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TOPOLOGICAL ∗-AUTONOMOUS CATEGORIES

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ABSTRACT. Given an additive equational category with a closed symmetric monoidal structure and a potential dualizing object, we find sufficient conditions that the category of topological objects over that category has a good notion of full subcategories of strong and weakly topologized objects and show that each is equivalent to the chu category of the original category with respect to the dualizing object.

1. Introduction

In [Mackey, 1945], Mackey introduced the category of pairs of vector spaces, equipped with a bilinear pairing into the ground field. It is likely that he viewed this abstract duality as a replacement for the topology. See also [Mackey, 1946], the review of the latter paper by Dieudonné as well as Dieudonné's review of [Arens, 1947], for a clear expression of this point of view.

In [Barr, 2000] I showed that the full subcategory of the category of (real or complex) topological vector spaces that consists of the Mackey spaces (defined in 3.3 below)is ∗-autonomous and equivalent to both the full subcategory of weakly topologized topological vector spaces and to the full subcategory of topological vector spaces topologized with the strong, or Mackey topology. This means, first, that those subcategories can, in principle at least, be studied without taking the topology onto consideration. Second it implies that both of those categories are ∗-autonomous.

André Joyal recently raised the question whether there was a similar result for vector spaces over the field $\mathbf{Q}_p$ of p-adic rationals. Thinking about this question, I realized that there is a useful general theorem that answers this question for any locally compact field and also for locally compact abelian groups. In the process, it emerges for the categories of vector spaces over a locally compact field that the structure of these categories, as categories, does not depend in any way on the topology of the ground field which we may as well suppose discrete.

1.1. TERMINOLOGY. We assume that all topological objects are Hausdorff. As we will see, each of the categories contains an object K with special properties. It will be convenient to call a morphism $T \longrightarrow K$ a **functional** on T. In the case of abelian groups, the word "character" would be more appropriate, but it is convenient to have one word. In a

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similar vein, we may refer to a mapping of topological abelian groups as "linear" to mean additive. We will be dealing with topological objects in categories of topological vector spaces and abelian groups. If T is such an object, we will denote by $|T|$ the underlying vector space or group.

If K is a topological field, we will say that a vector space is **linearly discrete** if it is a categorical sum of copies of the field.

2. An adjoint interlude

Before turning to the main results, there are some generalities on adjoints that we need. Suppose we have the situation

$$\mathcal{V}_w \underset{S}{\overset{I}{\rightleftarrows}} \mathcal{V} \underset{J}{\overset{T}{\rightleftarrows}} \mathcal{V}_m$$

in which $S \dashv I$ and $J \dashv T$. We also suppose that I and J are full inclusions so that $SI \cong \mathrm{Id}$ and $TJ \cong \mathrm{Id}$. Let $\sigma = IS$ and $\tau = JT$ so that σ and τ are idempotent endofunctors of $\mathcal{V}$. Till now there is no connection between the two full subcategories $\mathcal{V}_w$ and $\mathcal{V}_m$ of $\mathcal{V}$. The connection we make is that $\tau \dashv \sigma$, which is equivalent to a natural isomorphism

$$\mathrm{Hom}(T(V), T(V')) \cong \mathrm{Hom}(S(V), S(V'))$$

for all V, $V' \in \mathcal{V}$.

It is clear that $SJ \dashv TI$.

2.1. Proposition. $JTI \dashv S$.

Proof. We have that

$$\mathrm{Hom}(JTI(V), V') \cong \mathrm{Hom}(TI(V), T(V')) \cong \mathrm{Hom}(SI(V), S(V')) \cong \mathrm{Hom}(V, S(V')) \quad \blacksquare$$

Notice that since I, the right adjoint of S, is full and faithful, so is its left adjoint JTI.

Another simple observation is that if $\mathcal{A} \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{V}_w$ are inverse equivalences, then $JTIP \dashv QS \dashv IP$.

