Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 14 (2019), 61 – 108

SURVEY ON COMULTIPLICATION MODULES

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Abstract. The concept of comultiplication (as a dual notion of multiplication) modules was introduced and studied by H. Ansari-Toroghy and F. Farshadifar in 2007. This notion has obtained a great attention by many authors and now there is a considerable amount of research concerning this class of modules. The main purpose of this paper is to collect these results and provide a useful source for those who are interested in research in this field.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity and "$\subset$" will denote the strict inclusion. Further, $\mathbb{Z}$ and $\mathbb{N}$ will denote the ring of integers and the set positive integers, respectively. We use $N \leq M$ to indicate that N is a submodule of a module M. For any unexplained notions or terminology please see [8], [52], [55], [61], [62], or [67].

Multiplication rings are introduced by W. Krull in 1925 as a generalization of Dedekind domains [49]. In 1981, Barnard [29] has given the concept of multiplication modules. An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [29]. There is a large body of research concerning multiplication modules. H. Ansari-Toroghy and F. Farshadifer introduced the notion comultiplication module as a dual notion of multiplication module in [10] and investigated some main properties of this class of modules [11-27]. We mention that dual of not every result related to multiplication R-module is true. For example, it is well-known that every cyclic R-module is a multiplication module. Although the dual of this result was true in some special cases (see Theorems 83, 84, and 85), however, it was not known whether this is true in general. In fact this was a question posed in [10]. Later a counter example showed that this is not true in

2010 Mathematics Subject Classification: 13C13; 13C99.

Keywords: Comultiplication module; completely irreducible submodule; finitely generated; Goldie dimension; complements; endomorphism; strongly comultiplication module; copure submodule; cocyclic module; P-cotorsion module; P-cocyclic module; Fitting ideal; graded comultiplication module; fully coidempotent module; second submodule; weak comultiplication module.

general. In other words, not every cocylic R-module is a comultiplication R-module in general (see Example 86). Further we know that every multiplication R-module over an Artinian ring is cyclic. But dual of this result is not true in general. That is, not every comultiplication R-module over an Artinian ring is cocyclic [17, 3.5]. There is plenty of useful information which has been investigated by many authors [1, 6, 7, 28, 30, 37, 48, 56, 58, 59, 60, 65, 66]. The main purpose of this paper is to collect these results and provide a useful source for those who are interested in research in this field.

2 Comultiplication modules

Definition 1. *[10, 3.1] An R-module M is said to be a* comultiplication module *if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$.*

The following example shows that not every comultiplication R-module is a multiplication R-module.

Example 2. *[10, 3.2] Let p be a prime number and consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Choose $N = \langle 1/p^i + \mathbb{Z}\rangle$ and Set $I = \mathbb{Z}p^i, i \geq 0$. It is clear that $N = (0 :_M I)$. Therefore, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is a comultiplication module. But $\mathbb{Z}_{p^\infty}$ is not a multiplication $\mathbb{Z}$-module.*

The following example shows that not every multiplication R-module is a comultiplication R-module.

Example 3. *[10, 3.9] For a submodule $2\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ we have*

$$(0 :_{\mathbb{Z}} Ann_{\mathbb{Z}}(2\mathbb{Z})) = \mathbb{Z}.$$

Therefore, $\mathbb{Z}$ is not a comultiplication module. But $\mathbb{Z}$ is a multiplication $\mathbb{Z}$-module.

Lemma 4. *[6, 1.2] Let M be an R-module such that $AM = 0$ for some ideal A of R. Then the R-module M is a comultiplication module if and only if the (R/A)-module M is a comultiplication module.*

Recall that an R-module M is a *self-cogenerator*, provided that for each submodule N of M, the factor module M/N embeds in the direct product M^Λ of copies of M, for some index set Λ. We shall call an R-module M *strongly self-cogenerated* provided for each submodule N of M there exists a family $\varphi_\lambda (\lambda \in \Lambda)$ of trivial endomorphisms of M, for some index set Λ, such that $N = \cap_{\lambda\in\Lambda} ker\varphi_\lambda$ [6].

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{\lambda\in\Lambda} N_\lambda$, where $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of submodules of M, implies that $N = N_\lambda$ for some $\lambda \in \Lambda$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [44].

Recall that $Soc_R(M)$ denotes the sum of all minimal submodules of M [8].

Comultiplication modules can be characterized in various ways as we demonstrate in the following theorem.

Theorem 5. *[6, 1.5], [10, 3.7, 3.10] , and [13, 3.10]. Let M be an R-module. Then the following statements are equivalent.*

(a) M is a comultiplication module.

(b) For each completely irreducible submodule L of M there exists an ideal I of R such that $L = (0 :_M I)$.

(c) Given submodules N, K of M, $Ann_R(N) = Ann_R(K)$ implies that $N = K$.

(d) $N = (0 :_M Ann_R(N))$ for every submodule N of M.

(e) $Soc_R((0 :_M Ann_R(N))/N) = 0$ for every submodule N of M.

(f) Given submodules N, K of M, $Ann_R(N) \subseteq Ann_R(K)$ implies that $K \subseteq N$.

(g) Given any submodule N of M and $m \in M$, $Ann_R(N) \subseteq Ann_R(Rm)$ implies that $m \in N$.

(h) Given any submodule N of M and $m \in M$, $Ann_R(N) \subseteq Ann_R(Rm)$ implies that $(N :_R m)$ is not a maximal ideal of R.

(i) $(K :_R N) = (Ann_R(N) :_R Ann_R(L))$ for all submodules K and N of M.

(j) M is strongly self-cogenerated.

(k) For every submodule N of M and each ideal I of R with $N \subset (0 :_M I)$, there exists an ideal J of R such that $I \subset J$ and $N = (0 :_M J)$.

(l) For every submodule N of M and each ideal I of R with $N \subset (0 :_M I)$, there exists an ideal J of R such that $I \subset J$ and $N \subseteq (0 :_M J)$.

Proposition 6. *[10, 3.17] and [13, 3.7] Let M be a comultiplication R-module. Then the following assertions hold.*

(a) Every submodule of M is a comultiplication module.

(b) If R is a von Neumann regular ring, then every homomorphic image of M is a comultiplication R-module.

Let N be a non-zero submodule of an R-module M. Then N is said to be *large or essential* if for every non-zero submodule L of M, $N \cap L \neq 0$ [8].

Recall that an R-module M is said to be *cocyclic* if M has a simple essential socle.

Theorem 7. *[13, 3.1, 3.2] Let M be a comultiplication R-module. Then we have the following.*

(a) If P is a maximal ideal of R and $(0 :_M P) \neq 0$, then $(0 :_M P)$ is simple.

(b) If B is an ideal of R such that $(0 :_M B) = 0$, then $BM = M$.

(c) If B is an ideal of R such that $(0 :_M B) = 0$, then, for every element $m \in M$, there exists an element b of B such that $m = bm$.

(d) If M is a finitely generated R-module and B is an ideal of R such that $(0 :_M B) = 0$, then there exists $b \in B$ such that 1!' $- b \in Ann_R(M)$.

(e) Every non-zero submodule of M contains a minimal submodule of M.

(f) Let K be a submodule of M. Then K is a minimal submodule of M if and only if there exists a maximal ideal P of R such that $K = (0 :_M P) \neq 0$.

(g) If R has a unique maximal ideal, then M is a cocyclic R-module.

Corollary 8. *[13, 3.3] (A dual Nakayama lemma for comultiplication modules.) Let M be a comultiplication R-module and I be an ideal of R such that $I \subseteq Jac(R)$, where $Jac(R)$ denotes the Jacobson radical of R. If $(0 :_M I) = 0$, then $M = 0$.*

Definition 9. *[12, 3.1] R is said to be a comultiplication ring if, as an R-module, R is a comultiplication R-module.*

Example 10. *[12, 3.2] Every self-injective Noetherian ring is a comultiplication ring. In particular, every semi-simple ring is a comultiplication ring.*

Example 11. *[13, 3.8] Let n be a fixed number.*

(a) $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}$-module.

(b) $\mathbb{Z}_n$ is a comultiplication $\mathbb{Z}_n$-module. Furthermore, if G is a finite group, then the group ring $\mathbb{Z}_n(G)$ is a comultiplication $\mathbb{Z}_n(G)$-module.

Proposition 12. *[12, 3.3] Let M be a comultiplication R-module.*

(a) If $\{M_\lambda\}_{\lambda \in \Lambda}$ is a collection of submodules of M, then

$$(0 :_M \bigcap_{\lambda \in \Lambda} Ann_R(M_\lambda)) = \sum_{\lambda \in \Lambda} (0 :_M Ann_R(M_\lambda)).$$

(b) If R is a comultiplication ring, then for each collection $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R, $(0 :_M \cap_{\lambda \in \Lambda} I_\lambda) = \sum_{\lambda \in \Lambda} (0 :_M I_\lambda)$.

Theorem 13. *[12, 3.5] Let R be a Noetherian ring and let M be an injective R-module. Then M is a comultiplication R-module if M is a multiplication R-module. Furthermore, if R is a comultiplication ring the converse is true.*

For an R-module M, $Coass_R(M)$ denotes the set of all prime ideals P of R such that there exists a cocyclic homomorphic image L of M with $Ann_R(L) = P$ [69].

Theorem 14. *[15, 2.2] Let M be a comultiplication R-module. Then we have the following.*

(a) $Ass_R(M) \subseteq Max(R)$.

(b) If $Ann_R(M)$ *is a prime ideal of* R*, then* $Coass_R(M) = \{Ann_R(M)\}$.

(c) If R *is a Noetherian ring and* M *is a faithful* R*-module, then* $Coass_R(M) = Ass_R(R)$.

Proposition 15. *[17, 3.3] Let R be an integral domain. Then we have the following.*

(a) Every comultiplication R-module is cyclic or torsion.

(b) If M is a faithful finitely generated comultiplication R-module, then M is cyclic.

(c) If there exists a faithful multiplication and comultiplication R-module, then R is a field.

Theorem 16. *[6, 1.8] Let M be a self-cogenerated R-module such that, for each finitely generated submodule N of M, every homomorphism $\varphi : N \to M$ is trivial. Then $K = (0 :_M Ann_R(K))$ for every finitely generated submodule K of M.*

Recall that an R-module M is called *nonsingular*, provided that $Am \neq 0$ for every essential ideal A of R and every non-zero element m of M.

Theorem 17. *[6, 1.7] Every nonsingular comultiplication R-module is semisimple and projective.*

Proposition 18. *[7, 3.2, 3.3, 3.5, 3.7, 3.8, 3.9] Let M be a comultiplication R-module. Then we have the following.*

(a) If K *and* L *are submodules of* M*, then* $(0 :_M (K :_R L)) = Ann_R(K)L$.

(b) $BM = Ann_R((0 :_M B))M$ *for every ideal* B *of* R.

(c) $Ann_R(L)K = Ann_R(L \cap K)K$ *for all submodules* K *and* L *of* M.

(d) If K *and* L *are submodules of* M*, then* $K \cap L = (0 :_K Ann_R(L))$.

(e) If K and L are submodules of M such that, for all ideals A and B of R, $AK \subseteq BK$ implies that $A \subseteq B + Ann_R(K)$, then $Ann_R(L) + Ann_R(K) = Ann_R(L \cap K)$.

(f) If L is any submodule and m is any element of M, then $Ann_R(L)+Ann_R(m) = Ann_R(Rm \cap L)$.

Theorem 19. *[7, 3.10] An R-module M is a comultiplication module if and only if*

(a) $Ann_R(L) + Ann_R(M) = Ann_R(Rm \cap L)$, and

(b) Rm is a comultiplication module, for each $m \in M$ and submodule L of M.

Let L be a submodule of an R-module M. Then a homomorphism $\varphi : L \to M$ will be called trivial if there exists an $r \in R$ such that $\varphi(x) = rx$ $(x \in L)$.

Corollary 20. *[7, 3.11] Let L be any finitely generated submodule of a comultiplication R-module M. Then every homomorphism $\varphi : L \to M$ is trivial.*

Corollary 21. *[7, 3.12] Every Noetherian comultiplication module over a commutative ring is an Artinian quasi-injective module.*

Recall that if $A, \acute{A}$ and B are submodules of M such that $\acute{A} \subseteq B$, $M = A + \acute{A}$ and $\acute{A}$ is minimal with respect to this property, then $\acute{A}$ is said to be a *supplement* of A in B (this is the dual notion of a complement of a submodule). In [42], M is said to be *amply supplemented* when for each pair of submodules A, B of M with $M = A + B$, A has a supplement in B.

