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MONADS AND INTERPOLADS IN BICATEGORIES

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Transmitted by R. J. Wood

ABSTRACT. Given a bicategory, $\boldsymbol{Y}$, with stable local coequalizers, we construct a bicategory of monads $\boldsymbol{Y}$-$\boldsymbol{mnd}$ by using lax functors from the generic 0-cell, 1-cell and 2-cell, respectively, into $\boldsymbol{Y}$. Any lax functor into $\boldsymbol{Y}$ factors through $\boldsymbol{Y}$-$\boldsymbol{mnd}$ and the 1-cells turn out to be the familiar bimodules. The locally ordered bicategory $\boldsymbol{rel}$ and its bicategory of monads both fail to be Cauchy-complete, but have a well-known Cauchy-completion in common. This prompts us to formulate a concept of Cauchy-completeness for bicategories that are not locally ordered and suggests a weakening of the notion of monad. For this purpose, we develop a calculus of general modules between unstructured endo-1-cells. These behave well with respect to composition, but in general fail to have identities. To overcome this problem, we do not need to impose the full structure of a monad on endo-1-cells. We show that associative coequalizing multiplications suffice and call the resulting structures interpolads. Together with structure-preserving i-modules these form a bicategory $\boldsymbol{Y}$-$\boldsymbol{int}$ that is indeed Cauchy-complete, in our sense, and contains the bicategory of monads as a not necessarily full sub-bicategory. Interpolads over $\boldsymbol{rel}$ are idempotent relations, over the suspension of $\boldsymbol{set}$ they correspond to interpolative semi-groups, and over $\boldsymbol{spn}$ they lead to a notion of "category without identities" also known as "taxonomy". If $\boldsymbol{Y}$ locally has equalizers, then modules in general, and the bicategories $\boldsymbol{Y}$-$\boldsymbol{mnd}$ and $\boldsymbol{Y}$-$\boldsymbol{int}$ in particular, inherit the property of being closed with respect to 1-cell composition.

Introduction

Part of the original motivation for this work was to better understand, why the bicategory $\boldsymbol{idl}$ of pre-ordered sets, order-ideals, and inclusions inherits good properties from the bicategory $\boldsymbol{rel}$ of sets, relations and inclusions. Of particular interest was closedness with respect to 1-cell composition, also known as the existence of all right liftings and right extensions.

The key observation is that pre-ordered sets can be viewed as monads in $\boldsymbol{rel}$. Benabou [1] explicitly designed what is now known as lax functors to subsume the notion of monad, in this case as a lax functor from the terminal bicategory $\boldsymbol{1}$ to $\boldsymbol{rel}$. To keep the paper reasonably self-contained, in Section 1 we recall the relevant definitions and establish our notation.

Order-ideals, *i.e.*, relations compatible with the orders on domain and codomain, do not correspond to the kind of morphisms that are usually considered between monads. Since ordinary morphisms in a category correspond to functors with domain $\boldsymbol{2}$, the two-element chain or generic 1-cell, we introduce 1-cells between monads in terms of

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lax functors with $\mathbf{2}$ as domain and call them "m-modules". These later turn out to be bimodules between monads as described in Section 4.1 of Carboni, Kasangian and Walters [3] and also known to Street [9]. Since the inclusions in $\boldsymbol{idl}$ correspond to lax functors from the generic 2-cell, we may interpret $\boldsymbol{idl}$ as "the bicategory of monads over $\boldsymbol{rel}$".

Rosebrugh and Wood [8] mention a basic defect of $\boldsymbol{rel}$: idempotents need not split. This also is true for $\boldsymbol{idl}$. Hence they work with the Karoubian envelope, or Cauchy-completion, $\boldsymbol{kar}$ of $\boldsymbol{rel}$ instead. It also has all right liftings and right extensions and happens to be the Cauchy completion of (and hence to contain) $\boldsymbol{idl}$ as well.

How can this situation be generalized to other well-behaved bicategories besides $\boldsymbol{rel}$? There are (at least) two aspects to this question. The objects of $\boldsymbol{kar}$ are sets equipped with an idempotent relation. This need not be reflexive but has to be transitive and to satisfy the so-called "interpolation property", which is just the converse of the inclusion required for transitivity, *cf.* Example 3.07.(2). Hence one may think of it as a "monad without unit", but with an extra condition on the multiplication that in the general case has yet to be determined. We chose the name "interpolad" for such a structure. Of course, morphisms between interpolads ought to be suitably weakened versions of m-modules or bimodules. In fact, no extra structure on endo-1-cells is necessary to get a theory of general modules off the ground. In Section 2 we describe this for a fixed bicategory $\boldsymbol{Y}$ with local stable coequalizers. General modules already admit the type of composition, by means of coequalizers, described for bimodules by Carboni et. al [3], and known in various special cases. Unfortunately, endo-1-cells and general modules do not form a bicategory for lack of identity modules.

This prompts us, in Section 3, to look for the minimal structure that needs to be imposed on endo-1-cells and to be preserved by suitable modules in order to guarantee the existence of identities. This leads to the official introduction of interpolads: endo-1-cells with an associative coequalizing multiplication. Together with structure-preserving "i-modules" and the obvious 2-cells they form a bicategory $\boldsymbol{Y}\text{-}\boldsymbol{int}$. The extent to which the notions of monad and m-module have been weakened can be made precise: the absolute coequalizers implicit in those notions are replaced by coequalizers, the preservation of which in certain cases we require explicitly. Hence we recover $\boldsymbol{Y}\text{-}\boldsymbol{mnd}$ as a not necessarily full sub-bicategory of $\boldsymbol{Y}\text{-}\boldsymbol{int}$.

Section 4 addresses the second aspect of our quest to generalize the relationship between $\boldsymbol{rel}$, $\boldsymbol{idl}$ and $\boldsymbol{kar}$: we introduce the notion of Cauchy-completeness for arbitrary bicategories. If the hom-categories are not just ordered, associativity becomes an issue when reasoning about idempotency of 1-cells. $\boldsymbol{Y}\text{-}\boldsymbol{int}$ turns out to be Cauchy-complete in this sense, and hence to fully contain the Cauchy-completion of $\boldsymbol{Y}$. The latter will usually be too small to contain $\boldsymbol{Y}\text{-}\boldsymbol{mnd}$. Although for distributive $\boldsymbol{Y}$ all interpolads arise when splitting idempotent m-modules in $\boldsymbol{Y}\text{-}\boldsymbol{int}$, the Cauchy-completion of $\boldsymbol{Y}\text{-}\boldsymbol{mnd}$ usually cannot be realized as a subcategory of $\boldsymbol{Y}\text{-}\boldsymbol{int}$, although there exists an obvious "forgetful" functor.

In Section 5 we return to our original question and consider closedness with respect to 1-cell composition in terms of right extensions and right liftings. Provided that $\boldsymbol{Y}$ has hom-categories with equalizers, we show that the existence of all right extensions in $\boldsymbol{Y}$ implies this property for general modules (no identities are needed for this), as well as for the bicategories $\boldsymbol{Y}\text{-}\boldsymbol{mnd}$ and $\boldsymbol{Y}\text{-}\boldsymbol{int}$. In fact, right extensions in $\boldsymbol{Y}\text{-}\boldsymbol{mnd}$ are formed just as for general modules (this recovers parts of a result by Street [9]), while in $\boldsymbol{Y}\text{-}\boldsymbol{int}$ their construction requires an additional step, the pre- and post-composition with the appropriate identity i-modules. Right liftings can be handled dually.