3. The strong and weak topologies

3.1. Theorem. *Suppose $\mathcal{A}$ is an additive equational closed symmetric monoidal category and $\mathcal{T}$ is the category of topological $\mathcal{A}$-algebras. Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$ that is closed under finite products and closed subobjects and $\mathcal{V}$ be the subobject and product closure of $\mathcal{S}$ in $\mathcal{T}$. Assume that $\mathcal{S}$ contains an object K for which*

1. *K is injective with respect to topological embeddings in $\mathcal{S}$;*

2. *K is a cogenerator in $\mathcal{S}$;*

3. there is a neighbourhood U of 0 in K such that

(a) U is compact;

(b) U contains no non-zero subobject;

(c) whenever $\varphi : T \longrightarrow K$ is such that $\varphi^{-1}(U)$ is open, then φ is continuous.

Then K is injective with respect to topological embeddings in $\mathcal{V}$.

Proof. If $V \subseteq V'$ and $V' \subseteq \prod_{i\in I} S_i$ with each $S_i \in \mathcal{S}$, it is sufficient to show that every functional on V extends to the product. Suppose $\varphi : V \longrightarrow K$ is given. Then there is a finite subset of I, that we will pretend is the set $\{1, \ldots, n\}$ and neighbourhoods $U_i \subseteq S_i$ of 0 for $i = 1, \ldots, n$ such that if $J = I - \{1, \ldots, n\}$, then

$$\varphi^{-1}(U) \supseteq \tau V \cap \left(U_1 \times \cdots \times U_n \times \prod_{i\in J} S_i\right)$$

It follows that

$$\varphi\left(0 \times \cdots \times 0 \times \prod_{i\in J} S_i\right) \subseteq U$$

and, since the left hand side is a subobject contained in U, it must be 0. Let

$$V_0 = \frac{V}{V \cap \left(0 \times \cdots \times 0 \times \prod_{i\in J} S_i\right)}$$

topologized as a subobject of $S_1 \times \cdots \times S_n$. Algebraically, φ induces a map ψ on V_0 and $\psi^{-1}(U) \supseteq V' \cap (U_1 \times \cdots \times U_n)$ which is open in the subspace topology and our hypothesis on U implies that ψ is continuous. Since U is compact, the set of all translates of U is a neighbourhood cover of K by compact sets, so that K is complete. Since continuous homomorphisms are uniformly continuous, it follows that ψ extends to the closure of V_0 which lies in $\mathcal{S}$ and, by injectivity, it extends to $S_1 \times \cdots \times S_n$ and therefore φ extends to $\prod_{i\in I} S_i$. ∎

3.2. Theorem. *Under the same hypotheses, for every object V of $\mathcal{V}$, there are continuous linear bijections $\tau V \longrightarrow V \longrightarrow \sigma V$ with the property that σV has the coarsest topology that has the same functionals as V and τV has the finest topology that has same functionals as V.*

Proof. The argument for σ is standard. Simply retopologize V as a subspace of $K^{\mathrm{Hom}(V,K)}$. To say that an objects V and V' have the same functionals is to say that there is an isomorphism $|V| \longrightarrow |V'|$ that induces $\sigma V \xrightarrow{\cong} \sigma V'$. It is evident that σ is left adjoint to the inclusion of weakly topologized objects into $\mathcal{V}$. Let $\{V_i \mid i \in I\}$ range over isomorphism classes of such objects. We define τV as the pullback in

$$\begin{array}{ccc} \tau V & \longrightarrow & \prod V_i \\ \big\downarrow & & \big\downarrow \\ \sigma V & \longrightarrow & (\sigma V)^I \end{array}$$

The bottom map is the diagonal and is a topological embedding so that the top map is too. We must show that every functional on τV is continuous on V. Suppose $\varphi : \tau V \longrightarrow K$ is given. Using the same argument as in the preceding proof, we see that there is a finite subset, say $\{1, \ldots, n\}$ of I such that φ factors through ψ on $V_1 \times \cdots \times V_n$ and then a functional on $\sigma(V_1 \times \cdots \times V_n)$. Since the category is additive and finite products are also sums, the fact that σ is a left adjoint implies that σ commutes with products and hence ψ is continuous on $\sigma(V)^n$ which implies it is continuous on $\sigma(V)$ and therefore on V. ■

3.3. Remark. We will call the topologies on σV and τV the **weak** and **strong** topologies, respectively. They are the coarsest and finest topology that have the same underlying $\mathcal{A}$ structure and the same functionals as V. The strong topology is also called the **Mackey topology**.

