}} \left(0 :_M a^l\right)$ is zero. Indeed, if $x \in \left(0 :_M a^k\right)$, then since

$$\left(0 :_M a^k\right) = \left(0 :_M a^{t+l}\right) = \left(0 :_M a^t\right),$$

we have $x \in \left(0 :_M a^t\right)$, so $a^{k-l}x = a^t x = 0$. This shows that the inverse system $\left\{H_1\left(a^k; M\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition. But $H_i\left(a^k; M\right) = 0$ for every $i \geq 2$, so the inverse system $\left\{H_i\left(a^k; M\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$.

Now we argue by induction on n. If $n = 1$, then the inverse system $\left\{H_i\left(a_1^k; R\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$ by the discussion above. Now assume that $n \geq 2$, and make the obvious induction hypothesis. There is an exact sequence of inverse systems

$$\left\{H_i\left(a_1^k, ..., a_{n-1}^k; R\right)\right\}_{k\in\mathbb{N}} \to \left\{H_0\left(a_n^k; H_i\left(a_1^k, ..., a_{n-1}^k; R\right)\right)\right\}_{k\in\mathbb{N}} \to 0 \qquad (2.1)$$

of R-modules for every $i \geq 0$. By the induction hypothesis, the inverse system $\left\{H_i\left(a_1^k, ..., a_{n-1}^k; R\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$, so the exact sequence (2.1) shows that the inverse system

$$\left\{H_0\left(a_n^k; H_i\left(a_1^k, ..., a_{n-1}^k; R\right)\right)\right\}_{k\in\mathbb{N}}$$

satisfies the Mittag-Leffler condition for every $i \geq 1$. On the other hand, there is a short exact sequence of inverse systems

$$0 \to \left\{H_0\left(a_n^k; H_i\left(a_1^k, ..., a_{n-1}^k; R\right)\right)\right\}_{k\in\mathbb{N}} \to \left\{H_i\left(a_1^k, ..., a_n^k; R\right)\right\}_{k\in\mathbb{N}} \to$$
$$\left\{H_1\left(a_n^k; H_{i-1}\left(a_1^k, ..., a_{n-1}^k; R\right)\right)\right\}_{k\in\mathbb{N}} \to 0 \tag{2.2}$$

of R-modules for every $i \geq 0$. Since $H_{i-1}\left(a_1^k, ..., a_{n-1}^k; R\right)$ is a finitely generated R-module for every $i \geq 1$, the discussion above shows that

$$\left\{H_1\left(a_n^k; H_{i-1}\left(a_1^k, ..., a_{n-1}^k; R\right)\right)\right\}_{k\in\mathbb{N}}$$

satisfies the Mittag-Leffler condition for every $i \geq 1$. Therefore, the short exact sequence (2) shows that the inverse system $\left\{H_i\left(a_1^k, ..., a_n^k; R\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$. □

The category $\mathcal{C}(R)$ of R-complexes enjoys direct limits and inverse limits. However, the derived category $\mathcal{D}(R)$ does not support the notions of direct limits and inverse limits. But this situation is remedied by the existence of homotopy direct limits and homotopy inverse limits as defined in triangulated categories with countable products and coproducts.

Remark 10. *Let $\left\{X^\alpha, \varphi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ be a direct system of R-complexes, and $\left\{Y^\alpha, \psi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ an inverse system of R-complexes. Then we have:*

(i) The direct limit of the direct system $\left\{X^\alpha, \varphi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ is an R-complex $\varinjlim X^\alpha$ given by $\left(\varinjlim X^\alpha\right)_i = \varinjlim X_i^\alpha$ and $\partial_i^{\varinjlim X^\alpha} = \varinjlim \partial_i^{X^\alpha}$ for every $i \in \mathbb{Z}$. Indeed, it is easy to see that $\varinjlim X^\alpha$ satisfies the universal property of direct limits in a category.

(ii) The homotopy direct limit of the direct system $\left\{X^\alpha, \varphi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ is given by $\underrightarrow{\mathrm{holim}}\, X^\alpha = \mathrm{Cone}(\vartheta)$, where the morphism $\vartheta : \bigoplus_{\alpha=1}^\infty X^\alpha \to \bigoplus_{\alpha=1}^\infty X^\alpha$ is given by $\vartheta_i\left((x_i^\alpha)\right) = \iota_i^\alpha(x_i^\alpha) - \iota_i^{\alpha+1}\left(\varphi_i^{\alpha,\alpha+1}(x_i^\alpha)\right)$ for every $i \in \mathbb{Z}$. Indeed, it is easy to see that the morphism ϑ fits into a distinguished triangle

$$\bigoplus_{\alpha=1}^{\infty} X^\alpha \to \bigoplus_{\alpha=1}^{\infty} X^\alpha \to \underrightarrow{\mathrm{holim}}\, X^\alpha \to .$$

(iii) The inverse limit of the inverse system $\left\{Y^\alpha, \psi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ is an R-complex $\varprojlim Y^\alpha$ given by $\left(\varprojlim Y^\alpha\right)_i = \varprojlim Y_i^\alpha$ and $\partial_i^{\varprojlim Y^\alpha} = \varprojlim \partial_i^{Y^\alpha}$ for every $i \in \mathbb{Z}$. Indeed, it is easy to see that $\varprojlim X^\alpha$ satisfies the universal property of inverse limits in a category.

(iv) *The homotopy inverse limit of the inverse system* $\left\{Y^\alpha, \psi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ *is given by* $\underleftarrow{\operatorname{holim}}\, Y^\alpha = \Sigma^{-1}\operatorname{Cone}(\varpi)$, *where the morphism* $\varpi : \prod_{\alpha=1}^{\infty} Y^\alpha \to \prod_{\alpha=1}^{\infty} Y^\alpha$ *is given by* $\varpi_i\left((y_i^\alpha)\right) = \left(y_i^\alpha - \psi_i^{\alpha+1,\alpha}(y_i^{\alpha+1})\right)$ *for every* $i \in \mathbb{Z}$*. Indeed, it is easy to see that the morphism* ϖ *fits into a distinguished triangle*

$$\underleftarrow{\operatorname{holim}}\, Y^\alpha \to \prod_{\alpha=1}^{\infty} Y^\alpha \to \prod_{\alpha=1}^{\infty} Y^\alpha \to .$$

The Mittag-Leffler condition forces many limits to vanish.

Lemma 11. *Let* $\{M_\alpha, \varphi_{\alpha\beta}\}_{\alpha\in\mathbb{N}}$ *be an inverse system of* R*-modules that satisfies the trivial Mittag-Leffler condition, and* $\mathcal{F} : \mathcal{M}(R) \to \mathcal{M}(R)$ *an additive contravariant functor. Then the following assertions hold:*

(i) $\underleftarrow{\lim}\, M_\alpha = 0 = \underleftarrow{\lim}^1 M_\alpha$.

(ii) $\underrightarrow{\lim}\, \mathcal{F}(M_\alpha) = 0$.

Proof. (i): Let $\varpi : \prod_{\alpha\in\mathbb{N}} M_\alpha \to \prod_{\alpha\in\mathbb{N}} M_\alpha$ be an R-homomorphism given by $\varpi\left((x_\alpha)\right) = \left(x_\alpha - \varphi_{\alpha+1,\alpha}(x_{\alpha+1})\right)$. We show that ϖ is an isomorphism. Let $(x_\alpha) \in \prod_{\alpha\in\mathbb{N}} M_\alpha$ be such that $x_\alpha = \varphi_{\alpha+1,\alpha}(x_{\alpha+1})$ for every $\alpha \in \mathbb{N}$. Fix $\alpha \in \mathbb{N}$, and by the trivial Mittag-Leffler condition choose $\gamma \geq \alpha$ such that $\varphi_{\gamma\alpha} = 0$. Then we have

$$\begin{aligned} x_\alpha &= \varphi_{\alpha+1,\alpha}(x_{\alpha+1}) \\ &= \varphi_{\alpha+1,\alpha}\left(\varphi_{\alpha+2,\alpha+1}\left(\ldots\left(\varphi_{\gamma,\gamma-1}(x_\gamma)\right)\right)\right) \\ &= \varphi_{\gamma\alpha}(x_\gamma) \\ &= 0. \end{aligned}$$

Hence $(x_\alpha) = 0$, and thus ϖ is injective. Now let $(y_\alpha) \in \prod_{\alpha\in\mathbb{N}} M_\alpha$. For any $\beta \in \mathbb{N}$, we set $x_\beta := \sum_{\alpha=\beta}^{\infty} \varphi_{\alpha\beta}(y_\alpha)$ which is a finite sum by the trivial Mittag-Leffler condition. Then we have

$$\begin{aligned} \varphi_{\beta+1,\beta}(x_{\beta+1}) &= \varphi_{\beta+1,\beta}\left(\sum_{\alpha=\beta+1}^{\infty} \varphi_{\alpha,\beta+1}(y_\alpha)\right) \\ &= \sum_{\alpha=\beta+1}^{\infty} \varphi_{\alpha\beta}(y_\alpha) \\ &= \sum_{\alpha=\beta}^{\infty} \varphi_{\alpha\beta}(y_\alpha) - \varphi_{\beta\beta}(y_\beta) \\ &= x_\beta - y_\beta. \end{aligned}$$

Therefore, we have

$$\varpi\left((x_\alpha)\right) = \left(x_\alpha - \varphi_{\alpha+1,\alpha}(x_{\alpha+1})\right) = (y_\alpha),$$

so ϖ is surjective. It follows that ϖ is an isomorphism. Therefore, $\varprojlim M_\alpha \cong \ker \varpi = 0$ and $\varprojlim^1 M_\alpha \cong \operatorname{coker} \varpi = 0$.

