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# THE CATEGORY OF OPETOPES AND THE CATEGORY OF OPETOPIC SETS

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Abstract.

We give an explicit construction of the category **Opetope** of opetopes. We prove that the category of opetopic sets is equivalent to the category of presheaves over **Opetope**.

## Introduction

In [3], Baez and Dolan give a definition of weak $n$-category in which the underlying shapes of cells are 'opetopes' and the underlying data is given by 'opetopic sets'. The idea is that opetopic sets should be presheaves over the category of opetopes. However Baez and Dolan do not explicitly construct the category of opetopes, so opetopic sets are defined directly instead. A relationship between this category of opetopic sets and a category of presheaves is alluded to but not proved.

The main result of this paper is that the category of opetopic sets is equivalent to the category of presheaves over the category of opetopes. However, we do not use the opetopic definitions exactly as given in [3] but continue to use the modifications given in our earlier work [4, 5]. In these papers we use a generalisation along lines which the original authors began, but chose to abandon as they thought it would lead to an incorrect definition of braided monoidal categories. This question remainds unresolved. However, the generalisation enables us, in [4], to exhibit a relationship with the work of Hermida, Makkai and Power [7] and, in [5], with the work of Leinster [13]. Given these useful results, we continue to study the modified theory in this work.

We begin in Section 1 by giving an explicit construction of the category of opetopes. The idea is as follows. In [4] we constructed, for each $k \geq 0$, a category $\mathcal{O}_k$ of $k$-opetopes. For the category **Opetope** of opetopes of all dimensions, each category $\mathcal{O}_k$ should be a full subcategory of **Opetope**; furthermore there should be 'face maps' exhibiting the constituent $m$-opetopes, or 'faces' of a $k$-opetope, for $m \leq k$. We refer to the $m$-opetope faces as $m$-faces. Note that there are no degeneracy maps.

The $(k-1)$-faces of a $k$-opetope $\alpha$ should be the $(k-1)$-opetopes of its source and target; these should all be distinct. Then each of these faces has its own $(k-2)$-faces, but all these $(k-2)$-opetopes should not necessarily be considered as distinct $(k-2)$-faces in $\alpha$. For $\alpha$ is a configuration for composing its $(k-1)$-faces at their $(k-2)$-faces, so

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the $(k-2)$-faces should be identified with one another at places where composition is to occur. That is, the face maps from these $(k-2)$-opetopes to $\alpha$ should therefore be equal. Some further details are then required to deal with isomorphic copies of opetopes.

Recall that a 'configuration' for composing $(k-1)$-opetopes is expressed as a tree (see [4]) whose nodes are labelled by the $(k-1)$-opetopes in question, with the edges giving their inputs and outputs. So composition occurs along each edge of the tree, via an object-morphism label, and thus the tree tells us which $(k-1)$-opetopes are identified. In order to express this more precisely, we use a more formal characterisation of such a 'configuration'; see [6] for a complete treatment of this subject.

In Section 2, we examine the theory of opetopic sets. We begin by following through our modifications to the opetopic theory to include the theory of opetopic sets. (Our previous work has only dealt with the theory of opetopes.) We then use results of [12] to prove that the category of opetopic sets is indeed equivalent to the category of presheaves on **Opetope**. This is the main result of this work.

Finally, a comment is due on the notion of 'multitope' as defined in [7]. In this work, Hermida, Makkai and Power begin a definition of $n$-category explicitly analogous to that of [3], the analogous concepts being 'multitopes' and 'multitopic sets'. In [4] we prove that 'opetopes and multitopes are the same up to isomorphism', that is, for each $k \geq 0$ the category of $k$-opetopes is equivalent to the (discrete) category of $k$-multitopes. In [7], Hermida, Makkai and Power do go on to give an explicit definition of the analogous category **Multitope**, of multitopes. Given the above equivialences, and assuming the underlying idea is the same, this would be equivalent to the category **Opetope**, but we do not attempt to prove it in this work.

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## 1. The category of opetopes

In this section we give an explicit construction of the category **Opetope** of opetopes. This construction will enable us, in Section 2, to prove that the category of opetopic sets is in fact a presheaf category.

Opetopes themselves are constructed by iterating a 'slice' construction on a certain symmetric multicategory. We begin this section with a terse account of the definitions required for this theory. We include it for the sake of completeness; none of the material in Sections 1.1–1.3 is new, and it is treated in full in [4].

1.1. SYMMETRIC MULTICATEGORIES. We write $\mathcal{F}$ for the 'free symmetric strict monoidal category' monad on **Cat**, and $\mathbf{S}_k$ for the group of permutations on $k$ objects; we also write $\iota$ for the identity permutation.

A *symmetric multicategory* $Q$ with a category of objects is given by the following data

1) A category $o(Q) = \mathbb{C}$ of objects. We refer to $\mathbb{C}$ as the *object-category*, the morphisms of $\mathbb{C}$ as *object-morphisms*, and if $\mathbb{C}$ is discrete, we say that $Q$ is *object-discrete*.

2) For each $p \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$, a set $Q(p)$ of arrows. Writing
$$p = (x_1, \ldots, x_k; x),$$
an element $f \in Q(p)$ is considered as an arrow with source and target given by
$$\begin{array}{rcl} s(f) & = & (x_1, \ldots, x_k) \\ t(f) & = & x. \end{array}$$

3) For each object-morphism $f : x \longrightarrow y$, an arrow $\iota(f) \in Q(x; y)$. In particular we write $1_x = \iota(1_x) \in Q(x; x)$.

