Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2003, pp. 96–104.
Submitted: September 24, 2003. Published: December 31, 2003.

MULTIPLE RADIAL SYMMETRIC SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS OF p-LAPLACIAN

Qian Zhou and Yuanyuan Ke

Abstract. We discuss the existence of multiple radial symmetric solutions for nonlinear boundary value problems of p-Laplacian, based on Leggett-Williams's fixed point theorem.

A.M.S. (MOS) Subject Classification Codes. 35J40, 35J65, 35J67.

Key Words and Phrases. Multiple radial symmetric solutions, p-Laplacian equation, Leggett-Williams's fixed point theorem.

1. Introduction.

In this paper, we consider the existence of multiple radial symmetric solutions of the p-Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = g(x)f(x,u), \quad x \in \Omega, \tag{1.1}$$

subject to the nonlinear boundary value condition

$$B\left(\frac{\partial u}{\partial \nu}\right) + u = 0, \quad x \in \partial\Omega \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is the unit ball centered at the origin, ν denotes the unit outward normal to the boundary $\partial\Omega$, $g(x)$, $f(x,s)$ and $B(s)$ are all the given functions. In order to discuss the radially symmetric solutions, we assume that $g(x)$ and $f(x,s)$ are radially symmetric, namely, $g(x) = g(|x|)$, $f(x,s) = f(|x|,s)$. Let $w(t) \equiv u(|x|)$ with $t = |x|$ be a radially symmetric solution. Then a direct calculation shows that

$$(t^{n-1}\varphi(w'(t)))' + t^{n-1}g(t)f(t,w(t)) = 0, \qquad 0 < t < 1, \tag{1.3}$$

Department of Mathematics, Jilin University, P. R. China.
E-mail Address: keyy@email.jlu.edu.cn

where $\varphi(s) = |s|^{p-2}s$ and $p > 1$, with the boundary value condition

$$w'(0) = 0, \tag{1.4}$$

$$w(1) + B(w'(1)) = 0. \tag{1.5}$$

Such a problem arises in many different areas of applied mathematics and the fields of mechanics, physics and has been studied extensively, see [1]–[6]. In particular, the Leggett-Williams fixed point theorem has been used to discuss the multiplicity of solutions. For example, He, Ge and Peng [1] considered the following ordinary differential equation

$$(\varphi(y'))' + g(t)f(t, y) = 0, \qquad 0 < t < 1,$$

which corresponds to the special case $n = 1$ of the equation (1.3), with the boundary value conditions

$$y(0) - B_0(y'(0)) = 0,$$
$$y(1) - B_1(y'(1)) = 0.$$

They used the Leggett-Williams fixed point theorem and proved the existence of multi-nonnegative solutions.

In this paper, we extent the results in [1] with $n \geq 1$. We want to use Leggett-Williams's fixed-point theorem to search for solutions of the problem (1.3)–(1.5) too.

This paper is organized as follows. Section 2 collects the preliminaries and statements of results. The proofs of theorems will be given subsequently in Section 3.

2. Preliminaries and Main Results

As a preliminary, we first assume that the given functions satisfy the following conditions Preliminaries and Main Results

(A1) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function.

(A2) $g : (0, 1) \to [0, +\infty)$ is continuous and is allowed to be singular at the end points of $(0, 1)$, $g(t) \not\equiv 0$ on any subinterval of $(0, 1)$. In addition,

$$0 < \int_0^1 g(r)dr < +\infty.$$

(A3) $B(s)$ is a continuous, nondecreasing, odd function, defined on $(-\infty, +\infty)$. And there exists a constant $m > 0$, such that

$$0 \leq B(s) \leq ms, \qquad s \geq 0.$$

In order to prove the existence of the multi-radially symmetric solutions of the problem (1.3)–(1.5), we need some lemmas.