It is known that the bicategory of monads over $\boldsymbol{spn}$, the bicategory of sets, spans or matrices of sets, and span morphisms or matrices of functions, is essentially the bicategory of small categories and profunctors, *cf.* Example 3.07.(4). The bicategory of interpolads over $\boldsymbol{spn}$ seems to be the largest reasonable extension to a bicategory with objects that resemble categories but may fail to have identities. We propose to call them "taxonomies", even though this term has been introduced by Paré and Wood for a weaker concept (no further conditions on the associative multiplication).

The results of this paper seem to suggest that the unit condition for a binary multiplication may be "less fundamental" than the associativity condition. Indeed, both basic diagrammatic representations of bicategorical data, pasting diagrams and string diagrams, can account for the unit isomorphisms more directly than for the associativity isomorphisms. But this needs to be studied more closely in the future.

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1. Bicategories and lax functors

Bicategories and morphisms between them were introduced by Benabou [1]. Recently, Borceux [2] has covered some of this material in a textbook. Roughly speaking, the notion of bicategory arises by replacing the hom-sets of ordinary categories by hom-categories, the composition functions by functors, and by then relaxing the categorical axioms for morphisms to hold "up to coherent isomorphism". While the idea is straightforward, the actual formulation of "coherent isomorphism" requires some effort. Hence we briefly recall the definitions of bicategory and of lax functor and introduce our preferred notation (*cf.* Conventions 1.01. and 1.05. below).

1.00. Definition. A *bicategory* $\boldsymbol{X}$ consists of

- a class $\underline{\boldsymbol{X}}$ of *objects* or *0-cells*
- for all 0-cells A, B a small category $\langle A, B\rangle\boldsymbol{X}$; its objects (denoted by single arrows) are called *1-cells*, and its morphisms (denoted by double arrows) are called *2-cells*; we refer to the composition of 2-cells in $\langle A, B\rangle\boldsymbol{X}$ as *serial* or *vertical composition*, and write $f \xRightarrow{\sigma\,;\,\tau} h$ for the composite of $f \xRightarrow{\sigma} g$ and $g \xRightarrow{\tau} h$; [1]
- for all 0-cells A, B, and C a functor $\langle A, B\rangle\boldsymbol{X} \times \langle B, C\rangle\boldsymbol{X} \xrightarrow{\langle A,B,C\rangle\boldsymbol{c}} \langle A, C\rangle\boldsymbol{X}$, referred to as *parallel* or *horizontal composition*; we concatenate the arguments, *e.g.*, $\varphi\psi$ instead of $\langle\varphi, \psi\rangle\,(\langle A, B, C\rangle\boldsymbol{c})$; notationally, this binds stronger than " ; ";

[1] Readers unhappy with the order of composition can obtain a version of this paper in backwards notation from `http://www.iti.cs.tu-bs.de/TI-INFO/koslowj/koslowski.html` or from the author.

- for all composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ a natural *associativity isomorphism* $f(gh) \xRightarrow{\langle f,g,h\rangle\mathfrak{a}} (fg)h$;
- for each 0-cell A a distinguished *identity 1-cell* in $\langle A,A\rangle\boldsymbol{X}$, usually identified with the object A , or, if necessary, notationally disambiguated as 1_A ;
- for each 1-cell $A \xrightarrow{f} B$ natural *unit isomorphisms* $Af \xRightarrow{f\mathfrak{u}_\diamond} f$ and $fB \xRightarrow{f\mathfrak{u}^\diamond} f$

subject to the following coherence axioms:

(B0) for all composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ we have

$$\begin{tikzcd} & (fg)(hk) \arrow[dr, Rightarrow, "\langle fg,h,k\rangle\mathfrak{a}"] & \\ f(g(hk)) \arrow[ur, Rightarrow, "\langle f,g,hk\rangle\mathfrak{a}"] \arrow[d, Rightarrow, "f((\langle g,h,k\rangle\mathfrak{a}))"'] & & ((fg)h)k \\ f((gh)k) \arrow[rr, Rightarrow, "\langle f,gh,k\rangle\mathfrak{a}"'] & & (f(gh))k \arrow[u, Rightarrow, "((\langle f,g,h\rangle\mathfrak{a}))k"'] \end{tikzcd}$$

For later reference we call this property of the composition functors and the associativity isomorphisms *the essential associativity of horizontal composition.*

(B1) for all composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ we have

$$\begin{tikzcd} f(Bg) \arrow[rr, Rightarrow, "\langle f,B,g\rangle\mathfrak{a}"] \arrow[dr, Rightarrow, "f(g\mathfrak{u}_\diamond)"'] & & (fB)g \arrow[dl, Rightarrow, "(f\mathfrak{u}^\diamond)g"] \\ & fg & \end{tikzcd}$$

If there is no danger of confusion, we may drop the arguments to $\mathfrak{a}$, $\mathfrak{u}_\diamond$, and $\mathfrak{u}^\diamond$. If all these *structural* 2-cells are identities, we call $\boldsymbol{X}$ a 2-category.

DUAL NOTIONS: In the bicategories $\boldsymbol{X}^{\mathrm{op}}$ and $\boldsymbol{X}^{\mathrm{co}}$, the 1-cells and the 2-cells are reversed, respectively. *I.e.*, $\langle A,B\rangle\boldsymbol{X}^{\mathrm{op}} = \langle B,A\rangle\boldsymbol{X}$, and $\langle A,B\rangle\boldsymbol{X}^{\mathrm{co}} = (\langle A,B\rangle\boldsymbol{X})^{\mathrm{op}}$.

1.01. CONVENTION. We usually denote 0-cells by upper case letters, 1-cells with lower case letters, and 2-cells with lower case Greek letters. Exceptions are the structural associativity and unit isomorphisms, which we distinguish from "ordinary" 2-cells by means of bold Fraktur letters. The identity 2-cell of $A \xrightarrow{r} B$ is usually identified with r , or, if necessary, notationally disambiguated as $\boldsymbol{id}_r$. Diamonds $\diamond$ as subscripts and superscripts always indicate left and right actions, respectively.

1.02. REMARK. The coherence axioms guarantee, among other things, that

(0) for every object A the unit isomorphisms $AA \xRightarrow{A\mathfrak{u}_\diamond} A$ and $AA \xRightarrow{A\mathfrak{u}^\diamond} A$ agree; this justifies dropping the diamond, *i.e.*, we define $A\mathfrak{u} := A\mathfrak{u}_\diamond = A\mathfrak{u}^\diamond$;

(1) for any two composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ we have

$$
\begin{tikzcd}
A(fg) \arrow[rr, Rightarrow, "\langle A,f,g\rangle\mathfrak{a}"] \arrow[dr, Rightarrow, "(fg)\mathfrak{u}_\diamond"'] & & (Af)g \arrow[dl, Rightarrow, "(f\mathfrak{u}_\diamond)g"] \\
& fg &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
f(gC) \arrow[rr, Rightarrow, "\langle f,g,C\rangle\mathfrak{a}"] \arrow[dr, Rightarrow, "f(g\mathfrak{u}^\diamond)"'] & & (fg)C \arrow[dl, Rightarrow, "(fg)\mathfrak{u}^\diamond"] \\
& fg &
\end{tikzcd}
\tag{1-00}
$$

1.03. Examples. (0) Every "ordinary" category can be viewed as a bicategory with discrete hom-categories. Since it will always be clear from the context, which interpretation is intended, we will use the same name, *e.g.*, $\boldsymbol{set}$, in both cases.

(1) One of the simplest bicategories is $\boldsymbol{rel}$ with sets as objects, binary relations as 1-cells and inclusions as 2-cells. 1-cell composition is the usual relation product. Since the hom-categories are partially ordered, this is in fact a 2-category.

