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NEW ASPECTS OF TWO HESSIAN-RIEMANNIAN METRICS IN PLANE

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Abstract. Due to the importance of Hessian structures we express some algebraic and geometric features of two such semi-Riemannian metrics in dimension two. For this purpose we use the separable coordinate systems of the Euclidean plane. Several properties are expressed with the Pauli matrices and their associated quadratic forms.

1 Introduction

The setting of this note is a triple (M^n, g, f) with (M, g) a (semi-) Riemannian manifold of dimension n and $f \in C^\infty(M)$ a smooth function on it which we call *potential*. Let ∇^g be the Levi-Civita connection of g. The starting point is the assumption that the Hessian of f, [4]:

$$H_f \in \mathcal{T}^0_{2,sym}(M), \quad H_f(X,Y) := g(\nabla_X \nabla f, Y) = H_f(Y,X) \tag{1.1}$$

is non-degenerate and of constant signature; functions with H_f identically zero are potentials of g-Killing vector fields. It follows a new (semi-) Riemannian metric on M called *of Hessian type* and some properties of these structures are studied by professors Constantin Udrişte, Mihai Postolache and their co-workers in [1], [2], [11] and [12]. For example, the paper [11] is cited in [8] where the Hessian structures are complemented by Golden endomorphisms ([7]) and [1] is cited in [10] where Norden structures of Hessian type are studied. More recently, holomorphic Hamiltonian functions are used in [9] to construct anti-Hermitian metrics of Hessian type.

The aim of present work is to express some two-dimensional Euclidean examples of these papers in new coordinates especially the complex and polar.

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2 Hessian-Riemannian metrics in plane

As usual, fix the complex number $z = x + iy$ with its associated conjugate $\bar{z} = x - iy$. The inverse map is:

$$x = \Re z = \frac{1}{2}(z + \bar{z}), \quad y = \Im z = \frac{i}{2}(\bar{z} - z) \tag{2.1}$$

while the polar coordinates (r, φ) on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ give:

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{2.2}$$

In $\mathbb{R}^2$ in addition to Cartesian and polar coordinates there are other two *separable coordinates systems*, [3, p. 143-144]. These are as follows:

i) parabolic coordinates: $x = \frac{1}{2}(u^2 - v^2)$, $y = uv$,

ii) elliptic coordinates: $x^2 = c^2(\alpha - 1)(\beta - 1)$, $y^2 = -c^2\alpha\beta$.

All these coordinates systems are used recently in [6] to express the Euclidean Killing tensor fields of rank two.

Recall also that the 4-dimensional space of Hermitian 2×2 matrices has the basis:

$$I_2, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

with $\sigma_{1,2,3}$ the *Pauli matrices*. Its 3-dimensional subspace of symmetric real matrices of 2×2 type has the basis $\{I_2, \sigma_1, \sigma_3\}$ while the entire space $M_2(\mathbb{R})$ has the basis $\{I_2, \sigma_1, \sigma_3, \sigma_3 \cdot \sigma_1\}$. Let $q_A(x, y)$ be the real quadratic expression provided by a matrix A from the last basis:

$$q_A(x, y) = (x, y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \quad q_{I_2}(x, y) = x^2 + y^2,$$

$$q_{\sigma_1}(x, y) = 2xy, \quad q_{\sigma_3}(x, y) = x^2 - y^2. \tag{2.4}$$

Example 1. This is provided by Example 3.1 of [1, p. 10] and it starts with the Euclidean plane $\mathbb{E}^2 := (\mathbb{R}^2, g = \delta)$. The considered manifold is the domain $M := \{(x, y) \in \mathbb{R}^2; x^2 - y^2 > 0\}$ bounded from down and up by the first and second bisectrix:

$$B_\pm : y = \pm x. \tag{2.5}$$

The function is $f^1(x, y) = f(x, y) = \ln(x^2 - y^2) = \ln q_{\sigma_3}(x, y)$ with:

$$H_f(x, y) = \frac{2}{(x^2 - y^2)^2} \begin{pmatrix} -(x^2 + y^2) & 2xy \\ 2xy & -(x^2 + y^2) \end{pmatrix} =$$

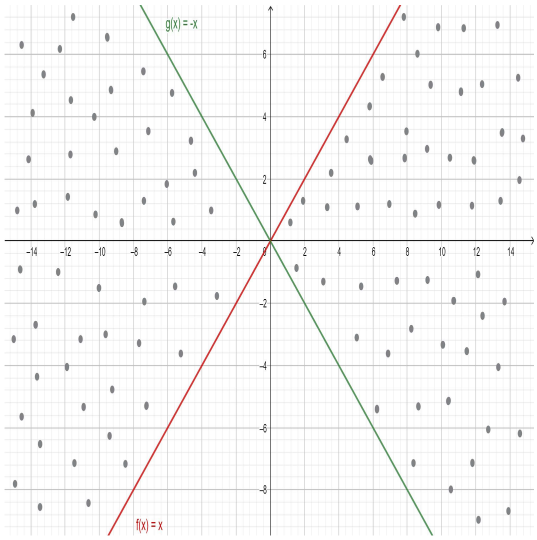

Figure 1: The manifold M

$$= \frac{-2(x^2+y^2)}{(x^2-y^2)^2} I_2 + \frac{4xy}{(x^2-y^2)^2}\sigma_1, \tag{2.6}$$