4) Composition: for any $f \in Q(x_1, \ldots, x_k; x)$ and $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$ for $1 \leq i \leq k$, a composite
$$f \circ (g_1, \ldots, g_k) \in Q(x_{11}, \ldots, x_{1m_1}, \ldots, x_{k1}, \ldots, x_{km_k}; x)$$

5) Symmetric action: for each permutation $\sigma \in \mathbf{S}_k$, a map
$$\begin{array}{rccc} \sigma : & Q(x_1, \ldots, x_k; x) & \longrightarrow & Q(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x) \\ & f & \longmapsto & f\sigma \end{array}$$

satisfying the following axioms:

1) Unit laws: for any $f \in Q(x_1, \ldots, x_m; x)$, we have
$$1_x \circ f = f = f \circ (1_{x_1}, \ldots, 1_{x_m})$$

2) Associativity: whenever both sides are defined,
$$\begin{array}{l} f \circ (g_1 \circ (h_{11}, \ldots, h_{1m_1}), \ldots, g_k \circ (h_{k1}, \ldots, h_{km_k})) = \\ \qquad (f \circ (g_1, \ldots, g_k)) \circ (h_{11}, \ldots, h_{1m_1}, \ldots, h_{k1}, \ldots, h_{km_k}) \end{array}$$

3) For any $f \in Q(x_1, \ldots, x_m; x)$ and $\sigma, \sigma' \in \mathbf{S}_k$,
$$(f\sigma)\sigma' = f(\sigma\sigma')$$

4) For any $f \in Q(x_1, \ldots, x_k; x)$, $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$ for $1 \leq i \leq k$, and $\sigma \in \mathbf{S}_k$, we have
$$(f\sigma) \circ (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = f \circ (g_1, \ldots, g_k) \cdot \rho(\sigma)$$
where $\rho : \mathbf{S}_k \longrightarrow \mathbf{S}_{m_1 + \ldots + m_k}$ is the obvious homomorphism.

5) For any $f \in Q(x_1, \ldots, x_k; x)$, $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$, and $\sigma_i \in \mathbf{S}_{m_i}$ for $1 \leq i \leq k$, we have
$$f \circ (g_1\sigma_1, \ldots, g_k\sigma_k) = (f \circ (g_1, \ldots, g_k))\sigma$$
where $\sigma \in \mathbf{S}_{m_1+\cdots+m_k}$ is the permutation obtained by juxtaposing the $\sigma_i$.

6) $\iota(f \circ g) = \iota(f) \circ \iota(g)$

1.2. The slice multicategory. We now define the 'slice', a way of moving up one dimension. Let $Q$ be a symmetric multicategory with a category $\mathbb{C}$ of objects, so $Q$ may be considered as a functor $Q : \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{Set}$ with certain extra structure. Then we can form $\mathrm{elt}\,Q$, the category of elements of the functor $Q$. $\mathrm{elt}\,Q$ has as objects pairs $(p, g)$ with $p \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ and $g \in Q(p)$; a morphism $\alpha : (p, g) \longrightarrow (p', g')$ is an arrow $\alpha : p \longrightarrow p' \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ such that
$$\begin{array}{rccc} Q(\alpha): & Q(p) & \longrightarrow & Q(p') \\ & g & \longmapsto & g' \;. \end{array}$$

The *slice multicategory* $Q^+$ is given by:

- Objects: $o(Q^+) = \mathrm{elt}(Q)$
- Arrows: $Q^+(f_1, \ldots, f_n; f)$ is given by the set of 'configurations' for composing $f_1, \ldots, f_n$ as arrows of $Q$, to yield $f$.

Writing $f_i \in Q(x_{i1}, \ldots x_{im_i}; x_i)$ for $1 \leq i \leq n$, such a configuration is given by $(T, \rho, \tau)$ where

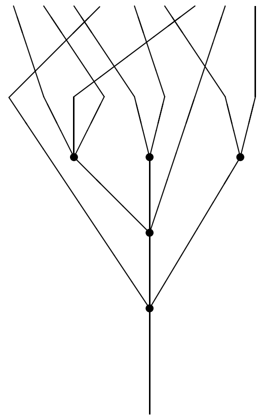
1) $T$ is a tree with $n$ nodes. Each node is labelled by one of the $f_i$, and each edge is labelled by an object-morphism of $Q$ in such a way that the (unique) node labelled by $f_i$ has precisely $m_i$ edges going in from above, labelled by $a_{i1}, \ldots, a_{im_i} \in \mathrm{arr}(\mathbb{C})$, and the edge coming out is labelled $a_i \in a(\mathbb{C})$, where $\mathrm{cod}(a_{ij}) = x_{ij}$ and $\mathrm{dom}(a_i) = x_i$.

2) $\rho \in \mathbf{S}_k$ where $k$ is the number of leaves of $T$.

3) $\tau : \{\text{nodes of } T\} \longrightarrow [n] = \{1, \ldots, n\}$ is a bijection such that the node $N$ is labelled by $f_{\tau(N)}$. (This specification is necessary to allow for the possibility $f_i = f_j,\ i \neq j$.)

The arrow resulting from this composition is given by composing the $f_i$ according to their positions in $T$, with the $a_{ij}$ acting as arrows $\iota(a_{ij})$ of $Q$, and then applying $\rho$ according to the symmetric action on $Q$. This construction uniquely determines an arrow $(T, \rho, \tau) \in Q^+(f_1, \ldots, f_n; f)$.

Note that $(T, \rho)$ may be considered as a 'combed tree', that is, a tree with a 'twisting' of branches at the top given by $\rho$. An example of such a tree is given below:



Note that there is a 'null tree' with no nodes



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- Composition

When it can be defined, $(T_1, \rho_1, \tau_1) \circ_m (T_2, \rho_2, \tau_2) = (T, \rho, \tau)$ is given by

1) $(T, \rho)$ is the combed tree obtained by replacing the node ${\tau_1}^{-1}(m)$ by the tree $(T_2, \rho_2)$, composing the edge labels as morphisms of $\mathbb{C}$, and then 'combing' the tree so that all twists are at the top.

2) $\tau$ is the bijection which inserts the source of $T_2$ into that of $T_1$ at the $m$th place.

- Identities: given an object-morphism

$$\alpha = (\sigma, f_1, \dots, f_m; f) : g \longrightarrow g',$$

$\iota(\alpha) \in Q^+(g; g')$ is given by a tree with one node, labelled by $g$, twist $\sigma$, and edges labelled by the $f_i$ and $f$ as in the example above.