(2) A very important bicategory, $\boldsymbol{bim}$, has unitary rings as objects. A 1-cell from A to B is an $\langle A, B\rangle$-bimodule, *i.e.*, an abelian group R equipped with the structures of a left-A-module and a right-B-module subject to the compatibility condition

$$a(rb) = (ar)b$$

for all $a \in A$, $r \in R$ and $b \in B$, *cf.* Examples 3.07.(0) and (1). 2-cells between $\langle A, B\rangle$-bimodules are group homomorphisms that are A-linear on the left and B-linear on the right. Bimodules are composed by means of their tensor product. For a commutative unitary ring R, the hom-category $\langle R, R\rangle\boldsymbol{bim}$ contains the familiar category of R-modules as a full subcategory. Hence $\boldsymbol{bim}$ is a true bicategory that is not a 2-category. Much of the terminology derives from this particular example.

(3) For any object A of a bicategory $\boldsymbol{X}$ the hom-category $\langle A, A\rangle\boldsymbol{X}$ carries the structure of a monoidal category, the tensor product being the composition of 1-cells. In fact, every monoidal category $\boldsymbol{M}$ may be viewed as the hom-category of a bicategory with one object $\{*\}$. The Australian school has coined the term *suspension of* $\boldsymbol{M}$ for the resulting bicategory $\boldsymbol{M}\Sigma$ that satisfies $\langle *, *\rangle\,(\boldsymbol{M}\Sigma) = \boldsymbol{M}$.

1.04. Definition. For bicategories $\boldsymbol{X}$ and $\boldsymbol{Y}$, a *lax functor* $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ consists of

- an object-function or *carrier* $\underline{\boldsymbol{X}} \xrightarrow{F} \underline{\boldsymbol{Y}}$;
- a family of functors $\langle A, B\rangle\boldsymbol{X} \xrightarrow{\langle A,B\rangle F} \langle AF, BF\rangle\,\boldsymbol{Y}$, where A and B are $\boldsymbol{X}$-objects; we usually write $f\boldsymbol{F}$ instead of $f\langle A, B\rangle F$;
- a family of natural 2-cells $(f\boldsymbol{F})\,(g\boldsymbol{F}) \xRightarrow{\langle f,g\rangle\mathfrak{q}} (fg)\boldsymbol{F}$, where $A \xrightarrow{f} B \xrightarrow{g} C$ are composable 1-cells in $\boldsymbol{X}$;
- a family of natural 2-cells $AF \xRightarrow{A\mathfrak{d}} A\boldsymbol{F}$, where A is an $\boldsymbol{X}$-object

subject to the following coherence axioms:

(L0) for all composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\boldsymbol{X}$ we have

$$\begin{tikzcd}
(f\boldsymbol{F})\big((g\boldsymbol{F})(h\boldsymbol{F})\big) \arrow[r, Rightarrow, "(f\boldsymbol{F})(\langle g,h\rangle\mathfrak{q})"] \arrow[d, Rightarrow, "\langle f\boldsymbol{F},g\boldsymbol{F},h\boldsymbol{F}\rangle\mathfrak{a}"'] & (f\boldsymbol{F})\big((gh)\boldsymbol{F}\big) \arrow[r, Rightarrow, "\langle f,gh\rangle\mathfrak{q}"] & \big(f(gh)\big)\boldsymbol{F} \arrow[d, Rightarrow, "(\langle f,g,h\rangle\mathfrak{a})\boldsymbol{F}"] \\
\big((f\boldsymbol{F})(g\boldsymbol{F})\big)(h\boldsymbol{F}) \arrow[r, Rightarrow, "(\langle f,g\rangle\mathfrak{q})(h\boldsymbol{F})"'] & \big((fg)\boldsymbol{F}\big)(h\boldsymbol{F}) \arrow[r, Rightarrow, "\langle fg,h\rangle\mathfrak{q}"'] & \big((fg)h\big)\boldsymbol{F}
\end{tikzcd}$$

(L1) for each 1-cell $A \xrightarrow{f} B$ in $\boldsymbol{X}$ we have

$$\begin{tikzcd}
(AF)(f\boldsymbol{F}) \arrow[r, Rightarrow, "(f\boldsymbol{F})\mathfrak{u}_\diamond"] \arrow[d, Rightarrow, "(A\mathfrak{d})(f\boldsymbol{F})"'] & f\boldsymbol{F} \\
(A\boldsymbol{F})(f\boldsymbol{F}) \arrow[r, Rightarrow, "\langle A,f\rangle\mathfrak{q}"'] & (Af)\boldsymbol{F} \arrow[u, Rightarrow, "(f\mathfrak{u}_\diamond)\boldsymbol{F}"']
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
(f\boldsymbol{F})(BF) \arrow[r, Rightarrow, "(f\boldsymbol{F})\mathfrak{u}^\diamond"] \arrow[d, Rightarrow, "(f\boldsymbol{F})(B\mathfrak{d})"'] & f\boldsymbol{F} \\
(f\boldsymbol{F})(B\boldsymbol{F}) \arrow[r, Rightarrow, "\langle f,B\rangle\mathfrak{q}"'] & (fB)\boldsymbol{F} \arrow[u, Rightarrow, "(f\mathfrak{u}^\diamond)\boldsymbol{F}"']
\end{tikzcd}$$

A lax functor $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ is called *normalized* or *unitary*, if all 2-cells of the form $A\mathfrak{d}$ are invertible, and a *bifunctor*, if is normalized and all 2-cells of the form $\langle f,g\rangle\mathfrak{q}$ are invertible as well. Normalized lax functors and bifunctors are called *strict*, if the respective structural 2-cells required to be invertible are in fact identities.

DUAL NOTION: For an *oplax functor* $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ the structural 2-cells are reversed:

$$(fg)\boldsymbol{F} \xRightarrow{\langle f,g\rangle\mathfrak{p}} (f\boldsymbol{F})(g\boldsymbol{F}) \quad \text{instead of} \quad (f\boldsymbol{F})(g\boldsymbol{F}) \xRightarrow{\langle f,g\rangle\mathfrak{q}} (fg)\boldsymbol{F}$$

and

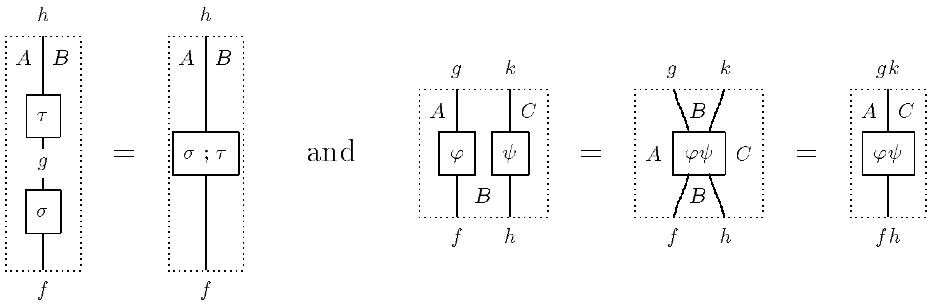
$$A\boldsymbol{F} \xRightarrow{A\mathfrak{b}} AF \quad \text{instead of} \quad AF \xRightarrow{A\mathfrak{d}} A\boldsymbol{F}$$

have to satisfy the duals of axioms (L0) and (L1), respectively.

1.05. CONVENTION. Again, we use boldface Fraktur letters for the structural 2-cells. The price we have to pay for abbreviating $f\langle A,B\rangle F$ to $f\boldsymbol{F}$ is that we need to notationally distinguish the action of $\boldsymbol{F}$ on 0-cells from that of $\boldsymbol{F}$ on identity 1-cells. This can either be done by distinguishing 0-cells and identity 1-cells (*e.g.*, A *vs.* 1_A, respectively), or by using different symbols for the functor action (F and $\boldsymbol{F}$, respectively), depending on the nature of the argument. Here we have opted for the second solution.

