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COMMUTATOR THEORY IN STRONGLY PROTOMODULAR CATEGORIES

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday.

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Abstract. We show that strongly protomodular categories (as the category Gp of groups for instance) provide an appropriate framework in which the commutator of two equivalence relations do coincide with the commutator of their associated normal subobjects, whereas it is not the case in any semi-abelian category.

1. Introduction

The notion of commutator of two congruences in Mal'cev varieties has been set up by J.D.H. Smith [24]. Then the concept of Mal'cev category was introduced by Carboni, Lambek and Pedicchio [13] (see also [14]), and the construction of the commutator of two equivalence relations in this more general setting was investigated by the last author ([22] and [23]), in a way mimicking the original construction of Smith.

We gave in [9] a new construction of different nature. The idea behind that came from two directions. The first one dealt with the notion of connector ([10] and [11]), which is equivalent to the notion of centralizing relation. The second one dealt with the notion of unital category where there is an intrinsic notion of commutativity and centrality [7], a setting in which, provided that moreover regularity and finite cocompleteness hold, there is a very natural categorical way to force a pair of morphisms with the same codomain to commute.

The relationship between these two directions is that a category $\mathbb{C}$ is Mal'cev if and only if the associated fibration of pointed objects $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ (see definition below) has its fibres unital [4]. And it was consequently possible to translate the unital means to force commutation from the unital setting to the Mal'cev one.

Since a pointed Mal'cev category is unital, there is consequently at the same time a way to assert when two maps having the same codomain commute (thanks to the notion of cooperator) and a way to assert when two equivalence relations on a same object commute (thanks to the notion of connector).

On the other hand, with any equivalence relation is associated a normal subobject, and any connector between two equivalence relations induces necessarily a cooperator between

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their associated normal subobjects. So that, provided that the pointed Mal'cev category is moreover regular and finitely cocomplete, there is a natural comparison between the commutator of these associated normal subobjects and the commutator of the original equivalence relations.

In this article:
1) we give an example of a pointed Mal'cev category (actually a semi-abelian one [17]) where these two commutators do not coincide.
2) we show that in any pointed regular and finitely cocomplete strongly protomodular category these two constructions do coincide.
3) we introduce an intrinsic definition of the center of an object in any unital category.
We refer to [9] and [3] for the main examples and references.

2. Unital categories

We shall suppose that $\mathbb{C}$ is a pointed category, i.e. a finitely complete category with a zero object. We shall denote by $\alpha_X : 1 \to X$ and $\tau_X : X \to 1$ the initial and terminal maps. The zero map $0 : X \to Y$ is then $\alpha_Y.\tau_X$.

2.1. Definition. *A* punctual span *in the pointed category $\mathcal{C}$ is a diagram of the form*

$$\begin{tikzcd} X \arrow[r, shift left, "s"] & Z \arrow[l, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, shift right, "t"'] \end{tikzcd}$$

with $f.s = 1_X, g.t = 1_Y, g.s = 0, f.t = 0$. A punctual relation is a punctual span such that the pair of maps (f, g) is jointly monic.

2.2. Example. For any pair (X, Y) of objects in $\mathbb{C}$, there is a canonical punctual relation which is called the coarse relation:

$$\begin{tikzcd} X \arrow[r, shift left, "l_X"] & X \times Y \arrow[l, shift left, "p_X"] \arrow[r, shift right, "p_Y"'] & Y \arrow[l, shift right, "r_Y"'] \end{tikzcd}$$

where $l_X = (1_X, 0)$ and $r_Y = (0, 1_Y)$.

2.3. Definition. *A pointed category $\mathbb{C}$ is called* unital*, see [7], when for each pair (X, Y) of objects in $\mathbb{C}$, the pair of maps (l_X, r_Y) is jointly strongly epic.*

In any unital category, there are no other punctual relations but the coarse ones.

2.4. Example. A variety $\mathcal{V}$ is unital if and only if it is Jonsson-Tarski, see [3]. This means that the theory of $\mathcal{V}$ contains a unique constant 0 and a binary term + satisfying $x + 0 = x = 0 + x$. In particular, the categories Mag, Mon, CoM, Gp, Ab, Rng of respectively unitary magmas, monoids, commutative monoids, groups, abelian groups, (non unitary) rings are unital.