First, we introduce some denotations. Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ is a cone. By a nonnegative continuous concave functional α on P, we mean a mapping $\alpha : P \to [0, +\infty)$ that is α is continuous and

$$\alpha(tw_1 + (1-t)w_2) \geq t\alpha(w_1) + (1-t)\alpha(w_2),$$

for all $w_1, w_2 \in P$, and all $t \in [0,1]$. Let $0 < a < b$, $r > 0$ be constants. Denote

$$P_r = \{w \in P | \|w\| < r\},$$

and

$$P(\alpha, a, b) = \{w \in P | a \leq \alpha(w), \|w\| \leq b\}.$$

We need the following two useful lemmas.

Lemma 2.1 (Leggett-Williams's fixed point theorem)*Let* $T : \overline{P}_c \to \overline{P}_c$ *be completely continuous and* α *be a nonnegative continuous concave functional on* P *such that* $\alpha(w) \leq \|w\|$, *for all* $w \in \overline{P}_c$. *Suppose there exist* $0 < a < b < d \leq c$ *such that*

(B1) $\{w \in P(\alpha, b, d) | \alpha(w) > b\} \neq \emptyset$ *and* $\alpha(Tw) > b$, *for* $w \in P(\alpha, b, d)$,

(B2) $\|Tw\| < a$, *for* $\|w\| \leq a$, *and*

(B3) $\alpha(Tw) > b$, *for* $w \in P(\alpha, b, c)$ *with* $\|Tw\| > d$.

Then, T *has at least three fixed points* w_1, w_2 *and* w_3 *satisfying*

$$\|w_1\| < a, \quad b < \alpha(w_2), \quad \text{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.$$

Lemma 2.2 *Let* $w \in P$ *and* $\delta \in (0, 1/2)$. *Then*

(C1) *If* $0 < \sigma < 1$,

$$w(t) \geq \begin{cases} \dfrac{\|w\|t}{\sigma}, 0 \leq t \leq \sigma, \\ \dfrac{\|w\|(1-t)}{(1-\sigma)}, \sigma \leq t \leq 1. \end{cases}$$

(C2) $w(t) \geq \delta\|w\|$, *for all* $t \in [\delta, 1-\delta]$.

(C3) $w(t) \geq \|w\|t$, $0 \leq t \leq 1$, *if* $\sigma = 1$.

(C4) $w(t) \geq \|w\|(1-t)$, $0 \leq t \leq 1$, *if* $\sigma = 0$.

Here $\sigma \in [0,1]$, *such that*

$$w(\sigma) = \|w\| \equiv \sup_{t \in [0,1]} |w(t)|.$$

We want to use the fixed-point theorem in Lemma 2.1 to search for solutions of the problem (1.3)–(1.5). By (A2), there exists a constant $\delta \in (0, 1/2)$, so that

$$L(x) \equiv \psi\Big(\int_\delta^x g(t)dt\Big) + \psi\Big(\int_x^{1-\delta} g(t)dt\Big), \quad \delta \leq x \leq 1-\delta,$$

is a positive and continuous function in $[\delta, 1-\delta]$, where $\psi(s) \equiv |s|^{\frac{1}{(p-1)}} \operatorname{sgn} s$ is the inverse function of $\varphi(s) = |s|^{p-2}s$. For convenience, we set

$$L \equiv \min_{\delta \leq x \leq 1-\delta} L(x),$$

and

$$\lambda = (m+1)\psi\Big(\int_0^1 g(r)dr\Big).$$

And in this paper, we set the Banach space $E = C[0,1]$ with the norm defined by

$$\|w\| = \sup_{t \in [0,1]} |w(t)|, \quad w \in E.$$

The cone $P \subset E$ is specified as,

$$P = \{w \in E | w \text{ is a nonnegative concave function in } [0,1]\}.$$

Furthermore, we define the nonnegative and continuous concave function α satisfying

$$\alpha(w) = \frac{w(\delta) + w(1-\delta)}{2}, \quad w \in P.$$

Obviously,

$$\alpha(w) \le \|w\|, \quad \text{for all } w \in P.$$

Under all the assumptions (A1)–(A3), we can get the main result as follows

Theorem 2.1 *Let a, b, d, δ be given constants with $0 < a < \delta b < b < b/\delta \le d$, and let the following conditions on f and φ are fulfilled:*