- Symmetric action: $(T, \rho, \tau)\sigma = (T, \rho, \sigma^{-1}\tau)$

1.3. Opetopes. For any symmetric multicategory $Q$ we write $Q^{k+}$ for the $k$th iterated slice of $Q$, that is

$$Q^{k+} = \begin{cases} Q & k = 0 \\ \left(Q^{(k-1)+}\right)^+ & k \geq 1 \end{cases}$$

Let $I$ be the symmetric multicategory with precisely one object, precisely one (identity) object-morphism, and precisely one (identity) arrow. A *$k$-dimensional opetope*, or simply *$k$-opetope*, is defined in [3] to be an object of $I^{k+}$. We write $\mathcal{O}_k = o(I^{k+})$, the category of $k$-opetopes.

1.4. Formal description of trees. In this section we give a formal description of the above trees, that will enable us, in Section 1.7, to determine which faces of faces are identified in an opetope. We only give the details necessary for the construction of the category of opetopes; for a full treatment of trees in this way, we refer the reader to [6].

We consider a tree with $k$ nodes $N_1, \ldots, N_k$ where $N_i$ has $m_i$ inputs and one output. Let $N$ be a node with $(\sum_i m_i) - k + 1$ inputs; $N$ will be used to represent the leaves and root of the tree.

Then a tree gives a bijection

$$\coprod_i \{\text{inputs of } N_i\} \coprod \{\text{output of } N\} \longrightarrow \coprod_i \{\text{output of } N_i\} \coprod \{\text{inputs of } N\}$$

since each input of a node is either connected to a unique output of another node, or it is a leaf, that is, input of $N$. Similarly each output of a node is either attached to an input of another node, or it is the root, that is, output of $N$.

We express this formally as follows.

1.5. Lemma. *Let $T$ be a tree with nodes $N_1, \ldots, N_k$, where $N_i$ has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output $x_i$. Let $N$ be a node with inputs $\{z_1, \ldots, z_l\}$ and output $z$, with*

$$l = (\sum_{i=1}^{k} m_i) - k + 1.$$

*Then $T$ gives a bijection*

$$\alpha \ : \ \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}.$$

In fact for the construction of opetopes we require the 'labelled' version of these trees. A tree labelled in a category $\mathbb{C}$ is a tree as above, with each edge labelled by a morphism of $\mathbb{C}$ considered to be pointing 'down' towards the root.

1.6. Proposition. *Let $T'$ be a labelled tree whose underlying tree is a tree $T$ as above. Then $T'$ is gives a bijection as above, together with, for each*

$$y \in \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\}$$

*a morphism $f \in \mathbb{C}$ giving the label of the edge joining $y$ and $\alpha(y)$. Then $y$ is considered to be labelled by the object* $\mathrm{cod}(f)$ *and $\alpha(y)$ by the object* $\mathrm{dom}(f)$.

1.7. The category of opetopes. In our earlier work [4] we constructed for each $k \geq 0$ the category $\mathcal{O}_k$ of $k$-opetopes. We now construct a category **Opetope** of opetopes of all dimensions whose morphisms are, essentially, face maps. Each category $\mathcal{O}_k$ is to be a full subcategory of **Opetope**, and there are no morphisms from an opetope to one of lower dimension. For brevity we write **Opetope** $= \mathcal{O}$ and construct this category as follows.

For the objects:

$$\text{ob } \mathcal{O} = \coprod_{k \geq 0} \mathcal{O}_k.$$

The morphisms of $\mathcal{O}$ are given by generators and relations as follows.

- Generators

1) For each morphism $f : \alpha \longrightarrow \beta \in \mathcal{O}_k$ there is a morphism

$$f : \alpha \longrightarrow \beta \in \mathcal{O}.$$

2) Let $k \geq 1$ and consider $\alpha \in \mathcal{O}_k = o(I^{k+}) = \text{elt}\,(I^{(k-1)+})$. Write $\alpha \in I^{(k-1)+}(x_1, \ldots, x_m; x)$, say. Then for each $1 \leq i \leq m$ there is a morphism

$$s_i : x_i \longrightarrow \alpha \in \mathcal{O}$$

and there is also a morphism

$$t : x \longrightarrow \alpha \in \mathcal{O}.$$

We write $G_k$ for the set of all generating morphisms of this kind.

Before giving the relations on these morphisms we make the following observation about morphisms in $\mathcal{O}_k$. Consider

$$\begin{array}{ccl} \alpha & \in & I^{(k-1)+}(x_1, \ldots, x_m; x) \\ \beta & \in & I^{(k-1)+}(y_1, \ldots, y_m; y) \end{array}$$

A morphism $\alpha \xrightarrow{g} \beta \in \mathcal{O}_k$ is given by a permutation $\sigma$ and morphisms

$$\begin{array}{ccccc} x_i & \xrightarrow{f_i} & y_{\sigma(i)} & & \\ x & \xrightarrow{f} & y & \in & \mathcal{O}_{k-1} \end{array}$$

So for each face map $\gamma$ there is a unique 'restriction' of $g$ to the specified face, giving a morphism $\gamma g$ of $(k-1)$-opetopes.

Note that, to specify a morphism in the category $\mathcal{F}\mathcal{O}_{k-1}{}^{\text{op}} \times \mathcal{O}_{k-1}$ the morphisms $f_i$ above should be in the direction $y_{\sigma(i)} \longrightarrow x_i$, but since these are all unique isomorphisms the direction does not matter; the convention above helps the notation. We now give the relations on the above generating morphisms.