$$\det H_f(x,y) = \frac{4}{(x^2-y^2)^2} = \left(\frac{2}{q_{\sigma_3}(x,y)}\right)^2 > 0,$$

$$H_f^{-1}(x,y) = -\begin{pmatrix} \frac{x^2+y^2}{2} & xy \\ xy & \frac{x^2+y^2}{2} \end{pmatrix} = -\frac{x^2+y^2}{2} I_2 - xy\sigma_1. \tag{2.7}$$

The Hessian-Riemannian metric H_f has constant signature $(0,2)$ and vanishing sectional curvature. Also, we remark:

$$-2H_f^{-1}(x,y) = q_{I_2}(x,y)I_2 + q_{\sigma_1}(x,y)\sigma_1 \tag{2.8}$$

and in order to eliminate the minus sign from most of above expressions the same example is considered in [12, p. 138] but with $f(x,y) = -\ln(x^2-y^2)$.

In our new coordinates it follows:

$$M = \{z \in \mathbb{C}; z^2 + \bar{z}^2 > 0\} = \{(r,\varphi); r > 0, \varphi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)\}, \tag{2.9}$$

$$f(z,\bar{z}) = \ln[\frac{1}{2}(z^2+\bar{z}^2)], \quad f(r,\varphi) = \ln(r^2\cos 2\varphi) = 2\ln r + \ln(\cos 2\varphi), \tag{2.10}$$

$$H_f(z,\bar{z}) = \frac{-4}{(z^2+\bar{z}^2)^2}\begin{pmatrix} 2|z|^2 & i(z^2-\bar{z}^2) \\ i(z^2-\bar{z}^2) & 2|z|^2 \end{pmatrix} =$$

$$= \frac{-8|z|^2}{(z^2+\bar{z}^2)^2} I_2 + \frac{-4i(z^2-\bar{z}^2)}{(z^2+\bar{z}^2)^2}\sigma_1, \tag{2.11}$$

$$\det H_f(z,\bar{z}) = \frac{16}{(z^2+\bar{z}^2)^2},$$

$$H_f^{-1}(z,\bar{z}) = \frac{1}{4}\begin{pmatrix} -2|z|^2 & i(z^2-\bar{z}^2) \\ i(z^2-\bar{z}^2) & -2|z|^2 \end{pmatrix} = -\frac{|z|^2}{2} I_2 + \frac{i(z^2-\bar{z}^2)}{4}\sigma_1, \tag{2.12}$$

$$H_f(r,\varphi) = \frac{2}{(r\cos 2\varphi)^2}\begin{pmatrix} -1 & \sin 2\varphi \\ \sin 2\varphi & -1 \end{pmatrix} = -\frac{2}{(r\cos 2\varphi)^2} I_2 + \frac{2\sin 2\varphi}{(r\cos 2\varphi)^2}\sigma_1, \tag{2.13}$$

$$\det H_f(r,\varphi) = \frac{4}{(r^2\cos 2\varphi)^2},$$

$$H_f^{-1}(r,\varphi) = \frac{-r^2}{2}\begin{pmatrix} 1 & \sin 2\varphi \\ \sin 2\varphi & 1 \end{pmatrix} = -\frac{r^2}{2} I_2 - \frac{r^2\sin 2\varphi}{2}\sigma_1, \tag{2.14}$$

$$f(u,v) = \ln[(u^2-v^2)^2 - 4u^2v^2] - 2\ln 2,$$

$$\det H_f(u,v) = \frac{4^3}{[(u^2-v^2)^2-4u^2v^2]^2}, \tag{2.15}$$

$$H_f(u,v) = \frac{2}{[(u^2-v^2)^2-4u^2v^2]^2}\begin{pmatrix} -(u^2+v^2)^2 & 4uv(u^2-v^2) \\ 4uv(u^2-v^2) & -(u^2+v^2)^2 \end{pmatrix}, \tag{2.16}$$

$$f(\alpha,\beta) = \ln c^2 + \ln(2\alpha\beta - \alpha - \beta + 1),$$

$$\det H_f(\alpha,\beta) = \frac{4}{c^4(\alpha\beta-\alpha-\beta+1)^2}, \tag{2.17}$$

$$\frac{c^2(\alpha\beta-\alpha-\beta+1)^2}{2} H_f(\alpha,\beta) =$$

$$\begin{pmatrix} \alpha+\beta-1 & (\alpha\beta(1-\alpha)(\beta-1))^{\frac{1}{2}} \\ (\alpha\beta(1-\alpha)(\beta-1))^{\frac{1}{2}} & \alpha+\beta-1 \end{pmatrix}. \tag{2.18}$$

We point out from the second part of (2.10) that the potential is expressed in separated variables when working in polar coordinates: $f(r,\varphi) = f_1(r) + f_2(\varphi) = 2\ln r + \ln(\cos 2\varphi)$.