3.4. Theorem. *The following are examples in which the hypotheses of Theorem 3.1 are satisfied.*

$\mathcal{A}$	K	$\mathcal{S}$
Real vector spaces	$\mathbf{R}$	*normed spaces*
Complex vector spaces	$\mathbf{C}$	*normed spaces*
K*-vector spaces*	K	*linearly discrete spaces*
Abelian groups	$\mathbf{R}/\mathbf{Z}$	*locally compact abelian groups*

Notice that the first two examples are not special cases of the third since the subcategory $\mathcal{S}$ is different. The locally compact fields have been completely classified: discrete fields, $\mathbf{R}$, $\mathbf{C}$ and finite algebraic extensions of $\mathbf{Q}_p$, and finite extensions of $\mathbf{S}_p$, the field of Laurent series in one variable over the finite field $\mathbf{Z}/p\mathbf{Z}$. See the German edition of [Pontrjagin, 1957, 1958] for details (the chapter on topological fields was omitted from the English translation, but was included in the German. Of course, the original Russian is definitive.) In any case, I have not been able to consult either one, so cannot provide a more detailed citation.

Proof. Most of this is well known. The Hahn-Banach theorem provides the injectivity of K in the first two cases and Pontrjagin duality in the last. A locally compact field has a compact neighbourhood of 0 and this contains no non-zero subspace. For the real or complex numbers, let U be the unit disk. We choose U as the ring of p-adic integers in $\mathbf{Q}_p$

and in $\mathbf{S}_p$ we choose the ring of formal power series over $\mathbf{Z}/p\mathbf{Z}$. In all cases, the set of all sets of the form aU as a ranges over the non-zero elements of the field is a neighbourhood base at 0. Thus if $\varphi : V \longrightarrow K$ is an algebraic homomorphism such that $\varphi^{-1}(U)$ is a neighbourhood of 0 in V, we see that $\varphi^{-1}(aU) = a\varphi^{-1}(U)$ is also a neighbourhood of 0 by the continuity of scalar multiplication, in this case by a^{-1}.

The argument is a bit different for abelian groups. In this case, any sufficiently small neighbourhood U of 0 is such that if $2x \in U$, then either $x \in U/2$ or $x + \frac{1}{2} \in U/2$. The fact that a map $\varphi : V \longrightarrow K$ is continuous as soon as $\varphi^{-1}(U)$ is a neighbourhood of 0 follows from the easily seen fact that if you define $U_1 = U$ and let U_n be a neighbourhood of 0 such that $U_n \subseteq U_{n-1}$ and $U_n + U_n \subseteq U_{n-1}$, the sequence so defined is a neighbourhood base at 0. ■

Say that an abstract homomorphism $V \longrightarrow V'$ is weakly continuous if the induced $\sigma V \longrightarrow \sigma V'$ is continuous. This is equivalent to saying that for every functional $V' \longrightarrow K$, the composite $V \longrightarrow V' \longrightarrow K$ is a functional on V.

3.5. Theorem. *Suppose that $\mathcal{S}$ is such that any weakly continuous $S \longrightarrow S'$ is continuous. Then $\tau S = S$ for every $S \in \mathcal{S}$.*

Proof. Suppose $S \longrightarrow V$ is a weakly continuous bijection. By assumption, there is an embedding $S \longrightarrow \prod S_i$ with each $S_i \in \mathcal{S}$. But then each composite $S \longrightarrow V \longrightarrow S_i$ is weakly continuous and, by hypothesis, continuous. Hence $S \longrightarrow V$ is continuous. Since the topology on τS is the coarsest for which all the weakly continuous bijections are continuous, we see that that topology is that of S. ■

For topological vector spaces, this is automatically satisfied by Mackey spaces. Since every map out of a linearly discrete space is continuous, it is satisfied there too. Finally, it follows from [Glicksberg, 1962, Theorem 1.1], that it is satisfied by locally compact groups.