(D1) For all $(t, w) \in [0,1] \times [0,a]$, $f(t,w) < \varphi\left(\dfrac{a}{\lambda}\right)$;

(D2) Either

i) $\limsup\limits_{w \to +\infty} \frac{f(t,w)}{w^{p-1}} < \varphi\left(\frac{1}{\lambda}\right)$, uniformly all $t \in [0,1]$, or

ii) $f(t,w) \le \varphi\left(\frac{\eta}{\lambda}\right)$, for all $(t,w) \in [0,1] \times [0,\eta]$ with some $\eta \ge d$, $\lambda > 0$;

(D3) $f(t,w) > \varphi\left(\dfrac{2b}{\delta L}\right)$, for $(t,w) \in [\delta, 1-\delta] \times [\delta b, d]$ with some $L > 0$.

Then, the problem (1.3)–(1.5) have at least three radially symmetric solutions w_1, w_2 and w_3, such that

$$\|w_1\| < a, \quad \alpha(w_2) > b, \quad \textit{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.$$

3. Proofs of the Main Results

We are now in a position to prove our main results.

Proof of Theorem 2.1. Define $T : P \to E$, $w \mapsto W$, where W is determined by

$$\begin{aligned} W(t) =& (Tw)(t) \\ \triangleq& B \circ \psi\Big(\int_0^1 r^{n-1} g(r) f(r, w(r)) dr\Big) \\ &+ \int_t^1 \psi\Big(s^{-(n-1)} \int_0^s r^{n-1} g(r) f(r, w(r)) dr\Big) ds, \qquad t \in [0,1], \end{aligned}$$

for each $w \in P$.

First we prove each fixed point of W in P is a solution of (1.3)– (1.5). By the definition of W, we have

$$W'(t) = (Tw)'(t) = -\psi\Big(t^{-(n-1)} \int_0^t r^{n-1} g(r) f(r, w(r)) dr\Big).$$

Noticing that

$$\begin{aligned} &\Big| -\psi\Big(t^{-(n-1)} \int_0^t r^{n-1} g(r) f(r, w(r)) dr\Big) \Big| \\ =& \Big| -\psi\Big(\int_0^t \left(\frac{r}{t}\right)^{n-1} g(r) f(r, w(r)) dr\Big) \Big| \\ \le& \Big| -\psi\Big(\int_0^t g(r) f(r, w(r)) dr\Big) \Big|, \end{aligned}$$

and by the integrability of g and f, we have

$$\lim_{t\to 0^+} W'(t) = \lim_{t\to 0^+} \psi\Big(\int_0^t g(r)f(r,w(r))dr\Big) = 0. \tag{3.1}$$

Considering

$$W'(0) = \lim_{t\to 0} \frac{W(t)-W(0)}{t},$$

and

$$\begin{aligned} &W(t)-W(0)\\ =&\int_t^1 \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\ &\quad - \int_0^1 \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\ =&-\int_0^t \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds, \end{aligned}$$

and by using L′Hospital's rule, we get

$$\begin{aligned} W'(0) =& \lim_{t\to 0}\frac{W(t)-W(0)}{t}\\ =&\lim_{t\to 0}\Big(W(t)-W(0)\Big)'\\ =&-\lim_{t\to 0}\psi\Big(t^{-(n-1)}\int_0^t r^{n-1}g(r)f(r,w(r))dr\Big)ds\\ =&0. \end{aligned}$$

Recalling (3.1), we know that $W'(t)$ is right-continuous at the point $t=0$, and $W'(0)=0$, namely, the fixed point of W satisfies (1.4). By the assumption (A1) and (A2), we also have

$$W'(t) = (Tw)'(t) \le 0.$$

Then $\|Tw\| = (Tw)(0)$. On the other hand, since

$$W(1) = B\psi\left(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\right),$$

and

$$B(W'(1)) = -B\psi\left(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\right),$$

we see that

$$W(1) + B(w'(1)) = 0,$$

namely, the fixed point of W also satisfies (1.5).