Example 2. This is provided by Example 3.2 of [1, p. 11] and it starts also with the Euclidean plane $\mathbb{E}^2 := (\mathbb{R}^2, g = \delta)$. The considered manifold is exactly the punctured plane $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and the function is $f^2(x,y) = f(x,y) = \ln(x^2+y^2) = \ln q_{I_2}(x,y)$ with:

$$H_f(x,y) = \frac{2}{(x^2+y^2)^2}\begin{pmatrix} y^2-x^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix} =$$

$$= -\frac{4xy}{(x^2+y^2)^2}\sigma_1 - \frac{2(x^2-y^2)}{(x^2+y^2)^2}\sigma_3, \tag{2.19}$$

$$\det H_f(x,y) = -\frac{4}{(x^2+y^2)^2} < 0 \tag{2.20}$$

which means that H_f is a Lorentzian metric i.e. it has the signature $(1,1)$; also its sectional curvature is zero. We present the graph of the Monge surface $z := f(x,y) = \ln(x^2+y^2)$. Its Gaussian curvature is:

$$K(x,y) = \frac{-4}{(x^2+y^2+4)^2} < 0. \tag{2.21}$$

We have also:

$$f(z,\bar{z}) = \ln(|z|^2), \quad f(r,\varphi) = 2\ln r = f_1(r), \tag{2.22}$$

$$H_f(z,\bar{z}) = \frac{1}{|z|^4}\begin{pmatrix} -(z^2+\bar{z}^2) & i(z^2-\bar{z}^2) \\ i(z^2-\bar{z}^2) & z^2+\bar{z}^2 \end{pmatrix} = \frac{i(z^2-\bar{z}^2)}{|z|^4}\sigma_1 - \frac{z^2+\bar{z}^2}{|z|^4}\sigma_3,$$

$$\det H_f(z,\bar{z}) = -\frac{1}{|z|^4}, \quad \det H_f(r,\varphi) = -\frac{1}{r^4}, \tag{2.23}$$

$$H_f(r,\varphi) = \frac{-2}{r^2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} = \frac{-2\sin 2\varphi}{r^2}\sigma_1 - \frac{-2\cos 2\varphi}{r^2}\sigma_3, \tag{2.24}$$

$$f(u,v) = 2[\ln(u^2+v^2) - \ln 2], \quad \det H_f(u,v) = -\frac{4^3}{(u^2+v^2)^4}, \tag{2.25}$$

$$H_f(u,v) = \frac{8}{(u^2+v^2)^2}\begin{pmatrix} 4u^2v^2-(u^2-v^2)^2 & 4uv(v^2-u^2) \\ 4uv(v^2-u^2) & (u^2-v^2)^2-4u^2v^2 \end{pmatrix}, \tag{2.26}$$

$$f(\alpha,\beta) = \ln c^2 + \ln(1-\alpha-\beta), \quad \det H_f(\alpha,\beta) = -\frac{4}{c^2(1-\alpha-\beta)^2}, \tag{2.27}$$

$$\frac{c^2(1-\alpha-\beta)^2}{2}H_f(\alpha,\beta) =$$

$$= \begin{pmatrix} \alpha+\beta-1-2\alpha\beta & 2[\alpha\beta(\alpha-1)(1-\beta)]^{\frac{1}{2}} \\ 2[\alpha\beta(\alpha-1)(1-\beta)]^{\frac{1}{2}} & 1-\alpha-\beta+2\alpha\beta \end{pmatrix}. \tag{2.28}$$

Locking again to the polar coordinate we remark that both f and $\det H_f$ are rotationally symmetric i.e. does not depend on the angle φ and we point out the occurrence of an orthogonal matrix with determinant (-1) from the expression of H_f.

Remark 3. i) We point out that the restriction to the real axis ($y = 0$) of the functions from examples above are equal: $f^1(x,0) = f^2(x,0) = x^2 = f_1(x)$. Also, the restrictions of the Hessian to the same axis are: $H_{f^1}(x,0) = -\frac{2}{x^2}I_2$, $H_{f^2}(x,0) = -\frac{2}{x^2}\sigma_3$.

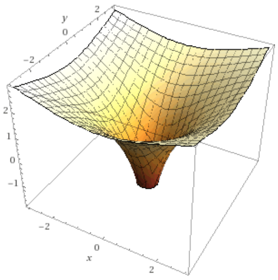

Figure 2: The Monge surface of $f(x, y) = \ln(x^2 + y^2)$

ii) The Hessian matrix of both examples is of special type. The matrix of the second example is a traceless one while the matrix of the first example has the form:

$$H_f = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

and hence is in relationship with the algebra $\mathbb{A}$ of paracomplex numbers $z = a + jb$ with the paracomplex unit $j^2 = 1$; see [5] for details. If $z \in \mathbb{A}$ is the first column of the matrix above then its second column is exactly jz.

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