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MATRIX POWER MEANS AND PÓLYA–SZEGÖ TYPE INEQUALITIES

Mohsen Kian and Fatemeh Rashid

Abstract. It has been shown that if μ is a compactly supported probability measure on $\mathbb{M}_n^+$, then for every unit vector $\eta \in \mathbb{C}^n$, there exists a compactly supported probability measure (denoted by $\langle \mu\eta, \eta \rangle$) on $\mathbb{R}^+$ so that the inequality

$$\langle P_t(\mu)\eta, \eta\rangle \le P_t(\langle \mu\eta, \eta\rangle) \qquad (t \in (0, 1])$$

holds. In particular, we consider a reverse of the above inequality and present some Pólya–Szegö type inequalities for power means of probability measures on positive matrices.

1 Introduction and preliminaries

In what follows, assume that $\mathbb{M}_n$ is the algebra of all $n \times n$ matrices with complex entries and $\mathbb{H}_n$ is the real subspace of all Hermitian matrices in $\mathbb{M}_n$. A matrix $A \in \mathbb{H}_n$ is called positive semi-definite (positive definite) and denoted by $A \ge 0$ ($A > 0$) if all of its eigenvalues are non-negative (positive). We denote by $\mathbb{M}_n^+$ the set of all positive definite matrices. The well-known Loewner partial order on $\mathbb{H}_n$ is defined by

$$A \le B \quad \Longleftrightarrow \quad B - A \ge 0, \quad (A, B \in \mathbb{H}_n).$$

In particular, if δ is an scalar, then we mean by $A \le \delta$ that $A \le \delta I$, where I denotes the identity matrix.

Matrix means have raised in the matrix theory as non-commutative extensions of scalar-valued means. Some of the most familiar matrix means are $A\nabla_t B = (1 - t)A + tB$ (weighted arithmetic mean), $A\sharp_t B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}$ (weighted geometric mean) and $A!_t B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$ (weighted harmonic mean), where $t \in [0, 1]$ and A, B are positive matrices, see e.g. [3].

Let $a = (a_1, \cdots, a_k)$ be a k-tuple of positive real numbers, $t \in (0, 1]$ and let $\omega = (\omega_1, \cdots, \omega_k)$ be a weight vector. The weighted power mean of $a_1, \cdots, a_k$ is

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defined by $P_t(\omega; a) = \left(\sum_{i=1}^{k} \omega_i a_i^t\right)^{\frac{1}{t}}$. It turns out to be the unique positive solution of the equation $x = \sum_{i=1}^{k} \omega_i x^{1-t} a_i^t$. When $t \to 0$, the power mean converges to the geometric mean of $a_1, \cdots, a_k$.

The matrix arithmetic and harmonic means can be naturally extended to a k-tuple $\mathbb{A} = (A_1, \cdots, A_k)$ of positive matrices,

$$\nabla(\omega; \mathbb{A}) = \sum_{i=1}^{k} \omega_i A_i, \qquad !(\omega; \mathbb{A}) = \left(\sum_{i=1}^{k} \omega_i A_i^{-1}\right)^{-1}.$$

Recently, there have been several works regarding extension of the matrix geometric mean to several variables. The notion of power means for positive matrices $\mathbb{A} = (A_1, \cdots, A_k)$ denoted by $P_t(\omega; \mathbb{A})$ has been introduced in [7] as the unique positive invertible solution of the non-linear matrix equation

$$X = \sum_{i=1}^{k} \omega_i (X \sharp_t A_i) \qquad (t \in (0, 1]) \tag{1.1}$$

For $t \in [-1, 0)$, put $P_t(\omega; \mathbb{A}) := P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \ldots, A_k^{-1})$.

