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ON QUANTIC CONUCLEI IN ORTHOMODULAR LATTICES

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ABSTRACT. In this paper we study the lattice of conuclei for orthomodular lattices. We show that under certain conditions we can get a complete characterization of all quantic conuclei. The thing to note is that we use a non-commutative, non-associative disjunction operator which can be thought of as from non-commutative, non-associative linear logic.

Introduction

The purpose of this note is to begin the study of quantic conuclei in several lattices. Quantic conuclei were first introduced in [NR] and later were also introduced in the book [R] written by K. Rosenthal. The main motivation of our work is as follows. In [RR2] the first author and Beatriz Rumbos gave a complete description of the lattice of all quantic nuclei of an orthomodular lattice L. The first author knew the *dual* concept of a quantic nucleus but there was no obvious connection between quantic nucleus and quantic conucleus for orthomodular lattices.

Since the birth of linear logic we know that dual properties could be useful in order to understand some lattice structures. Orthomodular lattices have a nice dual operator. If we take the standard example, the lattice of all closed subspaces of a Hilbert space, the dual is nothing but the complement of a subspace. Hence, a natural question is: suppose L is an orthomodular lattice then what can we say about the lattice of quantic conuclei (denoted by $CN(L)$). We shall see that under certain assumptions we can actually get a *complete dual* characterization of $CN(L)$. We say dual, since it will be the dual concept of quantic nucleus.

In Section 1 we introduce the main definition we need in order to state the main results of this paper.

Section 2 talks about the characterization of $CN(L)$ and uses a new kind of "disjunction" operator which for our purposes is crucial. We must say that this is the first time we found a non trivial application of this "disjunction" operator for orthomodular lattices. We knew the existence of such operator but had never been used in a nontrivial application.

Finally, the first author wants to express his sincere thanks to the second author for listening carefully at the begining of this project. We plan to continue the study of quantic

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conuclei in other lattices and also take a look at the lattice structure of quantic nucleus in the case of quantales.

1. The category of orthomodular lattices

Throughout this work L will denote an *orthomodular lattice*, that is, a lattice $(L, \wedge, \vee, 0, 1)$ with an unary operation $\perp\colon L \to L$ satisfying:

i) $a \leq b \implies b^\perp \leq a^\perp$

ii) $a^\perp \vee a = 1$

iii) $a^\perp \wedge a = 0$

for all a, b in L. Moreover, L satisfies the following weak modularity property:

$$a \leq b \iff b = a \vee (a^\perp \wedge b)\,.$$

These lattices have traditionally been associated with "quantum logic", since the propositions for a quantum system correspond with closed subspaces of a Hilbert space and these constitute an orthomodular lattice.

As pointed out in [RR1] a natural choice for an implication conjunction pair is to take the *Sasaki arrow* defined by

$$a \longrightarrow b = (a \wedge b) \vee a^\perp \quad \text{for all} \quad a, b \in L$$

together with the conjunction "&" given as

$$a \,\&\, b = \left(a \vee b^\perp\right) \wedge b \quad \text{for all} \quad a, b \in L\,.$$

The relation $a \,\&\, b \leq c$ iff $a \leq b \to c$ holds for any elements a, b, c in L. In categorical terms, if L is seen as a category, then this is just saying that the functor $- \,\&\, b$ is a left adjoint to $b \to -$ for any $b \in L$. An immediate result from this is that the operator & distributes over arbitrary joins (on the left), i.e., it is "left distributive". Moreover & can be thought of as a noncommutative nonassociative meet operator. In fact, if & is either associative or commutative then L will turn out to be a Boolean algebra (see [RR1] for more details).

We define another noncommutative operator in an orthomodular lattice,and denoted it by $*$. Here is the definition.

1.1. Definition. *Let $(L, \wedge, \vee, 0, 1)$ be an arbitrary orthomodular lattice. If a, b are arbitrary elements of L define $a * b$ by the following rule:*

$$a * b = \left(a^\perp \,\&\, b^\perp\right)^\perp$$

Once again, the operator $*$ can be viewed as a non commutative, non associative union. This operator, was unfortunately never studied in detail but plays a crucial role in this paper. There are some obvious consequences of the definitions of & and $*$, that we list for later reference.