- Relations

1) For any morphism

$$\alpha \xrightarrow{g} \beta \ \in \mathcal{O}_k$$

and face map

$$x_i \xrightarrow{s_i} \alpha$$

the following diagrams commute

$$\begin{tikzcd} x_i \arrow[r, "s_i"] \arrow[d, "s_i(g)"'] & \alpha \arrow[d, "g"] \\ y_{\sigma(i)} \arrow[r, "s_{\sigma(i)}"'] & \beta \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "t"] \arrow[d, "t(g)"'] & \alpha \arrow[d, "g"] \\ y \arrow[r, "t"'] & \beta \end{tikzcd}$$

We write these generally as

$$\begin{tikzcd} x \arrow[r, "\gamma"] \arrow[d, "\gamma g"'] & \alpha \arrow[d, "g"] \\ y \arrow[r, "\gamma'"'] & \beta \end{tikzcd}$$

2) Faces are identified where composition occurs: consider $\theta \in \mathcal{O}_k$ where $k \geq 2$. Recall that $\theta$ is constructed as an arrow of a slice multicategory, so is given by a labelled tree, with nodes labelled by its $(k-1)$-faces, and edges labelled by object-morphisms, that is, morphisms of $\mathcal{O}_{k-2}$.

   So by the formal description of trees (Section 1.4), $\theta$ is a certain bijection, and the elements that are in bijection with each other are the $(k-2)$-faces of the $(k-1)$-faces of $\theta$; they are given by composable pairs of face maps of the second kind above. That is, the node labels are given by face maps $\alpha \xrightarrow{\gamma} \theta$ and then the inputs and outputs of those are given by pairs

$$x \xrightarrow{\gamma_1} \alpha \xrightarrow{\gamma_2} \theta$$

   where $\gamma_2 \in G_k$ and $\gamma_1 \in G_{k-1}$. Now, if

$$\begin{array}{cccccc} & x & \xrightarrow{\gamma_1} & \alpha & \xrightarrow{\gamma_2} & \theta \\ \text{and} & y & \xrightarrow{\gamma_3} & \beta & \xrightarrow{\gamma_4} & \theta \end{array}$$

correspond under the bijection, there must be a unique object-morphism

$$f : x \longrightarrow y$$

labelling the relevant edge of the tree. Then for the composites in $\mathcal{O}$ we have the relation: the following diagram commutes

$$\begin{tikzcd} x \arrow[r, "\gamma_1"] \arrow[d, "f"'] & \alpha \arrow[dr, "\gamma_2"] & \\ & & \theta. \\ y \arrow[r, "\gamma_3"] & \beta \arrow[ur, "\gamma_4"'] & \end{tikzcd}$$

3) Composition in $\mathcal{O}_k$ is respected, that is, if $g \circ f = h \in \mathcal{O}_k$ then $g \circ f = h \in \mathcal{O}$.

4) Identities in $\mathcal{O}_k$ are respected, that is, given any morphism $x \xrightarrow{\gamma} \alpha \in \mathcal{O}$ we have $\gamma \circ 1_x = \gamma$.

Note that only the relation (2) is concerned with the identification of faces with one another; the other relations are merely dealing with isomorphic copies of opetopes.

We immediately check that the above relations have not identified any morphisms of $\mathcal{O}_k$.

1.8. Lemma. *Each $\mathcal{O}_k$ is a full subcategory of $\mathcal{O}$.*

Proof. Clear from definitions. ∎

We now check that the above relations have not identified any $(k-1)$-faces of $k$-opetopes.

1.9. Proposition. *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}_k$ and $\gamma_1, \gamma_2 \in G_k$ with*

$$\gamma_1,\ \gamma_2 : x \longrightarrow \alpha$$

*Then $\gamma_1 = \gamma_2 \in \mathcal{O} \implies \gamma_1 = \gamma_2 \in G_k$.*

We prove this by expressing all morphisms from $(k-1)$-opetopes to $k$-opetopes in the following "normal form"; this is a simple exercise in term rewriting (see [11]).

1.10. Lemma. *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}$. Then a morphism*

$$x \longrightarrow \alpha \;\in \mathcal{O}$$

*is uniquely represented by*

$$x \xrightarrow{\gamma} \alpha$$

*or a pair*

$$x \xrightarrow{f} y \xrightarrow{\gamma} \alpha$$

*where $f \in \mathcal{O}_{k-1}$ and $\gamma \in G_k$.*

Proof. Any map $x \longrightarrow \alpha$ is represented by terms of the form

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m \xrightarrow{\gamma} \alpha_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{j-1}} \alpha_j \xrightarrow{g_j} \alpha$$

where each $f_i \in \mathcal{O}_{k-1}$ and each $g_r \in \mathcal{O}_k$. Equalities are generated by equalities in components of the following forms:

1) $$\xrightarrow{\quad\gamma\quad}\;\xrightarrow{\quad g\quad} = \xrightarrow{\quad\gamma g\quad}\;\xrightarrow{\quad\gamma'\quad}$$

2) $$\xrightarrow{\quad f\quad}\;\xrightarrow{\quad f'\quad} = \xrightarrow{\quad f'\circ f\quad} \qquad \in \mathcal{O}_{k-1}$$

3) $$\xrightarrow{\quad g\quad}\;\xrightarrow{\quad g'\quad} = \xrightarrow{\quad g'\circ g\quad} \qquad \in \mathcal{O}_k$$

4) $$\xrightarrow{\quad 1\quad}\;\xrightarrow{\quad\gamma\quad} = \xrightarrow{\quad\gamma\quad}$$

where $\gamma \in G_k$ and $\gamma g$ and $\gamma'$ are as defined above. That is, equalities in terms are generated by equations $t = t'$ where $t'$ is obtained from $t$ by replacing a component of $t$ of a left hand form above, with the form in the right hand side, or vice versa.

We now orient the equations in the term rewriting style in the direction

$$\Longrightarrow$$

from left to right in the above equations. We then show two obvious properties:

1) Any reduction of $t$ by $\Longrightarrow$ terminates in at most $2j+m$ steps.

2) If we have

$$\begin{tikzcd} & t \arrow[dl] \arrow[dr] & \\ t' & & t'' \end{tikzcd}$$

then there exists $t'''$ with

$$\begin{tikzcd}
 & t \arrow[dl] \arrow[dr] & \\
t' \arrow[dr, dashed] & & t'' \arrow[dl, dashed] \\
 & t''' &
\end{tikzcd}$$

where the dashed arrows indicate a chain of equations (in this case of length at most 2).