(ii): First we note that $\{\mathcal{F}(M_\alpha), \psi_{\beta\alpha} := \mathcal{F}(\varphi_{\alpha\beta})\}_{\alpha\in\mathbb{N}}$ is a direct system of R-modules. Let $\psi_\alpha : \mathcal{F}(M_\alpha) \to \varinjlim \mathcal{F}(M_\alpha)$ be the natural injection of direct limit for every $\alpha \in \mathbb{N}$. We know that an arbitrary element of $\varinjlim \mathcal{F}(M_\alpha)$ is of the form $\psi_t(y)$ for some $t \in \mathbb{N}$ and some $y \in \mathcal{F}(M_t)$. By the trivial Mittag-Leffler condition, there is an integer $s \geq t$ such that $\varphi_{st} = 0$, so that $\psi_{ts} = \mathcal{F}(\varphi_{st}) = 0$. Then $\psi_t(y) = \psi_s\left(\psi_{ts}(y)\right) = 0$. Hence $\varinjlim \mathcal{F}(M_\alpha) = 0$. □

The next proposition collects some information on the homology of limits.

Proposition 12. *Let $\left\{X^\alpha, \varphi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ be a direct system of R-complexes, and let $\left\{Y^\alpha, \psi^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ be an inverse system of R-complexes. Then the following assertions hold for every $i \in \mathbb{Z}$:*

(i) There is a natural isomorphism $H_i\left(\varinjlim X^\alpha\right) \cong \varinjlim H_i(X^\alpha)$.

(ii) There is a natural isomorphism $H_i\left(\underrightarrow{\operatorname{holim}}\, X^\alpha\right) \cong \varinjlim H_i(X^\alpha)$.

(iii) If the inverse system $\left\{Y_i^\alpha, \psi_i^{\alpha\beta}\right\}_{\alpha\in\mathbb{N}}$ of R-modules satisfies the Mittag-Leffler condition for every $i \in \mathbb{Z}$, then there is a short exact sequence

$$0 \to \varprojlim{}^1 H_{i+1}(Y^\alpha) \to H_i\left(\varprojlim Y^\alpha\right) \to \varprojlim H_i(Y^\alpha) \to 0$$

of R-modules.

(iv) There is a short exact sequence

$$0 \to \varprojlim{}^1 H_{i+1}(Y^\alpha) \to H_i\left(\underleftarrow{\operatorname{holim}}\, Y^\alpha\right) \to \varprojlim H_i(Y^\alpha) \to 0$$

of R-modules.

Proof. (i): See [16, Theorem 4.2.4].

(ii): See the paragraph before [7, Lemma 0.1].

(iii): See [21, Theorem 3.5.8].

(iv): See the paragraph after [7, Lemma 0.1]. □

Now we are ready to present the following definitions.

Definition 13. *Let $\underline{a} = a_1, ..., a_n \in R$. Then:*

(i) *Define the Čech complex on the elements* $\underline{a}$ *to be* $\check{C}(\underline{a}) := \varinjlim \Sigma^{-n} K^R(\underline{a}^k)$.

(ii) *Define the stable Čech complex on the elements* $\underline{a}$ *to be*

$$\check{C}_\infty(\underline{a}) := \underrightarrow{\operatorname{holim}}\, \Sigma^{-n} K^R(\underline{a}^k).$$

We note that $\check{C}(\underline{a})$ is a bounded R-complex of flat modules concentrated in degrees $0, -1, ..., -n$, and $\check{C}_\infty(\underline{a})$ is a bounded R-complex of free modules concentrated in degrees $1, 0, ..., -n$. Moreover, it can be shown that there is a quasi-isomorphism $\check{C}_\infty(\underline{a}) \xrightarrow{\simeq} \check{C}(\underline{a})$, which in turn implies that $\check{C}_\infty(\underline{a}) \simeq \check{C}(\underline{a})$ in $\mathcal{D}(R)$. Therefore, $\check{C}_\infty(\underline{a})$ is a semi-projective approximation of the semi-flat R-complex $\check{C}(\underline{a})$.

The next proposition investigates the relation between local cohomology and local homology with Čech complex and stable Čech complex, and provides the first essential step towards the Greenlees-May Duality.

Proposition 14. *Let* $\mathfrak{a} = (a_1, ..., a_n)$ *be an ideal of* R, $\underline{a} = a_1, ..., a_n$, *and* M *an* R*-module. Then there are natural isomorphisms for every* $i \geq 0$*:*

(i) $H^i_{\mathfrak{a}}(M) \cong H_{-i}\left(\check{C}(\underline{a}) \otimes_R M\right) \cong H_{-i}\left(\check{C}_\infty(\underline{a}) \otimes_R M\right)$.

(ii) $H^{\mathfrak{a}}_i(M) \cong H_i\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), M\right)\right)$.

Proof. (i): Let $\mathcal{F}^i = H_{-i}\left(\check{C}(\underline{a}) \otimes_R -\right) : \mathcal{M}(R) \to \mathcal{M}(R)$ for every $i \geq 0$. Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules, since $\check{C}(\underline{a})$ is an R-complex of flat modules, the functor

$$\check{C}(\underline{a}) \otimes_R - : \mathcal{C}(R) \to \mathcal{C}(R)$$

is exact, whence we get a short exact sequence

$$0 \to \check{C}(\underline{a}) \otimes_R M' \to \check{C}(\underline{a}) \otimes_R M \to \check{C}(\underline{a}) \otimes_R M'' \to 0$$

of R-complexes, which in turn yields a long exact homology sequence in a functorial way. This shows that $\left(\mathcal{F}^i : \mathcal{M}(R) \to \mathcal{M}(R)\right)_{i \geq 0}$ is a cohomological covariant δ-functor. Moreover, using Proposition 12 (i), we have

$$\begin{aligned} \mathcal{F}^i &= H_{-i}\left(\check{C}(\underline{a}) \otimes_R -\right) \\ &= H_{-i}\left(\left(\varinjlim \Sigma^{-n} K^R(\underline{a}^k)\right) \otimes_R -\right) \\ &\cong \varinjlim H_{n-i}\left(K^R(\underline{a}^k) \otimes_R -\right) \\ &\cong \varinjlim H_{n-i}\left(\underline{a}^k; -\right) \end{aligned} \tag{2.3}$$

for every $i \geq 0$.

Let I be an injective R-module. Then by the display (2.3), we have

$$\begin{aligned}\mathcal{F}^i(I) &= \varinjlim H_{n-i}\left(\underline{a}^k; I\right) \\ &\cong \varinjlim H_{-i}\left(\operatorname{Hom}_R\left(K^R(\underline{a}^k), I\right)\right) \\ &\cong \varinjlim \operatorname{Hom}_R\left(H_i\left(K^R(\underline{a}^k)\right), I\right) \\ &\cong \varinjlim \operatorname{Hom}_R\left(H_i\left(\underline{a}^k; R\right), I\right).\end{aligned} \tag{2.4}$$

By Lemma 9, the inverse system $\left\{H_i\left(\underline{a}^k; R\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$. Now Lemma 11 (ii) implies that $\varinjlim \operatorname{Hom}_R\left(H_i\left(\underline{a}^k; R\right), I\right) = 0$, thereby the display (2.4) shows that $\mathcal{F}^i(I) = 0$ for every $i \geq 1$.

Let M be an R-module. Then by the display (2.3), we have the natural isomorphisms

$$\begin{aligned}\mathcal{F}^0(M) &\cong \varinjlim H_n\left(\underline{a}^k; M\right) \\ &\cong \varinjlim \left(0 :_M (\underline{a}^k)\right) \\ &\cong \varinjlim \operatorname{Hom}_R\left(R/(\underline{a}^k), M\right) \\ &\cong \varinjlim \operatorname{Hom}_R\left(R/\mathfrak{a}^k, M\right) \\ &\cong \Gamma_{\mathfrak{a}}(M) \\ &\cong H^0_{\mathfrak{a}}(M).\end{aligned}$$

It follows from Corollary 7 (ii) that $H^i_{\mathfrak{a}}(-) \cong \mathcal{F}^i$ for every $i \geq 0$.