The matrix power mean interpolates between the weighted harmonic and arithmetic means (see also [4]) and

$$\left(\sum_{i=1}^{k} \omega_i A_i^{-1}\right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{i=1}^{k} \omega_i A_i. \tag{1.2}$$

The notion of power mean for probability measures on $\mathbb{M}_n^+$ has also been studied [5]: If μ is a probability measure of compact support on $\mathbb{M}_n^+$ and $t \in (0, 1]$, then the equation

$$X = \int_{\mathbb{M}_n^+} X \sharp_t Z d\mu(Z)$$

has a unique solution in $\mathbb{M}_n^+$. It defines the power mean as a map P_t from the set of all probability measures of compact support on $\mathbb{M}_n^+$ into $\mathbb{M}_n^+$. In the case of $t \in [-1, 0)$, the power mean is defined by $P_t(\mu) = P_{-t}(\nu)^{-1}$, where $\nu(\mathcal{E}) = \mu(\mathcal{E}^{-1})$ for every measurable set $\mathcal{E}$. The above integral is in the sense of vector-valued. If f is a continuous function from a topological space $\mathcal{X}$ into a Banach space and μ is a probability measure of compact support on the Borel σ-algebra of $\mathcal{X}$, then

$$\int_X f d\mu = \lim_{m \to \infty} \sum_{i=1}^{N_m} f(a_i) \mu(B_{m,i})$$

in which $\{B_{m,i};\ i = 1, \cdots, N_m\}$ is a partition of *supp*(μ) and a_i is an arbitrary point in $B_{m,i}$.

It is known that a matrix mean σ has a monotonicity property via any positive unital linear mapping Φ, say $\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B)$, see [3, 9, 10]. In particular,

$$\langle (A\sigma B)\eta, \eta\rangle \leq \langle A\eta, \eta\rangle \sigma \langle B\eta, \eta\rangle \tag{1.3}$$

for every $\eta \in \mathbb{C}^n$, see [1, 2, 10].

In this paper, we present inequality (1.3) for power mean of probability measures. It provides some inequalities of type (1.3) and its reverses for matrix power means.

2 main results

Assume that t is a non-zero real number and μ is a probability measure of compact support on the positive half line. Consider the equation

$$x = \int_{\mathbb{R}^+} x^{1-t} z^t \, d\mu(z). \tag{2.1}$$

This equation has a unique solution, say

$$x = \left(\int_{\mathbb{R}^+} z^t \, d\mu(z) \right)^{\frac{1}{t}} \tag{2.2}$$

This unique solution, which we denote it by $P_t(\mu)$, can be regarded as a power mean and gives an extension of $P_t(\omega; a)$. If $a = (a_1, \cdots, a_k)$ is a k-tuple of positive real numbers, $\omega = (\omega_1, \cdots, \omega_k)$ is a weight vector and the measure μ is defined on the Borel σ-algebra of $\mathbb{R}^+$ satisfying that $\mu(\{a_i\}) = \omega_i$ for all $i = 1, \cdots, k$, then equation (2.1) turns to

$$x = \sum_{i=1}^{k} \omega_i x^{1-t} a_i^t$$

and

$$P_t(\mu) = \left(\sum_{i=1}^{k} \omega_i a_i^t \right)^{\frac{1}{t}} = P_t(\omega; a).$$

Suppose that μ is a probability measure of compact support on $\mathbb{M}_n^+$. Assume that $\mathcal{E}$ is a Borel subset of $\mathbb{R}^+$ and put $\overline{\mathcal{E}} = \{A \in \mathbb{M}_n^+;\ \langle A\eta, \eta\rangle \in \mathcal{E}\}$. We define a measure denoted by $\langle \mu\eta, \eta\rangle$ on $\mathbb{R}^+$ by $\langle \mu\eta, \eta\rangle(\mathcal{E}) = \mu(\overline{\mathcal{E}})$. It is easy to see that $\langle \mu\eta, \eta\rangle$ is a probability measure on $\mathbb{R}^+$. Now if f is a continuous function on $\mathbb{R}^+$ and integrable with respect to μ, then

$$\int_{\mathbb{M}_n^+} f(\langle Z\eta, \eta\rangle) d\mu(Z) = \int_{\mathbb{R}^+} f(z) d\langle \mu\eta, \eta\rangle(z). \tag{2.3}$$

We will use the following known result (see e.g., [3, 6])

Lemma 1 (Hölder–McCarthy inequality)**.** *Let $A \in \mathbb{M}_n$. If $\eta \in \mathbb{C}^n$ is a unit vector, then*

(i) $\langle A\eta, \eta\rangle^r \leq \langle A^r\eta, \eta\rangle$ *for all* $r > 1$;

(ii) $\langle A\eta, \eta\rangle^r \geq \langle A^r\eta, \eta\rangle$ *for all* $0 < r < 1$;

(iii) *If A is invertible, then* $\langle A\eta, \eta\rangle^r \leq \langle A^r\eta, \eta\rangle$ *for all* $r < 0$.