Theorem 22. *[7, 5.1] Let M be a comultiplication R-module such that $M = (0 :_M C) + (0 :_M D)$ for all ideals C and D of R with $C \cap D = Ann_R(M)$. Then M is amply supplemented.*

Let M be any R-module. It is well known that the collection of submodules of M forms a modular lattice $\mathcal{L}(_RM)$ with least element the zero submodule and greatest element M. Given two submodules N and L of M, the least upper bound of N and L in $\mathcal{L}(_RM)$ is $N + L$ and the greatest lower bound in $\mathcal{L}(_RM)$ is $N \cap L$. Now let $\mathcal{L}_M(_RR)$ denote the collection of ideals in R of the form $Ann_R(K)$ for some submodule K of M. Note that $\mathcal{L}_M(_RR)$ is a subset of $\mathcal{L}(_RR)$ but it need not be a sublattice even if M is a comultiplication module.

Theorem 23. *[7, 5.2] Let M be a comultiplication R-module. Then $\mathcal{L}_M(_RR)$ is a sublattice of $\mathcal{L}(_RR)$ if and only if $Ann_R(N \cap L) = Ann_R(N) + Ann_R(L)$ for all submodules N and L of M. Moreover, in this case, the mapping $\varphi : \mathcal{L}(_RM) \to \mathcal{L}_M(_RR)$, defined by $\varphi(K) = Ann_R(K)$ for every submodule K of M, is an antiisomorphism from the lattice $\mathcal{L}(_RM)$ to the lattice $\mathcal{L}_M(_RR)$.*

Theorem 24. *[13, 3.4] Let M be a faithful comultiplication R-module. Then consider the following statements.*

(a) M is finitely generated.

(b) $(0 :_M I) \neq 0$ for every proper ideal I of R.

(c) $(0 :_M P) \neq 0$ for every maximal ideal P of R.

(d) M is finitely cogenerated.

Then (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d).

Proposition 25. *[13, 3.5] Let M be a comultiplication R-module. Then we have the following.*

(a) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of submodules of M such that $\cap_{\lambda\in\Lambda} M_\lambda = 0$ and let $A = \sum_{\lambda\in\Lambda} Ann_R(M_\lambda)$. Then $R = Ann_R(X) + A$ for every finitely generated submodule X of M.

(b) Every finitely generated submodule of M is finitely cogenerated.

Lemma 26. *[10, 3.14] and [13, 3.5] Let M be a comultiplication R-module such that there exists a family of submodules L_λ ($\lambda \in \Lambda$) such that $\cap_{\lambda\in\Lambda} L_\lambda = 0$. Then we have the following.*

(a) $N = \cap_{\lambda\in\Lambda}(N + L_\lambda)$ for every submodule N of M.

(b) For each finitely generated submodule K of M there exists a finite subset Δ of Λ such that $R = \sum_{\delta\in\Delta} Ann_R(L_\delta) + Ann_R(K)$ and, hence, $K \cap (\cap_{\delta\in\Delta} L_\delta) = 0$.

Let L_λ ($\lambda \in \Lambda$) be the collection of all completely irreducible submodules of M. Then we define

$$\xi(M) = \sum_{\lambda\in\Lambda} Ann_R(L_\lambda).$$

Note that $\xi(M)$ is an ideal of R. By a minimal completely irreducible submodule of M we mean a completely irreducible submodule L of M such that there does not exist a completely irreducible submodule K of M with $K \subset L$.

Proposition 27. *[7, 1.12] and [13, 3.12] Let M be an R-module. Then we have the following.*

(a) If M is a comultiplication module, then $R = \xi(M) + Ann_R(m)$ for all $m \in M$.

(b) If $R = \xi(M) + Ann_R(m)$ for all $m \in M$, then $L = (0 :_M Ann_R(L))$ for every minimal completely irreducible submodule L of M.

(c) *If M is a comultiplication module and P is a maximal ideal of R with $\xi(M) \subseteq P$, then $M_P = 0$.*

Theorem 28. *[7, 1.13] Let M be a comultiplication R-module. Then there exist minimal completely irreducible submodules L_λ $(\lambda \in \Lambda)$ of M such that the following hold.*

(a) $\bigcap_{\lambda\in\Lambda} L_\lambda = 0$.

(b) $\bigcap_{\lambda\in\Lambda\setminus\{\delta\}} L_\lambda \neq 0$ *for all* $\delta \in \Lambda$.

(c) *For each completely irreducible submodule L of M there exists an $\lambda \in \Lambda$ such that $L_\lambda \subseteq L$.*

(d) $\xi(M) = \sum_{\lambda\in\Lambda} Ann_R(L_\lambda)$.

Elements m_λ $(\lambda \in \Lambda)$ of an R-module M are called *independent* provided the sum $\sum_{\lambda\in\Lambda} Rm_\lambda$ is direct.

Proposition 29. *[6, 2.4] Let M be a comultiplication R-module. Let n be a positive integer and let m_i $(1 \leq i \leq n)$ be independent elements of M. Then the submodule $Rm_1 \oplus ... \oplus Rm_n$ is cyclic.*

A proper submodule N of an R-module M is said to be *prime* if, for any $r \in R$ and any $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$ [35].

Theorem 30. *[7, 1.6] and [15, 2.3] Let M be a comultiplication R-module. Then we have the following.*

(a) *If the radical of M is zero, then M is a semisimple R-module.*

(b) *If $Ann_R(M)$ is a prime ideal of R and the intersection of all prime submodules of M is zero, then M is a semisimple R-module.*

Let N be a submodule of an R-module M. Then N is said to be *small* if for every proper submodule L of M, $L + N = M$ implies that $L = M$ [8].

An R-module M is said to be *uniform* if each of its non-zero submodules is large [8].

Theorem 31. *[12, 3.13, 3.12] and [18, 2.7] Let M be a faithful finitely generated comultiplication R-module. Then we have the following.*

(a) *A submodule N of M is essential if and only if there exists a small ideal I of R such that $N = (0 :_M I)$.*

(b) *M is uniform if and only if every proper ideal of R is small.*

(c) *If N is a direct summand of M, then $Ann_R(N)$ is a direct summand of R.*

Theorem 32. *[12, 3.14](Dual of Nakayama's lemma) For an ideal I of R, the following are equivalent.*

(a) $I \subseteq Jac(R)$;

(b) For every finitely cogenerated R-module M, if $(0 :_M I) = 0$, then $M = 0$;

(c) For every finitely cogenerated R-module M, $(0 :_M I)$ is large in M.

Remark 33. *[18, 2.4] It is well known that if M is a finitely generated multiplication R-module and I, J are ideals of R such that $IM \subseteq JM$, then $I \subseteq J + Ann_R(M)$. But the dual of this fact is not true in general. For example, let p be a prime number. Then the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a faithful Artinian comultiplication $\mathbb{Z}$-module such that $(0 :_{\mathbb{Z}_{p^\infty}} q\mathbb{Z}) = (0 :_{\mathbb{Z}_{p^\infty}} \mathbb{Z})$ for each prime number $q \neq p$, while $q\mathbb{Z} \neq \mathbb{Z}$. Next proposition shows that this is true for comultiplication modules under some restrictive conditions.*

Proposition 34. *[18, 2.5] Let M be a comultiplication R-module and $(0 :_M I) \subseteq (0 :_M J)$ for some ideals I and J of R. Then we have the following.*

(a) $J \subseteq I$ if there exists a finitely generated multiplication submodule N of M such that $Ann_R(N) \subseteq I$.

(b) $J \subseteq I$ if $I \in Supp_R(M)$.

Definition 35. *[56, 5.1] An R-module M is said to be a p-comultiplication module if, for each non-trivial submodule N of M, there is a prime ideal P of R, containing $Ann_R(M)$, such that $N = (0 :_M P)$.*

Lemma 36. *[56, 5.2] Suppose that $R = R_0 \times R_1$ and M is an R–module. Then M is p-comultiplication if and only if either (1) for some $i = 0, 1$, $R_iM = 0$ and M is a p-co-m R_i-module or (2) $M = M_0 \oplus M_1$ where M_i is a simple R_i-module such that $R_{1-i}M_i = 0$.*

Theorem 37. *[56, 5.5] Assume that M is a p-comultiplication module. In either of the following cases, M is cyclic.*

(a) M has a maximal submodule.

(b) $Ann_R(M)$ is not a prime ideal of R.

(c) $R/Ann_R(M)$ is an integral domain with finitely many height one primes such that every non-zero prime ideal of $R/Ann_R(M)$ contains a height one prime ideal; in particular, if $R/Ann_R(M)$ is a valuation domain.

(d) $R/Ann_R(M)$ is a Noetherian domain with Krull dimension ≤ 1, for example, a Dedekind domain.

Corollary 38. *[56, 5.6] A multiplication or finitely generated R-module M is a p-comultiplication module if and only if it is cyclic and $R/Ann_R(M)$ is a p-comultiplication module.*

Theorem 39. *[56, 5.8] A ring R is a p-comultiplication module over itself if and only if either R is a field or $R = F_1 \times F_2$, where F_i's are fields or R is an $SPIR$ with unique prime ideal R_p and $p^2 = 0$.*

Corollary 40. *[56, 5.9] A ring R is a p-comultiplication module over itself if and only if every nontrivial ideal of R is prime.*

Corollary 41. *[56, 5.10] A cyclic R-module M is p-comultiplication if and only if either $Ann_R(M)$ is a maximal ideal or an intersection of two maximal ideals or $Ann_R(M) = m^2$ for some maximal ideal M of R with $dim_{R/m} m/m^2 = 1$*

We now consider how comultiplication modules behave under localization. Let M be an R-module. For any prime ideal P of R we set

$$I_P = \{r \in R : rc = 0 \ for \ some \ c \in R \setminus P\}, \ and$$

$$T_P = \{m \in M : cm = 0 \ for \ some \ c \in R \setminus P\}.$$

Note that I_P is an ideal of R, T_P is a submodule of M and $I_P M \subseteq T_P$. We shall call the prime ideal P *good for* M if there exists $d \in R \setminus P$ such that $dT_P = 0$. For example, P is good for M, provided that T_P is finitely generated [6].

Theorem 42. *[6, 2.5] Let M be an R-module such that every maximal ideal of R is good for M. Then M is a comultiplication R-module if and only if M_P is a comultiplication R_P-module for every maximal ideal P of R.*

Corollary 43. *[6, 2.6] Let R be a Noetherian ring and let M be a comultiplication R-module. Then the R_P-module M_P is a comultiplication module for every prime ideal P of R.*

Let M be an R-module. Clearly

$$(0 :_M \sum_{\lambda \in \Lambda} A_\lambda) = \bigcap_{\lambda \in \Lambda} (0 :_M A_\lambda),$$

for any collection of ideals A_λ ($\lambda \in \Lambda$) of R. Moreover, if A and B are ideals of R, then $(0 :_M A) + (0 :_M B) \subseteq (0 :_M A \cap B)$. Note that if A and B are ideals of R such that $R = A + B$, then

$$(0 :_M A \cap B) = (0 :_M A \cap B)B + (0 :_M A \cap B)A \subseteq (0 :_M A) + (0 :_M B),$$

and hence, $(0 :_M A \cap B) = (0 :_M A) + (0 :_M B)$. Moreover, if U is a simple module and A and B are any ideals of R such that $(A \cap B)U = 0$, then $ABU = 0$ and hence,

$AU = 0$ or $BU = 0$, so that $(0:_U A \cap B) = (0:_U A) + (0:_U B)$. This can easily be extended to semisimple modules, so that $(0:_X A \cap B) = (0:_X A) + (0:_X B)$ for every semisimple R-module X and arbitrary ideals A and B of R. However, in general, $(0:_M A) + (0:_M B) \neq (0:_M A \cap B)$, as the following example shows [6].

Example 44. *[6, 2.7] Let K be any field, $K_i = K$ $(i \in \mathbb{N})$, and let T denote the direct product $\prod_{i\in\mathbb{N}} K_i$. Then T is a commutative ring. Let R denote the subring of T consisting of all elements $(k_1, k_2, k_3, ...)$, where $k_i \in K_i$ $(i \in \mathbb{N})$, such that there exists a positive integer n with $k_n = k_{n+1} = k_{n+2} = ...$. Then R is a commutative von Neumann regular ring with socle $S = \oplus_{i\in\mathbb{N}} K_i$ such that $(0:_R A \cap B) \neq (0:_R A) + (0:_R B)$ for some ideals A and B of R.*

A family $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of an R-module M is said to be an *inverse family of submodules of M* if the intersection of any two of its submodules again contains a module in $\{N_\lambda\}_{\lambda\in\Lambda}$. Also M, *satisfies the property $AB5^*$* if for every submodule K of M and every inverse family $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of M, $K + \cap_{\lambda\in\Lambda} N_\lambda = \cap_{\lambda\in\Lambda}(K + N_\lambda)$ [67].