Next we show that the conditions in Lemma 2.1 are satisfied. We first prove that condition (D2) implies the existence of a number c where $c > d$ such that

$$W : \overline{P}_c \to \overline{P}_c.$$

If ii) of (D2) holds, by the condition (A3), we see that

$$
\begin{aligned}
\|Tw\| =&(Tw)(0)\\
=&B\circ\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le& m\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(\int_0^s\left(\frac{r}{s}\right)^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le&(m+1)\psi\Big(\int_0^1 g(r)f(r,w(r))dr\Big)\\
\le&(m+1)\psi\Big(\int_0^1 g(r)\varphi\left(\frac{\eta}{\lambda}\right)dr\Big)\\
=&(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\psi\Big(\varphi\left(\frac{\eta}{\lambda}\right)\Big)\\
=&\frac{\eta}{\lambda}(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\\
=&\eta,\qquad\text{for } w\in\overline{P}_\eta.
\end{aligned}
$$

Then, if we select $c=\eta$, there must be $W:\overline{P}_c\to\overline{P}_c$.

If i) of (D2) is satisfied, then there must exist $D>0$ and $\epsilon<\varphi(1/\lambda)$, so that

$$\frac{f(t,w)}{w^{p-1}}<\epsilon,\quad\text{for } (t,w)\in[0,1]\times[D,+\infty). \tag{3.2}$$

Let $M=\max\{f(t,w)|\ 0\le t\le 1, 0\le w\le D\}$. By (3.2), we obtain

$$f(t,w)\le M+\epsilon w^{p-1},\quad\text{for } (t,w)\in[0,1]\times[0,+\infty). \tag{3.3}$$

Selecting a proper real number c, so that

$$\varphi(c)>\max\Big\{\varphi(d), M\Big(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\Big)^{-1}\Big\}. \tag{3.4}$$

Utilizing (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
\|Tw\| =&(Tw)(0)\\
=&B\circ\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le& m\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(\int_0^s\left(\frac{r}{s}\right)^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le&(m+1)\psi\Big(\int_0^1 g(r)f(r,w(r))dr\Big)\\
\le&(m+1)\psi\Big(\int_0^1 g(r)(M+\epsilon w^{p-1})dr\Big)\\
=&(m+1)\psi\left(\int_0^1 g(r)\left(M\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)^{-1}\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)+\epsilon w^{p-1}\right)dr\right)\\
\le&(m+1)\psi\left(\int_0^1 g(r)\left(\varphi(c)\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)+\epsilon c^{p-1}\right)dr\right)\\
=&(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\frac{c}{\lambda}=c, \qquad\qquad \text{for } w\in\overline{P}_c.
\end{aligned}$$

So we obtain $\|W\|\le c$, that is $W:\overline{P}_c\to\overline{P}_c$.

Then we want to verify that W satisfies the condition (B2) in Lemma 2.1. If $\|w\|\le a$, then by the condition (D1), we know

$$f(t,w)<\varphi\left(\frac{a}{\lambda}\right),\qquad \text{for } 0\le t\le 1,\quad 0\le w\le a.$$

We use the methods similarly to the above, and can get $\|W\|=\|Tw\|<a$, that is, W satisfies (B2).