The categories Mag($\mathbb{E}$), Mon($\mathbb{E}$), CoM($\mathbb{E}$), Gp($\mathbb{E}$), Ab($\mathbb{E}$), Rng($\mathbb{E}$) of respectively internal unitary magmas, monoids, commutative monoids, groups, abelian groups, rings in $\mathbb{E}$ are also unital, provided that $\mathbb{E}$ is finitely complete. In particular the categories Mon(Top) and Gp(Top) of topological monoids and topological groups are unital.

We have a non syntactic example with the dual Set_*^{op} of the category of pointed sets, and more generally with the dual of the category of pointed objects in any topos $\mathbb{E}$.

One of the main consequences of unitality is the fact that there is an intrinsic notion of commutativity and centrality. Indeed, given a unital category $\mathbb{C}$, the pair (l_X, r_Y), since it is jointly strongly epic, is actually jointly epic. Therefore a map $\varphi : X \times Y \to Z$ is uniquely determined by the pair of maps (f, g), $f : X \to Z$ and $g : Y \to Z$, with $f = \varphi.l_X$ and $g = \varphi.r_Y$. Accordingly the existence of such a map φ becomes a property of the pair (f, g). Whence the following definitions, see [7] and also [16]:

2.5. Definition. *Given a pair (f, g) of coterminal (=with the same codomain) morphisms in a unital category $\mathbb{C}$, when such a map φ exists, we say that the maps f and g* cooperate *and that the map φ is the* cooperator *of the pair (f, g). A map $f : X \to Y$ is* central *when f and 1_Y cooperate. An object X is called* commutative *when the map $1_X : X \to X$ is central.*

We shall suppose now that the unital category $\mathbb{C}$ is moreover finitely cocomplete. In this context, we gave in [9] the construction, from any pair $f : X \to Z$, $g : Y \to Z$ of coterminal maps, of a map which universally makes them cooperate. Indeed consider the following diagram, where $Q[\![f, g]\!]$ is the colimit of the diagram made of the plain arrows:

$$\begin{tikzcd}
 & X \arrow[dl, "l_X"'] \arrow[d, dotted, "\bar{\phi}_X"] \arrow[dr, "f"] & \\
X \times Y \arrow[r, dotted, "\bar{\phi}"'] & Q[\![f, g]\!] & Z \arrow[l, dotted, "\bar{\psi}"] \\
 & Y \arrow[ul, "r_Y"] \arrow[u, dotted, "\bar{\phi}_Y"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Clearly the maps $\bar{\phi}_X$ and $\bar{\phi}_Y$ are completely determined by the pair $(\bar{\phi}, \bar{\psi})$, and clearly the map $\bar{\phi}$ is the cooperator of the pair $(\bar{\psi}.f, \bar{\psi}.g)$. On the other hand, the map $\bar{\psi}$ is a strong epimorphism which measures the lack of cooperation of the pair (f, g), and we have [9]:

2.6. Proposition. *Suppose $\mathbb{C}$ finitely cocomplete and unital. Then $\bar{\psi}$ is the universal morphism which, by composition, makes the pair (f, g) cooperate. The map $\bar{\psi}$ is an isomorphism if and only if the pair (f, g) cooperates.*

Since the map $\bar{\psi}$ is a strong epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\bar{\psi}]$. Accordingly it is meaningful to introduce the following definition:

2.7. Definition. *Given any pair (f, g) of coterminal maps in a finitely cocomplete unital category $\mathbb{C}$, their commutator $[\![f, g]\!]$ is the kernel relation $R[\bar{\psi}]$.*

When the category $\mathbb{C}$ is moreover regular [1], i.e. such that the strong epimorphisms are stable by pullback and any effective equivalence relation admits a quotient, we can add some piece of information. First, any map $f : X \to Z$ has a canonical regular epi/mono factorization : $X \twoheadrightarrow f(X) \rightarrowtail Z$, and the map $f(X) \rightarrowtail Z$ is then called the image of the morphism f. Secondly, two maps f and g cooperate if and only if their images $f(X) \rightarrowtail Z$ and $g(Y) \rightarrowtail Z$ cooperate. Thirdly, particularizing the previous construction, we have the following corollary [9]:

2.8. Corollary. *Suppose $\mathbb{C}$ unital, finitely cocomplete and regular. Let $f : X \to Z$ be a map. Consider the following coequalizer $\bar{\phi}$:*

$$\begin{tikzcd} X \arrow[rr, "l_X"] \arrow[dr, "f"'] & & X \times Z \arrow[r, "\bar{\phi}"'] & Q[\![f, 1_Z]\!] \\ & Z \arrow[ur, "r_Z"'] & & \end{tikzcd}$$

then $\bar{\psi} = \bar{\phi}.r_Z$ is the universal map which makes the map f central by composition (on the left).