Theorem 45. *[6, 2.9] Let M be a comultiplication R-module such that $(0:_M A \cap B) = (0:_M A) + (0:_M B)$ for all ideals A and B of R. Then M is an $AB5^*$ module.*

Theorem 46. *[6, 3.1] Let P_λ $(\lambda \in \Lambda)$ be any non-empty collection of distinct maximal ideals of R, let $k(\lambda)$ $(\lambda \in \Lambda)$ be any collection of positive integers and let M_λ be any non-zero R-module such that $P_\lambda^{k(\lambda)} M_\lambda = 0$ for all $\lambda \in \Lambda$. Then the R-module $M = \oplus_{\lambda\in\Lambda} M_\lambda$ is a comultiplication module if and only if M_λ is a comultiplication module and $\cap_{\delta\in\Lambda\setminus\{\lambda\}} P_\delta^{k(\delta)} \nsubseteq P_\lambda$ for all $\lambda \in \Lambda$.*

Corollary 47. *[6, 3.2] Let an R-module $M = \oplus_{\lambda\in\Lambda} U_\lambda$ be a direct sum of simple submodules U_λ $(\lambda \in \Lambda)$, for some index set Λ. Then M is a comultiplication module if and only if $\cap_{\delta\in\Lambda\setminus\{\lambda\}} Ann_R(U_\delta) \nsubseteq Ann_R(U_\lambda)$ for all $\lambda \in \Lambda$.*

Corollary 48. *[6, 3.3] Let R be a semiprime ring with socle S. Then the R-module S is a comultiplication module.*

Example 49. *[6, 3.4] Let R and S be as in Example 44. Then S is a comultiplication R-module such that $Ann_R(N \cap L) \neq Ann_R(N) + Ann_R(L)$ for some submodules N and L of S.*

We now give an example to show that in Theorem 42 some condition is required on the maximal ideals of R. If p is any prime in the ring $\mathbb{Z}$ of integers, then $\mathbb{Z}_{(p)}$ will denote the localization of $\mathbb{Z}$ at the maximal ideal $\mathbb{Z}p$ and $M_{(p)}$ the localization of any $\mathbb{Z}$-module M at $\mathbb{Z}p$.

Example 50. *[6, 3.5] Let I denote an infinite set of primes in $\mathbb{Z}$ and let M denote the $\mathbb{Z}$-module $\oplus_{p\in I}(\mathbb{Z}/\mathbb{Z}p)$. Then $M_{(p)}$ is a simple (and hence comultiplication) $\mathbb{Z}_{(p)}$-module for every $p \in I$ but the $\mathbb{Z}$-module M is not a comultiplication module.*

Here we give a simple example to show that, in general, the condition that $\cap_{\delta\in\Lambda\setminus\{\lambda\}} P_\delta^{k(\delta)} \not\subseteq P_\lambda$ in Theorem 46 cannot be replaced by the simpler condition that $\cap_{\delta\in\Lambda\setminus\{\lambda\}} P_\delta \not\subseteq P_\lambda$.

Example 51. *[6, 3.5] Let R denote the polynomial ring $\mathbb{Z}[x]$ in an indeterminate x, let I be an infinite set of primes in $\mathbb{Z}$ and let M_p denote the maximal ideal $\mathbb{Z}p + Rx$ of R for each $p \in I$. Let $\{n(p) : p \in I\}$ be any unbounded collection of positive integers. Let Q be any maximal ideal of R such that $x \notin Q$. Then $\cap_{p\in I} M_p \not\subseteq Q$ but $\cap_{p\in I} M_p^{n(p)} \subseteq Q$.*

Lemma 52. *[6, 4.3] The following statements are equivalent for an R-module M.*

(a) For each finitely generated submodule L of M, every homomorphism $\beta : L \to M$ is trivial.

(b) $Rm = (0 :_M Ann_R(Rm))$ for all $m \in M$ and $Ann_R(N \cap K) = Ann_R(N) + Ann_R(K)$ for all finitely generated submodules N and K of M.

Theorem 53. *[6, 4.4] Let M be a Noetherian R-module such that*

(a) $Rm = (0 :_M Ann_R(Rm))$ for all $m \in M$, and

(b) $Ann_R(N \cap K) = Ann_R(N) + Ann_R(K)$ for all submodules N and K of M. Then M is quasi-injective.

Theorem 54. *[6, 4.6] Let M be a quasi-injective R-module. Then the following statements are equivalent.*

(a) $Rm = (0 :_M Ann_R(Rm))$ for all $m \in M$.

(b) $L = (0 :_M Ann_R(L))$ for every finitely generated submodule L of M.

Corollary 55. *[6, 4.7] Let M be a Noetherian quasi-injective R-module. Then M is a comultiplication module if and only if $Rm = (0 :_M Ann_R(Rm))$ for all $m \in M$.*

Corollary 56. *[6, 4.8] Let M be a Noetherian R-module such that $Ann_R(N \cap K) = Ann_R(N) + Ann_R(K)$ for all submodules N and K of M. Then M is a comultiplication module if and only if $Rm = (0 :_M Ann_R(Rm))$ for all $m \in M$.*

Given a submodule N of an R-module M, we know that there exists at least one complement K of N. However, K need not be unique. For example, if F is a field, V a two-dimensional vector space over F and U a one-dimensional subspace of V, then every one-dimensional subspace X of V other than U is a complement of U. In particular, if F is an infinite field, then there are an infinite number of one-dimensional subspaces of V, and hence there are an infinite number of complements of U in V. We shall say that an R-module M has unique complements, provided that, for each submodule N of M, there exists a unique complement of N in M. For example, simple modules have unique complements. More generally, uniform modules have unique complements. If U is a uniform module, then 0 is the unique complement of every non-zero submodule and U is the unique complement of 0. Thus, U has unique complements. In below we show that comultiplication modules over commutative rings have unique complements [7].

Theorem 57. *[7, 2.1] The following statements are equivalent for a submodule N of an R-module M.*

(a) N has a unique complement in M.

(b) $\{m \in M : mR \cap N = 0\}$ is a submodule of M.

(c) Given elements x and y in M with $xR \cap N = yR \cap N = 0$, then $(x+y)R \cap N = 0$.

(d) Given submodules K and L of M such that $K \cap N = L \cap N = 0$, then $(K + L) \cap N = 0$.

(e) Given submodules L_λ $(\lambda \in \Lambda)$ such that $N \cap L_\lambda = 0$ $(\lambda \in \Lambda)$, then $N \cap (\sum_{\lambda\in\Lambda} L_\lambda) = 0$.

Moreover, in this case, $\{m \in M : mR \cap N = 0\}$ is the unique complement of N in M.

Corollary 58. *[7, 2.2] An R-module M has unique complements if and only if for every submodule N of M the set $\{m \in M : mR \cap N = 0\}$ is a submodule of M.*

Corollary 59. *[7, 2.3] Let M be an R-module with unique complements. Then every submodule of M has unique complements.*

Let A be any ideal of R, and let M be an R-module. Then we define $T_A(M)$ to be the set of elements m in M such that $(1 - a)m = 0$ for some $a \in A$. Note that $T_A(M)$ is a submodule of M [7].

Lemma 60. *[7, 2.7] Let N be a submodule of a comultiplication R-module M, let $A = Ann_R(N)$ and let $m \in M$. Then*

$$\begin{aligned} T_A(M) =&\{m \in M \,:\, Rm \cap N = 0\} \\ =&\{m \in M \,:\, Hom_R(Rm, N) = 0\} \\ =&\{m \in M \,:\, (0 :_N Ann_R(m)) = 0\} \\ =&\{m \in M \,:\, Rz = Ann_R(m)z, \forall z \in N\}. \end{aligned}$$

Theorem 61. *[7, 2.8] Every comultiplication module has unique complements. Moreover, if N is any submodule of a comultiplication R-module M, then the unique complement of N in M is $T_A(M)$, where A is the ideal $Ann_R(N)$ of R.*

3 Endomorphism rings and Goldie dimension of comultiplication modules

Let M be an R-module and let $End_R(M)$ be the endomorphism ring of M. A submodule K of M is called *fully invariant* if $f(K) \subseteq K$ for every $f \in End_R(M)$.

Theorem 62. *[10, 3.17], [13, 3.19], [17, 3.3, 3.4], and [18, 2.1] Let M be a comultiplication R-module. Then the following assertions hold.*

(a) *Every submodule of M is fully invariant.*

(b) *If N is a submodule of M, then for each monomorphism $f : M \to M$, $f(N) = N$.*

(c) *$End_R(M)$ is a commutative ring.*

(d) *If R is an integral domain and M is a faithful R-module, then every non-zero endomorphism of M is an epimorphism.*

(e) *For each endomorphism f of M, we have $Im(f) = Ann_R(Ker(f))M$.*

(f) *If M is a semisimple module, then for each endomorphism f of M, we have $M = Ker(f) \oplus Im(f)$.*

Lemma 63. *[10, 2.3] Let M be an R-module and let $End_R(M)$ be a domain. Then $Ann_R(M)$ is a prime ideal of R.*

An R-module M is said to be *couniform* or *hollow* if each of its proper submodules is small [8].

Theorem 64. *[10, 3.24] Let M be a comultiplication R-module and let $End_R(M)$ be a domain. Then we have the following.*

(a) *Each non-zero endomorphism of M is an epimorphism.*

(b) *M is a couniform R-module.*

An R-module M is said to be *Hopfian* (resp. *generalized Hopfian* (gH for short)) if every surjective endomorphism f of M is an isomorphism (resp. has a small kernel) [45].

An R-module M is said to be *co-Hopfian* if every injective endomorphism f of M is an isomorphism [46].

Lemma 65. *[11, 3.1] and [11, 3.10]*

(a) *Every comultiplication R-module is co-Hopfian.*

(b) *Every comultiplication R-module is gH.*

Proposition 66. *[11, 3.9] and [66, 2.3] Let M be a comultiplication R-module and let N be a submodule of M such that M/N is a faithful R-module. Then we have the following.*

(a) M/N is a co-Hopfian R-module.

(b) M/N is gH.

We have shown that in the following examples every comultiplication (resp. Artinian) R-module is not an Artinian (resp. comultiplication) R-module.

Example 67. *[11, 3.2] Let p be a prime number and let R be the ring with underlying group*

$$R = End_{\mathbb{Z}}(\mathbb{Z}(p^\infty)) \oplus \mathbb{Z}(p^\infty),$$

and with multiplication

$$(n_1, q_1).(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1).$$

Osofsky has shown that R is a non-Artinian injective cogenerator [40, Exa. 24.34.1], and therefore, since R is a commutative ring, by [40, Prop. 23.13], R is a comultiplication R-module.

Example 68. *[11, 3.3] Let F be a field and let $M = \oplus_{i=1}^{n} F_i$, where $F_i = F$ for $i = 1, 2, ..., n$. Clearly M is Artinian non-comultiplication F-module.*

Let M be an R-module. Set

$$\Theta_R(M) = \{f : M \to M : f(rm) = rf(m), \forall r \in R, \forall m \in M\}.$$

Then M is said to be *semi-endomorphal* if $\Theta_R(M)$ is a ring [47].

Theorem 69. *[17, 3.7] and [15, 2.3] Let M be a comultiplication R-module. Then we have the following.*

(a) M is semi-endomorphal.

(b) If $End_R(M)$ is a division ring, then M is a simple R-module. That is, comultiplication R-module satisfies the converse of Schur's Lemma.

An R-module M is said to satisfy *Fitting's Lemma* if for each $f \in End_R(M)$ there exists an integer $n \geq 1$ such that $M = Ker(f^n) \bigoplus Im(f^n)$ [33].

Theorem 70. *[11, 3.4] Let M be a comultiplication R-module satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module. Then M satisfies Fitting's Lemma.*

Corollary 71. *[11, 3.5] Let M be an indecomposable comultiplication module satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module. Let $f \in End_R(M)$. Then the following are equivalent.*

(a) f is a monomorphism.

(b) f is an epimorphism.

(c) f is an automorphism.

(d) f is not nilpotent.