To fulfill condition (B1) of Lemma 2.1, we note that $w(t)\equiv(b+d)/2>b$, $0\le t\le 1$, is the member of $P(\alpha,b,d)$ and $\alpha(w)=\alpha((b+d)/2)>b$, hence $\{w\in P(\alpha,b,d)|\ \alpha(w)>b\}\neq\emptyset$. Now assume $w\in P(\alpha,b,d)$. Then

$$\alpha(w)=\frac{w(\delta)+w(1-\delta)}{2}\ge b,\quad \text{and } b\le\|w\|\le d.$$

Utilizing the condition (C2) in Lemma 2.2, we know that for all s, which satisfying $\delta\le s\le 1-\delta$, there has

$$\delta b\le\delta\|w\|\le w(s)\le d.$$

And meanwhile, we can select a proper ε, so that

$$\left(\frac{\varepsilon}{s}\right)^{n-1}>\left(\frac{\varepsilon}{1-\delta}\right)^{n-1}>\frac{1}{2}.$$

Combining the condition (D3), we can see

$$
\begin{aligned}
\alpha(Tw) &= \frac{(Tw)(\delta) + (Tw)(1-\delta)}{2} \\
&\geq (Tw)(1-\delta) \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_0^s \left(\frac{r}{s}\right)^{n-1} g(r) f(r, w(r)) dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_\varepsilon^s \left(\frac{r}{s}\right)^{n-1} g(r) f(r, w(r)) dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_\varepsilon^s \left(\frac{\varepsilon}{s}\right)^{n-1} g(r) f(r, w(r)) dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\left(\frac{\varepsilon}{s}\right)^{n-1} \int_\delta^{1-\delta} g(r) f(r, w(r)) dr\right) ds \\
&> \int_{1-\delta}^{1} \psi\left(\frac{1}{2} \int_\delta^{1-\delta} g(r) \varphi\left(\frac{2b}{\delta L}\right) dr\right) ds \\
&= \frac{1}{2}\delta\psi\left(\int_\delta^{1-\delta} g(r) dr\right) \frac{2b}{\delta L} \\
&\geq b.
\end{aligned}
$$

That is (B1) is well verified.

Finally, we prove (B3) of Lemma 2.1 is also satisfied. For $w \in P(\alpha, b, c)$, we have $\|Tw\| > d$. By using the condition (C2) in Lemma 2.2, we get

$$
\alpha(Tw) = \frac{(Tw)(\delta) + (Tw)(1-\delta)}{2} \geq \delta\|Tw\| > \delta d > b.
$$

Then, the condition (B3) in Leggett-Williams's fixed point theorem is well verified.

Using the above results and applying Leggett-Williams's fixed point theorem, we can see that the operator W has at least three fixed points, that is the problem (1.3)–(1.5) have at least three radially symmetric solutions w_1, w_2 and w_3, which satisfying

$$
\|w_1\| < a, \quad \alpha(w_2) > b, \quad \text{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.
$$

The proof is complete.

References

1. He, X. M., Ge, W. G. and Peng, M. S. : Multiple positive solution for one-dimensional p-Laplacian boundary value problems, *Applied Mathematics Letters,* **15**(2002), 937–943.
2. Wang, J. Y. : The existence of positive solution for one-dimensional p-Laplacian, *Proc. Amer. Math. Soc.,* **125**(1997), 2275–2283.
3. Yao, Q. L. and Lu, H. C. : Positive solution of one-dimensional singular p-Laplacian equations, (in Chinese), *Acta Mathematica Sinica,* **14**(1998), 1255–1264.

4. Henderson, J. and Thompson, H. B. : Multiple symmetric positive solutions for a second order boundary value problem, *Proc. Amer. Math. Soc.,* **128**(2000), 2372–2379.
5. Anderson, D. : Multiple positive solutions for a three-point boundary value problem, *Math. J. Comput. Modelling,* **27**(1998)(6), 49–57.
6. Avery, R. I. : Three symmetric positive solutions for a second-order boundary value problem, *Appl. Math. Lett.,* **13**(2000)(3), 1–7.