Let Z be any object of $\mathbb{C}$. Then its associated commutative object $\gamma(Z)$ is nothing but the codomain $Q[\![1_Z, 1_Z]\!]$ of the following coequalizer $\bar{\phi}$:

$$\begin{tikzcd} Z \arrow[r, shift left, "l_Z"] \arrow[r, shift right, "r_Z"'] & Z \times Z \arrow[r, "\bar{\phi}"'] & Q[\![1_Z, 1_Z]\!] \end{tikzcd}$$

In other words, the full inclusion of the commutative objects $Com(\mathbb{C}) \hookrightarrow \mathbb{C}$ admits a left adjoint γ.

We can now introduce the following dual notion:

2.9. Definition. *Let $\mathbb{C}$ be a unital category and $f : X \to Y$ any map in $\mathbb{C}$. Then the* centralizer *of the map f is the universal morphism $z : Z[f] \to X$ which makes the map f central by composition (on the right). The* center *of X is just $Z[1_X]$.*

Of course the centralizer z is always a monomorphism. The centralizers always exist in the categories Mag, Mon, Gp, Rng. In Mag for instance, we have $Z[f] = \{x \in X/ \ \forall y \in Y \ f(x).y = y.f(x); \ \forall (y, y') \in Y^2 \ f(x).(y.y') = (f(x).y).y', \ (y.y').f(x) = y.(y'.f(x))\}$. In Mon and Gp, these definitions coincide with the usual ones. In the category CoRng of (non unitary) commutative rings, the *center* is the annihilator. When $\mathbb{E}$ is a topos, a topos of sheaves for instance, the centralizers in Mon($\mathbb{E}$) and Gp($\mathbb{E}$) are obtained in the following way : first consider the subobject $i : C[f] \rightarrowtail X \times Y$, where $C[f] = \{(x, y)/f(x).y = y.f(x)\}$, then take z as $\Pi_{p_X}(i)$, where $p_X : X \times Y \to X$ is the canonical projection and Π_{p_X} denotes the right adjoint to the inverse image functor along p_X, see [19] for instance.

3. Mal'cev categories

Let us recall that a category $\mathbb{C}$ is Mal'cev when it is finitely complete and such that every reflexive relation is an equivalence relation, see [13] and [14]. Following the Mal'cev theorem [21], a variety $\mathcal{V}$ is Mal'cev if and only if its theory contains a ternary term p, satisfying : $p(x,y,y) = x = p(y,y,x)$ (called a Mal'cev operation). All the previous examples, except Mag, Mon, CoM, Mag($\mathbb{E}$), Mon($\mathbb{E}$) and CoM($\mathbb{E}$), are Mal'cev categories.

There is a strong connection with unital categories which is given by the following observation. Let $\mathbb{C}$ be a finitely complete category. We denote by Pt$\mathbb{C}$ the category whose objects are the split epimorphisms in $\mathbb{C}$ with a given splitting and morphisms the commutative squares between these data. We denote by $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ the functor associating its codomain with any split epimorphism. Since the category $\mathbb{C}$ has pullbacks, the functor π is a fibration which is called the *fibration of pointed objects*. The fibre above X is denoted $\mathrm{Pt}_X(\mathbb{C})$.

A finitely complete category $\mathbb{C}$ is Mal'cev if and only if the fibres of the fibration π are unital, see [4].

Now consider $(d_0, d_1) : R \rightrightarrows X$ an equivalence relation on the object X in $\mathbb{C}$. We shall denote by $s_0 : X \to R$ the inclusion arising from the reflexivity of the relation, and we shall write ΔX and ∇X respectively for the smallest $(1_X, 1_X) : X \rightrightarrows X$ and the largest $(p_0, p_1) : X \times X \rightrightarrows X$ equivalence relations on X.