4. Chu and chu

We briefly review the categories $\mathrm{Chu}(\mathcal{A}, K)$ and $\mathrm{chu}(\mathcal{A}, K)$. The first has a objects pairs (A, X) of objects of $\mathcal{A}$ equipped with a "pairing" $\langle -, - \rangle : A \otimes X \longrightarrow K$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of a map $f : A \longrightarrow B$ and a map $g : Y \longrightarrow X$ such that

$$\begin{array}{ccc} A \otimes Y & \xrightarrow{f \otimes Y} & B \otimes Y \\ {\scriptstyle A \otimes g}\downarrow & & \downarrow{\scriptstyle \langle -,- \rangle} \\ A \otimes X & \xrightarrow[\langle -,- \rangle]{} & K \end{array}$$

commutes. This says that $\langle fa, y \rangle = \langle a, gy \rangle$ for all $a \in A$ and $y \in Y$. This can be enriched over $\mathcal{A}$ by internalizing this definition as follows. Note first that the map

$A \otimes X \longrightarrow K$ induces, by exponential transpose, a map $X \longrightarrow A \multimap K$. This gives a map $Y \multimap X \longrightarrow Y \multimap (A \multimap K) \cong A \otimes Y \multimap K$. There is a similarly defined arrow $A \multimap B \longrightarrow A \otimes Y \multimap K$. Define $[(A, X), (B, Y)]$ so that

$$\begin{array}{ccc} [(A,X),(B,Y)] & \longrightarrow & A \multimap B \\ \downarrow & & \downarrow \\ Y \multimap X & \longrightarrow & A \otimes Y \multimap K \end{array}$$

is a pullback. Then define

$$(A, X) \multimap (B, Y) = ([(A, X), (B, Y)], A \otimes Y)$$

with $\langle (f, g), a \otimes y \rangle = \langle fa, y \rangle = \langle a, gy \rangle$ and

$$(A, X) \otimes (B, Y) = (A \otimes B, [(A, X), (Y, B)])$$

with pairing $\langle a \otimes b, (f, g) \rangle = \langle b, fa \rangle = \langle a, gb \rangle$. The duality is given by $(A, X)^* = (X, A) \cong (A, X) \multimap (K, \top)$ where $\top$ is the tensor unit of $\mathcal{A}$. Incidentally, the tensor unit of $\mathrm{Chu}(\mathcal{A}, K)$ is $(\top, K)$.

The category $\mathrm{Chu}(\mathcal{A}, K)$ is complete (and, of course, cocomplete). The limit of a diagram is calculated using the limit of the first coordinate and the colimit of the second. The full subcategory $\mathrm{chu}(\mathcal{A}, K) \subseteq \mathrm{Chu}(\mathcal{A}, K)$ consists of those objects (A, X) for which the two transposes of $A \otimes X \longrightarrow K$ are injective homomorphisms. When $A \rightarrowtail X \multimap K$, the pair is called separated and when $X \rightarrowtail A \multimap K$, it is called extensional. In the general case, one must choose a factorization system and assume that the exponential of an epic is monic, but here we are dealing with actual injective maps. The separated pairs form a reflective subcategory and the extensional ones a coreflective subcategory. The reflector and coreflector commute so that $\mathrm{chu}(\mathcal{A}, K)$ is complete and cocomplete. We know that when (A, X) and (B, Y) are separated and extensional, $(A, X) \multimap (B, Y)$ is separated but not necessarily extensional and, dually, $(A, X) \otimes (B, Y)$ is extensional, but not necessarily separated. Thus we must apply the reflector to the hom and the coreflector to the tensor, but everything works out and $\mathrm{chu}(\mathcal{A}, K)$ is also ∗-autonomous. See [Barr, 1998] for details.