Remark 72. *[11, 3.7] In the Corollary 71, the condition "M satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module" can not be omitted. For example, let p be a prime number. Then $M = \mathbb{Z}_{p^\infty}$ is an indecomposible comultiplication $\mathbb{Z}$-module but not satisfying ascending chain condition on submodules N such that M/N is a comultiplication $\mathbb{Z}$-module. Define $f : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ by $x \to px$. Clearly f is an epimorphism with $Kerf = \mathbb{Z}(1/p + \mathbb{Z})$. Hence, f is not a monomorphism.*

Theorem 73. *[13, 3.19] Let M be a comultiplication R-module. If M is an indecomposable R-module satisfying the ascending chain condition on submodules N such that M/N is a comultiplication R-module, then $End_R(M)$ is a local ring.*

Lemma 74. *[7, 4.1] Let R be a local ring. Then every non-zero comultiplication R-module M is uniform.*

Following [34, page 8], a non-empty family of submodules N_λ $(\lambda \in \Lambda)$ of an R-module M is called *coindependent*, provided that, for each non-empty finite subset Δ of Λ and element $\lambda \in \Lambda \setminus \Delta$,

$$N_\lambda + \cap_{\delta \in \Delta} N_\delta = M.$$

We shall say that module M has *finite dual Goldie dimension*, provided that, M does not contain an infinite coindependent family of proper submodules. In this case, there exists a unique positive integer k, called the *dual Goldie dimension* of M, denoted here by $dGdimM$, such that k is the supremum of the cardinalities of coindependent families of non-zero submodules [34, 5.2].

Theorem 75. *[7, 4.2] Let M be a comultiplication R-module such that the submodules Rm_i $(1 \le i \le n)$ are independent for some positive integer n and non-zero elements $m_i \in M$ $(1 \le i \le n)$. Then $Ann_R(m_i)$ $(1 \le i \le n)$ is a coindependent family of proper ideals of R.*

A ring R is called *semilocal* if it contains only a finite number of maximal ideals, say P_i $(1 \leq i \leq n)$. In this case, if k is a positive integer and A_i $(1 \leq i \leq k)$ any coindependent collection of proper ideals of R, then $R = A_i + A_j$ for all $1 \leq i < j \leq k$. Thus, for each $1 \leq i \leq n$, there exists a unique integer j with $1 \leq j \leq k$ and $A_j \subseteq P_i$. Thus, $k \leq n$. It follows that the R-module R has finite dual Goldie dimension. On the other hand, if R is any ring and Q_i $(1 \leq i \leq t)$ any collection of distinct maximal ideals of R, for some positive integer t, then clearly Q_i $(1 \leq i \leq t)$ are coindependent submodules of the R-module R. Thus, a ring R is semilocal if and only if the R-module R has finite dual Goldie dimension and, in this case, $dGdimR$ is precisely the number of distinct maximal ideals of R [7].

It is proved in Theorem 24 that every finitely generated comultiplication module is finitely cogenerated and hence, has finite Goldie dimension. On the other hand, Theorem 24 also shows that every comultiplication module has essential socle, so that if it has finite Goldie dimension, then it is finitely cogenerated. Moreover, in [6, Corollary 2.2], it is proved that, if M is a non-zero finitely generated comultiplication module over a ring R, then the ring $R/Ann_R(M)$ is semilocal. Now note the following corollary of Theorem 75.

Corollary 76. *[7, 4.3] Let R be a semilocal ring and let M be a comultiplication R-module. Then M has finite Goldie dimension. Moreover, $GdimM \leq dGdimR$.*

Corollary 77. *[7, 4.4] Let M be a non-zero finitely generated comultiplication R-module. Then M has finite Goldie dimension if and only if the ring $R/Ann_R(M)$ is semilocal.*

If M is a Noetherian comultiplication module, then M is Artinian and hence, M has finite hollow dimension [34, 5.2]. We next investigate when comultiplication modules have finite hollow dimension. In particular, we would like to know whether Corollary 76 has an analogue for hollow dimension. Recall that an ideal A of a ring R is called *(meet) irreducible*, provided that A is a proper ideal of R and $A \neq B \cap C$ for any ideals B and C, both properly containing A. Clearly, A is an irreducible ideal of R if and only if the R-module R/A is uniform [7].

Lemma 78. *[7, 4.5] Let M a comultiplication R-module such that $Ann_R(M)$ is an irreducible ideal of R. Then M is a hollow module.*

Proposition 79. *[7, 4.6] Let M be a comultiplication R-module such that $R/Ann_R(M)$ has finite Goldie dimension and $Ann_R(K \cap L) = Ann_R(K) + Ann_R(L)$ for all submodules K and L of M. Then M has finite hollow dimension and, moreover, $dGdimM \leq Gdim(R/(0 : RM))$.*

4 P-cotorsion, P-cocyclic, and comultiplication modules

Every multiplication module over an Artinian ring is cyclic [39]. In below, it is shown that the dual of this fact is not true in general.

Theorem 80. *[17, 3.5] Let R be an Artinian non-local ring and M be a faithful comultiplication R-module. Then M is not a cocyclic R-module.*

Example 81. *[17, 3.6] $\mathbb{Z}_6$ is an Artinian non-local ring and $\mathbb{Z}_6$ (as a $\mathbb{Z}_6$-module) is a faithful comultiplication module. But $\mathbb{Z}_6$ is not a cocyclic $\mathbb{Z}_6$-module.*

An R-module M is said to be *coprimal* if $M \neq 0$ and $Zd_R(M)$ is an ideal of R, where $Zd_R(M)$ is the set of all zero divisors of M [36].

Theorem 82. *[15, 2.3] Let M be a comultiplication R-module. Then we have the following.*

(a) If M is a finitely generated R-module, then M is cocyclic or $R/AnnR(M)$ is a decomposable ring.

(b) If M is a coprimal R-module, then M is cocyclic.

Theorem 83. *[10, 3.17] Every cocyclic module over a complete Noetherian local ring is a comultiplication module.*

Theorem 84. *[15, 2.5] Let R be a Noetherian ring and let M be a finitely generated cocyclic R-module. Then M is a comultiplication R-module. In particular, every cocyclic module over an Artinian ring is a comultiplication module.*

Theorem 85. *[66, 2.11] Let M be a cocyclic $\mathbb{Z}$-module, then M is a comultiplication module.*

In the following example we see that not every cocyclic R-module is a comultiplication R-module in general.

Example 86. *[6, 3.7] Let S be any non-zero integral domain which is not a local ring (e.g., S could be the ring $\mathbb{Z}$). Let U be any simple S-module and let E be the injective envelope of U. Let $R = E \oplus S$ be the trivial extension of E by S. Then the R-module R is a cocyclic module which is not a comultiplication module.*

Theorem 87. *[6, 3.9] Let R be a Dedekind domain. Then a non-zero R-module M is a comultiplication module if and only if M is cocyclic or there exist positive integers $n, k(1), ..., k(n)$ and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R such that $M \cong (R/P_1^{k(1)} \oplus ... \oplus R/P_1^{k(1)})$.*

Let M be an R-module and let P be a maximal ideal of R. Then the set

$$T_P(M) = \{m \in M \mid (1-p)m = 0 \ for\ some\ p \in P\}$$

is a submodule of M. M is said to be P-*torsion module* if $T_P(M) = M$ [63]. M is said to be P-*cyclic module* if there exist $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$ [39].

In [39], El-Bast and Smith have given a characterization of multiplication modules which is essentially a useful method for studying multiplication modules. They showed that an R-module M is a multiplication module if and only if, for each maximal ideal P of R, M is either P-torsion module or P-cyclic module. Dually, in this regard, for a maximal ideal P of R, the notion of P-cotorsion (resp. P-cocyclic) modules (see Definitions 90 and 88) were introduced and proved that if M is a comultiplication R-module then, for any maximal ideal P of R, M is P-torsion module or P-cocyclic module. Moreover, the converse holds if M is Noetherian (see Theorem 93).

Definition 88. *[15, 2.6] Let M be an R-module and let P be a maximal ideal of R. We say that M is P-cocyclic module, provided that there exist $p \in P$ and completely irreducible submodule L of M such that $(1-p)L = 0$. This is, in fact, a dual notion of P-cyclic modules.*

Definition 89. *[19, 2.7] Let P be a prime ideal of R and let N be a submodule of an R-module M.* The P-interior of N, relative to M, *is defined as the set*

$$I_P^M(N) = \cap\{L \mid L \textit{ is a completely irreducible submodule of } M \textit{ and}$$

$$rN \subseteq L \textit{ for some } r \in R - P\}.$$

Definition 90. *[15, 2.7] Let M be an R-module and let P be a maximal ideal of R. We say that M is P-cotorsion module if $I_P^M(N) = 0$. This can be regarded as a dual notion of P-torsion modules.*

Example 91. *[15, 2.8]*

(a) Let P be a maximal ideal of R. Then every cocyclic R-module is a P-cocyclic R-module. In particular, for each prime number q of $\mathbb{Z}$ the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a $q\mathbb{Z}$-cocyclic $\mathbb{Z}$-module.

(b) For each prime number p of $\mathbb{Z}$, we have $\mathbb{Z}$ is a $p\mathbb{Z}$-cotorsion $\mathbb{Z}$-module.

(c) Let p be a prime number. Then the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is neither $p\mathbb{Z}$-cocyclic nor $p\mathbb{Z}$-cotorsion.

Theorem 92. *[15, 2.9]*

(a) Let M be an Artinian P-cotorsion R-module for some maximal ideal P of R. Then M is P-cocyclic.

(b) Let M be a non-zero comultiplication R-module. Then M is a P-cocyclic R-module for some maximal ideal P of R.

(c) Let M be a Noetherian P-cocyclic R-module for each maximal ideal P of R. Then M is an Artinian comultiplication R-module.

Theorem 93. *[15, 2.10] Let M be an R-module. If M is a comultiplication module, then, for any maximal ideal P of R, M is a P-torsion module or P-cocyclic module. Moreover, the converse holds if M is Noetherian.*

Corollary 94. *[15, 2.11] Let M be a finitely generated comultiplication R-module. Then M is a P-cocyclic module for every maximal ideal P of R.*

Proposition 95. *[7, 1.9] Let M be an R-module such that every cocyclic homomorphic image of M is a comultiplication module. Suppose further that, for every maximal ideal P of R, the module M is a P-torsion module or a P-cocyclic module. Then M is a comultiplication module.*

Theorem 96. *[15, 2.12] Let I be an ideal of R and let M be a comultiplication R-module. Then $M/(0:_M I)$ is a comultiplication R-module if I is a multiplication ideal of R and M is a Noetherian R-module.*

Corollary 97. *[15, 2.13] Let R be a multiplication ring, m a maximal ideal of R and let M be a Noetherian comultiplication R-module. Then, for each integer n, the factor module $(0:_M m^{n+1})/(0:_M m^n)$ is simple.*

Theorem 98. *[15, 2.14] Let M be an R-module and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a collection of submodules of M such that M/M_λ is a comultiplication R-module and $\cap_{\lambda\in\Lambda}M_\lambda = 0$. If M is a comultiplication R-module, then, for each maximal ideal P of R, M is either a P-torsion module or there exist a completely irreducible submodule L/M_λ of M/M_λ for some $\lambda \in \Lambda$ and $p \in P$, with $(1-p)L = 0$. The converse holds if M is a Noetherian R-module.*

In [63], Smith proved that if N_1 and N_2 are submodules of an R-module M such that N_1, N_2 and $N_1 + N_2$ are all multiplication modules, then $N_1 \cap N_2$ is a multiplication module. The following theorem is the dual of this fact under a restrictive condition.

Theorem 99. *[15, 2.15] Let M be a Noetherian R-module and let N_1 and N_2 be two submodules of M such that $M/N_1, M/N_2$, and $M/(N_1\cap N_2)$ are comultiplication R-modules. Then $M/(N_1 + N_2)$ is a comultiplication R-module.*

5 Strong comultiplication modules and copure submodules

An R-module M satisfies the *double annihilator conditions* (DAC for short) if, for each ideal I of R, we have $I = Ann_R(0:_M I)$ [41].

Definition 100. *[14, 2.1] We say that an R-module M is a* strong comultiplication module *if M is a comultiplication R-module and satisfies the DAC conditions.*

Example 101. *[14, 2.2] Every cocyclic R-module over a complete Noetherian local ring is a strong comultiplication R-module.*

Example 102. *(a) For each positive integer n, the $\mathbb{Z}_n$-module $\mathbb{Z}_n$ is a strong comultiplication module.*

(b) Let $R = \mathbb{Z}_2[x, y, z]$ be the polynomial ring over a field $\mathbb{Z}_2$ in indeterminates x, y, z. Then $\bar{R} = R/(x^2, y^2, z^2)$ is a strong comultiplication $\bar{R}$-module.

(c) The $\mathbb{Z}_6$-module $\mathbb{Z}_2$ is not a strong comultiplication module.