3.1. Remark. Since the category $\mathbb{C}$ is Mal'cev, to give an equivalence relation $(d_0, d_1) : R \rightrightarrows X$ on the object X in $\mathbb{C}$ is equivalent to give, in the fibre $\mathrm{Pt}_X(\mathbb{C})$ above X, an inclusion of the object $((d_0, s_0) : R \rightleftarrows X)$ into the object $((p_0, s_0) : X \times X \rightleftarrows X)$:

$$\begin{tikzcd}
R \arrow[r, "{(d_0,d_1)}"] \arrow[dr, shift left, "d_0"] & X \times X \arrow[d, shift left, "p_0"] \\
& X \arrow[ul, shift left, "s_0"] \arrow[u, shift left, "s_0"]
\end{tikzcd}$$

So, by abuse of notation, we shall often identify the equivalence relation R with the subobject $((d_0, s_0) : R \rightleftarrows X)$ in the fibre $\mathrm{Pt}_X(\mathbb{C})$, and conversely.

Now consider $(d_0, d_1) : R \rightrightarrows X$ and $(d_0, d_1) : S \rightrightarrows X$ two equivalence relations on the same object X in $\mathbb{C}$. Then take the following pullback:

$$\begin{tikzcd}
R \times_X S \arrow[r, shift left, "p_S"] \arrow[d, shift right, "p_R"'] & S \arrow[l, shift left, "r_S"] \arrow[d, shift right, "d_{0,S}"'] \\
R \arrow[r, shift left, "d_{1,R}"] \arrow[u, shift right, "l_R"'] & X \arrow[l, shift left, "s_{0,R}"] \arrow[u, shift right, "s_{0,S}"']
\end{tikzcd}$$

where l_R and r_S are the sections induced by the maps $s_{0,R}$ and $s_{0,S}$.

Let us recall the following definition, see [10] (see also [20], [18], [14], [22]):

3.2. Definition. *In a Mal'cev category $\mathbb{C}$, a connector on the pair (R, S) is a morphism*

$$p : R \times_X S \to X, \ (xRySz) \mapsto p(x, y, z)$$

which satisfies the identities : $p(x, y, y) = x$ and $p(y, y, z) = z$.

This notion actually makes sense in any finitely complete category provided that some further conditions are satisfied [10], which are always fulfilled in a Mal'cev category. Moreover, in a Mal'cev category, a connector is necessarily unique when it exists, and thus the existence of a connector becomes a property. We say then that R and S are *connected*.

3.3. Example. By Proposition 3.6, Proposition 2.12 and Definition 3.1 in [22], two relations R and S in a Mal'cev variety $\mathcal{V}$ are connected if and only if $[R, S] = 0$ in the sense of Smith [24], see also [15]. Accordingly we shall denote a connected pair of equivalence relations by the formula $[R, S] = 0$.

The fibre $\mathrm{Pt}_X(\mathbb{C})$ being unital, it is natural to ask when, given two equivalence relations R and S on X, the associated subobjects R and S of $((p_0, s_0) : X \times X \rightleftarrows X)$ (in the fibre $\mathrm{Pt}_X(\mathbb{C})$) cooperate in this fibre. Let us observe that:

3.4. Proposition. *Let $\mathbb{C}$ be a Mal'cev category, the subobjects R and S cooperate in the fibre $\mathrm{Pt}_X(\mathbb{C})$ if and only if the equivalence relations R and S are connected in $\mathbb{C}$.*

Proof. Let us consider the product of R and S in $\mathrm{Pt}_X(\mathbb{C})$. It is given by the following pullback in $\mathbb{C}$:

$$\begin{tikzcd} R \times_0 S \arrow[r] \arrow[d] & S \arrow[d, "d_0"] \\ R \arrow[r, "d_0"'] & X \end{tikzcd}$$

A cooperator between R and S in $\mathrm{Pt}_X(\mathbb{C})$ is thus a map $\phi : R \times_0 S \to X \times X$ such that $\phi(x, y, x) = (x, y)$ and $\phi(x, x, z) = (x, z)$. But ϕ is a morphism in the fibre and is necessarily of the form $\phi(x, y, z) = (x, q(x, y, z))$. Accordingly a cooperator between R and S is just given by a map $q : R \times_0 S \to X$ such that $q(x, y, x) = y$ and $q(x, x, z) = z$. Consequently to set $p(u, v, w) = q(v, u, w)$ is to define a bijection between the cooperators and the connectors. ∎