The first part is clear from the definitions; for the second part the only non-trivial case is for a component of the form

$$\xrightarrow{\gamma} \quad \xrightarrow{g_1} \quad \xrightarrow{g_2} \; .$$

This reduces uniquely to

$$\xrightarrow{\gamma(g_2 \circ g_1)} \quad \xrightarrow{\gamma'}$$

since 'restriction' is unique, as discussed earlier.

It follows that, for any terms $t$ and $s$, $t = s$ if and only if $t$ and $s$ reduce to the same normal form as above. ∎

PROOF of Proposition 1.9. $\gamma_1$ and $\gamma_2$ are in normal form. ∎

## 2. Opetopic Sets

In this section we examine the theory of opetopic sets. We begin by following through our modifications to the opetopic theory to include the theory of opetopic sets. We then use results of [12] to prove that the category of opetopic sets is indeed equivalent to the category of presheaves on $\mathcal{O}$, the category of opetopes defined in Section 1.

Recall that, by the equivalences proved in [4] and [5], we have equivalent categories of opetopes, multitopes and Leinster opetopes. So we may define equivalent categories of opetopic sets by taking presheaves on any of these three categories. In the following definitions, although the opetopes we consider are the 'symmetric multicategory' kind, the concrete description of an opetopic set is not *precisely* as a presheaf on the category of these opetopes. The sets given in the data are indexed not by opetopes themselves but by *isomorphism classes* of opetopes; so at first sight this resembles a presheaf on the category of Leinster opetopes. However, we do not pursue this matter here, since the equivalences proved in our earlier work are sufficient for the purposes of this article.

We adopt this presentation in order to avoid naming the same cells repeatedly according to the symmetries; that is, we do not keep copies of cells that are isomorphic by the symmetries.

In [3], weak $n$-categories are defined as opetopic sets satisfying certain universality conditions. However, opetopic sets are defined using only symmetric multicategories with a *set* of objects; in the light of the results of our earlier work, we seek a definition using symmetric multicategories with a *category* of objects. The definitions we give here are those given in [3] but with modifications as demanded by the results of our previous work.

The underlying data for an opetopic $n$-category are given by an opetopic set. Recall that, in [3], given a symmetric multicategory $Q$ a $Q$-opetopic set $X$ is given by, for each $k \geq 0$, a symmetric multicategory $Q(k)$ and a set $X(k)$ over $o(Q(k))$, where

$$\begin{aligned} Q(0) &= Q \\ \text{and} \quad Q(k+1) &= Q(0)_{X(0)}{}^{+}. \end{aligned}$$

An opetopic set is then an $I$-opetopic set, where $I$ is the symmetric multicategory with one object and one (identity) arrow.

The idea is that the category of opetopic sets should be equivalent to the presheaf category

$$[\mathbf{Opetope}^{\mathrm{op}}, \mathbf{Set}]$$

and we use this to motivate our generalisation of the Baez-Dolan definitions.

Recall that we have for each $k \geq 0$ a category $\mathcal{O}_k$ of $k$-opetopes, and each $\mathcal{O}_k$ is a full subcategory of **Opetope**. A functor

$$\mathbf{Opetope}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

may be considered as assigning to each opetope a set of 'labels'.

Recall that for each $k$, $\mathcal{O}_k$ is equivalent to a discrete category. So it is sufficient to specify 'labels' for each isomorphism class of opetopes.

Recall [4] that we call a symmetric multicategory $Q$ *tidy* if it is freely symmetric with a category of objects $\mathbb{C}$ equivalent to a discrete category. Throughout this section we say '$Q$ has object-category $\mathbb{C}$ equivalent to $S$ discrete' to mean that $S$ is the set of isomorphism classes of $\mathbb{C}$, so $\mathbb{C}$ is equipped with a morphism $\mathbb{C} \xrightarrow{\sim} S$. We begin by defining the construction used for 'labelling' as discussed above. The idea is to give a set of labels as a set over the isomorphism classes of objects of $Q$, and then to 'attach' the labels using the following pullback construction.

2.1. Pullback multicategory. Let $Q$ be a tidy symmetric multicategory with category of objects $\mathbb{C}$ equivalent to $S$ discrete. Given a set $X$ over $S$, that is, equipped with a function $f : X \longrightarrow S$, we define the *pullback multicategory* $Q_X$ as follows.

- Objects: $o(Q_X)$ is given by the pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "\sim"] \\ X \arrow[r, "f"'] & S \end{tikzcd} .$$

Observe that the morphism on the left is an equivalence, so $o(Q_X)$ is equivalent to $X$ discrete. Write $h$ for this morphism.

- Arrows: given objects $a_1, \dots a_k, a \in o(Q_X)$ we have
$$Q_X(a_1, \dots, a_k; a) \cong Q(fh(a_1), \dots, fh(a_k); fh(a)).$$
- Composition, identities and symmetric action are then inherited from $Q$.

We observe immediately that since $Q$ is tidy, $Q_X$ is tidy. Also note that if $Q$ is object-discrete this definition corresponds to the definition of pullback symmetric multicategory given in [3].

We are now ready to describe the construction of opetopic sets.

2.2. $Q$-OPETOPIC SETS. Let $Q$ be a tidy symmetric multicategory with object-category $\mathbb{C}$ equivalent to $S$ discrete. A *$Q$-opetopic set* $X$ is defined recursively as a set $X(0)$ over $S$ together with a ${Q_{X(0)}}^+$-opetopic set $X_1$.

So a $Q$-opetopic set consists of, for each $k \geq 0$:

- a tidy symmetric multicategory $Q(k)$ with object-category $\mathbb{C}(k)$ equivalent to $S(k)$ discrete
- a set $X(k)$ and function $X(k) \xrightarrow{f_k} S(k)$

where

$$\begin{array}{rrcl} & Q(0) & = & Q \\ \text{and} & Q(k+1) & = & {Q(k)_{X(k)}}^+. \end{array}$$

We refer to $X_1$ as the underlying ${Q(0)_{X(0)}}^+$-opetopic set of $X$.