For the second isomorphism, using the display (2.3) and Proposition 12 (ii), we have the natural isomorphisms

$$\begin{aligned}H^i_{\mathfrak{a}}(M) &\cong \mathcal{F}^i(M) \\ &\cong \varinjlim H_{n-i}\left(\underline{a}^k; M\right) \\ &\cong \varinjlim H_{n-i}\left(K^R(\underline{a}^k) \otimes_R M\right) \\ &\cong H_{n-i}\left(\underrightarrow{\operatorname{holim}}\left(K^R(\underline{a}^k) \otimes_R M\right)\right) \\ &\cong H_{-i}\left(\left(\underrightarrow{\operatorname{holim}}\, \Sigma^{-n} K^R(\underline{a})\right) \otimes_R M\right) \\ &\cong H_{-i}\left(\check{C}_\infty(\underline{a}) \otimes_R M\right)\end{aligned}$$

for every $i \geq 0$.

(ii): Let $\mathcal{F}_i = H_i\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), -\right)\right) : \mathcal{M}(R) \to \mathcal{M}(R)$ for every $i \geq 0$. Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

R-modules, since $\check{C}_\infty(\underline{a})$ is an R-complex of free modules, the functor

$$\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), -\right) : \mathcal{C}(R) \to \mathcal{C}(R)$$

is exact, whence we get a short exact sequence

$$0 \to \operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), M'\right) \to \operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), M\right) \to \operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), M''\right) \to 0$$

of R-complexes, which in turn yields a long exact homology sequence in a functorial way. It follows that $\left(\mathcal{F}_i : \mathcal{M}(R) \to \mathcal{M}(R)\right)_{i\geq 0}$ is a homological covariant δ-functor. Moreover, using the self-duality property of Koszul complex, we have

$$\begin{aligned} \mathcal{F}_i &= H_i\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), -\right)\right) \\ &= H_i\left(\operatorname{Hom}_R\left(\underrightarrow{\operatorname{holim}}\, \Sigma^{-n} K^R(\underline{a}^k), -\right)\right) \\ &\cong H_i\left(\underleftarrow{\operatorname{holim}}\, \Sigma^{n} \operatorname{Hom}_R\left(K^R(\underline{a}^k), -\right)\right) \\ &\cong H_i\left(\underleftarrow{\operatorname{holim}}\left(K^R(\underline{a}^k) \otimes_R -\right)\right) \end{aligned} \tag{2.5}$$

for every $i \geq 0$.

Let M be an R-module. By Proposition 12 (iv), we get a short exact sequence

$$\begin{gathered} 0 \to \underleftarrow{\lim}^1 H_{i+1}\left(K^R(\underline{a}^k) \otimes_R M\right) \to H_i\left(\underleftarrow{\operatorname{holim}}\left(K^R(\underline{a}^k) \otimes_R M\right)\right) \to \\ \underleftarrow{\lim}\, H_i\left(K^R(\underline{a}^k) \otimes_R M\right) \to 0, \end{gathered}$$

which implies the short exact sequence

$$0 \to \underleftarrow{\lim}^1 H_{i+1}\left(\underline{a}^k; M\right) \to \mathcal{F}_i(M) \to \underleftarrow{\lim}\, H_i\left(\underline{a}^k; M\right) \to 0$$

of R-modules for every $i \geq 0$.

Let F be a free R-module. If $i \geq 1$, then the inverse system $\left\{H_i\left(\underline{a}^k; R\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition by Lemma 9. But

$$H_i\left(\underline{a}^k; F\right) \cong H_i\left(\underline{a}^k; R\right) \otimes_R F,$$

so it straightforward to see that the inverse system $\left\{H_i\left(\underline{a}^k; F\right)\right\}_{k\in\mathbb{N}}$ satisfies the trivial Mittag-Leffler condition for every $i \geq 1$. Therefore, Lemma 11 (i) implies that

$$\underleftarrow{\lim}^1 H_i\left(\underline{a}^k; F\right) = 0 = \underleftarrow{\lim}\, H_i\left(\underline{a}^k; F\right)$$

for every $i \geq 1$. It follows from the above short exact sequence that $\mathcal{F}_i(F) = 0$ for every $i \geq 1$.

Upon setting $i = 0$, the above short exact sequence yields

$$0 = \underleftarrow{\lim}^1 H_1\left(\underline{a}^k; F\right) \to \mathcal{F}_0(F) \to \underleftarrow{\lim}\, H_0\left(\underline{a}^k; F\right) \to 0.$$

Thus we get the natural isomorphisms

$$
\begin{aligned}
\mathcal{F}_0(F) &\cong \varprojlim H_0\left(\underline{a}^k; F\right) \\
&\cong \varprojlim F/(\underline{a}^k)F \\
&\cong \varprojlim F/\mathfrak{a}^k F \\
&= \widehat{F}^{\mathfrak{a}} \\
&\cong H_0^{\mathfrak{a}}(F).
\end{aligned}
$$

It now follows from Corollary 7 (i) that $H_i^{\mathfrak{a}}(-) \cong \mathcal{F}_i$ for every $i \geq 0$. □

Remark 15. *One should note that* $H_i^{\mathfrak{a}}(M) \not\cong H_i\left(\operatorname{Hom}_R\left(\check{C}(\underline{a}), M\right)\right)$.

3 Complex Prerequisites

In this section, we commence on developing the requisite tools on complexes which are to be deployed in Section 4. For more information on the material in this section, refer to [1], [8], [5], [11], and [19].

The derived category $\mathcal{D}(R)$ is defined as the localization of the homotopy category $\mathcal{K}(R)$ with respect to the multiplicative system of quasi-isomorphisms. Simply put, an object in $\mathcal{D}(R)$ is an R-complex X displayed in the standard homological style

$$X = \cdots \to X_{i+1} \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \to \cdots ,$$

and a morphism $\varphi : X \to Y$ in $\mathcal{D}(R)$ is given by the equivalence class of a pair (f, g) of morphisms $X \xleftarrow{g} U \xrightarrow{f} Y$ in $\mathcal{C}(R)$ with g a quasi-isomorphism, under the equivalence relation that identifies two such pairs (f, g) and (f', g'), whenever there is a diagram in $\mathcal{C}(R)$ as follows which commutes up to homotopy:

$$
\begin{tikzcd}
 & U \arrow[dl, "g"', "\simeq"] \arrow[dr, "f"] & \\
X & V \arrow[l, "\simeq"'] \arrow[u] \arrow[r] \arrow[d] & Y \\
 & U' \arrow[ul, "g'", "\simeq"'] \arrow[ur, "f'"'] &
\end{tikzcd}
$$

The isomorphisms in $\mathcal{D}(R)$ are marked by the symbol $\simeq$.

The derived category $\mathcal{D}(R)$ is triangulated. A distinguished triangle in $\mathcal{D}(R)$ is a triangle that is isomorphic to a triangle of the form

$$X \xrightarrow{\mathfrak{L}(f)} Y \xrightarrow{\mathfrak{L}(\varepsilon)} \operatorname{Cone}(f) \xrightarrow{\mathfrak{L}(\varpi)} \Sigma X,$$

for some morphism $f : X \to Y$ in $\mathcal{C}(R)$ with the mapping cone sequence

$$0 \to Y \xrightarrow{\varepsilon} \operatorname{Cone}(f) \xrightarrow{\varpi} \Sigma X \to 0,$$

in which $\mathfrak{L} : \mathcal{C}(R) \to \mathcal{D}(R)$ is the canonical functor that is defined as $\mathfrak{L}(X) = X$ for every R-complex X, and $\mathfrak{L}(f) = \varphi$ where φ is represented by the morphisms $X \xleftarrow{1^X} X \xrightarrow{f} Y$ in $\mathcal{C}(R)$. We note that if f is a quasi-isomorphism in $\mathcal{C}(R)$, then $\mathfrak{L}(f)$ is an isomorphism in $\mathcal{D}(R)$. We sometimes use the shorthand notation

$$X \to Y \to Z \to$$

for a distinguished triangle.

We let $\mathcal{D}_{\sqsubset}(R)$ (res. $\mathcal{D}_{\sqsupset}(R)$) denote the full subcategory of $\mathcal{D}(R)$ consisting of R-complexes X with $H_i(X) = 0$ for sufficiently large (res. small) i, and $D_{\square}(R) := \mathcal{D}_{\sqsubset}(R) \cap \mathcal{D}_{\sqsupset}(R)$. We further let $\mathcal{D}^f(R)$ denote the full subcategory of $\mathcal{D}(R)$ consisting of R-complexes X with finitely generated homology modules. We also feel free to use any combination of the subscripts and the superscript as in $\mathcal{D}^f_{\square}(R)$, with the obvious meaning of the intersection of the two subcategories involved.

We recall the resolutions of complexes.

Definition 16. *We have:*

(i) An R-complex P of projective modules is said to be semi-projective if the functor $\operatorname{Hom}_R(P, -)$ preserves quasi-isomorphisms. By a semi-projective resolution of an R-complex X, we mean a quasi-isomorphism $P \xrightarrow{\simeq} X$ in which P is a semi-projective R-complex.

(ii) An R-complex I of injective modules is said to be semi-injective if the functor $\operatorname{Hom}_R(-, I)$ preserves quasi-isomorphisms. By a semi-injective resolution of an R-complex X, we mean a quasi-isomorphism $X \xrightarrow{\simeq} I$ in which I is a semi-injective R-complex.