From this observation, and the universal construction of the first section, we derived a new construction of the commutator of two equivalence relations [9]. We shall suppose from now on, in this section, that the category $\mathbb{C}$ is finitely cocomplete, Mal'cev and regular. In a regular Mal'cev category, given a regular epi $f : X \to Y$, any equivalence relation R on X has a *direct image* $f(R)$ along f on Y. It is given by the regular epi/mono factorization of the map $(f.d_0, f.d_1) : R \twoheadrightarrow f(R) \rightarrowtail Y \times Y$. Clearly in any regular category $\mathbb{C}$, the relation $f(R)$ is reflexive and symmetric; when moreover $\mathbb{C}$ is Mal'cev, $f(R)$ is an equivalence relation.

Now let us consider the following diagram where $Q[R,S]$ is the colimit of the plain arrows:

$$\begin{tikzcd} & R \arrow[dl, "l_R"'] \arrow[d, dotted, "\phi_R"'] \arrow[dr, "d_{0,R}"] & \\ R \times_X S \arrow[r, dotted, "\phi"'] & Q[R,S] & X \arrow[l, dotted, "\psi"] \\ & S \arrow[ul, "r_S"] \arrow[u, dotted, "\phi_S"] \arrow[ur, "d_{1,S}"'] & \end{tikzcd}$$

Notice that, here, in consideration of the pullback defining $R \times_X S$, the roles of the projections d_0 and d_1 have been interchanged. As in the section above, the maps ϕ_R and ϕ_S are completely determined by the pair (ϕ, ψ). This map ψ measures the lack of connection between R and S, see [9]:

3.5. Theorem. *Let the category $\mathbb{C}$ be finitely cocomplete, regular and Mal'cev. Then the map ψ is the universal regular epimorphism which makes the direct images $\psi(R)$ and $\psi(S)$ connected. The equivalence relations R and S are connected (i.e. $[R,S]=0$) if and only if ψ is an isomorphism.*

Since the map ψ is a regular epi, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\psi]$. Accordingly, it is meaningful to introduce the following definition:

3.6. Definition. *Let the category $\mathbb{C}$ be finitely cocomplete, regular and Mal'cev. Let two equivalence relations $(d_0, d_1) : R \rightrightarrows X$ and $(d_0, d_1) : S \rightrightarrows X$ be given on the same object X in $\mathbb{C}$. The kernel relation $R[\psi]$ of the map ψ is called the commutator of R and S. It will be classically denoted by $[R,S]$.*

3.7. Example. If we suppose moreover the category $\mathbb{C}$ exact [1], namely such that any equivalence relation is effective, i.e. the kernel relation of some map, then, thanks to Theorem 3.9 in [22], the previous definition is equivalent to the definition of [22], and accordingly to the definition of Smith [24] in the Mal'cev varietal context. On the other hand, one of the advantages of this definition is that it extends the meaning of commutator from the exact Mal'cev context to the regular Mal'cev one, enlarging the range of examples to the Mal'cev quasi-varieties, or to the case of the topological groups for instance.

4. Normal subobjects

Let us recall that a map $j : I \to X$ in a finitely complete category $\mathbb{C}$ is *normal to the equivalence relation* R on X when : (1) $j^{-1}(R)$ is the relation ∇I on I, and : (2) the induced map $I \to R$ in the category $Rel\mathbb{C}$ of internal equivalence relations in $\mathbb{C}$ is a discrete fibration, see [5]. This means that (1) there is a (certainly unique) map $\tilde{j} : I \times I \to R$ in $\mathbb{C}$ such that the following diagram commutes, and (2) any of the commutative squares is a pullback:

$$\begin{tikzcd} I \times I \arrow[r, "\tilde{j}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ I \arrow[r, "j"'] & X \end{tikzcd}$$

This implies that the map j is necessarily a monomorphism. This definition gives an intrinsic way to express that the object I is an equivalence class of R. Clearly left exact functors preserve this kind of monomorphism. It is the case, in particular, for the Yoneda embedding. Of course a map j could be normal to different equivalence relations, even in a Mal'cev category. When the category $\mathbb{C}$ is moreover pointed, then with any equivalence relation R is associated a normal subobject $j(R)$ given by the following left hand side pullback where s_1 is, according to the simplicial notations, the map $(\alpha_X \times X)$:

$$\begin{tikzcd} I(R) \arrow[r, "j(R)"] \arrow[d] & X \arrow[r, "\tau_X"] \arrow[d, "s_1"'] & 1 \arrow[d, "\alpha_X"] \\ R \arrow[r, "(d_0,d_1)"'] & X \times X \arrow[r, "p_0"'] & X \end{tikzcd}$$