2.3. MORPHISMS OF $Q$-OPETOPIC SETS. We now define morphisms of opetopic sets. Suppose we have opetopic sets $X$ and $X'$ with notation as above, together with a morphism of symmetric multicategories
$$F : Q \longrightarrow Q'$$
and a function
$$F_0 : X(0) \longrightarrow X'(0)$$
such that the following diagram commutes

$$\begin{tikzcd} X(0) \arrow[r, "f_0"] \arrow[d, "F_0"'] & S(0) \arrow[d, "F"] \\ X'(0) \arrow[r, "f'_0"'] & S'(0) \end{tikzcd}$$

where the morphism on the right is given by the action of $F$ on objects. This induces a morphism

$$Q_{X(0)} \longrightarrow Q'_{X'(0)}$$

and so a morphism

$$Q_{X(0)}{}^{+} \longrightarrow Q'_{X'(0)}{}^{+}.$$

We make the following definition.

A *morphism of $Q$-opetopic sets*

$$F : X \longrightarrow X'$$

is given by:

- an underlying morphism of symmetric multicategories and function $F_0$ as above
- a morphism $X_1 \longrightarrow X'_1$ of their underlying opetopic sets, whose underlying morphism is induced as above.

So $F$ consists of

- a morphism $Q \longrightarrow Q'$
- for each $k \geq 0$ a function $F_k : X(k) \longrightarrow X'(k)$ such that the following diagram commutes

$$\begin{tikzcd}
X(k) \arrow[r, "f_k"] \arrow[d, "F_k"'] & S(k) \arrow[d] \\
X'(k) \arrow[r, "f'_k"'] & S'(k)
\end{tikzcd}$$

where the map on the right hand side is induced as appropriate.

Note that the above notation for a $Q$-opetopic set $X$ and morphism $F$ will be used throughout this section, unless otherwise specified.

2.4. Definition. *An* opetopic set *is an $I$-opetopic set. A* morphism of opetopic sets *is a morphism of $I$-opetopic sets. We write* **OSet** *for the category of opetopic sets and their morphisms.*

2.5. Intuitions. In this section we briefly discuss some of the intuition behind the idea of an opetopic set.

Eventually, a weak $n$-category is defined as an opetopic set with certain properties. The idea is that $k$-cells have underlying shapes given by the objects of $I^{k+}$. These are 'unlabelled' cells. To make these into fully labelled $k$-cells, we first give labels to the 0-cells, via the function $X(0) \longrightarrow S(0)$, and then to 1-cells via $X(1) \longrightarrow S(1)$, and so on. This idea may be captured in the following 'schematic' diagram.

$$
\begin{tikzcd}
 & & Q(0)_{X(0)} \arrow[r] \arrow[d, "+"] & Q(0) = I \arrow[d, "+"] \\
 & Q(1)_{X(1)} \arrow[r] \arrow[d, "+"] & Q(1) = {Q(0)_{X(0)}}^{+} & I^{+} \arrow[d, "+"] \\
Q(2)_{X(2)} \arrow[r] \arrow[d, "+"] & Q(2) = {Q(1)_{X(1)}}^{+} & & I^{2+} \arrow[d, "+"] \\
{Q(2)_{X(2)}}^{+} & & & I^{3+} \arrow[d, "+"] \\
 & & \vdots & {}
\end{tikzcd}
$$

Bearing in mind our modified definitions, we use the Baez-Dolan terminology as follows.

- A *$k$-dimensional cell* (or *$k$-cell*) is an element of $X(k)$ (i.e. an isomorphism class of objects of $Q(k)_{X(k)}$ ).
- A *$k$-frame* is an isomorphism class of objects of $Q(k)$ (i.e. an isomorphism class of arrows of $Q(k-1)_{X(k-1)}$ ).
- A *$k$-opening* is an isomorphism class of arrows of $Q(k-1)$, for $k \geq 1$.

So a $k$-opening may acquire $(k-1)$-cell labels and become a $k$-frame, which may itself acquire a label and become a $k$-cell. We refer to such a cell and frame as being *in* the original $k$-opening.

On objects, the above schematic diagram becomes:

$$
\begin{tikzcd}
 & & & \text{0-cells} \arrow[r] \arrow[d, "+"] & \text{0-opetopes} \arrow[d, "+"] \\
 & & \text{1-cells} \arrow[r] \arrow[d, "+"] & \text{1-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{1-opetopes} \arrow[d, "+"] \\
 & \text{2-cells} \arrow[r] \arrow[d, "+"] & \text{2-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{2-openings} \arrow[r, dotted] \arrow[d, dotted] & \text{2-opetopes} \arrow[d, "+"] \\
{} \arrow[r] & \text{3-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{3-openings} \arrow[r, dotted] \arrow[d, dotted] & \cdot \arrow[r, dotted] \arrow[d, dotted] & \text{3-opetopes} \arrow[d, "+"] \\
 & {} & {} & {} & {}
\end{tikzcd}
$$

$$
\xrightarrow[\textit{labels for 2-cells}]{} \triangleright \qquad \xrightarrow[\textit{labels for 1-cells}]{} \triangleright \qquad \xrightarrow[\textit{labels for 0-cells}]{} \triangleright
$$

Horizontal arrows represent the process of labelling, as shown; vertical arrows represent the process of 'moving up' dimensions. Starting with a $k$-opetope, we have from right to left the progressive labelling of 0-cells, 1-cells, and so on, to form a $k$-cell at the far left, the final stages being:

$$
\begin{tikzcd}
k\text{-opening} \arrow[d, Rightarrow, "\text{labels for constituent } (k-1)\text{-cells}"] \\
k\text{-frame} \arrow[d, Rightarrow, "\text{label for } k\text{-cell itself}"] \\
k\text{-cell}
\end{tikzcd}
$$

A $k$-opening acquires labels as an arrow of $Q(k-1)$, becoming a $k$-frame as an arrow of $Q(k-1)_{X(k-1)}$. That is, it has $(k-1)$-cells as its source and a $(k-1)$-cell as its target.