The next theorem gives inequality (1.3) for power means.

Theorem 2. *Let μ be a probability measure of compact support on $\mathbb{M}_n^+$. If η is a unit vector in $\mathbb{C}^n$ and $t \in (0, 1]$, then*

$$\langle P_t(\mu)\eta, \eta\rangle \leq P_t(\langle \mu\eta, \eta\rangle). \tag{2.4}$$

If $t \in [-1, 0)$, then a reverse inequality holds.

Proof. Let $t \in (0, 1]$ and let the function f be defined on $\mathbb{M}_n^+$ by $f(X) = \int_{\mathbb{M}_n^+} X\sharp_t Z d\mu(Z)$. Then

$$f(X) = \lim_{m\to\infty} \sum_{i=1}^{N_m} X\#_t Z_i \mu(B_{m,i}),$$

where $\{B_{m,i};\ \ i = 1, \cdots, N_n\}$ is a Borel partition of $supp(\mu)$ and Z_i is an arbitrary point in $B_{m,i}$, see [5]. If $\eta \in \mathbb{C}^n$ is a unit vector, then

$$\begin{aligned}
\langle f(X)\eta, \eta\rangle &= \langle \int_{\mathbb{M}_n^+} X\#_t Z\mu(dZ)\eta, \eta\rangle \\
&= \langle \lim_{m\to\infty} \sum_{i=1}^{N_m} X\#_t Z_i \mu(B_{m,i})\eta, \eta\rangle \\
&= \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) \langle (X\#_t Z_i)\eta, \eta\rangle \\
&= \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) \langle (X^{-1/2} Z_i X^{-1/2})^t X^{1/2}\eta, X^{1/2}\eta\rangle \\
&= \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) ||X^{1/2}\eta||^2 \langle (X^{-1/2} Z_i X^{-1/2})^t \frac{X^{1/2}\eta}{||X^{1/2}\eta||}, \frac{X^{1/2}\eta}{||X^{1/2}\eta||}\rangle.
\end{aligned}$$

Since $t \in (0, 1]$, using Lemma 1 we get

$$\begin{aligned}
\langle f(X)\eta, \eta\rangle &\leq \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) ||X^{1/2}\eta||^{2(1-t)} \langle Z_i\eta, \eta\rangle^t \\
&= \langle X\eta, \eta\rangle^{1-t} \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) \langle Z_i\eta, \eta\rangle^t.
\end{aligned}$$

Set $C = \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) \langle Z_i\eta, \eta\rangle^t$, so that

$$\langle f(X)\eta, \eta\rangle \le \langle X\eta, \eta\rangle^{1-t} C. \tag{2.5}$$

It follows from (2.5) that

$$\langle f^2(X)\eta, \eta\rangle \le \langle f(X)\eta, \eta\rangle^{1-t} C \quad \text{and} \quad \langle f(X)\eta, \eta\rangle^{1-t} \le \langle X\eta, \eta\rangle^{(1-t)^2} C^{1-t}. \tag{2.6}$$

Combining two inequalities in (2.6) we get

$$\langle f^2(X)\eta, \eta\rangle \le \langle X\eta, \eta\rangle^{(1-t)^2} C^{1+(1-t)}. \tag{2.7}$$

By using an induction process, we reach

$$\begin{aligned} \langle f^\ell(X)\eta, \eta\rangle &\le \langle X\eta, \eta\rangle^{(1-t)^\ell} C^{1+(1-t)+(1-t)^2+\ldots+(1-t)^{\ell-1}} \\ &= \langle X\eta, \eta\rangle^{(1-t)^\ell} C^{\frac{1-(1-t)^\ell}{1-(1-t)}} \qquad\qquad (\ell \in \mathbb{N}). \end{aligned}$$