i) $a \wedge b \leq a \,\&\, b, \quad a * b \leq a \vee b.$

ii) $a \,\&\, b \leq b,\ a * b \geq b.$

iii) $1 \,\&\, a = a = a \,\&\, 1,\ 0 * a = a = a * 0.$

iv) $a \,\&\, a = a,\ a * a = a.$

v) $a \,\&\, a^{\perp} = 0,\ a * a^{\perp} = 1.$

It is clear that for any $a \in L$ the orthocomplement of a can be written as $a \to 0$ and satisfies the following property:

$$x \,\&\, b = 0 \quad \text{iff} \quad x \leq b^{\perp}$$

As we noted in the introduction we are interested in the study of quantic conuclei. Since our methods also use the concept of a quantic nucleus we shall introduce both concepts. In order to do that we have to make a choice, i.e., introduce the definitions only for orthomodular lattices or in a more general context, namely, in any *quantic lattice.* We believe that the last idea is the best choice since we can in fact introduce the concept of quantale. So, let us begin with the definition of a quantic lattice.

1.2. Definition. *Let Q be a complete lattice $(Q, \leq)$, then Q is a* quantic lattice *iff there exists a binary operation $\& : Q \times Q \to Q$ satisfying:*

i) $- \,\&\, q : Q \to Q$ is a poset morphism for all q in Q.

ii) $- \,\&\, q : Q \to Q$ has a right adjoint $q \to (-)$ for all q in Q.

The reader can find some interesting examples of such lattices in [RR3]. Here we just mention some of them:

1) Any locale with $a \,\&\, b = a \wedge b, \quad \forall\, a, b \in L.$

2) Any complete orthomodular lattice, where $\&$ is the operation already defined.

3) Any quantale (see [NR] for details).

Quantic nuclei and conuclei were first introduced by [NR]. Our definition does not differ from theirs.

1.3. Definition. *Let Q be any quantic lattice. If $j : Q \to Q$ is an order preserving function then j is a* quantic nucleus *if and only if the following is satisfied:*

i) $a \leq ja$.

ii) $j^2 a = ja$.

iii) $j(a) \,\&\, j(b) \leq j(a \,\&\, b)$.

for all a, b in Q.

We shall say $j: Q \to Q$ is a quantic conucleus *if and only if j satisfies:*

i) $ja \leq a$.

ii) $j^2 a = ja$.

iii) $j(a) \,\&\, j(b) \leq j(a \,\&\, b)$.

for all a, b in Q.

We shall say $j: Q \to Q$ is a strict *quantic nucleus (conucleus) if $j(a \,\&\, b) = ja \,\&\, jb$.*

EXAMPLE. Quantic nuclei were characterized in [RR2] when Q is an orthomodular lattice. All quantic nuclei $j: Q \to Q$ are of the form $u_z: Q \to Q$ where

$$u_z(a) = a \vee z$$

$z = j(0)$ and z belongs to the *center* of Q.

By the center of Q we mean the following subset of Q.

Suppose Q is an orthomodular lattice then the center of Q, denoted by $Z(Q)$ is given by the following condition:

$$Z(Q) = \{x \in Q \mid x \,\&\, a = a \,\&\, x, \quad \forall\, a \in Q\} .$$

$Z(Q)$ is not only a suborthomodular lattice of Q, in fact is a Boolean algebra, as the reader can prove easily.

For quantic conuclei, suppose Q is a Boolean algebra and a is an arbitrary element of Q, then the function $m_a: Q \to Q$ defined by

$$m_a(x) = x \wedge a$$

is easily shown to be a quantic conucleus, in fact is a strict one, i.e.,

$$m_a(x) \wedge m_a(y) = m_a(x \wedge y)$$

Notice that in order for m_a to be a quantic conucleus we must use that the operator $\wedge$ is commutative and idempotent.

Since the operator $\&$ already defined for an orthomodular lattice is not necessarily commutative (unless Q is a Boolean algebra) the problem seems to be different from the Boolean case. However, we shall see that this is not the case. Indeed, a similar result (as in Boolean algebras) will hold. We have the following.

1.4. LEMMA. *If L is a (complete) orthomodular lattice and z is an arbitrary element of $Z(L)$, then the operator $m_z: L \to L$ is a strict quantic conucleus.*

The only thing to be noted is that as soon as you have an element the center of L, then the *distributive law* holds, that is, given x, y arbitrary elements of L and z an element of $Z(L)$, then

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) .$$

What about the converse of Lemma 1.4? In contrast with the case of quantic nucleus we have to made some assumptions; we shall do it in the next section.

2. Quantic conuclei for Boolean algebras and orthomodular lattices

As we pointed out in the introduction our main target is to give a characterization of quantic nuclei in terms of the meet operator. We must say that the condition we found is rather surprising. For the sake of simplicity we begin first with the case of a Boolean algebra and later we shall deal with the case of orthomodular lattices. First of all, we introduce some notation. Denote by $CN(L)$ the set of all quantic conuclei. We can state now the following.