Example 103. *[14, 2.3] Let p be a prime number and n be a positive integer. Then $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_n$ are comultiplication $\mathbb{Z}$-modules but they are not strong comultiplication $\mathbb{Z}$-modules.*

It is possible, for a comultiplication R-module M, to have a submodule N for which there exist two ideals $I \neq J$ with the property $(0 :_M I) = N = (0 :_M J)$. For example, if $M = \mathbb{Z}_{2^\infty}$, then $(0 :_M 2\mathbb{Z}) = (0 :_M 6\mathbb{Z})$. It is easy to see that, for each submodule N of an R-module M, there exists a unique ideal I of R such that $N = (0 :_M I)$ if and only if M is strong comultiplication R-module [56].

Proposition 104. *[14, 2.4] Let M be an R-module. Then we have the following.*

(a) Let M be a faithful cogenerator for R and let $S = End_R(M)$. If every $f \in S$ is trivial, then M is a strong comultiplication R-module.

(b) If R is a Noetherian ring and M is a strong comultiplication R-module, then M is an injective R-module.

Theorem 105. *[14, 2.5] Let M be a strong comultiplication R-module and let I be an ideal of R. Let N be a submodule of M. Then we have the following.*

(a) M/N is a comultiplication R-module if and only if $Ann_R(N)Ann_R(K/N) = Ann_R(K)$ for each submodule K of M with $N \subseteq K$.

(b) If M/N is a comultiplication R-module, then $Ann_R(N)$ is a multiplication ideal of R.

(c) If $M/(0 :_M I)$ is a comultiplication R-module, then I is a multiplication ideal of R.

Example 106. *[14, 2.6] Let $A = K[x, y]$ be the polynomial ring over a field K in two indeterminates x, y. Then $\bar{A} = A/(x^2, y^2)$ is a strong comultiplication $\bar{A}$-module. But $\bar{A}/\bar{A}\bar{x}\bar{y}$ is not a comultiplication $\bar{A}$-module, by [40, 24.4]. Furthermore, this example shows that not every homomorphic image of a strong comultiplication module is a comultiplication module, in general.*

Theorem 107. *[56, 4.4] Assume that M is a strong comultiplication R-module. Then we have the following.*

(a) M is finitely cogenerated and both M and R are amply supplemented R-modules.

(b) R is semilocal.

Recall that a *reduced ring* is one with no nilpotents.

Corollary 108. *[56, 4.5] If M is a strong comultiplication module, having a maximal submodule over a reduced ring R, then $M \cong R$ and R is semisimple.*

Definition 109. *[14, 2.7] We say that a submodule N of an R-module M is* copure *if $(N :_M I) = N + (0 :_M I)$ for each ideal I of R.*

Example 110. *[14, 2.8] Every submodule of $\mathbb{Z}_k$ $(k \in \mathbb{N})$ as a $\mathbb{Z}$-module is copure.*

Theorem 111. *[14, 2.9] Let M be an R-module and let N and K be submodules of M such that $N \subseteq K \subseteq M$. Then we have the following.*

(a) If K is a copure submodule of M and N is a copure submodule of K, then N is a copure submodule of M.

(b) If N is a copure submodule of M, then N is a copure submodule of K.

(c) If K is a copure submodule of M, then K/N is a copure submodule of M/N.

(d) If N is a copure submodule of M and $K/N$is a copure submodule of M/N, then K is a copure submodule of M.

(e) If N is a copure submodule of M, then there is a bijection between the copure submodules of M containing N and the copure submodules of M/N.

Theorem 112. *[14, 2.10] For an exact sequence*

$$0 \longrightarrow N \xrightarrow{\psi} L \xrightarrow{\phi} K \longrightarrow 0$$

of R-modules and R-homomorphisms, the following assertions are equivalent.

(a) For every ideal I of R, the following sequence is exact.

$$0 \longrightarrow Hom_R(R/I, N) \xrightarrow{\overline{\psi}} Hom_R(R/I, L) \xrightarrow{\overline{\phi}} Hom_R(R/I, K) \longrightarrow 0.$$

(b) $\psi(N)$ is a copure submodule of L.

Proposition 113. *[14, 2.11] Let M be an R-module. Then we have the following.*

(a) If M is a comultiplication module and N is a large and pure submodule of M, then $N = M$.

(b) If N and K are submodules of M such that $N \cap K$ and $N + K$ are copure submodules of M, then N is a copure submodule of M.

(c) If $\{M_\lambda\}_\Lambda$ is a family of submodules of M with copure submodules $N_\lambda \subseteq M_\lambda$, then $\sum_{\lambda\in\Lambda} N_\lambda$ is a copure submodule of $\sum_{\lambda\in\Lambda} M_\lambda$.

Theorem 114. *[14, 2.12] Let R be a principal ideal domain and let M be an R-module. Then we have the following.*

(a) Every submodule of M is a pure submodule of M if and only if it is a copure submodule of M.

(b) If M is a prime module, then every copure submodule of M is a prime submodule of M.

Theorem 115. *[14, 2.13] Let M be a strong comultiplication R-module. Then we have the following.*

(a) N is a copure submodule of M if and only if $Ann_R(N)$ is a pure ideal of R.

(b) An ideal I of R is pure if and only if $(0 :_M I)$ is a copure submodule of M.

(c) If N is a copure submodule of M, then, for every non-empty collection $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of R, we have

$$\sum_{\lambda\in\Lambda}(N :_M I_\lambda) = (N :_M \cap_{\lambda\in\Lambda} I_\lambda).$$

(d) If N is a copure submodule of M, then $Ann_R(N)$ is the intersection of all ideals I of R such that $N = (N :_M I)$.

Proposition 116. *[14, 2.13], [18, 2.7], and [56, 4.3] Let M be an R-module. Then we have the following.*

(a) If M is a comultiplication R-module and $Soc_R(M)$ is a pure submodule of M, then $M = Soc_R(M)$. In particular if R is a local ring, then M is simple.

(b) If M is a comultiplication R-module and N is a copure submodule of M such that M/N is a finitely generated R-module, then N is a direct summand of M.

(c) A non-zero multiplication R-module M is strong comultiplication module if and only if it is finitely generated faithful R-module and R is a comultiplication ring.

Proposition 117. *[66, 2.4] Let M be a faithful comultiplication R-module with the property $(0:_M I)+(0:_M J)=(0:_M (I\cap J))$ for any two ideals I and J of R, and $N\leq M$. Then N is a small submodule of M if and only if there exists a large ideal I of R such that $N=(0:_M I)$.*

Theorem 118. *[66, 2.5, 2.8, 2.10, 2.7] Let M be a strong comultiplication R-module. Then we have the following.*

(a) A submodule N of M is large if and only if there exists a small ideal I of R such that $N=(0:_M I)$.

(b) M is uniform if and only if R is hollow.

(c) M is a semisimple module if and only if R is a semisimple ring.

(d) $Soc_R(M)=(0:_M Jac(R))$.

6 Comultiplication modules over special rings and Fitting ideals

The following example shows that it is possible that every localization of an R-module M is a comultiplication module, without M being so.

Example 119. *[56, 5.4] Let $R=\mathbb{Z}$ and $M=\oplus_{p\in P}\mathbb{Z}_p$, where P is the set of positive prime integers. Clearly $N=\oplus_{2\neq p\in P}\mathbb{Z}_p$ is a maximal (and hence prime) submodule of M and $N\neq(0:_M Ann_R(N))$. Therefore, M is not a comultiplication R-module. But for each maximal ideal of R such as $m=Rp$ ($p\in P$), $M_m\cong\mathbb{Z}_p$ as R_m-module and hence is a simple and comultiplication R_m-module. Notice that $M_0=0$ is trivially a comultiplication $\mathbb{Q}$-module.*

Lemma 120. *[37, 2.4] Every non-zero comultiplication module over a discrete valuation domain R is indecomposable.*

Theorem 121. *[37, 2.5] Let R be a discrete valuation domain with a unique maximal ideal $P=Rp$. Then the comultiplication modules over R are:*

(a) R/P^n, $n\geq 1$;

(b) $E(R/P)$, the injective hull of R/P.

Lemma 122. *[56, 2.1] Suppose that M is an R-module and $R=R_1\times R_2$, where R_1 and R_2 are non-trivial rings. Then $M=M_1\oplus M_2$, where M_1 is an R_1-module and M_2 is an R_2-module. Also, in this case, M is comultiplication module if and only if both M_1 and M_2 are so.*

Theorem 123. *[56, 2.2] If M is a faithful comultiplication R-module with a maximal submodule N and R is a reduced ring with a decomposition as a finite direct product of indecomposable rings, then $M \cong R$ and R is semisimple.*

Corollary 124. *[56, 2.3] Assume that M is a comultiplication R-module having a maximal submodule (for example, if M is finitely generated) and $m = Ann_R(M)$. Then m is a prime ideal if and only if m is a maximal ideal and $M \cong R/m$ is a simple module.*

Corollary 125. *[56, 2.4] If M is a finitely generated comultiplication R-module with $Ann_R(M)$ a radical ideal, then M is cyclic and $R/Ann_R(M)$ is a semisimple ring.*

A chained ring is a ring in which every two ideals are comparable. For example, localization of $\mathbb{Z}$ at any prime ideal or, more generally, every valuation domain is a chained ring.

Lemma 126. *[56, 2.5] If R is a chained ring and M is a comultiplication R-module having a maximal submodule N, then M is cyclic.*

A ring, in which every non-zero proper ideal is a product of prime ideals, is called a Zerlegung Primideale ring (ZPI-ring).

A principal ideal rings with exactly one prime ideal is called a special principal ideal ring ($SPIR$).

Corollary 127. *[56, 2.6] Suppose that R is a ZPI-ring and M is a finitely generated R-module, then M is comultiplication R-module if and only if M is cyclic and $R/Ann_R(M)$ is a finite direct product of $SPIR$s.*

In what follows, by a *semi-non-torsion* R-module M, we mean a module, which is a non-torsion module over $R/Ann_R(M)$. The following remark states some other conditions under which a comultiplication module must be cyclic.

Remark 128. *[56, 2.7]*

(a) If M is a semi-non-torsion comultiplication R-module, then M is cyclic.

(b) If M is a finitely generated comultiplication R-module and $Ann_R(M)$ is irreducible, then M is cyclic.

(c) If R is a finitely cogenerated ring with irreducible zero ideal and M is a faithful comultiplication R-module, then M is cyclic.

Proposition 129. *[56, 2.10] The following are equivalent for the ring R.*

(a) R is a comultiplication ring.

(b) Every faithful multiplication R-module is a comultiplication R-module.

(c) There exists a finitely generated faithful comultiplication R-module.

Theorem 130. *[56, 2.11] Suppose that $R = \prod_{a\in A} R_a$, where R_a's are non-trivial rings. A faithful R-module M is a comultiplication module if and only if $M = \oplus_{a\in A} M_a$, where each M_a is a comultiplication R_a-module and $R_b M_a = 0$ for $a \neq b \in A$. In particular, R is a comultiplication ring if and only if $|A| < \infty$ and each R_a is a comultiplication ring.*

Corollary 131. *[56, 2.12] Let R be a reduced Noetherian ring. Then R is comultiplication if and only if it is a finite direct product of fields.*

We know that if a Noetherian ring is a comultiplication ring, then it is Artinian. Because an Artinian ring is a finite direct product of some Artinian local rings, to know which Noetherian rings are comultiplication, it suffices to consider Artinian local rings. Not all Artinian local rings are comultiplication rings, as the following example shows [56].

Example 132. *[56, 2.15] Set $R_0 = K[X;Y]$, where K is a field and $m =< X, Y >$. Let $R = R_0/m^2$. Then clearly R is an Artinian local ring. But R is not a comultiplication ring.*

Example 133. *[56, 2.17] Set $R_0 = \mathbb{Z}_3[X, Y]$ and $R = R_0/I$, where $I =< XY, X^2 - Y^2 >$. Then R is an Artinian local comultiplication ring, but not an $SPIR$.*

Example 134. *[56, 2.18] Set $R_0 = \mathbb{Z}_3[X, Y]$ and $R = R_0/I$, where $I =< XY, X^2 - Y^2 >$. Let x, y denote the images of X, Y in R, respectively and $m =< x, y >$. Then the maximal ideal m of R, being a submodule of a comultiplication module, is itself a finitely generated comultiplication module which is not cyclic.*

For a set $\{x_1, ..., x_n\}$ of generators of an R-module M, there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0,$$

where R^n is a free R-module with the set $\{e_1, ..., e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + ... + a_{n\lambda}e_n$, with λ in some index set Λ. Let $Fitt_i(M)$ be the ideal of R, generated by the minors of size $n - i$ of the matrix

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

For $i > n$, $Fitt_i(M)$ is defined to be R, and for $i < 0$, $Fitt_i(M)$ is defined to be the zero ideal. It is known that $Fitt_i(M)$ is an invariant ideal, determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [43]. The ideal $Fitt_i(M)$ will be called the i-th Fitting ideal of the module M. It follows, from the definition of $Fitt_i(M)$, that $Fitt_i(M) \subseteq Fitt_{i+1}(M)$. Moreover, it is shown that $Fitt_0(M) \subseteq Ann_R(M)$ and $(Ann_R(M))^n \subseteq Fitt_0(M)$ (M is generated by n elements) [38]. The most important Fitting ideal of M is the first of the $Fitt_j(M)$ that is non-zero. We shall denote this Fitting ideal by $I(M)$.