We close this section with a few remarks about the presentation of bicategorical data. Instead of the familiar *pasting diagrams*, we prefer *string diagrams*. Joyal and Street [10] attribute the first use of this technique for the manipulation of tensors to Penrose in 1971. For the author the experience of converting proofs from pasting diagrams to string diagrams was illuminating. The clear directional division of the two types of composition available in a bicategory – horizontally from left to right, and vertically from bottom to top – subjectively made the comprehension of string diagrams easier. We hope the reader will agree.

The basic entities of a bicategory are its 2-cells; they are to be thought of as labeled points. Here we have chosen to "blow up" the points to rectangular boxes with a label inside. 1-cells correspond to labeled essentially vertical lines (no horizontal tangent lines allowed), while 0-cells or objects correspond to labeled regions. Concretely, the vertical composition of two 2-cells $f \overset{\sigma}{\Longrightarrow} g$, and $g \overset{\tau}{\Longrightarrow} h$ inside a hom-category $\langle A, B\rangle \boldsymbol{X}$ and the horizontal composition of 2-cells $f \overset{\varphi}{\Longrightarrow} g$ in $\langle A, B\rangle \boldsymbol{X}$ and $h \overset{\psi}{\Longrightarrow} k$ in $\langle B, C\rangle \boldsymbol{X}$ can be depicted by either one of the first two or one of the last three diagrams, respectively:

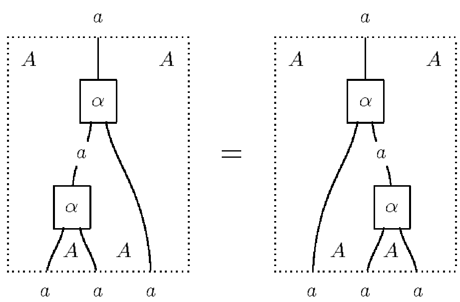


Notice how easily string diagrams can be combined horizontally and vertically, provided the appropriate domains and codomains match. Also, since φ and ψ are independent, we may change their respective heights, which leads to $\varphi\psi = f\psi \,;\, \varphi k = \varphi h \,;\, g\psi$.

2. The general calculus of modules in a bicategory

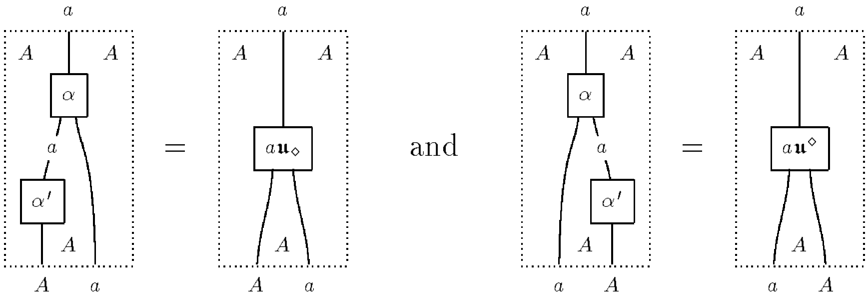
The following crucial example was one of the reasons for Benabou [1] to introduce what we now call lax functors instead of the more restrictive concept of bifunctors.

2.00. Example. Monads may be viewed as lax functors with domain $\boldsymbol{1}$, the singleton bicategory with a single element 0 that serves triple duty as a 0-cell, an identity 1-cell, and an identity 2-cell. Indeed, let $\boldsymbol{1} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ be a lax functor with $0F = A$ and write a for the endo-1-cell $0\boldsymbol{F}$ on A. By axiom (L0) the multiplication $\alpha := \langle 0, 0\rangle \mathfrak{q}$ from aa to a is associative, *i.e.*,



(2-00)

And by axiom (L1) the 2-cell $\alpha' := 0\mathfrak{d}$ from A to a serves as two-sided unit:



(2-01)

For a lax functor $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ each 0-cell XF in $\boldsymbol{Y}$ carries a monad structure, namely $\langle X\boldsymbol{F}, \langle X,X\rangle \mathfrak{q}, X\mathfrak{d}\rangle$. We wish to construct a "bicategory of monads" over $\boldsymbol{Y}$, through which $\boldsymbol{F}$ factors. Which 1-cells and 2-cells are appropriate? Recall that the morphisms of an ordinary category correspond to functors with domain $\boldsymbol{2}$.

2.01. Definition. (0) By a *morphism of monads*, or *m-module* for short, we mean a lax functor with domain the generic 1-cell $\boldsymbol{2}$,

$$0 \xrightarrow{\quad k \quad} 1$$

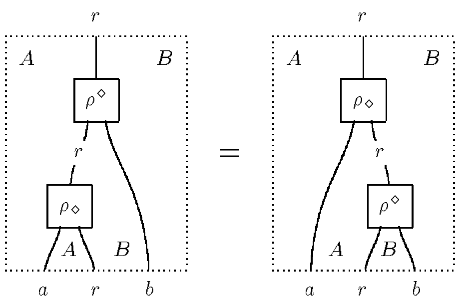
(1) By an *m-module homomorphism*, or *modism* for short, we mean a lax functor with domain the generic 2-cell ***two***

$$\begin{tikzcd} 0 \arrow[r, bend left, "m"] \arrow[r, bend right, "n"'] \arrow[r, phantom, "\Downarrow\psi"] & 1 \end{tikzcd}$$

Before giving a more explicit description of these concepts, let us explore a simpler notion than that of m-module, and study the composition of such entities.

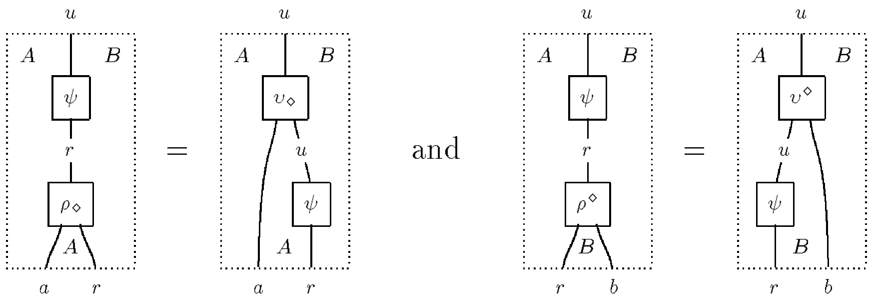
2.02. Convention. From now on let $\boldsymbol{Y}$ be a fixed bicategory. To simplify the notation, we write $[A,B]$ for the hom-category $\langle A,B\rangle\boldsymbol{Y}$. If $A' \xrightarrow{f} A$ and $B \xrightarrow{g} B'$ are 1-cells, the functor $[A,B] \xrightarrow{[f,g]} [A',B']$ pre-composes with f and post-composes with g.