2.6. **OSet** IS A PRESHEAF CATEGORY. We now prove the main result of this work, that the category of opetopic sets is a presheaf category, and moreover, that it is equivalent to the presheaf category

$$[\mathbf{Opetope}^{\mathrm{op}}, \mathbf{Set}] = [\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

To prove this we use [12], Theorem 5.26, in the case $\mathcal{V} = \mathbf{Set}$. This theorem is as follows.

2.7. THEOREM. *Let $\mathcal{C}$ be a $\mathcal{V}$-category. In order that $\mathcal{C}$ be equivalent to $[\mathcal{E}^{op}, \mathcal{V}]$ for some small category $\mathcal{E}$ it is necessary and sufficient that $\mathcal{C}$ be cocomplete, and that there be a set of small-projective objects in $\mathcal{C}$ constituting a strong generator for $\mathcal{C}$.*

We see from the proof of this theorem that if $E$ is such a set and $\mathcal{E}$ is the full subcategory of $\mathcal{C}$ whose objects are the elements of $E$, then

$$\mathcal{C} \simeq [\mathcal{E}^{\mathrm{op}}, \mathcal{V}].$$

We prove the following propositions; the idea is to "realise" each isomorphism class of opetopes as an opetopic set; the set of these opetopic sets constitutes a strong generator as required.

2.8. PROPOSITION. **OSet** *is cocomplete.*

2.9. PROPOSITION. *There is a full and faithful functor*

$$G : \mathcal{O} \longrightarrow \mathbf{OSet}.$$

2.10. PROPOSITION. *Let $\alpha \in \mathcal{O}$. Then $G(\alpha)$ is small-projective in* **OSet**.

2.11. PROPOSITION. *Let*

$$E = \coprod \{G(\alpha) \mid \alpha \in \mathcal{O}\} \subseteq \mathbf{OSet}.$$

*Then $E$ is a strongly generating set for* **OSet**.

2.12. COROLLARY. **OSet** *is a presheaf category.*

2.13. COROLLARY.

$$\mathbf{OSet} \simeq [\mathcal{O}^{op}, \mathbf{Set}].$$

Proof of Proposition 2.8. Consider a diagram

$$D : \mathbb{I} \longrightarrow \mathbf{OSet}$$

where $\mathbb{I}$ is a small category. We seek to construct a limit $Z$ for $D$; the set of cells of $Z$ of shape $\alpha$ is given by a colimit of the sets of cells of shape $\alpha$ in each $D(I)$.

We construct an opetopic set $Z$ as follows. For each $k \geq 0$, $Z(k)$ is a colimit in **Set**:

$$Z(k) = \int^{I \in \mathbb{I}} D(I)(k).$$

Now for each $k$ we need to give a function

$$F(k) : Z(k) \longrightarrow o(Q(k))$$

where

$$Q(k) = Q(k-1)_{Z(k-1)}{}^{+}$$
$$Q(0) = I.$$

That is, for each $\alpha \in Z(k)$ we need to give its frame. Now

$$Z(k) = \left. \coprod_{I \in \mathbb{I}} D(I)(k) \middle/ \sim \right.$$

where $\sim$ is the equivalence relation generated by

$$D(u)(\alpha_{I'}) \sim \alpha_I \quad \text{for all } u : I \longrightarrow I' \in \mathbb{I} \text{ and } \alpha_I \in D(I)(k).$$

So $\alpha \in Z(k)$ is of the form $[\alpha_I]$ for some $\alpha_I \in D(I)(k)$ where $[\alpha_I]$ denotes the equivalence class of $\alpha_I$ with respect to $\sim$.

Now suppose the frame of $\alpha_I$ in $D(I)$ is

$$(\beta_1, \ldots, \beta_j) \xrightarrow{?} \beta$$

where $\beta_i, \beta \in D(I)(k-1)$ label some $k$-opetope $x$. We set the frame of $[\alpha_I]$ to be

$$(\ [\beta_1], \ldots, [\beta_j]\ ) \xrightarrow{?} [\beta]$$

labelling the same opetope $x$. This is well-defined since a morphism of opetopic sets preserves frames of cells, so the frame of $D(u)(\alpha_I)$ is

$$(\ D(u)(\beta_1)\ , \ldots,\ D(u)(\beta_j)\ ) \xrightarrow{?} D(u)(\beta)$$

also labelling $k$-opetope $x$. It follows from the universal properties of the colimits in **Set** that $Z$ is a colimit for $D$, with coprojections induced from those in **Set**. Then, since **Set** is cocomplete, **OSet** is cocomplete. ∎

Proof of Proposition 2.9. Let $\alpha$ be a $k$-opetope. We express $\alpha$ as an opetopic set $G(\alpha) = \hat{\alpha}$ as follows, using the usual notation for an opetopic set. The idea is that the $m$-cells are given by the $m$-faces of $\alpha$.

For each $m \geq 0$ set

$$X(m) = \{\ [(x,f)] \ \mid \ \begin{array}{l} x \in \mathcal{O}_m \text{ and } x \xrightarrow{f} \alpha \in \mathcal{O} \\ \text{where } [\ ] \text{ denotes isomorphism class in } \mathcal{O}/\alpha \}. \end{array}$$

So in particular we have

$$X(k) = \{[(\alpha, 1)]\}$$

and for all $m > k$, $X(m) = \emptyset$. It remains to specify the frame of $[(x,f)]$. The frame is an object of

$$Q(m) = Q(m-1)_{X(m-1)}{}^{+}$$

so an arrow of

$$Q(m-2)_{X(m-2)}{}^{+}$$

labelled with elements of $X(m-1)$. Now such an arrow is a configuration for composing arrows of $Q(m-2)_{X(m-2)}$; for the frame as above, this is given by the opetope $x$ as a labelled tree. Then the $(m-1)$-cell labels are given as follows. Write

$$x : y_1, \ldots, y_j \longrightarrow y$$

say, and so we have for each $i$ a morphism

$$y_i \longrightarrow x$$

and a morphism

$$y \longrightarrow x \in \mathcal{O}.$$

Then the labels in $X(m-1)$ are given by

$$[y_i \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1)$$

and

$$[y \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1).$$

Now, given a morphism

$$h : \alpha \longrightarrow \beta \in \mathcal{O}$$

we define

$$\hat{h} : \hat{\alpha} \longrightarrow \hat{\beta} \in \mathbf{OSet}$$

by

$$[(x,f)] \mapsto [(x, h \circ f)]$$

which is well-defined since if $(x,f) \cong (x',f')$ then $(x,hf) \cong (x',hf')$ in $\mathcal{O}/\alpha$. This is clearly a morphism of opetopic sets.