In the chu category, one sees immediately that in a map $(f, g) : (A, X) \longrightarrow (B, Y)$, f and g determine each other uniquely. So a map could just as well be described as an $f : A \longrightarrow B$ such that $x.\tilde{y} \in X$ for every $y \in Y$. Here $\tilde{y} : B \longrightarrow K$ is the evaluation at $y \in Y$ of the exponential transpose $Y \longrightarrow B \multimap K$.

Although the situation in the category of abelian groups is as described, in the case of vector spaces over a field, the hom and tensor of two separated extensional pairs turns out to be separated and extensional ([Barr, 1996]).

5. The main theorem

5.1. Theorem. *Under the hypotheses of Theorem 3.2, the categories of weak spaces and strong spaces are equivalent to each other and to chu*$(\mathcal{A}, K)$ *and are thus* $*$*-autonomous.*

Proof. Define $F : \mathcal{V} \longrightarrow \text{chu}$ by $F(V) = (|V|, \text{Hom}(V, K))$ with evaluation as pairing. We first define the right adjoint R of F. Let $R(A, X)$ be the object A, topologized as a subobject of K^X. Since it is already inside a power of K, it has the weak topology. Let $f : |V| \longrightarrow A$ be a homomorphism such that for all $x \in X$, $\tilde{x}.f \in \text{Hom}(V, K)$. This just means that the composite $V \longrightarrow R(A, X) \longrightarrow K^X \xrightarrow{\pi_x} K$ is continuous for all $x \in X$, exactly what is required for the map into $R(A, X)$ to be continuous. The uniqueness of f is clear and this establishes the right adjunction.

We next claim that $FR \cong \text{Id}$. That is equivalent to showing that $\text{Hom}(R(A, X), K) = X$. The argument is very similar to the proof that τV and V have the same maps to K. Every map $R(A, X) \longrightarrow K$ extends to a map $K^X \longrightarrow K$. By continuity, it passes to a finite power, meaning it is represented by a finite sum of elements of X, but that finite sum restricted to $R(A, X)$ is a single element of X. In particular, R is full and faithful. A similar argument shows that the image of R consists of the weakly topologized spaces, which is thereby a $*$-autonomous category.

Now we let $L = \tau R$. If $f : (A, X) \longrightarrow F(V) = (|V|, \text{Hom}(V, K))$ is a map in chu, then f is a homomorphism in $\mathcal{A}$ with the property that when $\varphi : V \longrightarrow K$ is continuous, then $f.\varphi = \tilde{x}$ for some $x \in X$. This implies that $L(A, X) \longrightarrow \sigma V$ is continuous and so is $L(A, X) = \tau L(A, X) \longrightarrow \tau V$ which, together with the continuity of $\tau V \longrightarrow V$ shows that $L(A, X) \longrightarrow V$ is continuous. Again the uniqueness of f is clear. Obviously $FL \cong \text{Id}$. If $V = \tau V$ is strongly topologized, then $RF(V)\sigma V$ and $LF(V) = \tau\sigma V = \tau V = V$ so the image of L consists of the strongly topologized objects, so that category is also $*$-autonomous. ∎

The fact that the categories of weak and Mackey spaces are equivalent was shown, for the case of B (Banach) spaces in [Dunford & Schwartz, 1958, Theorem 15, p. 422]. Presumably, the general case has also been long known, but I am not aware of a reference.

6. Concluding remarks

There is a curious conclusion to be drawn from all this. Consider, for example, the case of real vector spaces. We could treat this using for $\mathcal{S}$ the category of normed vector spaces. Or we could use the category of finite dimensional vector spaces with the usual Euclidean topology. Or we could just treat $\mathbf{R}$ as an abstract field with the discrete topology. For each choice we get strong and weak topologies. This gives us six distinct $*$-autonomous categories (at least, there may be other possibilities). But all of them are equivalent, because they are all equivalent to $\text{chu}(\mathcal{V}, \mathbf{R})$, where $\mathcal{V}$ is the category of real vector spaces. In other words, the topology of the reals and of various vector spaces plays no role in the structure of these categories, qua categories. The same thing happens with the complex numbers and with the other locally compact fields. There are also some choices with topological abelian groups (see [Barr & Kleisli, 2001]) that all lead to chu(Ab, K).