2.03. Definition. Let $A \xrightarrow{a} A$ and $B \xrightarrow{b} B$ be endo-1-cells. A *module* $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ from a to b consists of a 1-cell $A \xrightarrow{r} B$, the *carrier*, equipped with a left action $ar \xRightarrow{\rho_\diamond} r$ and a right action $rb \xRightarrow{\rho^\diamond} r$ subject to the following *stacking condition*



(2-02)

We write aR and Rb for the modules $\langle a\rho_\diamond, ar, a\rho^\diamond\rangle$ and $\langle \rho_\diamond b, rb, \rho^\diamond b\rangle$, respectively. If $U = \langle v_\diamond, u, v^\diamond\rangle$ is another module from a to b, by a *modism* $R \overset{\psi}{\Longrightarrow} U$ we mean a 2-cell $r \overset{\psi}{\Longrightarrow} u$ compatible with actions in the following sense

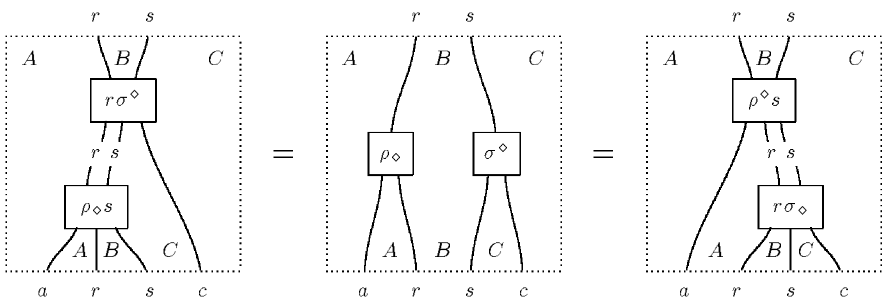


(2-03)

The category of modules from a to b with modisms as arrows is denoted by $\langle a,b\rangle\boldsymbol{mod}$.

2.04. Proposition. *If a, b and c are endo-1-cells on objects A, B and C, respectively, the composition $[A,B]\times[B,C] \xrightarrow{\langle A,B,C\rangle\boldsymbol{c}} [A,C]$ restricts to a functor from $\langle a,b\rangle\boldsymbol{mod} \times \langle b,c\rangle\boldsymbol{mod}$ to $\langle a,c\rangle\boldsymbol{mod}$ that maps $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ and $S = \langle \sigma_\diamond, s, \sigma^\diamond\rangle$ to $RS := \langle \rho_\diamond s, rs, r\sigma^\diamond\rangle$. The 2-cells $rbs \overset{\rho^\diamond s}{\Longrightarrow} rs$ and $rbs \overset{r\sigma_\diamond}{\Longrightarrow} rs$ are modisms from $RbS = (Rb)S = R(bS)$ to RS that constitute the $\langle R,S\rangle$-components of two natural transformations with codomain $\langle A,B,C\rangle\boldsymbol{c}$.*

Proof. The stacking condition for RS is trivial, since $\rho_\diamond$ and $\sigma^\diamond$ are independent:



Establishing that modisms $R \overset{\kappa}{\Longrightarrow} R'$ and $S \overset{\nu}{\Longrightarrow} S'$ yield a modism $RS \overset{\kappa\nu}{\Longrightarrow} R'S'$ is similarly straightforward.

The stacking conditions for R and S, and the independence of $\rho^\diamond$ and $\sigma^\diamond$ as well as $\rho_\diamond$ and $\sigma_\diamond$ imply that $\rho^\diamond s$ and $r\sigma_\diamond$ are modisms from RbS to RS. Naturality turns out to be a direct consequence of the modism axioms. ∎

The composition operations for modules defined in Proposition 2.04., although clearly associative, are not very interesting, since they ignore the right action of the first factor and the left action of the second factor. How to define a more meaningful composition operation for modules $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ and $S = \langle \sigma_\diamond, s, \sigma^\diamond\rangle$ is indicated by the second part of this proposition: form the coequalizer of $\rho^\diamond s$ and $r\sigma_\diamond$ in $\langle a,b\rangle\boldsymbol{mod}$. Of course, this approach requires the existence of the appropriate coequalizers.

2.05. Definition. We say that $\boldsymbol{Y}$ *has stable local colimits* of a given type, provided these colimits exist in every hom-category, and 1-cell composition from either side preserves them. If $\boldsymbol{Y}$ has stable local small colimits, Carboni et. al. [3] call $\boldsymbol{Y}$ *distributive.*

2.06. Assumption. *$\boldsymbol{Y}$ has stable local coequalizers.*

For technical reasons we assume a certain choice of these coequalizers has been made. Hence we can refer to *the* coequalizer of a parallel pair. Although such a choice need not be canonical in any sense, we still may assume that 1-cell composition from either side preserves the chosen coequalizers.

2.07. Proposition. *For endo-1-cells a and b the category $\langle a,b\rangle\boldsymbol{mod}$ has coequalizers.*

Proof. For modules $R=\langle\rho_\diamond,r,\rho^\diamond\rangle$ and $U=\langle\upsilon_\diamond,u,\upsilon^\diamond\rangle$ from a to b consider modisms ψ and χ from R to U. Let φ be the coequalizer of ψ and χ in $[A,B]$. Since composition from either side preserved coequalizers, the modism axioms for ψ and χ allow us to obtain actions $aw \xRightarrow{\omega_\diamond} w$ and $wb \xRightarrow{\omega^\diamond} w$ via the following diagram

$$\begin{tikzcd}
rb \arrow[r, Rightarrow, shift left, "\psi b"] \arrow[r, Rightarrow, shift right, "\chi b"'] \arrow[d, Rightarrow, "\rho^\diamond"'] & ub \arrow[r, Rightarrow, "\varphi b"] \arrow[d, Rightarrow, "\upsilon^\diamond"] & wb \arrow[d, Rightarrow, dotted, "\omega^\diamond"] \\
r \arrow[r, Rightarrow, shift left, "\psi"] \arrow[r, Rightarrow, shift right, "\chi"'] & u \arrow[r, Rightarrow, "\varphi"] & w \\
ar \arrow[u, Rightarrow, "\rho_\diamond"] \arrow[r, Rightarrow, shift left, "a\psi"] \arrow[r, Rightarrow, shift right, "a\chi"'] & au \arrow[u, Rightarrow, "\upsilon_\diamond"'] \arrow[r, Rightarrow, "a\varphi"'] & aw \arrow[u, Rightarrow, dotted, "\omega_\diamond"']
\end{tikzcd}$$

Since $a\varphi b$ is a coequalizer, and hence epi, we can derive the stacking condition for $\omega_\diamond$ and $\omega^\diamond$ directly from the stacking condition for $\upsilon_\diamond$ and $\upsilon^\diamond$. ∎

2.08. Definition. For modules $R=\langle\rho_\diamond,r,\rho^\diamond\rangle$ from a to b and $S=\langle\sigma_\diamond,s,\sigma^\diamond\rangle$ from b to c let $\langle\varphi,R\bullet S\rangle$ be the coequalizer in $\langle a,b\rangle\boldsymbol{mod}$ of $\rho^\diamond s$ and $r\sigma_\diamond$. We denote the carrier of $R\bullet S$ by $r\bullet s$, and the left and right actions by $\rho_\diamond\bullet s$ and $r\bullet\sigma^\diamond$, respectively.

2.09. Proposition. *The operation $\bullet$ on modules extends to a family of functors*

$$\langle a,b\rangle\boldsymbol{mod}\times\langle b,c\rangle\boldsymbol{mod}\xrightarrow{\bullet}\langle a,c\rangle\boldsymbol{mod}$$

that is essentially associative as specified by Axiom (B0).