Theorem 135. *[48, 1.3] Let M be a finitely generated comultiplication R-module. If R is an integral domain, then $I(M) = Fitt_0(M)$ or $M \cong R$.*

Proposition 136. *[48, 1.6] Let M be a finitely generated comultiplication module over an integral domain R. If $I(M)$ is a prime ideal of R, then M is a simple R-module.*

Proposition 137. *[48, 1.7] Every finitely generated comultiplication module over a valuation ring is cyclic.*

Lemma 138. *[48, 1.8] Let M be a finitely generated comultiplication R-module. If R is a Dedekind domain, then M is cyclic.*

Lemma 139. *[48, 1.9] Let M be a finitely generated comultiplication R-module. If $M = \langle x_1, ..., x_n\rangle$ and $\cap_{i=1}^{n} Rx_i = 0$, then $Fitt_{n-1}(M) = R$.*

Proposition 140. *[48, 1.11] Let M be a decomposable comultiplication R-module. If M is generated by two elements, then $Fitt_0(M) = Ann_R(M)$.*

Theorem 141. *[48, 1.12] Let M be a finitely generated R-module.*

(a) If $I(M)$ is a prime ideal of R, then $Ann_R(M) \subseteq I(M)$.

(b) If $I(M) = Q_1...Q_n$ such that Q_i are distinct maximal ideals of R, then we have $Ann_R(M) \subseteq I(M)$.

(c) If $Ann_R(M) = Q^n$ for some maximal ideal Q of R and positive integer n, then $I(M) = R$ or $I(M)$ is a Q-primary ideal of R.

Proposition 142. *[48, 1.13] Let M be a comultiplication R-module. If M is a decomposable module and $M = \langle x_1, ..., x_n\rangle$, then $(Ann_R(M))^{n-1} \subseteq Fitt_0(M)$.*

Theorem 143. *[48, 1.14] Let M be a finitely generated comultiplication R-module. If $Ann_R(M) = Q_1, ..., Q_n$, where Q_i, $1 \leq i \leq n$, are distinct maximal ideals of R, then $M \cong R/Q_1 \oplus ... \oplus R/Q_n$.*

Proposition 144. *[48, 1.15] Let M be a finitely generated comultiplication R-module. If R is a von Neumann regular ring, then $I(M) = Q_1...Q_n$, where Q_i are maximal ideals of R, $1 \leq i \leq n$.*

Theorem 145. *[48, 1.16] Let M be a finitely generated comultiplication R-module. If $Fitt_0(M) = Q_1...Q_n$, where Q_i, $1 \leq i \leq n$, are distinct maximal ideals of R, then M is a semisimple module.*

Lemma 146. *[48, 1.17] Let M be a finitely generated R-module. If $Ann_R(M) = \langle e \rangle$, where e is a non-zero idempotent element of R, then $I(M) = Ann_R(M)$.*

Theorem 147. *[48, 1.18] Let M be a finitely generated comultiplication R-module. If there is a submodule N of M such that $Ann_R(N) = \langle e \rangle$, where e is an idempotent element of R, then N is a direct summand of M and $I(M) \subseteq \langle e \rangle$.*

Corollary 148. *[48, 1.19] Let M be a finitely generated strong comultiplication R-module. If e is an idempotent element of R, then $e \in Ann_R(M)$ or $1-e \in Ann_R(M)$.*

Proposition 149. *[48, 1.20] Let M be a finitely generated module over a Prüfer domain R and Q be a maximal ideal of R. Then $Ann_R(M) = Q_n$ for some positive integer n if and only if $Fitt_0(M) = Q_k$ for some $k \in N$.*

Theorem 150. *[48, 1.21] Let M be a finitely generated comultiplication R-module over a Prüfer domain R. If $Fitt_0(M) = Q_n$, where Q is a maximal ideal of R and n is a positive integer, then M is cyclic.*

Theorem 151. *[48, 1.22] Let M be a finitely generated comultiplication R-module. Then $R/Fitt_0(M)$ is a semilocal ring.*

Corollary 152. *[48, 1.23] Let M be a finitely generated comultiplication R-module. If R is not a semilocal ring, then $I(M) = Fitt_0(M)$.*

Let G be an abelian group with identity e. The ring R, graded by the group G, will be denoted by $R = \oplus_{g \in G} R_g$, where R_g is an additive subgroup of R and $R_g.R_h \subseteq R_{gh}$ for every g, h in G. If the inclusion is an equality, then the ring is called *strongly graded*. If an element of R belongs to $\cup_{g \in G} R_g = h(R)$, then it is called *homogeneous* and any $x_g \in R_g$ is said to have degree g. Now, let R be a graded ring. Then an R-module M is said to be a *graded module* if $M = \oplus_{g \in G} M_g$ for a family of subgroups $\{M_g\}_{g \in G}$ of M such that $R_g.M_h \subseteq M_{hg}$ for every g, h in G. Analogously is defined *strongly graded module*. A *graded submodule* N of M is a submodule, verifying $N = \oplus_{g \in G}(N \cap M_g)$.

In the remainder of this section, R will denote a commutative G-graded ring with identity.

Definition 153. *[16, 3.1] We say that a graded R-module M is a* comultiplication graded module *(*gr-comultiplication module*) if, for every graded submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$.*

Remark 154. *[16, 3.2] It is clear that every comultiplication R-module, which is a graded module, is a gr-comultiplication R-module. Furthermore, if $N = (0 :_M I)$ for some ideal I of R, then $N = (0 :_M Ann_R(N))$. Thus, the ideal I of the definition can be taken graded. We will show that there is an example of a gr-comultiplication module that is not comultiplication module (see Example 155 (d)).*

Example 155. *[16, 3.3]*

(a) Let G be a finite group. Then, for each positive integer n, the group ring $R = \mathbb{Z}_n[G]$ is a gr-comultiplication R-module [7, 15.26 (5)].

(b) Let K be a field, $A = K[x, y]$ be the polynomial ring over a field K in two indeterminates x, y. Then $\bar{A} = A/(x^2, y^2)$ is a gr-comultiplication $\bar{A}$-module.

(c) Let K be a field, $R = K[x]$, where x is an indeterminate and let $M = K[x^{-1}, x]$. Then R is a graded R-submodule of M and $R \neq (0 :_M Ann_R(R)) = M$. Thus, M is not a gr-comultiplication R-module.

(d) If we take the graded ring $R = K[x, x^{-1}](= K[x]_x)$, where K is a field and x is an indeterminate, then R is a gr-comultiplication R-module, which is not a comultiplication R-module.

Theorem 156. *[16, 3.4] Let R be a strongly graded ring and let M be a graded R-module. Then M is a gr-comultiplication module if and only if M_e is a comultiplication module as an R_e-module.*

Lemma 157. *[16, 3.5] An R-module M is a gr-comultiplication R-module if and only if for all graded submodules N and K of M with $Ann_R(N) = Ann_R(K)$, we have $N = K$.*

Theorem 158. *[16, 3.6] Let R be a strongly graded ring and let M be a graded R-module. Then we have the following.*

(a) If M is a comultiplication R_e-module, then $M = M_e$.

(b) If M is a gr-multiplication R-module, then $J^{gr}(M) \cap M_e = J(M_e)$.

Theorem 159. *[16, 3.7] Let M be a gr-comultiplication R-module. Then we have the following.*

(a) Every graded submodule of M is a gr-comultiplication R-module.

(b) If P is a gr-maximal ideal of R and $(0 :_M P) \neq 0$, then $(0 :_M P)$ is a gr-simple R-submodule.

(c) If I and J are graded ideals of R such that $(0 :_M I) = (0 :_M J)$, then $IM = JM$.

(d) *If B is a graded ideal of R such that $(0 :_M B) = 0$, then, for each homogeneous element $m \in M$, there exists an element $b \in B$ such that $m = bm$.*

(e) *If $B \subseteq J^{gr}(R)$ and $(0 :_M B) = 0$, then $M = 0$.*

Theorem 160. *[16, 3.8] Let M be a gr-comultiplication R-module. Then we have the following.*

(a) *If $\{M_\lambda\}_{\lambda\in\Lambda}$ is a collection of graded submodules of M, then*

$$(0 :_M \cap_{\lambda\in\Lambda} Ann_R(M_\lambda)) = \sum_{\lambda\in\Lambda}(0 :_M Ann_R(M_\lambda)).$$

(b) *If $\{M_\lambda\}_{\lambda\in\Lambda}$ is a family of submodules of M with $\cap_{\lambda\in\Lambda} M_\lambda = 0$, then, for every graded submodule N of M, we have*

$$N = \cap_{\lambda\in\Lambda}(N + M_\lambda).$$

(c) *If M is finitely generated and B is a graded ideal of R such that $(0 :_M B) = 0$, then there exists $b \in B$ such that $1 - b \in Ann_R(M)$.*

(d) *If P is a gr-minimal ideal of R such that $(0 :_M P) = 0$, then M is gr-cyclic.*

(e) *If R is a ring, satisfying the descending (res. ascending) chain condition on graded ideals containing $Ann_R(M)$, then M is a gr-Noetherian (resp. a gr-Artinian) module.*

Theorem 161. *[16, 3.9] Let M be a gr-comultiplication R-module. Then we have the following.*

(a) *Every non-zero graded submodule of M contains a graded minimal submodule of M.*

(b) *Let K be a graded submodule of M. Then K is a gr-minimal submodule of M if and only if there exists a gr-maximal ideal P of R such that $K = (0 :_M P) \neq 0$.*

(c) *If R has a unique gr-maximal ideal, then M is a gr-cocyclic R-module.*

(d) *If M is a faithful finitely generated R-module, then $(0 :_M I) \neq 0$ for every proper graded ideal I of R.*

(e) *If R is a strongly graded ring, and for each family $\{N_\lambda\}_{\lambda\in\Lambda}$ of graded submodules of M, we have $(\sum_{\lambda\in\Lambda} N_\lambda) \cap M_e = \sum_{\lambda\in\Lambda}(N_\lambda \cap M_e)$, then $Soc^{gr}(M) \cap M_e = Soc(M_e)$.*

Theorem 162. *[16, 3.10] Let M be a gr-comultiplication R-module. Then we have the following.*

(a) Every graded submodule of M is fully invariant.

(b) For every graded endomorphism f of M, there exists an ideal I of R such that $Im(f) = IM$.

(c) M is graded co-Hopfian.

(d) If M a is finitely generated R-module, then M is co-Hopfian.

(e) If M is graded semisimple R-module, then, for each graded endomorphism f of M, we have $M = Ker(f) \oplus Im(f)$.

Theorem 163. *[16, 3.11] Let M be a faithful gr-comultiplication R-module. Then we have the following.*

(a) $W^{gr}(M) = Z^{gr}(R)$, where

$$W^{gr}(M) = \{a \in h(R) : \text{ the homothety } M \xrightarrow{a} M \text{ is not surjective}\}$$

and

$$Z^{gr}(R) = \{a \in h(R) : \text{ the homothety } R \xrightarrow{a} R \text{ is not injective}\}.$$

(b) M is gr-divisible.

(c) For each graded submodule N of M, we have $Ann_R(Ann_R(M/N)) = Ann_R(N)$.

(d) If M is finitely generated, then M is gr-uniform if and only if every proper graded ideal of R is gr-small.

(e) If M is finitely generated, then a graded submodule N of M is gr-large if and only if there exists a gr-small ideal I of R such that $N = (0 :_M I)$.

Theorem 164. *[16, 3.12] Let M be a gr-comultiplication R-module. If $W^{gr}(M) = 0$ and M has a gr-maximal submodule, then M is gr-simple R-module.*

Lemma 165. *[16, 3.13] Let R be a gr-Noetherian ring and let M be a graded R-module. Then we have the following.*

(a) If M is a gr-comultiplication R-module and $S \subseteq h(R)$ is a multiplicatively closed subset of R, then $S^{-1}M$ is a gr-comultiplication $S^{-1}R$-module.

(b) If M is a finitely generated R-module, then M is a gr-comultiplication R-module if and only if M_P is a gr-comultiplication R_P-module for every gr-prime ideal P of R.