Letting $\ell \to \infty$ and noting that $f^\ell(X) \to P_t(\mu)$ we observe that $\langle P_t(\mu)\eta, \eta\rangle \le C^{1/t}$. It follows from the definition of the vector-valued integrals that

$$\begin{aligned} C = \lim_{m\to\infty} \sum_{i=1}^{N_m} \mu(B_{m,i}) \langle Z_i\eta, \eta\rangle^t &= \int_{\mathbb{M}_n^+} \langle Z\eta, \eta\rangle^t d\mu(Z) \\ &= \int_{\mathbb{R}^+} z^t d\langle \mu\eta, \eta\rangle(z), \end{aligned}$$

where the last equality follows from (2.3). Therefore,

$$C^{1/t} = \left(\int_{\mathbb{R}^+} z^t d\langle \mu\eta, \eta\rangle(z) \right)^{1/t} = P_t(\langle \mu\eta, \eta\rangle).$$

This gives (2.4).

Now assume that $t \in [-1, 0)$ and $\nu(\mathcal{E}) = \mu(\mathcal{E}^{-1})$ for every measurable set $\mathcal{E}$. Inequality (2.4) then implies that $\langle P_{-t}(\nu)\eta, \eta\rangle \le P_{-t}(\langle \nu\eta, \eta\rangle)$. It follows from Lemma 1 that

$$P_t(\langle \mu\eta, \eta\rangle) = P_{-t}(\langle \nu\eta, \eta\rangle)^{-1} \le \langle P_{-t}(\nu)\eta, \eta\rangle^{-1} \le \langle P_{-t}(\nu)^{-1}\eta, \eta\rangle = \langle P_t(\mu)\eta, \eta\rangle.$$

This completes the proof. □

It is known that a reverse of (1) holds as follows:

Lemma 3. *[8] Let $0 < r < 1$ and m, M be two positive real numbers . If A is a positive definite matrix with $0 < m \leq A \leq M$, then*

$$\langle A^r \eta, \eta\rangle \geq \alpha(m, M, r)\langle A\eta, \eta\rangle^r, \tag{2.8}$$

where $\alpha(m, M, r) = \frac{Mm^r - mM^r}{(1-r)(M-m)}\left(\frac{1-r}{r}\frac{M^r - m^r}{Mm^r - mM^r}\right)^r$. If $r \in [-1, 0)$, then a reverse inequality holds in (2.8).

Utilizing Lemma 3 and an argument as in the proof of Theorem 2 we obtain the next result. We omit the proof.

Proposition 4. *Let μ be a probability measure of compact support on $\mathbb{M}_n^+$ and η be a unit vector in $\mathbb{C}^n$. If*

$$mP_t(\mu) \leq Z \leq MP_t(\mu), \qquad (Z \in supp(\mu)),$$

then

$$\langle P_t(\mu)\eta, \eta\rangle \geq \alpha(m, M, t)P_t(\langle \mu\eta, \eta\rangle)$$

for every $t \in (0, 1]$. If $t \in [-1, 0)$, then a reverse inequality holds.

Let $\mathbb{A} = (A_1, \ldots, A_k)$ be a k-tuple of positive definite matrices and let $\omega = (\omega_1, \ldots, \omega_k)$ be a weight vector. Consider the probability measure μ on the set $\{A_1, \ldots, A_k\} \subseteq \mathbb{M}_n^+$ by $\mu(\{A_i\}) = \omega_i$ for every $i = 1, \cdots, k$. If $X_t = P_t(\mu)$, then

$$X_t = \int_{\mathbb{M}_n^+} X_t \sharp_t Z d\mu(Z) = \sum_{i=1}^{k} \omega_i X_t \sharp_t A_i = P_t(\omega; \mathbb{A}).$$

Therefore, we have the next corollary.

Corollary 5. *Let $\mathbb{A} = (A_1, \ldots, A_k)$ be a k-tuple of positive definite matrices and let $\omega = (\omega_1, \ldots, \omega_k)$ be a weight vector. Then*

$$\langle P_t(\omega; \mathbb{A})\eta, \eta\rangle \leq P_t(\omega; \langle A_1\eta, \eta\rangle, \cdots, \langle A_k\eta, \eta\rangle) \qquad (t \in (0, 1]) \tag{2.9}$$

for every $\eta \in \mathbb{C}^n$. If in addition $m \leq A_i \leq M$ for two positive real numbers m, M, then

$$\langle P_t(\omega; \mathbb{A})\eta, \eta\rangle \geq \alpha(m, M, t)P_t(\omega; \langle A_1\eta, \eta\rangle, \cdots, \langle A_k\eta, \eta\rangle) \qquad (t \in (0, 1]). \tag{2.10}$$

If $t \in [-1, 0)$, then inequalities (2.9) and (2.10) are reversed.