Observe that any morphism $\hat{\alpha} \longrightarrow \hat{\beta}$ must be of this form since the faces of $\alpha$ must be preserved. Moreover, if $\hat{h} = \hat{g}$ then certainly $[(\alpha,h)] = [(\alpha,g)]$. But this gives $(\alpha,h) = (\alpha,g)$ since there is a unique morphism $\alpha \longrightarrow \alpha \in \mathcal{O}$ namely the identity. So $G$ is full and faithful as required. ∎

PROOF of Proposition 2.10. For any $\alpha \in \mathcal{O}_k$ we show that $\hat{\alpha}$ is small-projective, that is that the functor

$$\psi = \mathbf{OSet}(\hat{\alpha}, -) : \mathbf{OSet} \longrightarrow \mathbf{Set}$$

preserves small colimits. First observe that for any opetopic set $X$

$$\begin{array}{rcl} \psi(X) = \mathbf{OSet}(\hat{\alpha}, X) & \cong & \{k\text{-cells in } X \text{ whose underlying } k\text{-opetope is } \alpha\} \\ & \subseteq & X(k) \end{array}$$

and the action on a morphism $F : X \longrightarrow Y$ is given by

$$\begin{array}{rccc} \psi(F) = \mathbf{OSet}(\hat{\alpha}, F) : & \mathbf{OSet}(\hat{\alpha}, X) & \longrightarrow & \mathbf{OSet}(\hat{\alpha}, Y) \\ & x & \mapsto & F(x). \end{array}$$

So $\psi$ is the 'restriction' to the set of cells of shape $\alpha$. This clearly preserves colimits since the cells of shape $\alpha$ in the colimit are given by a colimit of the sets cells of shape $\alpha$ in the original diagram. ■

PROOF of Proposition 2.11. First note that

$$\hat{\alpha} = \hat{\beta} \iff \alpha \cong \beta \in \mathcal{O}$$

so

$$E \cong \coprod_k S_k$$

where for each $k$, $S_k$ is the set of $k$-dimensional Leinster opetopes. Since each $S_k$ is a set it follows that $E$ is a set.

We need to show that, given a morphism of opetopic sets $F : X \longrightarrow Y$, we have

$$\mathbf{OSet}(\hat{\alpha}, F) \text{ is an isomorphism for all } \hat{\alpha} \implies F \text{ is an isomorphism.}$$

Now, we have seen above that

$$\mathbf{OSet}(\hat{\alpha}, X) \cong \{\text{cells of } X \text{ of shape } \alpha\}$$

so

$$\mathbf{OSet}(\hat{\alpha}, F) = F|_\alpha = F \text{ restricted to cells of shape } \alpha.$$

So

$$\begin{array}{l} \mathbf{OSet}(\hat{\alpha}, F) \text{ is an isomorphism for all } \hat{\alpha} \\ \qquad \iff F|_\alpha \text{ is an isomorphism for all } \alpha \in \mathcal{O} \\ \qquad \iff F \text{ is an isomorphism.} \end{array}$$

■

PROOF of Corollary 2.12. Follows from Propositions 2.8, 2.9, 2.10, 2.11 and [12] Theorem 5.26. ■

Proof of Corollary 2.13. Let $\mathcal{E}$ be the full subcategory of **OSet** whose objects are those of $E$. Since $G$ is full and faithful, $\mathcal{E}$ is the image of $G$ and we have

$$\mathcal{O} \simeq \mathcal{E}$$

and hence

$$\mathbf{OSet} \simeq [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}] \simeq [\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

■

## References

[1] John Baez. An introduction to $n$-categories. In *7th Conference on Category Theory and Computer Science*, volume 1290 of *Springer Lecture Notes in Computer Science*, 1997. Also available via `http://math.ucr.edu/home/baez`.

[2] John Baez and James Dolan. Letter to R. Street, Nov. 30, 1995. Corrected version as of Dec. 3, 1995. Available via `http://math.ucr.edu/home/baez`.

[3] John Baez and James Dolan. Higher-dimensional algebra III: $n$-categories and the algebra of opetopes. *Adv. Math.*, 135(2):145–206, 1998. Also available via `http://math.ucr.edu/home/baez`.

[4] Eugenia Cheng. Weak $n$-categories: opetopic and multitopic foundations, October 2002. To appear in *Journal of Pure and Applied Algebra*; also available via E-print `math.CT/0304277`.

[5] Eugenia Cheng. Weak $n$-categories: comparing opetopic foundations, October 2002. To appear in *Journal of Pure and Applied Algebra*; also available via E-print `math.CT/0304279`.

[6] Eugenia Cheng. A relationship between trees and Kelly-Mac Lane graphs, October 2002. E-print `math.CT/0304287`.

[7] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories, 1997. Available via `http://triples.math.mcgill.ca`.

[8] Claudio Hermida, Michael Makkai, and John Power. Higher dimensional multigraphs. *Thirteenth Annual IEEE Symposium on Logic in Computer Science*, pages 199–206, 1998. IEEE Computer Soc., Los Alamitos, CA.

[9] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories I: Part 1. *Journal of Pure and Applied Algebra*, 154:221–246, 2000.

[10] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories I – 2. *Journal of Pure and Applied Algebra*, 157:247–277, 2001.

[11] R. C. De Vrijer J. W. Klop. *Term Rewriting Systems.* Number 25 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1999.

[12] G. M. Kelly. *Basic Concepts of Enriched Category Theory.* Number 64 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1982.

[13] Tom Leinster. *Operads in Higher Dimensional Category Theory.* PhD thesis, University of Cambridge, July 2000. E-print `math.CT/0011106`.

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