Since the right hand side diagram is also a pullback, the normal subobject $j(R)$ is just the image by the change of base functor $\alpha_X^* : \mathrm{Pt}_X(\mathbb{C}) \to \mathrm{Pt}_1(\mathbb{C}) = \mathbb{C}$ of the subobject $(d_0, d_1) : R \to \nabla X$ in the fibre $\mathrm{Pt}_X(\mathbb{C})$, see Remark 2.1. We shall need also the following equivalent construction: take the following pullback, and set $j(R) = d_1.a(R)$:

$$\begin{tikzcd} I(R) \arrow[r, "a(R)"] \arrow[d] & R \arrow[r, "d_1"] \arrow[d, "d_0"] & X \\ 1 \arrow[r, "\alpha_X"'] & X & \end{tikzcd}$$

Let us bring in our basic observation, see also [10]:

4.1. Proposition. *Given a pointed Mal'cev category $\mathbb{C}$, when two equivalence relations (R, S) on the same object X are connected, then their associated normal subobjects $(j(R), j(S))$ cooperate.*

Proof. According to Proposition 2.1, the connector p determines a cooperator ϕ between R ans S in the fibre $\mathrm{Pt}_X(\mathbb{C})$. Then $\alpha_X^*(\phi)$ is a cooperator between $j(R)$ and $j(S)$. ∎

5. Commutators in pointed finitely cocomplete regular Mal'cev categories

Let us begin by the following observation:

5.1. Proposition. *Let $\mathbb{C}$ be a pointed regular Mal'cev category. Let $f : X \twoheadrightarrow Y$ be a regular epimorphism and R an equivalence relation on X. Then the normal subobject $j(f(R))$ is equal to $f(j(R))$.*

Proof. Consider the following diagram, where the commutative lower right hand part with vertex P is a pullback:

$$\begin{tikzcd}
I(R) \arrow[rr, "a(R)"] \arrow[dr, "\bar{f}"] \arrow[dd] & & R \arrow[rr, "\tilde{f}"] \arrow[dr, "\chi"] \arrow[dd, "d_0"'] & & f(R) \arrow[dd, "d_0"] \\
& I(f(R)) \arrow[rr, "\gamma"'] \arrow[dl] & & P \arrow[ur, "\check{f}"'] \arrow[dl, "\delta_0"'] & \\
1 \arrow[rr, "\alpha_X"'] & & X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

The two downward vertical arrows d_0 are split and the two horizontal ones f and $\tilde{f}$ are regular epis. Since the category $\mathbb{C}$ is regular and Mal'cev, the factorization $\chi : R \to P$ is a regular epi according to Proposition 3.2 in [8]. On the other hand, since $\alpha_Y = f.\alpha_X$, there is by definition of $I(f(R))$ a factorization $\gamma : I(f(R)) \to P$ which makes the upper left hand side part of the diagram a pullback. Consequently the map $\bar{f}$ is also a regular epi. Finally:

$$f.j(R) = f.d_1.a(R) = d_1.\tilde{f}.a(R) = d_1.\check{f}.\gamma.\bar{f} = d_1.a(f(R)).\bar{f} = j(f(R)).\bar{f}$$

and then $j(f(R))$ is equal to $f(j(R))$. ∎

Whence the following comparison between the two notions of commutator:

5.2. Corollary. *Let $\mathbb{C}$ be a pointed regular and finitely cocomplete Mal'cev category. Then, given any pair (R, S) of equivalence relations on an objet X, there is a natural comparison $\zeta : Q[\![j(R), j(S)]\!] \to Q[R, S]$, and consequently we have always $[\![j(R), j(S)]\!] \leq [R, S]$.*

Proof. Consider the map $\psi : X \to Q[R, S]$. Then $[\psi(R), \psi(S)] = 0$, and according to the Propositions 4.1 and 3.1 $[\![\psi(j(R)), \psi(j(S))]\!] = [\![j(\psi(R)), j(\psi(S))]\!] = 0$. Now thanks to the universal property of the map $\bar{\psi} : X \to Q[\![j(R), j(S)]\!]$, there is a unique factorization $\zeta : Q[\![j(R), j(S)]\!] \to Q[R, S]$, and thus an inclusion $[\![j(R), j(S)]\!] \leq [R, S]$ between the two kinds of commutator. ∎