As a simple example, let $A, B > 0$ and $\omega \in [0,1]$. Then

$$P_t(\omega; A, B) = A^{\frac{1}{2}} \left((1-\omega)I + \omega \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right)^t \right)^{\frac{1}{t}} A^{\frac{1}{2}}.$$

Consider

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Assume that $t = -1/2$, $\omega = 1/2$ and $\eta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\langle P_t(\omega; A, B)\eta, \eta \rangle = \left\langle P_{-t} \left(\omega; A^{-1}, B^{-1}\right)^{-1} \eta, \eta \right\rangle = 1$$

and

$$P_t(\omega; \langle A\eta, \eta\rangle, \langle B\eta, \eta\rangle) = 0.615.$$

Remark 6. *If $\mathbb{A} = (A_1, \ldots, A_k)$ is a k-tuple of commuting positive matrices, then $P_t(\omega; \mathbb{A}) = \left(\sum_{i=1}^k \omega_i A_i^t\right)^{\frac{1}{t}}$. Corollary 5 implies that*

$$\left\langle \left(\sum_{i=1}^k \omega_i A_i^t \right)^{\frac{1}{t}} \eta, \eta \right\rangle \le \left(\sum_{i=1}^k \omega_i \langle A_i \eta, \eta \rangle^t \right)^{\frac{1}{t}}.$$

Theorem 7. *Let μ be a probability measure of compact support on $\mathbb{M}_n^+$ and η be a unit vector in $\mathbb{C}^n$. If*

$$m \le Z \le M \qquad \text{for every } Z \in supp(\mu),$$

then

$$P_t(\langle \mu\eta, \eta\rangle) - \langle P_t(\mu)\eta, \eta\rangle \le (\sqrt{M} - \sqrt{m})^2 \qquad (t \in (0,1]).$$

Proof. The power means $P_t(\mu)$ satisfy [5] the inequality

$$\left(\int_{\mathbb{M}_n^+} Z^{-1} d\mu(Z) \right)^{-1} \le P_t(\mu) \le \int_{\mathbb{M}_n^+} Z d\mu(Z), \tag{2.11}$$

where $P_{-1}(\mu) = \left(\int_{\mathbb{M}_n^+} Z^{-1} d\mu(Z)\right)^{-1}$ is the harmonic mean and $P_1(\mu) = \int_{\mathbb{M}_n^+} Z d\mu(Z)$ is the arithmetic mean for every compactly supported probability measure μ on $\mathbb{M}_n^+$.

Assume that $\{B_{m,i};\ i=1,\cdots,N_m\}$ is a partition of $supp(\mu)$ and Z_i is an arbitrary point in $B_{m,i}$ for $i=1,\cdots,N_m$. Then

$$\begin{aligned}\int_{\mathbb{M}_n^+} Zd\mu(Z)-P_t(\mu) &\le \int_{\mathbb{M}_n^+} Zd\mu(Z)-\left(\int_{\mathbb{M}_n^+} Z^{-1}d\mu(Z)\right)^{-1} \qquad \text{(by (2.11))}\\ &= \lim_{m\to\infty}\left[\sum_{i=1}^{N_m} Z_i\mu(B_{m,i})-\left(\sum_{i=1}^{N_m} Z_i^{-1}\mu(B_{m,i})\right)^{-1}\right].\end{aligned}$$

Define a unital positive linear mapping $T:\mathbb{M}_{nN_m}^+\to\mathbb{M}_n^+$ by $T(A_1\oplus\cdots\oplus A_{N_m})=\sum_{i=1}^{N_m}\mu(B_{m,i})A_i$. Then

$$\begin{aligned}\sum_{i=1}^{N_m} Z_i\mu(B_{m,i})-\left(\sum_{i=1}^{N_m} Z_i^{-1}\mu(B_{m,i})\right)^{-1} &= T(Z_1\oplus\cdots\oplus Z_{N_m})-T(Z_1^{-1}\oplus\cdots\oplus Z_{N_m}^{-1})^{-1}\\ &\le(\sqrt{M}-\sqrt{m})^2, \qquad (2.12)\end{aligned}$$

since $m\le Z_i\le M$ for $i=1,\cdots,N_n$. Therefore

$$\int_{\mathbb{M}_k^+} Zd\mu(Z)-P_t(\mu)\le(\sqrt{M}-\sqrt{m})^2 \qquad (2.13)$$