Proof. To establish the functoriality of module composition, let $R\xRightarrow{\kappa}\bar R$ and $S\xRightarrow{\nu}\bar S$ be modisms, where $a\xrightarrow{R}b$ and $b\xrightarrow{S}c$. If φ and $\bar\varphi$ are the coequalizers used to construct $R\bullet S$ and $\bar R\bullet\bar S$, respectively, the universal property of φ induces $\kappa\bullet\nu$ via

$$\begin{tikzcd}
RbS \arrow[r, Rightarrow, shift left, "\rho^\diamond s"] \arrow[r, Rightarrow, shift right, "r\sigma_\diamond"'] \arrow[d, Rightarrow, "\kappa b\nu"'] & RS \arrow[r, Rightarrow, "\varphi"] \arrow[d, Rightarrow, "\kappa\nu"] & R\bullet S \arrow[d, Rightarrow, dotted, "\kappa\bullet\nu"] \\
\bar Rb\bar S \arrow[r, Rightarrow, shift left, "\bar\rho^\diamond\bar S"] \arrow[r, Rightarrow, shift right, "\bar R\bar\sigma_\diamond"'] & \bar R\bar S \arrow[r, Rightarrow, "\bar\varphi"'] & \bar R\bullet\bar S
\end{tikzcd}$$

where the rectangles on the left commute because of naturality, *cf.* Proposition 2.04..

For another module $U = \langle \upsilon_\diamond, u, \upsilon^\diamond \rangle$ from c to d and the coequalizers φ, λ, ψ and μ used in the construction of $R \bullet S$, $S \bullet U$, $(R \bullet S) \bullet U$ and $R \bullet (S \bullet U)$, respectively, consider the diagram

$$
\begin{tikzcd}
 & & Rb(S \bullet U) \arrow[dr, Rightarrow, shift left, "\rho^\diamond(s \bullet u)"] \arrow[dr, Rightarrow, shift right, "r(\sigma_\diamond \bullet u)"'] & & \\
 & RbSU \arrow[ur, Rightarrow, "rb\lambda"] \arrow[dr, Rightarrow, shift left, "\rho^\diamond su"] \arrow[dr, Rightarrow, shift right, "r\sigma_\diamond u"'] & & R(S \bullet U) \arrow[r, Rightarrow, "\mu"] & R \bullet (S \bullet U) \arrow[dd, dotted, "\langle R,S,U \rangle \dot{\mathfrak{a}}"] \\
RbScU \arrow[ur, Rightarrow, shift left, "rb\sigma^\diamond u"] \arrow[ur, Rightarrow, shift right, "rbs\upsilon_\diamond"'] \arrow[dr, Rightarrow, shift left, "\rho^\diamond scu"] \arrow[dr, Rightarrow, shift right, "r\sigma_\diamond cu"'] & & RSU \arrow[ur, Rightarrow, "r\lambda"'] \arrow[dr, Rightarrow, "\varphi u"] & & \\
 & RScU \arrow[ur, Rightarrow, shift left, "r\sigma^\diamond u"] \arrow[ur, Rightarrow, shift right, "rs\upsilon_\diamond"'] \arrow[dr, Rightarrow, "\varphi cu"'] & & (R \bullet S)U \arrow[r, Rightarrow, "\psi"'] & (R \bullet S) \bullet U \\
 & & (R \bullet S)cU \arrow[ur, Rightarrow, shift left, "(r \bullet \sigma^\diamond)u"] \arrow[ur, Rightarrow, shift right, "(r \bullet s)\upsilon_\diamond"'] & &
\end{tikzcd}
\tag{2-04}
$$

Both $\varphi u \,;\, \psi$ and $r\lambda \,;\, \mu$ turn out to be multi-coequalizers of the four 2-cells $\rho^\diamond \sigma^\diamond u$, $\rho^\diamond s \upsilon_\diamond$, $r\sigma_\diamond \upsilon_\diamond$, and $r\sigma_\diamond cu \,;\, r\sigma^\diamond u = rb\sigma^\diamond u \,;\, r\sigma_\diamond u$ from $RbScU$ to RSU, which induces the iso $\langle R, S, U \rangle \dot{\mathfrak{a}}$ from $(R \bullet S) \bullet U$ to $R \bullet (S \bullet U)$. The universal property of colimits also allows us to establish the coherence axiom (B0) in a lengthy but straightforward diagram chase. ■

From now on we will no longer explicitly mention the associativity isos. In some sense it is taken care of by the geometry of the diagrams, this applies to pasting diagrams as well as to string diagrams. More formally, we utilize the fact that every bicategory is biequivalent to a 2-category, *e.g.*, Theorem 1.4 in [4].

3. Interpolads and monads

In general, endo-1-cells, modules and modisms do not form a bicategory, since identity modules may not exist. Identity modules must at least be idempotent under module composition. The only reasonable candidate for an identity module on $A \xrightarrow{a} A$ is a, equipped with suitable left and right actions. Of course, modules now have to be compatible with the extra structure.

3.00. Definition. Let $A \xrightarrow{a} A$ be a 1-cell in $\boldsymbol{Y}$. We refer to any 2-cell $aa \overset{\alpha}{\Longrightarrow} a$ as a *multiplication*, and we call α *interpolative*, and the pair $\mathcal{A} = \langle a, \alpha \rangle$ an *interpolad*, if α is a coequalizer of αa and $a\alpha$ (and hence in particular associative). If $\mathcal{A}$ and $\mathcal{B} = \langle b, \beta \rangle$ are interpolads, an *i-module* $R = \langle \rho_\diamond, r, \rho^\diamond \rangle$ from $\mathcal{A}$ to $\mathcal{B}$, denoted by $\mathcal{A} \xrightarrow{R} \mathcal{B}$, is a module from a to b that satisfies

- (I0) $\rho_\diamond$ is a coequalizer of αr and $a\rho_\diamond$ (not necessarily the chosen one);
- (I1) $\rho^\diamond$ is a coequalizer of $\rho^\diamond b$ and $r\beta$ (not necessarily the chosen one).

Modisms between i-modules are ordinary modisms.

It is clear that interpolads can be interpreted as their own identity i-modules, with the multiplication serving double duty as left and right action; the two i-module axioms together with the stacking condition (2-02) guarantee the properties of a left and right unit, respectively.

3.01. REMARKS. (0) Considering idempotent modules $\langle \alpha_\diamond, a, \alpha^\diamond \rangle$ instead of interpolads leads to no further generality. In this case for an i-module $R = \langle \rho_\diamond, r, \rho^\diamond \rangle$ from $\langle \alpha_\diamond, a, \alpha^\diamond \rangle$ to $\langle \beta_\diamond, b, \beta^\diamond \rangle$ the left action $\rho_\diamond$ ought to be a coequalizer of $\alpha^\diamond r$ and $a\rho_\diamond$, and the right action $\rho^\diamond$ ought to be a coequalizer of $\rho^\diamond b$ and $r\beta_\diamond$. For $\langle \alpha_\diamond, a, \alpha^\diamond \rangle$ to be an endo-i-module on itself, both $\alpha_\diamond$ and $\alpha^\diamond$ have to be coequalizers of $\alpha^\diamond a$ and $a\alpha_\diamond$.

(1) Another way to arrive at the notions of interpolad and i-module is to relax the notions of lax functor with domain $\boldsymbol{1}$, respectively $\boldsymbol{2}$, just enough as to keep the coequalizers previously guaranteed by implicit split coequalizers. [more precisely...]

(2) Each $\boldsymbol{Y}$-object A trivially is structured as an interpolad: with the identity 1-cell A as carrier and the multiplication $A\mathfrak{u}$ (*cf.* Remark 1.02.(0)). A 1-cell $A \xrightarrow{r} B$ can be viewed as a trivial i-module between trivial interpolads, by taking $Ar \xRightarrow{r\mathfrak{u}_\diamond} r$ and $rB \xRightarrow{r\mathfrak{u}^\diamond} r$ as actions. Naturality and coherence (use Remark 1.02.(1) here) guarantee that the diagram (2-02) commutes, and also the coequalizer conditions. Clearly, 2-cells of $\boldsymbol{Y}$ turn into modisms between such trivial i-modules.