(iii) An R-complex F of flat modules is said to be semi-flat if the functor $F \otimes_R -$ preserves quasi-isomorphisms. By a semi-flat resolution of an R-complex X, we mean a quasi-isomorphism $F \xrightarrow{\simeq} X$ in which F is a semi-flat R-complex.

Semi-projective, semi-injective, and semi-flat resolutions exist for any R-complex. Moreover, any right-bounded R-complex of projective (flat) modules is semi-projective (semi-flat), and any left-bounded R-complex of injective modules is semi-injective.

We now remind the total derived functors that we need.

Remark 17. *Let $\mathfrak{a}$ be an ideal of R, and X and Y two R-complexes. Then we have:*

(i) Each of the functors $\operatorname{Hom}_R(X,-)$ and $\operatorname{Hom}_R(-,Y)$ on $\mathcal{C}(R)$ enjoys a right total derived functor on $\mathcal{D}(R)$, together with a balance property, in the sense that $\mathbf{R}\operatorname{Hom}_R(X,Y)$ can be computed by

$$\mathbf{R}\operatorname{Hom}_R(X,Y) \simeq \operatorname{Hom}_R(P,Y) \simeq \operatorname{Hom}_R(X,I),$$

where $P \xrightarrow{\simeq} X$ is any semi-projective resolution of X, and $Y \xrightarrow{\simeq} I$ is any semi-injective resolution of Y. In addition, these functors turn out to be triangulated, in the sense that they preserve shifts and distinguished triangles. Moreover, we let

$$\operatorname{Ext}^i_R(X,Y) := H_{-i}\left(\mathbf{R}\operatorname{Hom}_R(X,Y)\right)$$

for every $i \in \mathbb{Z}$.

(ii) Each of the functors $X \otimes_R -$ and $- \otimes_R Y$ on $\mathcal{C}(R)$ enjoys a left total derived functor on $\mathcal{D}(R)$, together with a balance property, in the sense that $X \otimes^{\mathbf{L}}_R Y$ can be computed by

$$X \otimes^{\mathbf{L}}_R Y \simeq P \otimes_R Y \simeq X \otimes_R Q,$$

where $P \xrightarrow{\simeq} X$ is any semi-projective resolution of X, and $Q \xrightarrow{\simeq} Y$ is any semi-projective resolution of Y. Besides, these functors turn out to be triangulated. Moreover, we let

$$\operatorname{Tor}^R_i(X,Y) := H_i\left(X \otimes^{\mathbf{L}}_R Y\right)$$

for every $i \in \mathbb{Z}$.

(iii) The functor $\Gamma_{\mathfrak{a}}(-)$ on $\mathcal{M}(R)$ extends naturally to a functor on $\mathcal{C}(R)$. The extended functor enjoys a right total derived functor $\mathbf{R}\Gamma_{\mathfrak{a}}(-) : \mathcal{D}(R) \to \mathcal{D}(R)$, that can be computed by $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$, where $X \xrightarrow{\simeq} I$ is any semi-injective resolution of X. Besides, we define the ith local cohomology module of X to be

$$H^i_{\mathfrak{a}}(X) := H_{-i}\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X)\right)$$

for every $i \in \mathbb{Z}$. The functor $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ turns out to be triangulated.

(iv) The functor $\Lambda^{\mathfrak{a}}(-)$ on $\mathcal{M}(R)$ extends naturally to a functor on $\mathcal{C}(R)$. The extended functor enjoys a left total derived functor $\mathbf{L}\Lambda^{\mathfrak{a}}(-) : \mathcal{D}(R) \to \mathcal{D}(R)$, that can be computed by $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(P)$, where $P \xrightarrow{\simeq} X$ is any semi-projective resolution of X. Moreover, we define the ith local homology module of X to be

$$H^{\mathfrak{a}}_i(X) := H_i\left(\mathbf{L}\Lambda^{\mathfrak{a}}(X)\right)$$

for every $i \in \mathbb{Z}$. The functor $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ turns out to be triangulated.

We further need the notion of way-out functors for functors between the category of complexes.

Definition 18. *Let R and S be two rings, and $\mathcal{F} : \mathcal{C}(R) \to \mathcal{C}(S)$ a covariant functor. Then:*

(i) $\mathcal{F}$ is said to be way-out left if for every $n \in \mathbb{Z}$, there is an $m \in \mathbb{Z}$, such that for any R-complex X with $X_i = 0$ for every $i > m$, we have $\mathcal{F}(X)_i = 0$ for every $i > n$.

(ii) $\mathcal{F}$ is said to be way-out right if for every $n \in \mathbb{Z}$, there is an $m \in \mathbb{Z}$, such that for any R-complex X with $X_i = 0$ for every $i < m$, we have $\mathcal{F}(X)_i = 0$ for every $i < n$.

(iii) $\mathcal{F}$ is said to be way-out if it is both way-out left and way-out right.

The following lemma is the Way-out Lemma for functors between the category of complexes. We include a proof since there is no account of this version in the literature.

Lemma 19. *Let R and S be two rings, and $\mathcal{F}, \mathcal{G} : \mathcal{C}(R) \to \mathcal{C}(S)$ two additive covariant functors that commute with shift and preserve the exactness of degreewise split short exact sequences of R-complexes. Let $\sigma : \mathcal{F} \to \mathcal{G}$ be a natural transformation of functors. Then the following assertions hold:*

(i) If X is a bounded R-complex such that $\sigma^{X_i} : \mathcal{F}(X_i) \to \mathcal{G}(X_i)$ is a quasi-isomorphism for every $i \in \mathbb{Z}$, then $\sigma^X : \mathcal{F}(X) \to \mathcal{G}(X)$ is a quasi-isomorphism.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are way-out left, and X is a left-bounded R-complex such that $\sigma^{X_i} : \mathcal{F}(X_i) \to \mathcal{G}(X_i)$ is a quasi-isomorphism for every $i \in \mathbb{Z}$, then $\sigma^X : \mathcal{F}(X) \to \mathcal{G}(X)$ is a quasi-isomorphism.

(iii) If $\mathcal{F}$ and $\mathcal{G}$ are way-out right, and X is a right-bounded R-complex such that $\sigma^{X_i} : \mathcal{F}(X_i) \to \mathcal{G}(X_i)$ is a quasi-isomorphism for every $i \in \mathbb{Z}$, then $\sigma^X : \mathcal{F}(X) \to \mathcal{G}(X)$ is a quasi-isomorphism.

(iv) If $\mathcal{F}$ and $\mathcal{G}$ are way-out, and X is an R-complex such that $\sigma^{X_i} : \mathcal{F}(X_i) \to \mathcal{G}(X_i)$ is a quasi-isomorphism for every $i \in \mathbb{Z}$, then $\sigma^X : \mathcal{F}(X) \to \mathcal{G}(X)$ is a quasi-isomorphism.

Proof. (i): Without loss of generality we may assume that

$$X : 0 \to X_n \xrightarrow{\partial_n^X} X_{n-1} \to \cdots \to X_1 \xrightarrow{\partial_1^X} X_0 \to 0.$$

Let

$$Y : 0 \to X_{n-1} \xrightarrow{\partial_{n-1}^X} X_{n-2} \to \cdots \to X_1 \xrightarrow{\partial_1^X} X_0 \to 0.$$

Consider the degreewise split short exact sequence

$$0 \to Y \to X \to \Sigma^n X_n \to 0$$

of R-complexes, and apply $\mathcal{F}$ and $\mathcal{G}$ to get the following commutative diagram of S-complexes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(Y) \arrow[r] \arrow[d, "\sigma_Y"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\sigma_X"] & \Sigma^n \mathcal{F}(X_n) \arrow[r] \arrow[d, "\Sigma^n \sigma_{X_n}"] & 0 \\ 0 \arrow[r] & \mathcal{G}(Y) \arrow[r] & \mathcal{G}(X) \arrow[r] & \Sigma^n \mathcal{G}(X_n) \arrow[r] & 0 \end{tikzcd}$$

Note that $\Sigma^n \sigma_{X_n}$ is a quasi-isomorphism by the assumption. Hence to prove that σ_X is a quasi-isomorphism, it suffices to show that σ_Y is a quasi-isomorphism. Since Y is bounded, by continuing this process with Y, we reach at a level that we need σ_{X_0} to be a quasi-isomorphism, which holds by the assumption. Therefore, we are done.