7 Fully coidempotent modules

Let N and K be two submodules of an R-module M. The *product* of N and K is defined by $(N :_R M)(K :_R M)M$ and denoted by NK. Also, the *coproduct* of N and K is defined by $(0 :_M Ann_R(N)Ann_R(K))$ and denoted by $C(NK)$ [9].

In below, we recall the concept of idempotent submodules, which is introduced and investigated by some authors (see [5], [4], [34], and [51].)

In [34], a submodule N of an R-module M is called *idempotent*, provided that $N = Hom(M, N)N = \sum\{\varphi(N) : \varphi : M \to N\}$.

In [5], a submodule N of an R-module M is called *idempotent*, if $N = (N :_R M)N$.

Definition 166. *[21, 2.1] We say that a submodule N of an R-module M is* idempotent *if $N = N^2$.*

The following lemma and Example 168 show the relation between the above various concepts of idempotent submodules.

Lemma 167. *[21, 2.2] Let N be a submodule of an R-module M. Consider the following statements.*

(a) $N = N^2$.

(b) $N = (N : M)N$.

(c) $N = Hom_R(M, N)N = \sum\{\varphi(N) : \varphi : M \to N\}$.

Then $(a) \Leftrightarrow (b)$ and $(b) \Rightarrow (c)$.

Example 168. *[21, 2.3] For each prime number p, the submodule $N = \mathbb{Z}_p \oplus 0$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ is not idempotent but $N = Hom_{\mathbb{Z}}(M, N)N$.*

Definition 169. *[21, 2.4] An R-module M is said to be* fully idempotent *if every submodule of M is idempotent.*

Definition 170. *[21, 3.1] We say that a submodule N of an R-module M is* coidempotent *if $N = C(N^2)$.*

Definition 171. *[21, 3.2] An R-module M is said to be* fully coidempotent *if every submodule of M is coidempotent.*

Example 172. *[21, 3.3] For each prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is fully coidempotent. Moreover, $E(\mathbb{Z}_P) = \mathbb{Z}_{p^\infty}$ is not a fully coidempotent $\mathbb{Z}$-module.*

A non-zero submodule S of an R-module M is said to be *naturally semi-coprime* if, for a submodule N of M, the relation $S \subseteq C(N^2)$ implies that $S \subseteq N$ [9].

In the following proposition, we characterize the fully coidempotent R-modules.

Proposition 173. *[21, 3.4] Let M be an R-module. Then the following statements are equivalent.*

(a) M is a fully coidempotent module.

(b) Every completely irreducible submodule of M is coidempotent.

(c) Every non-zero submodule of M is naturally semi-coprime.

(d) For all submodules N and K of M, we have $N + K = C(NK)$.

Proposition 174. *[21, 3.5] Let M be a fully coidempotent R-module. Then we have the following.*

(a) M is a comultiplication R-module.

(b) Every submodule and every homomorphic image of M is fully coidempotent.

(c) If M is a finitely generated R-module, then M is a multiplication module.

(d) If R is a Noetherian ring and M is an injective R-module, then every submodule of M is also an injective R-module.

The following example shows that the converse of part (a) of the above proposition is not true in general.

Example 175. *[21, 3.6] $\mathbb{Z}_4$ is a comultiplication $\mathbb{Z}$-module, which is not fully coidempotent.*

Let M be an R-module and N be a submodule of M. The following example shows that if N and M/N are fully coidempotent modules, then M is not necessarily a fully coidempotent module.

Example 176. *[21, 3.7] Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z}$ and set $N = 2\mathbb{Z}/4\mathbb{Z}$. Then N and M/N are fully coidempotent $\mathbb{Z}$-modules, while M is not fully coidempotent.*

Theorem 177. *[21, 3.8] Let M be an R-module. Then we have the following.*

(a) If M is a Noetherian fully idempotent module, then M is a fully coidempotent module.

(c) If R is a von Neumann regular ring and M is a comultiplication R-module, then M is a fully coidempotent R-module.

(b) If M is a comultiplication module such that every completely irreducible submodule of M is a direct summand of M, then M is a fully coidempotent module.

(d) If M is a semisimple comultiplication module, then M is a fully coidempotent module.

Theorem 178. *[21, 3.9] Let M be a fully coidempotent R-module. Then we have the following.*

(a) M is Hopfian.

(b) If R is a domain and M is a faithful R-module, then M is simple.

Definition 179. *[21, 3.10] We say that an R-module M is* fully copure *if every submodule of M is copure.*

Lemma 180. *[21, 3.11] Let M be a semisimple R-module. Then M is fully copure.*

Theorem 181. *[21, 3.12] Let M be a comultiplication R-module and N be a submodule of M. Then the following statements are equivalent.*

(a) N is a copure submodule of M.

(b) M/N is a comultiplication R-module and N is a coidempotent submodule of M.

(c) M/N is a comultiplication R-module and $K = (N :_M Ann_R(K))$, where K is a submodule of M with $N \subseteq K$.

(d) M/N is a comultiplication R-module and $(N :_M Ann_R(K)) = (N :_M (N :_R K))$, where K is a submodule of M.

Corollary 182. *[21, 3.13] Let M be an R-module. Then we have the following.*

(a) If M is a fully coidempotent module, then M is fully copure.

(b) If M is a comultiplication fully copure module, then M is fully coidempotent.

The following example shows that in part (b) of the above corollary, the condition "M is a comultiplication module" can not be omitted.

Example 183. *[21, 3.14] Set $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then M, as a $\mathbb{Z}$-module, is fully copure, while M is not fully coidempotent.*

Proposition 184. *[21, 3.15] Let M be an R-module and N be a submodule of M. Then we have the following.*

(a) If M is a multiplication module and N is a copure submodule of M, then N is idempotent.

(b) If M is a comultiplication module and N is a pure submodule of M, then N is coidempotent.

Corollary 185. *[21, 3.16] Let M be an R-module. Then we have the following.*

(a) *If M is a multiplication fully copure module, then M is fully pure.*

(b) *If M is a comultiplication fully pure module, then M is fully copure.*

(c) *If M is a multiplication fully coidempotent module, then M is fully idempotent.*

(d) *If M is a comultiplication fully idempotent module, then M is fully coidempotent.*

The following example shows that in part (d) of the above corollary, the condition "M is a comultiplication module" can not be omitted.

Example 186. *[21, 3.17] Let*

$$R = \{(a_n) \in \prod_{i=1}^{\infty} \mathbb{Z}_2 : a_n \text{ is eventually constant}\}$$

and let

$$P = \{(a_n) \in R : a_n \text{ is eventually } 0\}.$$

Then R is a Boolean ring and P is a maximal ideal of R. Moreover, $Ann_R(P) = 0$. Hence, P is an idempotent submodule of R but it is not a coidempotent submodule of R. Thus, R is a fully idempotent R-module but it is not a fully coidempotent R-module.

Example 187. *[21, 3.18] $\mathbb{Z}_n$ is a fully idempotent and fully coidempotent $\mathbb{Z}_n$-module if and only if n is square free.*

Theorem 188. *[21, 3.19] Let M be a fully coidempotent R-module. Then we have the following.*

(a) *For each submodule K of M and each collection $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of M, $\cap_{\lambda\in\Lambda}(N_\lambda + K) = \cap_{\lambda\in\Lambda}N_\lambda + K$.*

(b) *If M is a finitely generated R-module, then M is a semisimple R-module.*

Corollary 189. *Let M be a fully coidempotent R-module. Then M is a distributive R-module.*

Proof. This follows from Theorem 188 (a) and [64, 2.3]. □

8 Second submodules and comultiplication modules

A non-zero submodule N of an R-module M is said to be *second* if, for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [68]. More information about this class of modules can be found in [19], [20], [23], [31], and [32].

Definition 190. *[19, 2.1] We say that a second submodule N of an R-module M is a* maximal second submodule *of a submodule K of M, if $N \subseteq K$ and there does not exist a second submodule L of M such that $N \subset L \subset K$.*

Theorem 191. *[19, 2.3] Let M be a finitely cogenerated comultiplication R-module which satisfies the property $AB5^*$. Suppose that for each maximal second submodule K of M, we have that M/K is Artinian. Then the number of maximal second submodules of M is finite.*

Theorem 192. *[20, 2.7] Let M be a faithful finitely generated comultiplication R-module, satisfying the descending chain condition on second submodules. Then R satisfies the ascending chain condition on prime ideals.*

Theorem 193. *[17, 3.1] Let M be a Noetherian comultiplication R-module. Then*

(a) M has a finite number of second submodules.

(b) Every second submodule of M is a minimal submodule of M.

Definition 194. *[13, 3.13] A submodule N of an R-module M is said to be completely coirreducible, if $N = \sum_{\lambda\in\Lambda} N_\lambda$, where $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of submodules of M, implies that $N = N_\lambda$ for some $\lambda \in \Lambda$.*

Remark 195. *[13, 3.13] Let M be an R-module. It is clear that every completely coirreducible submodule of M is a cyclic module. However, the converse is not true in general. For example, for the cyclic $\mathbb{Z}$-module $\mathbb{Z}$, we have $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$, but $\mathbb{Z} \neq 2\mathbb{Z}$ and $\mathbb{Z} \neq 3\mathbb{Z}$.*

Theorem 196. *[10, 3.11], [17, 3.7], and [13, 3.14] Let M be a comultiplication R-module. Then*

(a) If N is a submodule of M such that $Ann_R(N)$ is a prime ideal of R, then N is a second submodule of M.

(b) If R is a ring such that every prime ideal of R is contained in the unique maximal ideal of R, then every second submodule of M contains a unique minimal submodule of M.

(c) If S is a finitely generated second submodule of M, then S is completely coirreducible.

Corollary 197. *[13, 3.15] Let M be a comultiplication R-module and let S be a finitely generated second submodule of M. Then S is a multiplication R-module.*

Remark 198. *[13, 3.17] In the part (c) of Theorem 196, the condition, that S is finitely generated, can not be omitted. For example, let p be a prime number. Then $\mathbb{Z}_{p^\infty}$, as a $\mathbb{Z}$-module, is a comultiplication module and it is second. Also is not finitely generated and it is not completely coirreducible. (Note that $\mathbb{Z}_{p^\infty}$ is equal to the sum of all its submodules but it is not equal to any one of them.)*

Proposition 199. *[14, 2.12] Let R be a principal ideal domain and let M be an R-module. If M is a second module, then every pure submodule of M is a second submodule of M.*

Theorem 200. *[21, 33.9] Let M be a fully coidempotent R-module. Then every second submodule of M is a minimal submodule of M.*

Definition 201. *[16, 3.14] Let M be a graded R-module and let N be a non-zero graded submodule of M. We say that N is a* graded second *(*gr-second*) if, for each homogeneous element a of R, the endomorphism of M, given by multiplication by a, is either surjective or zero.*

Proposition 202. *[16, 3.15] Let M be a graded R-module and let N be a graded submodule of M. Then we have the following.*

(a) If N is a gr-second submodule of M, then $Ann_R(N)$ is a gr-prime ideal of R.

(b) If M is a gr-comultiplication R-module and $Ann_R(N)$ is a gr-prime ideal of R, then N is a gr-second submodule of M.

Theorem 203. *[16, 3.16] Let M be a Noetherian gr-comultiplication R-module. Then we have the following.*

(a) M has a finite number of gr-second submodules.

(b) Every gr-second submodule of M is a gr-minimal submodule of M.

An R-module M is said to be a *weak multiplication module* if M does not have any prime submodule or for every prime submodule N of M there exists an ideal I of R such that $N = IM$ [3].

The following definition can be regarded as a dual notion of weak multiplication module.

Definition 204. *[20, 3.1] We say that an R-module M is a* weak comultiplication module *if M does not have any second submodule or, for every second submodule S of M, we have $S = (0 :_M I)$, where I is an ideal of R.*

Remark 205. *[20, 3.2] It is clear that every comultiplication R-module is a weak comultiplication R-module. However, in general, the converse is not true. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is a weak comultiplication module which is not a comultiplication $\mathbb{Z}$-module.*

Lemma 206. *[20, 3.3] Let M be an R-module. Then we have the following.*

(a) M is a weak comultiplication module if and only if $S = (0 :_M Ann_R(S))$ for each second submodule S of M.

(b) *If M is a weak comultiplication module, then every submodule of M is a weak comultiplication module.*

Theorem 207. *[20, 3.4] Let M be an R-module. Then we have the following.*

(a) *If M is a weak comultiplication R-module and has finite length, then every second submodule of M is minimal.*

(b) *If M is a Noetherian weak comultiplication R-module, then M has a finite number of second submodules*

(c) *If M is an Artinian weak multiplication R-module, then M has a finite number of prime submodules.*

Lemma 208. *[20, 3.5] Let R be a Noetherian ring and let M be a finitely generated R-module. Then we have the following.*

(a) *If S is a multiplicatively closed subset of R and N is a second submodule of M such that $Ann_R(N) \cap S = \emptyset$, then $S^{-1}N$ is a second submodule of $S^{-1}M$.*

(b) *If, for every maximal ideal P of R, M_P is a weak comultiplication R_P-module, then M is a weak comultiplication R-module.*

Theorem 209. *[20, 3.6] Let (R, P) be a Noetherian local ring and let M be a finite length weak comultiplication R-module. Then M is a comultiplication R-module.*

$X^s = Spec^s(M)$ will denote the set of all second submodules of an R-module M.