Moreover,

$$\begin{aligned}P_t(\langle\mu\eta,\eta\rangle)-\langle P_t(\mu)\eta,\eta\rangle &\le \int_{\mathbb{R}^+} zd\langle\mu\eta,\eta\rangle(z)-\langle P_t(\mu)\eta,\eta\rangle\\ &= \int_{\mathbb{M}_k^+}\langle Z\eta,\eta\rangle d\mu(Z)-\langle P_t(\mu)\eta,\eta\rangle \qquad \text{(by (2.3))}\\ &= \left\langle\int_{\mathbb{M}_k^+} Zd\mu(Z)\eta,\eta\right\rangle-\langle P_t(\mu)\eta,\eta\rangle\\ &\le(\sqrt{M}-\sqrt{m})^2,\end{aligned}$$

where the last inequality follows from (2.13). □

Corollary 8. *Let $\mathbb{A}=(A_1,\ldots,A_k)$ be a k-tuple positive matrices and let $\omega=(\omega_1,\ldots,\omega_k)$ be a weight vector. If $0<m\le A_i\le M$ $(i=1,\cdots,k)$ for some scalars $0<m\le M$, then*

$$\langle A_1\eta,\eta\rangle^{\omega_1}\cdots\langle A_k\eta,\eta\rangle^{\omega_k}-\langle P_t(\omega,\mathbb{A})\eta,\eta\rangle\le(\sqrt{M}-\sqrt{m})^2$$

for every $\eta\in\mathbb{C}^n$.

Proof. It follows from the arithmetic-geometric mean inequality that

$$\langle A_1\eta,\eta\rangle^{\omega_1}\cdots\langle A_k\eta,\eta\rangle^{\omega_k}\le\sum_{i=1}^{k}\omega_i\langle A_i\eta,\eta\rangle=\left\langle\left(\sum_{i=1}^{k}\omega_iA_i\right)\eta,\eta\right\rangle.$$

The desired inequality therefore follows from (2.13).

□

Let $A,B\in\mathbb{M}_n$ be positive matrices with $0<m\le A,B\le M$ and $\omega\in[0,1]$. It follows from Corollary 8 that

$$\langle A\eta,\eta\rangle^{\omega}\langle B\eta,\eta\rangle^{1-\omega}\le\langle P_t(\omega,A,B)\eta,\eta\rangle+(\sqrt{M}-\sqrt{m})^2 \tag{2.14}$$

for every unit vector $\eta\in\mathbb{C}^n$. Assume that $a_1,\cdots,a_n$ and $b_1,\cdots,b_n$ are positive scalars such that $m\le a_i,b_i\le M\quad(i=1,\cdots,k)$. Put $\eta=\frac{1}{\sqrt{n}}(1,\cdots,1)\in\mathbb{C}^n$, $A=\mathrm{diag}(a_1,\cdots,a_n)$ and $B=\mathrm{diag}(b_1,\cdots,b_n)$. Inequality (2.14) implies that

$$\left(\sum_{i=1}^{n}a_i\right)^{\omega}\left(\sum_{i=1}^{n}b_i\right)^{1-\omega}\le\sum_{i=1}^{n}\left(\omega a_i^t+(1-\omega)b_i^t\right)^{\frac{1}{t}}+n(\sqrt{M}-\sqrt{m})^2,$$

which is a reverse Hölder type inequality.

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Mohsen Kian
Department of Mathematics University of Bojnord,
P. O. Box 1339, Bojnord 94531, Iran.
e-mail: kian@ub.ac.ir
https://ub.ac.ir/ kian

Fatemeh Rashid
Department of Mathematics University of Bojnord,
P. O. Box 1339, Bojnord 94531, Iran.
e-mail: f_rashid@yahoo.com

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