(ii): Without loss of generality we may assume that

$$X : 0 \to X_n \xrightarrow{\partial_n^X} X_{n-1} \to \cdots .$$

Let $i \in \mathbb{Z}$. We show that $H_i(\sigma_X) : H_i(\mathcal{F}(X)) \to H_i(\mathcal{G}(X))$ is an isomorphism. Since $\mathcal{F}$ and $\mathcal{G}$ are way-out left, we can choose an integer $j \in \mathbb{Z}$ corresponding to $i-2$. Let

$$Z : 0 \to X_n \xrightarrow{\partial_n^X} X_{n-1} \to \cdots \to X_{j+1} \xrightarrow{\partial_{j+1}^X} X_j \to 0$$

and

$$Y : 0 \to X_{j-1} \xrightarrow{\partial_{j-1}^X} X_{j-2} \to \cdots .$$

Then there is a degreewise split short exact sequence

$$0 \to Y \to X \to Z \to 0$$

of R-complexes. Apply $\mathcal{F}$ and $\mathcal{G}$ to get the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}(Y) \arrow[r] \arrow[d, "\sigma_Y"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\sigma_X"] & \mathcal{F}(Z) \arrow[r] \arrow[d, "\sigma_Z"] & 0 \\ 0 \arrow[r] & \mathcal{G}(Y) \arrow[r] & \mathcal{G}(X) \arrow[r] & \mathcal{G}(Z) \arrow[r] & 0 \end{tikzcd}$$

From the above diagram, we get the following commutative diagram of S-modules with exact rows:

$$\begin{tikzcd}
0 = H_i\left(\mathcal{F}(Y)\right) \arrow[r] & H_i\left(\mathcal{F}(X)\right) \arrow[r] \arrow[d, "H_i(\sigma_X)"] & H_i\left(\mathcal{F}(Z)\right) \arrow[r] \arrow[d, "H_i(\sigma_Z)"] & H_{i-1}\left(\mathcal{F}(Y)\right) = 0 \\
0 = H_i\left(\mathcal{G}(Y)\right) \arrow[r] & H_i\left(\mathcal{G}(X)\right) \arrow[r] & H_i\left(\mathcal{G}(Z)\right) \arrow[r] & H_{i-1}\left(\mathcal{G}(Y)\right) = 0
\end{tikzcd}$$

where the vanishing is due to the choice of j. Since Z is bounded, it follows from (i) that $H_i(\sigma_Z)$ is an isomorphism, and as a consequence, $H_i(\sigma_X)$ is an isomorphism.

(iii): Without loss of generality we may assume that

$$X : \cdots \to X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \to 0.$$

Let $i \in \mathbb{Z}$. We show that $H_i(\sigma_X) : H_i\left(\mathcal{F}(X)\right) \to H_i\left(\mathcal{G}(X)\right)$ is an isomorphism. Since $\mathcal{F}$ and $\mathcal{G}$ are way-out right, we can choose an integer $j \in \mathbb{Z}$ corresponding to $i+2$. Let

$$Y : 0 \to X_{j-1} \xrightarrow{\partial^X_{j-1}} X_{j-2} \to \cdots \to X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \to 0$$

and

$$Z : \cdots \to X_{j+1} \xrightarrow{\partial^X_{j+1}} X_j \to 0.$$

Then there is a degreewise split short exact sequence

$$0 \to Y \to X \to Z \to 0$$

of R-complexes. Apply $\mathcal{F}$ and $\mathcal{G}$ to get the following commutative diagram of S-complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(Y) \arrow[r] \arrow[d, "\sigma_Y"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\sigma_X"] & \mathcal{F}(Z) \arrow[r] \arrow[d, "\sigma_Z"] & 0 \\
0 \arrow[r] & \mathcal{G}(Y) \arrow[r] & \mathcal{G}(X) \arrow[r] & \mathcal{G}(Z) \arrow[r] & 0
\end{tikzcd}$$

From the above diagram, we get the following commutative diagram of S-modules with exact rows:

$$\begin{tikzcd}
0 = H_{i+1}\left(\mathcal{F}(Z)\right) \arrow[r] & H_i\left(\mathcal{F}(Y)\right) \arrow[r] \arrow[d, "H_i(\sigma_Y)"] & H_i\left(\mathcal{F}(X)\right) \arrow[r] \arrow[d, "H_i(\sigma_X)"] & H_i\left(\mathcal{F}(Z)\right) = 0 \\
0 = H_{i+1}\left(\mathcal{G}(Z)\right) \arrow[r] & H_i\left(\mathcal{G}(Y)\right) \arrow[r] & H_i\left(\mathcal{G}(X)\right) \arrow[r] & H_i\left(\mathcal{G}(Z)\right) = 0
\end{tikzcd}$$

where the vanishing is due to the choice of j. Since Y is bounded, it follows from (i) that $H_i(\sigma_Y)$ is an isomorphism, and as a consequence, $H_i(\sigma_X)$ is an isomorphism.

(iv): Let

$$Y : 0 \to X_0 \xrightarrow{\partial_0^X} X_{-1} \to \cdots$$

and

$$Z : \cdots \to X_2 \xrightarrow{\partial_2^X} X_1 \to 0.$$

Then there is a degreewise split short exact sequence

$$0 \to Y \to X \to Z \to 0$$

of R-complexes. Applying $\mathcal{F}$ and $\mathcal{G}$, we get the following commutative diagram of S-complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(Y) \arrow[r] \arrow[d, "\sigma_Y"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\sigma_X"] & \mathcal{F}(Z) \arrow[r] \arrow[d, "\sigma_Z"] & 0 \\
0 \arrow[r] & \mathcal{G}(Y) \arrow[r] & \mathcal{G}(X) \arrow[r] & \mathcal{G}(Z) \arrow[r] & 0
\end{tikzcd}$$

Since Y is left-bounded, σ_Y is a quasi-isomorphism by (ii), and since Z is right-bounded, σ_Z is a quasi-isomorphism by (iii). Therefore, σ_X is a quasi-isomorphism. □

Although $\check{C}_\infty(\underline{a})$ is suitable in Proposition 14, it is not applicable in the next proposition due to the fact that it is concentrated in degrees $1, 0, ..., -n$. What we really need here is a semi-projective approximation of $\check{C}(\underline{a})$ of the same length, i.e. concentrated in degrees $0, -1, ..., -n$. We proceed as follows.

Given an element $a \in R$, consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R[X] \oplus R \arrow[r, "f_a"] \arrow[d, "\pi"] & R[X] \arrow[r] \arrow[d, "g_a"] & 0 \\
0 \arrow[r] & R \arrow[r, "\lambda_R^a"] & R_a \arrow[r] & 0
\end{tikzcd}$$

in which, $f_a\,(p(X), b) = (aX - 1)p(X) + b$, $\pi\,(p(X), b) = b$, λ_R^a is the localization map, and $g_a\,(p(X)) = \frac{b_k}{a^k} + \cdots + \frac{b_1}{a} + \frac{b_0}{1}$ where $p(X) = b_kX^k + \cdots + b_1X + b_0 \in R[X]$. Let $L^R(a)$ denote the R-complex in the first row of the diagram above concentrated in degrees $0, -1$. Since the second row is isomorphic to $\check{C}(a)$, it can be seen that the diagram above provides a quasi-isomorphism $L^R(a) \xrightarrow{\simeq} \check{C}(a)$. Hence $L^R(a) \xrightarrow{\simeq} \check{C}(a)$ is a semi-projective resolution of $\check{C}(a)$. Now for the elements $\underline{a} = a_1, ..., a_n \in R$, let

$$L^R(\underline{a}) = L^R(a_1) \otimes_R \cdots \otimes_R L^R(a_n).$$

Then $L^R(\underline{a})$ is an R-complex of free modules concentrated in degrees $0, -1, ..., -n$, and $L^R(\underline{a}) \xrightarrow{\simeq} \check{C}(\underline{a})$ is a semi-projective resolution of $\check{C}(\underline{a})$.

The next proposition inspects the relation between derived torsion functor and derived completion functor with Čech complex, and provides the second crucial step towards the Greenlees-May Duality.

Proposition 20. *Let $\mathfrak{a} = (a_1, ..., a_n)$ be an ideal of R, $\underline{a} = a_1, ..., a_n$, and $X \in \mathcal{D}(R)$. Then there are natural isomorphisms in $\mathcal{D}(R)$:*

(i) $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} X \simeq \check{C}_\infty(\underline{a}) \otimes_R^{\mathbf{L}} X$.

(ii) $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\operatorname{Hom}_R\left(\check{C}(\underline{a}), X\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), X\right)$.

Proof. (i): Let $X \xrightarrow{\simeq} I$ be a semi-injective resolution of X. Then $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \Gamma_{\mathfrak{a}}(I)$, and

$$\check{C}(\underline{a}) \otimes_R^{\mathbf{L}} X \simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} I \simeq \check{C}(\underline{a}) \otimes_R I,$$

since $\check{C}(\underline{a})$ is a semi-flat R-complex. Hence it suffices to establish a quasi-isomorphism $\Gamma_{\mathfrak{a}}(I) \to \check{C}(\underline{a}) \otimes_R I$.