(3) The terms "interpolative" and "interpolad" are inspired by the *interpolation property* for binary relations that is of some interest in domain theory. A relation $a \subseteq A \times A$ has the interpolation property, if it satisfies $a \subseteq aa$, *i.e.*, the converse of the transitivity requirement. The latter, in general, corresponds to the associativity of the multiplication, while the interpolation property may be viewed as a couniversal property of the associativity diagram. We propose the slogan:

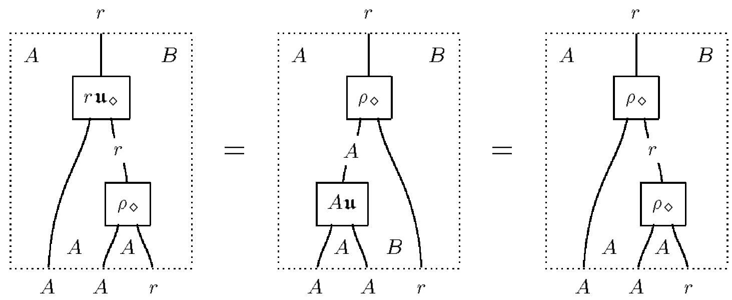
$$\text{associative} + \text{interpolation property} = \text{interpolative}$$

3.02. THEOREM. *Interpolads in $\boldsymbol{Y}$, i-modules and modisms form a bicategory $\boldsymbol{Y}$-$\boldsymbol{int}$ that contains $\boldsymbol{Y}$ as a full sub-bicategory.*

PROOF. I-modules between two interpolads and their modisms clearly form a category. A simple coequalizer argument shows that i-modules are closed under module composition.

For an i-module $\mathcal{A} \xrightarrow{R} \mathcal{B}$ let $ar \xRightarrow{\varphi} a \bullet r$ be the coequalizer of αr and $a\rho_\diamond$. Since $\rho_\diamond$ is a coequalizer of this pair as well, we obtain an iso $\mathcal{A} \bullet R \xRightarrow{R\dot{\mathfrak{u}}_\diamond} R$. Similarly, we get an iso $R \bullet \mathcal{B} \xRightarrow{R\dot{\mathfrak{u}}^\diamond} R$. These isos together with the associativity isos $\dot{\mathfrak{a}}$ defined in the proof of Proposition 2.09. clearly satisfy coherence axiom (B1). This establishes the bicategory $\boldsymbol{Y}$-$\boldsymbol{int}$.

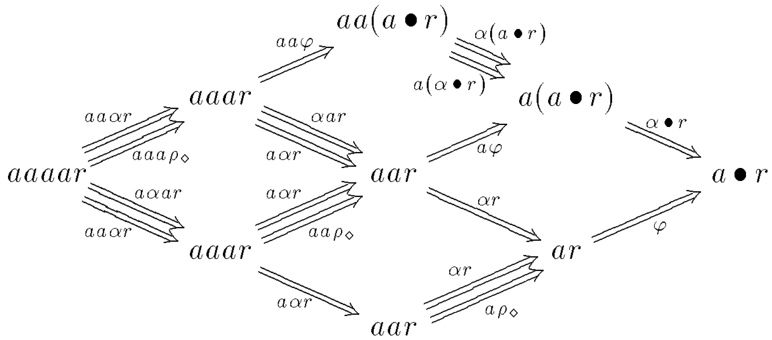
Naturality of $\mathfrak{u}_\diamond$ and (I0) for an i-module $\langle \rho_\diamond, r, \rho^\diamond \rangle$ from $\langle A, A\mathfrak{u} \rangle$ to $\langle B, B\mathfrak{u} \rangle$ imply



Since $A\rho_\diamond$ is epi, we get $\rho_\diamond = r\mathfrak{u}_\diamond$. A similar argument shows $\rho^\diamond = r\mathfrak{u}^\diamond$. Hence we can identify modules from $A\mathfrak{u}$ to $B\mathfrak{u}$ with 1-cells from A to B. Coherence condition (B1) implies that composition of trivial i-modules reduces to 1-cell composition in $\boldsymbol{Y}$. ■

3.03. Proposition. *If $\mathcal{A}$ and $\mathcal{B}$ are interpolads with carriers a and b, respectively, and if $a \xrightarrow{R} b$ is any module, then $\mathcal{A} \bullet R \bullet \mathcal{B}$ is an i-module from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Let $R = \langle \rho_\diamond, r, \rho^\diamond \rangle$ be a module from a to b, and consider the following diagram, where φ is the chosen coequalizer of αr and $a\rho_\diamond$ used in the construction of $a \bullet r$:



Since 1-cell composition preserves coequalizers, and since α is a coequalizer of αa and $a\alpha$, it follows immediately that the left action $\alpha \bullet r$ of $\mathcal{A} \bullet R$ is a coequalizer of $\alpha(a \bullet r)$ and $a(\alpha \bullet r)$ and hence satisfies condition (I0). Similarly, we see that the left action $\alpha \bullet r \bullet b$ of $\mathcal{A} \bullet R \bullet \mathcal{B}$ satisfies (I0), and that the right action $a \bullet r \bullet \beta$ of this module satisfies (I1), *i.e.*, $\mathcal{A} \bullet R \bullet \mathcal{B}$ is an i-module from $\mathcal{A}$ to $\mathcal{B}$. ■

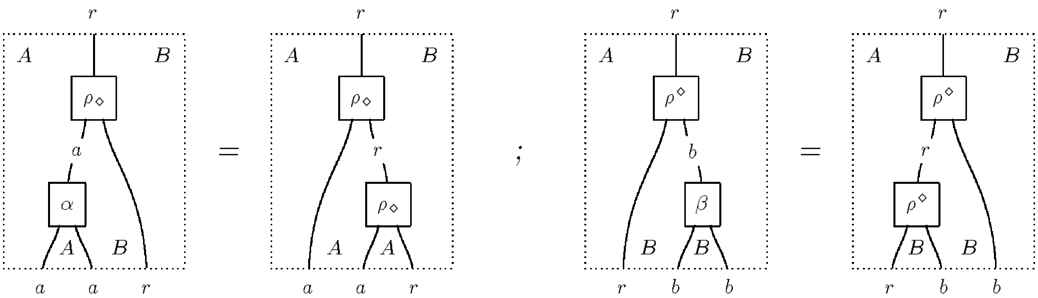
Clearly, an interpolative multiplication is the minimal structure necessary to support identity modules, and hence yield a bicategory. Adding more structure will lead to sub-bicategories of $\boldsymbol{Y}$-$\boldsymbol{int}$ that in general will not be full, if the modules have to preserve this additional structure. The modisms will be the same in all cases.

The obvious extra structure to add is a two-sided unit for the multiplication. Then we obtain monads instead of interpolads. The interpolation property for the multiplication then follows from the unit axioms, hence only associativity has to be required.