If $X^s \neq \emptyset$, then, for every $S \in X^s$, the map $\psi^s : X^s \to Spec^s(R/Ann_R(M))$, defined by $S \mapsto Ann_R(S)/Ann_R(M)$, will be called *the natural map of X^s*.

Lemma 210. *[22, 2.10] Let M be an R-module. Then we have the following.*

(a) *If M is a finitely generated comultiplication module and P is a prime ideal of R, containing $Ann_R(M)$, then $(0 :_M P)$ is a second submodule of M.*

(b) *If M is a finitely generated comultiplication module, then the natural map ψ^s of $Spec^s(M)$ is surjective.*

(c) *If the natural map ψ^s of $Spec^s(M)$ is surjective and I is an ideal of R, containing $Ann_R(M)$, then $Ann_R((0 :_M \sqrt{I})) = \sqrt{I}$.*

The intersection of all prime submodules of an R-module M, containing N, is said to be the *prime radical* of N and denoted by $rad_M N$ (or simply by $rad(N)$). In case N is not contained in any prime submodule, the radical of N is defined to be M [53].

For a submodule N of an R-module M, the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M, contained in N, and it is denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0). $N \neq 0$ is said to be a *second radical submodule of M* if $sec(N) = N$ ([31] and [23]).

Theorem 211. *[22, 2.11] Let M be a faithful R-module such that the natural map ψ^s of $Spec^s(M)$ is surjective. Consider the following equalities:*

(a) $sec((0 :_M I)) = (0 :_M \sqrt{I})$ *for each ideal I of R.*

(b) $sec(N) = (0 :_M \sqrt{Ann_R(N)})$ *for each submodule N of M.*

(c) $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$ *for each submodule N of M.*

(d) $Ann_R(sec((0 :_M I))) = \sqrt{I}$ *for each ideal I of R.*

Then $(b) \Rightarrow (c) \Rightarrow (d)$ *and* $(b) \Rightarrow (a) \Rightarrow (d)$. *Furthermore, if M is a comultiplication module, then (a), (b), (c) and (d) are all equivalent.*

Proposition 212. *[22, 2.6] Let M be a comultiplication R-module. If S is a second submodule of M such that $S \subseteq N + K$ for any pair of submodules N and K of M, then either $S \subseteq N$ or $S \subseteq K$. Consequently,*

$$sec(N + K) = sec(N) + sec(K)$$

for every pair of submodules N and K of M.

Theorem 213. *[22, 2.12] Let N and K be two submodules of a finitely generated comultiplication R-module M. Then the following hold.*

(a) $sec(N) = (0 :_M \sqrt{Ann_R(N)})$.

(b) $Ann_R(sec(N)) = \sqrt{Ann_R(N)}$.

(c) If $Ann_R(K) = \sqrt{Ann_R(K)}$ *and* $Ann_R(N) = \sqrt{Ann_R(N)}$, *then* $Ann(sec(N + K)) = Ann_R(N + K)$.

(d) If N, K are secondary submodules of M with $sec(N) = sec(K)$, then $N + K$ is a secondary submodule of M.

Corollary 214. *[22, 2.13] If Q is a secondary submodule of a finitely generated comultiplication R-module M, then $sec(Q)$ is a second submodule of M.*

Definition 215. *[23, 3.2] Let M be an R-module. We define $V^s(N) = \{S \in Spec^s(M) : Ann_R(N) \subseteq Ann_R(S)\}$. Then*

(i) $V^s(M) = Spec^s(M)$ *and* $V^s(0) = \emptyset$,

(ii) $\cap_{\lambda\in\Lambda} V^s(N_\lambda) = V^s(\cap_{\lambda\in\Lambda}(0 :_M Ann_R(N_\lambda)))$ *for every* $N_\lambda \leq M$, $\lambda \in \Lambda$,

(iii) $V^s(N) \cup V^s(K) = V^s(N + K)$, *for every* $N, K \leq M$.

Set $\zeta^s(M) := \{V^s(N) : N \leq M\}$. Then from (i), (ii), and (iii) we see that always there exists a topology, say τ^s, on $X^s = Spec^s(M)$, having ζ^s as the family of all closed sets. We call the topology τ^s the Zariski topology *on X^s.*

Definition 216. *[23, 1.3] Let M be an R-module. Set $V^{s*}(N) = \{S \in Spec^s(M) : S \subseteq N\}$. Then*

(i) $V^{s*}(M) = Spec^s(M)$ *and* $V^{s*}(0) = \emptyset$,

(ii) $\cap_{\lambda \in \Lambda} V^{s*}(N_\lambda) = V^{s*}(\cap_{\lambda \in \Lambda} N_\lambda)$ *for every* $N_\lambda \leq M$, $\lambda \in \Lambda$,

(iii) $V^{s*}(N) \cup V^{s*}(K) \subseteq V^{s*}(N + K)$ *for every* $N, K \leq M$.

Put $\zeta^{s*}(M) = \{V^{s*}(N) : N \leq M\}$. $\zeta^{s*}(M)$ is not closed under finite union in general. Following [2], M is called a *top*^{s}*-module* (or *cotop module*, for convenience) if $\zeta^{s*}(M)$ induces a topology on X^s. When this is the case, we call the topology τ^{s*} the *quasi-Zariski topology* on X^s.

Example 217. *[23, 3.5] Every comultiplication module is a cotop module so that* $\tau^{s*} = \tau^s$.

For a prime ideal p of R, $X^s_p(M) = Spec^s_p(M)$ denotes the collection of second submodules N of an R-module M such that $Ann_R(N) = p$.

For any set Y, $|Y|$ will denote the cardinality of Y.

Theorem 218. *[23, 2.11] Consider the following statements for an R-module M.*

(a) M *is a comultiplication module.*

(b) For every submodule N of M there exists an ideal I of R such that $V^{s*}(N) = V^{s*}((0 :_M I))$.

(c) $|X^s_p(M)| \leq 1$ *for every prime ideal p of R.*

(d) $(0 :_M p)$ *is a cocyclic module for every maximal ideal p of R.*

Then (a) ⇒ (b) and (c) ⇒ (d). In case M is Artinian, (b) ⇒ (c). Moreover, (d) ⇒ (a) if R is a Noetherian ring and M has finite length.

Example 219. *[23, 2.12] Let p be any prime integer and let M denote the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_p$. Then M is not a comultiplication module but for every submodule N of M there exists an ideal I of R such that* $V^{s*}(N) = V^{s*}((0 :_M I))$.

Example 220. *[23, 2.14] Let M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Q}$. Then* $(0 :_M p) = 0$ *for every prime p in $\mathbb{Z}$ but there exist distinct second submodules K and N of M such that* $0 = Ann_R(K) = Ann_R(N)$*. In particular, M is not a comultiplication module.*

Proposition 221. *[23, 3.7] Let M be an R-module. Then the following statements are equivalent.*

(a) The natural map $\psi^s : X^s \to X^{\overline{R}}$ *is injective.*

(b) If $V^s(S_1) = V^s(S_2)$, then $S_1 = S_2$ for any $S_1, S_2 \in X^s$.

(c) $|X^s_p(M)| \leq 1$ for every $p \in Spec(R)$.

Theorem 222. *[23, 6.3] Let M be an R-module and ψ^s the natural map of $X^s = Spec^s(M)$. Consider the following cases:*

(1) X^s is a T_0-space;

(2) X^s is a spectral space;

(3) X^s is homeomorphic to $Spec(\overline{R})$ under ψ^s;

(4) M is a comultiplication R-module.

Then all conditions in Proposition 221 are equivalent to part (1) (resp., parts (2) and (3), if ψ^s is surjective). Moreover, if R is a Noetherian ring and $0 \neq M$ has finite length, then all the conditions in Proposition 221 are equivalent to parts (1)-(4).

Corollary 223. *[23, 6.4] If M is a cotop module (in particular, if M is a comultiplication module), then $Spec^s(M)$ is a T_0-space for both the Zariski topology τ^s and the quasi-Zariski topology τ^{s*}.*

Proposition 224. *[26, 3.8] Let R be a Noetherian ring and let M be a cotop R-module with finite length. Then M is a comultiplication R-module.*

A proper ideal I of a ring R is called *pseudo-irreducible*, if it satisfies the following equivalent conditions:

(a) For all ideals J, K of R, if $I = JK$ and $J + K = R$, then $J = R$ or $K = R$;

(b) For all $x \in R$, if $x(x-1) \in I$, then $x \in I$ or $x - 1 \in I$;

(c) The ring R/I is indecomposable.

Definition 225. *[59, 2.2] A nonzero R-module M is called an API-module if $Ann_R(M)$ is a pseudo-irreducible ideal of R.*

The following lemma gives some useful characterizations of API-modules.

Lemma 226. *[59, 2.4] Let M be a non-zero R-module. Then the following are equivalent:*

(a) M is an API-module.

(b) For all $r \in R$, if $r(r-1)M = 0$, then $rM = 0$ or $(r-1)M = 0$.

(c) For all submodules A, B of M, if $M = A + B$ and $Ann_R(A) + Ann_R(B) = R$, then $M = A$ and $B = 0$ or $M = B$ and $A = 0$.

Let M be a non-zero R-module. By a *comaximal decomposition* of M we mean an expression $M = \oplus_{i=1}^{n} N_i$, where the $Ann_R(N_i)$'s are pairwise comaximal proper ideals of R. We call this comaximal decomposition *complete* if the N_i's are API-modules [59].

Lemma 227. *[59, 3.1] Let M be a finitely generated comultiplication R-module and N be a non-zero submodule of M. Then every finite direct sum decomposition of N into non-zero submodules is a comaximal decomposition. Therefore, N is an API-module if and only if it is indecomposable.*

Theorem 228. *[59, 3.2] Let M be a finitely generated comultiplication R-module. Then every non-zero submodule of M can be written uniquely as a finite direct sum of non-zero indecomposable submodules.*

Proposition 229. *[59, 3.3] Let M be a strong comultiplication R-module. Then every nonzero submodule of M has a complete comaximal decomposition.*

For a cotop R-module M, we consider $Min(M)$, the set of all minimal submodules of an R-module M, as a subspace of $Spec^s(M)$ with respect to the quasi-Zariski topology.

Theorem 230. *[59, 3.5] Let M be a comultiplication R-module. Then the following are equivalent:*

(a) Every nonzero submodule of M has a complete comaximal decomposition.

(b) M has only finitely many simple submodules.

(c) $Min(M)$ is a Noetherian topological space as a subspace of $Spec^s(M)$.

(d) M has no infinite collection of submodules with pairwise comaximal annihilators.

Now, let N be a submodule of M. We define $W^s(N) = Spec^s(M) - V^{s*}(N)$ and put $\Omega^s(M) = \{W^s(N) : N \leq M\}$. Let $\eta^s(M)$ be the topology on $Spec^s(M)$, defined by the sub-basis $\Omega^s(M)$. In fact, $\eta^s(M)$ is the collection U of all unions of finite intersections of elements of $\Omega^s(M)$ [54]. We call this topology the *second classical Zariski topology* of M. It is clear that if M is a cotop module, then its related topology, as it was mentioned in the above paragraph, coincides with the second classical Zariski topology [26].

Theorem 231. *[26, 2.9] Let M be a finite length module over a commutative Noetherian ring R such that $Spec^s(M)$ is a T_1-space. Then M is a comultiplication module.*

Lemma 232. *[26, 2.14] Let M be a finite length weak comultiplication module. Then $Spec^s(M)$ is a cofinite topology.*

Proposition 233. *[26, 3.9] Let M be a comultiplication R-module with finite length. Then $Spec^s(M)$ is a spectral space.*

Conclusion 234. *As we mentioned in the introduction, there is a large body of researches related to comultiplication modules since this notion has been introduced. Also, there is large open space for this notion parallel to researches on multiplication modules. In [27], the authors applied the notion of comultiplication modules in the graph theory. Also, this concept has been used in lattice theory [30, 57]. Moreover, the concept of comultiplication module can be applied in other fields such as Fuzzy theory.*

Acknowledgment. The authors would like to thank the referee for careful reading our manuscript and his/ her valuable comments which greatly improved this work.

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