Let Y be an R-complex and $i \in \mathbb{Z}$. Let $\sigma_i^Y : \Gamma_{\mathfrak{a}}(Y)_i \to \left(\check{C}(\underline{a}) \otimes_R Y\right)_i$ be the composition of the following natural R-homomorphisms:

$$\Gamma_{\mathfrak{a}}(Y)_i = \Gamma_{\mathfrak{a}}(Y_i) \xrightarrow{\cong} H^0_{\mathfrak{a}}(Y_i) \xrightarrow{\cong} H_0\left(\check{C}(\underline{a}) \otimes_R Y_i\right) = \ker\left(\partial_0^{\check{C}(\underline{a})} \otimes_R Y_i\right) \to \check{C}(\underline{a})_0 \otimes_R Y_i$$

$$\to \bigoplus_{s+t=i} \left(\check{C}(\underline{a})_s \otimes_R Y_t\right) = \left(\check{C}(\underline{a}) \otimes_R Y\right)_i$$

We note that the second isomorphism above comes from Proposition 14 (i). One can easily see that $\sigma^Y = (\sigma_i^Y)_{i\in\mathbb{Z}} : \Gamma_{\mathfrak{a}}(Y) \to \check{C}(\underline{a}) \otimes_R Y$ is a natural morphism of R-complexes.

Since I_i is an injective R-module for any $i \in \mathbb{Z}$, using Proposition 14 (i), we get

$$H_{-j}\left(\check{C}(\underline{a}) \otimes_R I_i\right) \cong H^j_{\mathfrak{a}}(I_i) = 0$$

for every $j \geq 1$. It follows that $\sigma^{I_i} : \Gamma_{\mathfrak{a}}(I_i) \to \check{C}(\underline{a}) \otimes_R I_i$ is a quasi-isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathfrak{a}}(I_i) \arrow[r] \arrow[d, "\sigma_0^{I_i}"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{C}(\underline{a})_0 \otimes_R I_i \arrow[r] & \check{C}(\underline{a})_{-1} \otimes_R I_i \arrow[r] & \cdots \arrow[r] & \check{C}(\underline{a})_{-n} \otimes_R I_i \arrow[r] & 0
\end{tikzcd}$$

In addition, it is easily seen that the functors $\Gamma_{\mathfrak{a}}(-) : \mathcal{C}(R) \to \mathcal{C}(R)$ and $\check{C}(\underline{a}) \otimes_R - : \mathcal{C}(R) \to \mathcal{C}(R)$ are additive way-out functors that commute with shift and preserve the exactness of degreewise split short exact sequences of R-complexes. Hence by Lemma 19 (iv), we conclude that $\sigma^I : \Gamma_{\mathfrak{a}}(I) \to \check{C}(\underline{a}) \otimes_R I$ is a quasi-isomorphism.

The second isomorphism is immediate since $\check{C}(\underline{a}) \simeq \check{C}_\infty(\underline{a})$ and $- \otimes_R^{\mathbf{L}} X$ is a functor on $\mathcal{D}(R)$.

(ii): We know that $L^R(\underline{a}) \simeq \check{C}(\underline{a}) \simeq \check{C}_\infty(\underline{a})$. Let $P \xrightarrow{\simeq} X$ be a semi-projective resolution of X. Then $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \Lambda^{\mathfrak{a}}(P)$, and

$$\mathbf{R}\operatorname{Hom}_R\left(\check{C}(\underline{a}), X\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(L^R(\underline{a}), P\right) \simeq \operatorname{Hom}_R\left(L^R(\underline{a}), P\right),$$

since $L^R(\underline{a})$ is a semi-projective R-complex. Moreover, we have

$$\operatorname{Hom}_R\left(L^R(\underline{a}), P\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(L^R(\underline{a}), P\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), P\right) \simeq \operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), P\right),$$

since $\check{C}_\infty(\underline{a})$ is a semi-projective R-complex. In particular, we get

$$H_i\left(\operatorname{Hom}_R\left(L^R(\underline{a}), P\right)\right) \cong H_i\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), P\right)\right) \tag{3.1}$$

for every $i \in \mathbb{Z}$. Now it suffices to establish a natural quasi-isomorphism

$$\operatorname{Hom}_R\left(L^R(\underline{a}), P\right) \to \Lambda^{\mathfrak{a}}(P).$$

Let Y be an R-complex and $i \in \mathbb{Z}$. Let $\varsigma_i^Y : \operatorname{Hom}_R\left(L^R(\underline{a}), Y\right)_i \to \Lambda^{\mathfrak{a}}(Y)_i$ be the composition of the following natural R-homomorphisms:

$$\operatorname{Hom}_R\left(L^R(\underline{a}), Y\right)_i = \prod_{s\in\mathbb{Z}} \operatorname{Hom}_R\left(L^R(\underline{a})_s, Y_{s+i}\right) \to \operatorname{Hom}_R\left(L^R(\underline{a})_0, Y_i\right)$$

$$\to \frac{\operatorname{Hom}_R\left(L^R(\underline{a})_0, Y_i\right)}{\operatorname{im}\left(\operatorname{Hom}_R\left(\partial_0^{L^R(\underline{a})}, Y_i\right)\right)} = H_0\left(\operatorname{Hom}_R\left(L^R(\underline{a}), Y_i\right)\right) \xrightarrow{\cong} H_0\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), Y_i\right)\right)$$

$$\xrightarrow{\cong} H_0^{\mathfrak{a}}(Y_i) \to \Lambda^{\mathfrak{a}}(Y_i) = \Lambda^{\mathfrak{a}}(Y)_i$$

We note that the first isomorphism above comes from the isomorphism (3.1) and the second comes from Proposition 14 (ii). One can easily see that $\varsigma^Y = (\varsigma_i^Y)_{i\in\mathbb{Z}} : \operatorname{Hom}_R\left(L^R(\underline{a}), Y\right) \to \Lambda^{\mathfrak{a}}(Y)$ is a natural morphism of R-complexes.

Since P_i is a projective R-module for any $i \in \mathbb{Z}$, using the isomorphism (3.1) and Proposition 14 (ii), we get

$$H_j\left(\operatorname{Hom}_R\left(L^R(\underline{a}), P_i\right)\right) \cong H_j\left(\operatorname{Hom}_R\left(\check{C}_\infty(\underline{a}), P_i\right)\right) \cong H_j^{\mathfrak{a}}(P_i) = 0$$

for every $j \geq 1$. It follows that $\varsigma^{P_i} : \operatorname{Hom}_R\left(L^R(\underline{a}), P_i\right) \to \Lambda^{\mathfrak{a}}(P_i)$ is a quasi-isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R\left(L^R(\underline{a})_{-n}, P_i\right) \arrow[r] \arrow[d] & \cdots \arrow[r] & \operatorname{Hom}_R\left(L^R(\underline{a})_{-1}, P_i\right) \arrow[r] \arrow[d] & \operatorname{Hom}_R\left(L^R(\underline{a})_{0}, P_i\right) \arrow[r] \arrow[d, "\varsigma_0^{P_i}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \Lambda^{\mathfrak{a}}(P_i) \arrow[r] & 0
\end{tikzcd}$$

In addition, it is easily seen that the functors $\operatorname{Hom}_R\left(L^R(\underline{a}), -\right) : \mathcal{C}(R) \to \mathcal{C}(R)$ and $\Lambda^{\mathfrak{a}}(-) : \mathcal{C}(R) \to \mathcal{C}(R)$ are additive way-out functors that commute with shift and preserve the exactness of degreewise split short exact sequences of R-complexes. Hence by Lemma 19 (iv), we conclude that $\varsigma^{P} : \operatorname{Hom}_R\left(L^R(\underline{a}), P\right) \to \Lambda^{\mathfrak{a}}(P)$ is a quasi-isomorphism.

The second isomorphism is immediate since $\check{C}(\underline{a}) \simeq \check{C}_{\infty}(\underline{a})$ and $\mathbf{R}\operatorname{Hom}_R(-, X)$ is a functor on $\mathcal{D}(R)$. □

We note that if $\mathfrak{a} = \langle a_1, ..., a_n \rangle$ is an ideal of R and $\underline{a} = a_1, ..., a_n$, then $\check{C}(\underline{a})$ as an element of $\mathcal{C}(R)$ depends on the generators $\underline{a}$. However, the proof of the next corollary shows that $\check{C}(\underline{a})$ as an element of $\mathcal{D}(R)$ is independent of the generators $\underline{a}$.

Corollary 21. *Let $\mathfrak{a}$ be an ideal of R. Then there are natural isomorphisms in $\mathcal{D}(R)$:*

(i) $\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes_R^{\mathbf{L}} X$.

(ii) $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R), X\right)$.

Proof. Suppose that $\mathfrak{a} = (a_1, ..., a_n)$, and $\underline{a} = a_1, ..., a_n$. By Proposition 20 (i), we have

$$\mathbf{R}\Gamma_{\mathfrak{a}}(R) \simeq \check{C}(\underline{a}) \otimes_R^{\mathbf{L}} R \simeq \check{C}(\underline{a}).$$

Now (i) and (ii) follow from Proposition 20. □

4 Greenlees-May Duality

Having the material developed in Sections 2 and 3 at our disposal, we are fully prepared to prove the celebrated Greenlees-May Duality Theorem.

Theorem 22. *Let $\mathfrak{a}$ be an ideal of R, and $X, Y \in \mathcal{D}(R)$. Then there is a natural isomorphism*

$$\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X), Y\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right)$$

in $\mathcal{D}(R)$.