Pairs equipped with a bilinear map were originally introduced by Mackey [Mackey, 1945]. Although his motivation is not entirely clear, it is at least plausible that in a pair (E, E') he thought of E' not so much as a set of continuous linear functionals with respect to a topology on E (although he did think of them that way) but perhaps even more as embodying a replacement for the very idea of a topology. However, he appears never to have defined what he would mean by a continuous homomorphism in this setting, although it is clear enough what it would have to be. Even less did he suggest any thought of a space of such transformations between two such spaces or of a tensor product of them.

6.1. Interpretation of the dual of an internal hom. These remarks are especially relevant to the vector spaces, although they are appropriate to the other examples. The fact that $(U \multimap V)^* \cong U \multimap V^*$ can be interpreted that the dual of $U \multimap V$ is a subspace of $V \multimap U$, namely those linear transformations of finite rank. An element of the form $u \otimes v^*$ acts as a linear transformation by the formula $(u \otimes v^*)(v) = \langle v, v^* \rangle u$. This is a transformation of row rank 1. Sums of these elements is similarly an element of finite rank.

This observation generalizes the fact that in the category of finite dimensional vector spaces, we have that $(U \multimap V)^* \cong V \multimap U$ (such a category is called a compact ∗-autonomous category). In fact, Halmos avoids the complications of the definition of tensor products in that case by *defining* $U \otimes V$ as the dual of the space of bilinear forms on $U \oplus V$, which is quite clearly equivalent to the dual of $U \multimap V^* \cong V \multimap U^*$ ([Halmos, 1958, Page 40]). (Incidentally, it might be somewhat pedantic to point out that Halmos's definition makes no sense since $U \oplus V$ is a vector space in its own right and a bilinear form on a vector space is absurd. It would have been better to use the equivalent form above or to define Bilin(U, V).)

Since linear transformations of finite rank are probably not of much interest in the theory of topological vector spaces, this may explain why the internal hom was not pursued.

References

[Arens, 1947] Arens, R. (1947), *Duality in linear spaces.* Duke Math. J. **14**, 787–794. M.R. 0022651.

[Barr, 1996] M. Barr (1996), *Separability of tensor in Chu categories of vector spaces.* Mathematical Structures Computer Science, **6**, 213–217.

[Barr, 1998] M. Barr (1998), *The separated extensional Chu category.* Theory and Applications of Categories, **4**, 127–137.

[Barr, 2000] M. Barr (2000), *On ∗-autonomous categories of topological vector spaces.* Cahiers de Topologie et Géométrie Différentielle Catégorique, **41**, 243–254.

[Barr & Kleisli, 2001] M. Barr and H. Kleisli (2001), *On Mackey topologies in topological abelian groups.* Theory and Applications of Categories, **9**, 54-62.

[Dunford & Schwartz, 1958] N. Dunford and J. Schwartz (1958), Linear Operators, Interscience.

[Glicksberg, 1962] I. Glicksberg (1962), *Uniform boundedness for groups.* Canad. Math. J. **14**, 269–276.

[Halmos, 1958] P. Halmos (1958), *Finite Dimensional Vector Spaces*, second edition. D. Van Nostrand.

[Mackey, 1945] G. Mackey (1945), *On infinite dimensional vector spaces.* Trans. Amer. Math. Soc. **57**, 155–207.

[Mackey, 1946] G. Mackey (1946), *On convext topological linear spaces.* Trans. Amer. Math. Soc. **66**, 519–537. M.R. 0020214.

[Pontrjagin, 1957, 1958] L.S. Pontrjagin (1957, 1958), **Topologische Gruppen, I and II**. B.G. Teubner Verlagsgesellschaft, Leipzig, 263 + 308 pages.

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