2.1. LEMMA. *If L is a (complete) Boolean algebra and j is an element of $CN(L)$, the following two conditions are equivalent:*

- *i) For all a, b in L, $j(a \vee b) = ja \vee jb$.*
- *ii) $j = m_a$ where $a = j(1)$.*

PROOF. i) $\Longrightarrow$ ii) Take any element j of $CN(L)$. We define $\varphi: L \to L$ as follows: $\varphi(a) = j(a^{\perp})^{\perp}$. We check φ is in fact a quantic nucleus. Clearly, φ is inflationary and idempotent. The only thing we have to check is $\varphi(a) \wedge \varphi(b) \leq \varphi(a \wedge b)$.

The LHS of the last inequality is equal to $\big(j(a^{\perp}) \vee j(b^{\perp})\big)^{\perp} = \big(j(a^{\perp} \vee b^{\perp})\big)^{\perp}$ by the hypothesis, and this of course is equal to the RHS of the last inequality. Therefore $\varphi = u_z$ where $z = \varphi(0) = j(1)^{\perp}$; i.e., if a is an arbitrary element of L we have $\varphi(a) = a \vee j(1)^{\perp}$ but $\varphi(a) = j(a^{\perp})^{\perp}$ hence $j(a^{\perp})^{\perp} = a \vee j(1)^{\perp}$ and $j(a) = a \wedge j(1) = m_{j(1)}$.

The converse is of course trivial. ∎

We want now to extend this result for the case of orthomodular lattices. As the reader can imagine the conditions will be similar. However, we will use our noncommutative, nonassociative join operator. Namely, the operator $*$, already defined in section 1. Notice that if $j: L \to L$ is an order preserving function and L has finite joins we always have the inequality $j(a) \vee j(b) \leq j(a \vee b)$, a and b arbitrary elements of L. If we take the operator $*$ we cannot claim for such inequality. However, we do not need any such a condition. Our next result will tell us what we need in order to obtain a similar result to Lemma 2.1.

2.2. THEOREM. *If L is a (complete) orthomodular lattice and j is an element of $CN(L)$, the following two conditions are equivalent :*

- *i) For all a, b in L, $j(a * b) \leq j(a) * j(b)$.*
- *ii) $j = m_a$ where $a = j(1)$ and is an element of the* center *of L.*

PROOF. Take any element j of $CN(L)$ and suppose j satisfies (i), define again $\varphi: L \to L$ as follows: $\varphi(a) = j(a^{\perp})^{\perp}$. Clearly, φ is inflationary and idempotent. We check $\varphi(a) \,\&\, \varphi(b) \leq \varphi(a \,\&\, b)$.

The LHS of the last inequality is equal to

$$j(a^{\perp})^{\perp} \,\&\, j(b^{\perp})^{\perp} = \big(j(a^{\perp}) * j(b^{\perp})\big)^{\perp}\ .$$

Now, the RHS of the last inequality is equal to

$$j\left((a \,\&\, b)^{\perp}\right)^{\perp} = j\left(a^{\perp} * b^{\perp}\right)^{\perp}.$$

By the assumption we know $j(a^{\perp} * b^{\perp}) \leq j(a^{\perp}) * j(b^{\perp})$ and the result follows. Hence φ is a quantic nucleus and by Theorem 2.4 of [RR2], $\varphi = u_z$ where z is an element of the center of L and $z = \varphi(0)$. A simple calculation shows that $a = j(1)$ is also an element of the center of L (since $Z(L)$ is a Boolean algebra) and $j = m_a$.

For the converse, the only thing to be noted is that $a = j(1)$ is an element of the center of L and therefore for any pair of arbitrary elements b, c of L, the triple $\{a, b, c\}$ is always distributive. ∎

Remark. Notice we used in a crucial way the result for quantic nuclei.

For some time we tried to find a straight proof of Theorem 2.2 without any success. Perhaps, it will be nice if we can actually get some proof in a different way but until now we do not know how to do it.

We close this paper with a comment. First of all, the lattice $N(L)$ has a very nice structure when L is an orthomodular lattice without any further assumption.

On the other hand, the structure of $CN(L)$ has a complicated structure even if L is a Boolean algebra as the reader just saw.

As is well known dualities have an important role in mathematics. We think this an example of when the dual concept of a mathematical structure gives a different thing.

In the case of an orthomodular lattice L if we want to obtain a duality between $CN(L)$ and $N(L)$, we must assume the condition of Theorem 2.2. However, we feel the condition is quite natural and simple.

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