6. Protomodular categories

In the protomodular context, a subobject is normal to at most one equivalence relation (of course, up to isomorphism), so that the normal subobjects characterize the equivalence relations [5]. Accordingly, in a pointed protomodular category, there is a bijection between the normal subobjects and the equivalence relations on a given object X. It is thus quite natural to ask whether a cooperator between two normal subobjects can be turned into a connector between their associated equivalence relations. This is classically the case in the category Gp of groups. We are going to show in this section that this is not the case in general.

Let us recall that a category $\mathbb{C}$ is protomodular [5] when it is finitely complete and such that the change of base functors with respect to the fibration of points $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ reflect the isomorphisms. Of course, the leading example of this kind of situation is the category Gp. On the other hand the protomodular varieties are characterized in [12]. In a pointed protomodular category, a morphism is a monomorphism if and only if its kernel is trivial; if the category is not pointed, then, equivalently, pullbacks reflect monomorphisms [5].

Let us now point out a very interesting example borrowed from G.Janelidze: let us denote by U : Gp $\to$ Set$*$ the forgetful functor towards the category of pointed sets, associating with each group its underlying set pointed by the unit element. Let us call the category of *digroups* the category defined by the following pullback:

$$\begin{tikzcd} \mathrm{DiGp} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathrm{Gp} \arrow[d, "U"] \\ \mathrm{Gp} \arrow[r, "U"'] & \mathrm{Set}* \end{tikzcd}$$

The functor p_0 is clearly left exact and conservative (i.e. it reflects the isomorphisms). Thus the category DiGp is protomodular. On the other hand, it is clearly pointed. What is important, here, is that the two group laws given on an object of DiGp are not coordinated by coherence axioms (except that they have the same unit element).

Now, let A be an abelian group, such that there is an element a with $a \neq -a$. Let us define $\theta : A \times A \to A \times A$ in the following way:

- if $x \neq a$, then $\theta(z, x) = (z, x)$
- if $x = a$ and $z \neq a$, $z \neq -a$, then $\theta(z, a) = (z, a)$
- while $\theta(a, a) = (-a, a)$ and $\theta(-a, a) = (a, a)$.

We have then: $\theta \neq Id$, $\theta^2 = Id$, and $p_1.\theta = p_1$. Now let $\#$ be the transform along θ of the ordinary product law on $A \times A$. Whence $(z, x)\#(z', x') = \theta(\theta(z, x) + \theta(z', x'))$. Now $(A \times A, +, \#)$ is a digroup and $p_1 : (A \times A, +, \#) \to (A, +, +)$ a digroup homomorphism which is split in DiGp by the homomorphism s, with $s(z) = (0, z)$. Thus (p_1, s) is an object in the fibre of PtDiGp above $(A,+,+)$. Moreover the kernel of p_1 is just the map $k : (A,+,+) \rightarrowtail (A \times A,+,\#)$ defined by $k(x) = (x,0)$, which is clearly the normal subobject associated with the kernel relation $R[p_1]$.

Since $(A,+,+)$ is a commutative object (see Definition 1.3) in DiGp, we have the following cooperator ϕ for the pair (k,k); so that we have $[\![k,k]\!] = 0$:

$$\phi : (A,+,+) \times (A,+,+) \to (A \times A,+,\#),\ (a,a') \mapsto (a+a',0)$$

We are now going to show that, however, $[R[p_1],R[p_1]] \neq 0$, i.e. that there is no connector

$$\pi : R[p_1] \times_{A\times A} R[p_1] = A \times A \times A \times A \to A \times A$$

between $R[p_1]$ and itself. Such a connector would be unique and would preserve the law $+$, so that its unique possible definition would be:

$$\pi(x,y,t,z) = (x-y+t,z)$$

Let us show that the function π does not preserve the law $\#$. For that, let us consider $t \in A$ such that $t \neq a$, $t \neq -a$. We have then:

$$(a,t,0,a)\#(-a,-t-2a,0,-a) = (-2a,-2a,0,0)$$

and $\pi(-2a,-2a,0,0) = (0,0)$. 0n the other hand $\pi(a,t,0,a) = (a-t,a)$ and $\pi(-a,-t-2a,0,-a) = (a+t,-a)$. But, when $t \neq 0$ and $t \neq 2a$, we have $(a-t,a)\#(a+t,-a) = (2a,0)$. Accordingly π does not preserve the law $\#$. Whence:

6.1. Proposition. *In the pointed protomodular category* DiGp*, a cooperator between two normal subobjects cannot in general be turned into a connector between their associated equivalence relations. Accordingly, for a pair (R,S) of equivalence relations on an object X, the commutator $[\![j(R),j(S)]\!]$ can be strictly smaller than $[R,S]$.*

It is worth emphasizing that DiGp is not only a pointed regular and finitely cocomplete protomodular category, but it is also Barr exact [1]; accordingly, it is semi-abelian in the sense of [17].

7. Strongly protomodular categories

There is a situation where a cooperator between two normal subobjects can be always turned into a connector between their associated equivalence relations. We are going to recall it now. Once the notion of normal subobject was introduced, it was quite natural to set the following definition [6]:

7.1. DEFINITION. *We shall call normal a left exact functor $F : \mathbb{C} \to \mathbb{C}'$ between finitely complete categories which reflects isomorphisms and normal monomorphisms: when j is such that $F(j)$ is normal to some equivalence relation S on $F(X)$, then there exists a (necessarily unique up to isomorphism) equivalence relation R on X such that j is normal to R and $F(R) = S$.*

Now a category $\mathbb{C}$ is strongly protomodular [6] when the change of base functors of the fibration $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ are not only conservative, but also normal. The category Gp of groups and the category Rng of (non unitary) rings are example of such categories. It is also the case for the categories Gp($\mathbb{E}$) and Rng($\mathbb{E}$) of respectively internal groups and rings in a finitely complete category $\mathbb{E}$. And in particular for the category Gp(Top) of topological groups.

We are then in position to recall the following result, see Theorem 6.1 in [10]:

7.2. THEOREM. *Let $\mathbb{C}$ be a pointed strongly protomodular category. Given two normal subobjects $j : I \to X$ and $j' : I' \to X$ associated with two equivalence relations R and R' on X. Then R and R' are connected if and only if j and j' cooperate.*

Whence the following major consequence:

7.3. THEOREM. *Let $\mathbb{C}$ be a pointed regular and finitely cocomplete strongly protomodular category. Then, given any pair (R, S) of equivalence relations on an object X, the natural comparison $\zeta : Q[\![j(R), j(S)]\!] \to Q[R, S]$ is an isomorphism, and consequently we have always $[\![j(R), j(S)]\!] \simeq [R, S]$.*

PROOF. Consider the map $\bar{\psi} : X \to Q[\![j(R), j(S)]\!]$ of the Corollary 4.1. Then $[\![\bar{\psi}(j(R)), \bar{\psi}(j(S)]\!] = 0$. Consequently, according to the Proposition 4.1, we have $[\![j(\bar{\psi}(R)), j(\bar{\psi}(S))]\!] = 0$. Now thanks to the previous theorem, we do have $[\bar{\psi}(R), \bar{\psi}(S)] = 0$. The universal property of the map $\psi : X \to Q[R, S]$ produces a unique factorization $\theta : Q[R, S] \to Q[\![j(R), j(S)]\!]$ which is necessarily an inverse of ζ (see Corollary 4.1), and thus also an isomorphism $[R, S] \simeq [\![j(R), j(S)]\!]$. Accordingly the two notions of commutator coincide. ∎

This is obviously a fortiori the case for the pointed Barr exact and finitely cocomplete strongly protomodular category. Considering [17], it seems then relevant to introduce the following definition:

7.4. DEFINITION. *A category $\mathbb{C}$ is said strongly semi-abelian when it is semi-abelian and strongly protomodular, what means pointed, Barr exact, finitely cocomplete and strongly protomodular.*

Of course, this is the case for the categories Gp and Rng, and more generally for any pointed strongly protomodular variety $\mathcal{V}$. If $\mathbb{C}$ is strongly semi-abelian, then any fiber of the fibration of points $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ is still strongly semi-abelian. When $\mathbb{E}$ is a topos with a natural number object, then the categories Gp($\mathbb{E}$) and Rng($\mathbb{E}$) are strongly semi-abelian.

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