Proof. Using Corollary 21 and the Adjointness Isomorphism, we have

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y\right) &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R)\otimes_R^{\mathbf{L}}X,Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(X,\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R),X\right)\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(X,\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right).\end{aligned}$$

□

Corollary 23. *Let $\mathfrak{a}$ be an ideal of R, and $X,Y\in\mathcal{D}(R)$. Then there are natural isomorphisms:*

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{a}}\left(\mathbf{R}\operatorname{Hom}_R(X,Y)\right) &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{L}\Lambda^{\mathfrak{a}}(X),\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(X,\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),\mathbf{R}\Gamma_{\mathfrak{a}}(Y)\right).\end{aligned}$$

Proof. By Corollary 21, Adjointness Isomorphism, and Theorem 22, we have

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{a}}\left(\mathbf{R}\operatorname{Hom}_R(X,Y)\right) &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R),\mathbf{R}\operatorname{Hom}_R(X,Y)\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(R)\otimes_R^{\mathbf{L}}X,Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(X,\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right).\end{aligned}\tag{4.1}$$

Further, by Theorem 22, [2, Corollary on Page 6], and [11, Proposition 3.2.2], we have

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right) &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X)\right),Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),\mathbf{R}\Gamma_{\mathfrak{a}}(Y)\right).\end{aligned}\tag{4.2}$$

Moreover, by Theorem 22 and [2, Corollary on Page 6], we have

$$\begin{aligned}\mathbf{R}\operatorname{Hom}_R\left(\mathbf{L}\Lambda^{\mathfrak{a}}(X),\mathbf{L}\Lambda^{\mathfrak{a}}(Y)\right) &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}\left(\mathbf{L}\Lambda^{\mathfrak{a}}(X)\right),Y\right)\\ &\simeq \mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{a}}(X),Y\right).\end{aligned}\tag{4.3}$$

Combining the isomorphisms (4.1), (4.2), and (4.3), we get all the desired isomorphisms.
□

Now we turn our attention to the Grothendieck's Local Duality, and demonstrate how to derive it from the Greenlees-May Duality.

We need the definition of a dualizing complex.

Definition 24. *A dualizing complex for R is an R-complex $D \in \mathcal{D}^f_\square(R)$ that satisfies the following conditions:*

(i) The homothety morphism $\chi^D_R : R \to \mathbf{R}\operatorname{Hom}_R(D,D)$ is an isomorphism in $\mathcal{D}(R)$.

(ii) $\operatorname{id}_R(D) < \infty$.

Moreover, if R is local, then a dualizing complex D is said to be normalized if $\sup(D) = \dim(R)$.

It is clear that if D is a dualizing complex for R, then so is $\Sigma^s D$ for every $s \in \mathbb{Z}$, which accounts for the non-uniqueness of dualizing complexes. Further, $\Sigma^{\dim(R)-\sup(D)}D$ is a normalized dualizing complex.

Example 25. *Let $(R,\mathfrak{m},k)$ be a local ring with a normalized dualizing complex D. Then $\mathbf{R}\Gamma_\mathfrak{m}(D) \simeq E_R(k)$. For a proof, refer to [8, Proposition 6.1].*

The next theorem determines precisely when a ring enjoys a dualizing complex.

Theorem 26. *The the following assertions are equivalent:*

(i) R has a dualizing complex.

(ii) R is a homomorphic image of a Gorenstein ring of finite Krull dimension.

Proof. See [8, Page 299] and [10, Corollary 1.4]. □

Now we prove the Local Duality Theorem for complexes. We recall that given a local ring $(R,\mathfrak{m},k)$, we let $(-)^\vee := \operatorname{Hom}_R(-,E_R(k))$, where $E_R(k)$ is the injective envelope of k.

Theorem 27. *Let $(R,\mathfrak{m})$ be a local ring with a dualizing complex D, and $X \in \mathcal{D}^f_\square(R)$. Then*

$$H^i_\mathfrak{m}(X) \cong \operatorname{Ext}^{\dim(R)-i-\sup(D)}_R(X,D)^\vee$$

for every $i \in \mathbb{Z}$.

Proof. Clearly, we have

$$\operatorname{Ext}^{\dim(R)-i-\sup(D)}_R(X,D) \cong \operatorname{Ext}^{-i}_R\left(X, \Sigma^{\dim(R)-\sup(D)}D\right)$$

for every $i \in \mathbb{Z}$, and $\Sigma^{\dim(R)-\sup(D)}D$ is a normalized dualizing for R. Hence by replacing D with $\Sigma^{\dim(R)-\sup(D)}D$, it suffices to assume that D is a normalized dualizing complex and prove the isomorphism $H^i_\mathfrak{m}(X) \cong \operatorname{Ext}^{-i}_R(X,D)^\vee$ for every $i \in \mathbb{Z}$. By Theorem 22, we have

$$\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_\mathfrak{m}(X), E_R(k)) \simeq \mathbf{R}\operatorname{Hom}_R(X, \mathbf{L}\Lambda^\mathfrak{m}(E_R(k)))\,. \tag{4.4}$$

But since $E_R(k)$ is injective, it provides a semi-injective resolution of itself, so we have

$$\mathbf{R}\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{m}}(X), E_R(k)\right) \simeq \operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{m}}(X), E_R(k)\right). \tag{4.5}$$

Besides, by Example 25, [2, Corollary on Page 6], and [6, Proposition 2.7], we have

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{m}}\left(E_R(k)\right) &\simeq \mathbf{L}\Lambda^{\mathfrak{m}}\left(\mathbf{R}\Gamma_{\mathfrak{m}}(D)\right)\\ &\simeq \mathbf{L}\Lambda^{\mathfrak{m}}(D)\\ &\simeq D \otimes_R^{\mathbf{L}} \widehat{R}^{\mathfrak{m}}\\ &\simeq D \otimes_R \widehat{R}^{\mathfrak{m}}.\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5), and (4.6), we get

$$\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{m}}(X), E_R(k)\right) \simeq \mathbf{R}\operatorname{Hom}_R\left(X, D \otimes_R \widehat{R}^{\mathfrak{m}}\right).$$

Taking Homology, we obtain

$$\begin{aligned}\operatorname{Hom}_R\left(H^i_{\mathfrak{m}}(X), E_R(k)\right) &\cong \operatorname{Hom}_R\left(H_{-i}\left(\mathbf{R}\Gamma_{\mathfrak{m}}(X)\right), E_R(k)\right)\\ &\cong H_i\left(\operatorname{Hom}_R\left(\mathbf{R}\Gamma_{\mathfrak{m}}(X), E_R(k)\right)\right)\\ &\cong H_i\left(\mathbf{R}\operatorname{Hom}_R\left(X, D \otimes_R \widehat{R}^{\mathfrak{m}}\right)\right)\\ &\cong \operatorname{Ext}^{-i}_R\left(X, D \otimes_R \widehat{R}^{\mathfrak{m}}\right)\end{aligned} \tag{4.7}$$

for every $i \in \mathbb{Z}$.

Since $X \in \mathcal{D}^f_{\square}(R)$, we have $X \otimes_R \widehat{R}^{\mathfrak{m}} \in \mathcal{D}^f_{\square}\left(\widehat{R}^{\mathfrak{m}}\right)$, so $H^i_{\mathfrak{m}\widehat{R}^{\mathfrak{m}}}\left(X \otimes_R \widehat{R}^{\mathfrak{m}}\right)$ is an artinian $\widehat{R}^{\mathfrak{m}}$-module by [9, Proposition 2.1], and thus Matlis reflexive for every $i \in \mathbb{Z}$. Moreover, $D \otimes_R \widehat{R}^{\mathfrak{m}}$ is a normalized dualizing complex for $\widehat{R}^{\mathfrak{m}}$. Therefore, using the isomorphism (4.7) over the $\mathfrak{m}$-adically complete ring $\widehat{R}^{\mathfrak{m}}$, we obtain

$$\begin{aligned}H^i_{\mathfrak{m}}(X) &\cong H^i_{\mathfrak{m}}(X) \otimes_R \widehat{R}^{\mathfrak{m}}\\ &\cong H^i_{\mathfrak{m}\widehat{R}^{\mathfrak{m}}}\left(X \otimes_R \widehat{R}^{\mathfrak{m}}\right)\\ &\cong \operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}\left(\operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}\left(H^i_{\mathfrak{m}\widehat{R}^{\mathfrak{m}}}\left(X \otimes_R \widehat{R}^{\mathfrak{m}}\right), E_{\widehat{R}^{\mathfrak{m}}}(k)\right), E_{\widehat{R}^{\mathfrak{m}}}(k)\right)\\ &\cong \operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}\left(\operatorname{Ext}^{-i}_{\widehat{R}^{\mathfrak{m}}}\left(X \otimes_R \widehat{R}^{\mathfrak{m}}, D \otimes_R \widehat{R}^{\mathfrak{m}}\right), E_{\widehat{R}^{\mathfrak{m}}}(k)\right)\\ &\cong \operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}\left(\operatorname{Ext}^{-i}_R(X, D) \otimes_R \widehat{R}^{\mathfrak{m}}, E_{\widehat{R}^{\mathfrak{m}}}(k)\right)\end{aligned} \tag{4.8}$$

for every $i \in \mathbb{Z}$. However, $\mathbf{R}\operatorname{Hom}_R(X, D) \in \mathcal{D}^f_{\sqsubset}(R)$, so $\operatorname{Ext}^{-i}_R(X, D)$ is a finitely generated R-module for every $i \in \mathbb{Z}$. It follows that

$$\operatorname{Hom}_{\widehat{R}^{\mathfrak{m}}}\left(\operatorname{Ext}^{-i}_R(X, D) \otimes_R \widehat{R}^{\mathfrak{m}}, E_{\widehat{R}^{\mathfrak{m}}}(k)\right) \cong \operatorname{Hom}_R\left(\operatorname{Ext}^{-i}_R(X, D), E_R(k)\right) \tag{4.9}$$

for every $i \in \mathbb{Z}$. Combining (4.8) and (4.9), we obtain

$$H^i_{\mathfrak{m}}(X) \cong \operatorname{Hom}_R\left(\operatorname{Ext}^{-i}_R(X, D), E_R(k)\right)$$

for every $i \in \mathbb{Z}$ as desired. □

Our next goal is to obtain the Local Duality Theorem for modules. But first we need the definition of a dualizing module.

Definition 28. *Let $(R, \mathfrak{m})$ be a local ring. A dualizing module for R is a finitely generated R-module ω that satisfies the following conditions:*

(i) The homothety map $\chi^{\omega}_R : R \to \operatorname{Hom}_R(\omega, \omega)$, given by $\chi^{\omega}_R(a) = a1^{\omega}$ for every $a \in R$, is an isomorphism.

(ii) $\operatorname{Ext}^i_R(\omega, \omega) = 0$ for every $i \geq 1$.

(iii) $\operatorname{id}_R(\omega) < \infty$.

The next theorem determines precisely when a ring enjoys a dualizing module.

Theorem 29. *Let $(R, \mathfrak{m})$ be a local ring. Then the following assertions are equivalent:*

(i) R has a dualizing module.

(ii) R is a Cohen-Macaulay local ring which is a homomorphic image of a Gorenstein local ring.

Moreover in this case, the dualizing module is unique up to isomorphism.

Proof. See [20, Corollary 2.2.13] and [3, Theorem 3.3.6]. □

Since the dualizing module for R is unique whenever it exists, we denote a choice of the dualizing module by ω_R. It can be seen that R is Gorenstein if and only if $\omega_R \cong R$.

Proposition 30. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and ω a finitely generated R-module. Then the following assertions are equivalent:*

(i) ω is a dualizing module for R.

(ii) $\omega^{\vee} \cong H^{\dim(R)}_{\mathfrak{m}}(R)$.

Proof. See [4, Definition 12.1.2, Exercises 12.1.23 and 12.1.25, and Remark 12.1.26], and [3, Definition 3.3.1]. □

We can now derive the Local Duality Theorem for modules.

Theorem 31. *Let $(R, \mathfrak{m})$ be a local ring with a dualizing module ω_R, and M a finitely generated R-module. Then*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}_R^{\dim(R)-i}(M, \omega_R)^{\vee}$$

for every $i \geq 0$.

Proof. By Theorem 29, R is a Cohen-Macaulay local ring which is a homomorphic image of a Gorenstein local ring S. Since S is local, we have $\dim(S) < \infty$. Hence Theorem 26 implies that R has a dualizing complex D. Since R is Cohen-Macaulay, we have $H^i_{\mathfrak{m}}(R) = 0$ for every $i \neq \dim(R)$. On the other hand, by Theorem 27, we have

$$\begin{aligned} H^i_{\mathfrak{m}}(R) &\cong \operatorname{Ext}_R^{\dim(R)-i-\sup(D)}(R, D)^{\vee} \\ &\cong H_{-\dim(R)+i+\sup(D)}\left(\mathbf{R}\operatorname{Hom}_R(R, D)\right)^{\vee} \\ &\cong H_{-\dim(R)+i+\sup(D)}(D)^{\vee}. \end{aligned} \tag{4.10}$$

It follows from the display (4.10) that $H_{-\dim(R)+i+\sup(D)}(D) = 0$ for every $i \neq \dim(R)$, i.e. $H_i(D) = 0$ for every $i \neq \sup(D)$. Therefore, we have

$$D \simeq \Sigma^{\sup(D)} H_{\sup(D)}(D).$$

In addition, letting $i = \dim(R)$ in the display (4.10), we get $H^{\dim(R)}_{\mathfrak{m}}(R) \cong H_{\sup(D)}(D)^{\vee}$, which implies that $\omega_R \cong H_{\sup(D)}(D)$ by Proposition 30. It follows that $D \simeq \Sigma^{\sup(D)}\omega_R$.

Now let M be a finitely generated R-module. Then by Theorem 27, we have

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong \operatorname{Ext}_R^{\dim(R)-i-\sup(D)}(M, D)^{\vee} \\ &\cong H_{-\dim(R)+i+\sup(D)}\left(\mathbf{R}\operatorname{Hom}_R(M, D)\right)^{\vee} \\ &\cong H_{-\dim(R)+i+\sup(D)}\left(\mathbf{R}\operatorname{Hom}_R\left(M, \Sigma^{\sup(D)}\omega_R\right)\right)^{\vee} \\ &\cong H_{-\dim(R)+i}\left(\mathbf{R}\operatorname{Hom}_R(M, \omega_R)\right)^{\vee} \\ &\cong \operatorname{Ext}_R^{\dim(R)-i}(M, \omega_R)^{\vee}. \end{aligned}$$

□

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References

[1] L. Avramov and H-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra, **71**(2-3), (1991), 129-155. MR1117631. Zbl 0737.16002.

[2] L. Alonso Tarrío, A. Jeremías López and J. Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup., (4), **30**(1), (1997), 1-39. MR1422312. Zbl 0894.14002.

[3] W. Bruns, and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993. MR1251956. Zbl 0909.13005.

[4] M. Brodmann and R.Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics, **136**, Cambridge University Press, Cambridge, Second Edition (2013). MR1613627. Zbl 1303.00029.

[5] H-B. Foxby, *Hyperhomological algebra and commutative rings*, notes in preparation.

[6] A. Frankild, *Vanishing of local homology*, Math. Z., **244**(3), (2003), 615-630. MR1992028. Zbl 1020.13003.

[7] J.P.C. Greenlees and J.P. May, *Derived functors of I-adic completion and local homology*, J. Algebra, **149**(2), (1992), 438-453. MR1172439. Zbl 0774.18007.

[8] R. Hartshorne, *Residues and duality*, Lecture notes in mathematics, **20**, (1966). MR0222093. Zbl 0212.26101.

[9] M. Hatamkhani and K. Divaani-Aazar, *The derived category analogue of the Hartshorne-Lichtenbaum vanishing theorem*, Tokyo. J. Math., **36**(1), (2013), 195-205. MR3112383. Zbl 1277.13013.

[10] T. Kawasaki, *On arithmetic Macaulayfication of noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), 123-149. MR1859029. Zbl 1087.13502.

[11] J. Lipman, *Lectures on local cohomology and duality*, Local cohomology and its applications, Lecture notes in pure and applied mathematics, **226**, (2012), Marcel Dekker, Inc. MR1888195. Zbl 1011.13010.

[12] E. Matlis, *The Koszul complex and duality*, Comm. Algebra, **1**, (1974), 87-144. MR0344241. Zbl 0277.13011.

[13] M. Porta, L. Shaul and A. Yekutieli, *On the Homology of completion and torsion*, Algebr. Represent. Theor., **17**, (2014), 31-67. MR3160712. Zbl 1316.13020.

[14] J.J. Rotman, *An introduction to homological algebra*, Universitext. Springer, New York, second edition, 2009. MR2455920. Zbl 1157.18001.

[15] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand., **92**(2), (2003), 161-180. MR1973941. Zbl 1023.13011.

****

[16] P. Selick, *Introduction to homotopy theory*, Fields Institute Monographs, American Mathematical Society, (1997). MR1450595. Zbl 0883.55001.

[17] A.M. Simon, *Some homological properties of complete modules*, Math. Proc. Camb. Phil. Soc., **108**(2), (1990), 231-246. MR1074711. Zbl 0719.13007.

[18] A.M. Simon, *Adic-completion and some dual homological results*, Publ. Mat. Camb., **36**(2B), (1992), 965-979. MR1210029. Zbl 0839.13015.

[19] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math., **65**(2), (1988), 121-154. MR0932640. Zbl 0636.18006.

[20] S. Sather-Wagstaff, *Semidualizing modules*, notes in preparation.

[21] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math., vol. 38, Cambridge Univ. Press, Cambridge, 1994, xiv+450 pp. MR1269324. Zbl 0797.18001.

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