3.04. Proposition. *Let $\mathcal{A} = \langle a, \alpha, \alpha' \rangle$ and $\mathcal{B} = \langle b, \beta, \beta' \rangle$ be monads with units α' and β', respectively. For a module $R = \langle \rho_\diamond, r, \rho^\diamond \rangle$ from a to b the following are equivalent:*

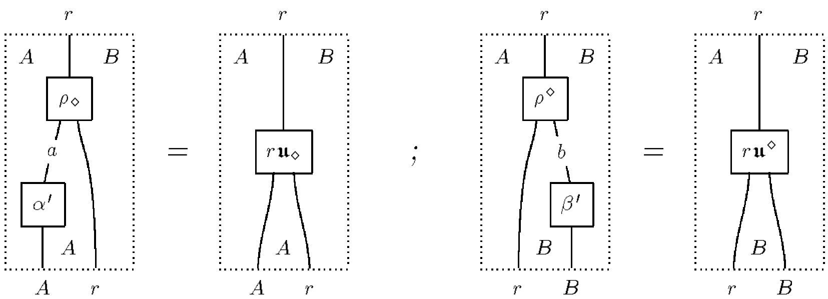
(a) *R is an m-module;*

(b) *R is a module that is compatible with the multiplication*



(3-00)

and with the units



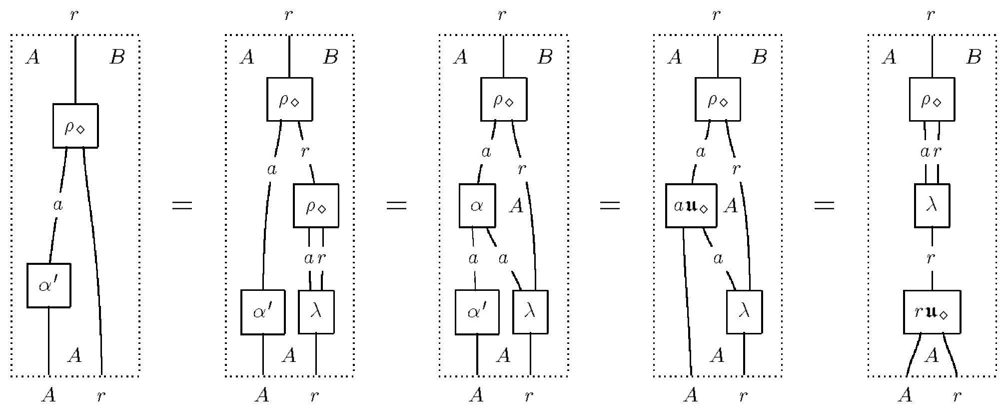
(3-01)

(c) *R is an i-module, and both actions are split epi.*

Proof. (b) $\Leftrightarrow$ (a) Let $\boldsymbol{F}$ be a lax functor from $\mathit{2}$ to $\boldsymbol{Y}$. Besides two monads $\mathcal{A}$ and $\mathcal{B}$ (based on $0\boldsymbol{F}$ and $1\boldsymbol{F}$, respectibely) $\boldsymbol{F}$ also specifies a module R based on $k\boldsymbol{F}$ subject to five compatibility conditions: the stacking condition (2-02) as well as the commutativity of the diagrams specified by (I0) and (I1), and the two equalities in (3-01).

(b) $\Rightarrow$ (c) By (3-01) the actions are split epi, which togther with (3-00) implies (I0) and (I1).

(c) $\Rightarrow$ (b) Let $r \overset{\lambda}{\Longrightarrow} ar$ be a left inverse for $\rho_\diamond$. Then we have



where the second step uses (I0), the third one uses (2-01), and the fourth one uses the naturality of $\mathfrak{u}_\diamond$. The other condition follows by symmetry. ∎

3.05. Remarks. (0) Condition (b) of Proposition 3.04. directly translates the notion of bimodule familiar from algebra, *cf.* Examples 1.03.(2), 3.07.(1) as well as [7] and [3]. We have dropped the prefix "bi", because of it's different connotation in "bicategory". The terms "distributor" and "profunctor" also appear in the literature.

(1) Remark 3.01.(2) carries over to the monad setting.

3.06. Theorem. *Monads in* $\boldsymbol{Y}$, *m-modules and modisms form a bicategory* $\boldsymbol{Y}$-$\boldsymbol{mnd}$ *that contains* $\boldsymbol{Y}$ *as a full sub-bicategory. Moreover, there is a forgetful lax functor* $\boldsymbol{Y}\text{-}\boldsymbol{mnd} \xrightarrow{\mathfrak{Y}} \boldsymbol{Y}$ *with the property that every lax functor* $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ *factors through* $\mathfrak{Y}$ *by means of a normalized lax functor.*

Proof. A simple calculation shows that m-modules are closed under module composition. Thus we obtain a bicategory $\boldsymbol{Y}$-$\boldsymbol{mnd}$. The lax functor $\mathfrak{Y}$ maps monads to their underlying objects, m-modules to their underlying 1-cells, and modisms to their underlying 2-cells. The structural 2-cells for composits are given by the coequalizers used to define the m-module composition, and by the units of the monads. Hence by construction any lax functor $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ factors through $\mathfrak{Y}$ in the specified fashion. ∎

In general, a lax functor $\boldsymbol{X} \xrightarrow{\boldsymbol{F}} \boldsymbol{Y}$ will not factor through $\mathfrak{Y}$ by means of a bifunctor. The price for upgrading a lax functor to a normalized lax functor is that 1-cell composition in $\boldsymbol{Y}$-$\boldsymbol{mnd}$ usually is more complicated than in $\boldsymbol{Y}$.

3.07. Examples. (0) Let $\boldsymbol{X}$ be the suspension of $\boldsymbol{set}$ (or of any elementary topos), *i.e.*, $\boldsymbol{X}$ has a single 0-cell $*$, sets as (endo-) 1-cells with cartesian product as horizontal composition, and functions as 2-cells. Here we use capital letters for sets, and small letters for elements. A module $\boldsymbol{R}$ from A to B is a set R equipped with left and right actions $A \times R \overset{\rho_\diamond}{\Longrightarrow} R$ and $R \times B \overset{\rho^\diamond}{\Longrightarrow} R$ that satisfy

$$(ar)b = a(rb) \tag{3-02}$$

for $a \in A$, $r \in R$, and $b \in B$, where we have used concatenation for both actions. Composing $\boldsymbol{R}$ with a module $\boldsymbol{S}$ from B to C results in the module $\boldsymbol{R} \bullet \boldsymbol{S}$ based on the set $(R \times S)/ \leftrightsquigarrow$, where $\leftrightsquigarrow$ is the smallest equivalence relation on $R \times S$ that contains all pairs $\langle\langle rb, s\rangle, \langle r, bs\rangle\rangle$, for $r \in R$, $b \in B$, and $s \in S$. Modisms are simply functions that commute with the actions.

Interpolads now are *interpolative semi-groups*, *i.e.*, semi-groups $\langle A, \circ\rangle$ with surjective multiplication such that the kernel pair of $\circ$ coincides with the equivalence relation generated by $\langle \circ \times A, A \times \circ\rangle$ (this automatically implies associativity). The actions of an i-module from $\langle A, \circ\rangle$ to an interpolative semi-group $\langle B, \cdot\rangle$ besides (3-02) have to satisfy

$$(a' \circ a)r = a'(ar) \quad \text{and} \quad (rb)b' = r(b \cdot b') \tag{3-03}$$

Moreover, they have to be surjective, and their kernel pairs have to agree with the equivalence relations generated by $\langle \circ \times R, A \times \rho_\diamond\rangle$ and $\langle \rho^\diamond \times B, R \times \cdot\rangle$, respectively.

Semi-groups with identity, *i.e.*, monoids, automatically are interpolative and correspond to monads. The actions of an m-module from a monoid $\langle A, \circ, e\rangle$ to a monoid $\langle B, \cdot, i\rangle$ besides satisfying (3-02) and (3-03) have to preserve the identities, *i.e.*,

$$er = r \quad \text{and} \quad ri = r \tag{3-04}$$

which again implies the interpolation properties.

If $\langle \rho_\diamond, R, \rho^\diamond\rangle$ is an m-module from $\langle A, \circ, e\rangle$ to $\langle B, \cdot, i\rangle$, then invertible elements $x \in A$ and $y \in B$ give rise to i-modules between these monoids that need not be m-modules: just defined new left and right actions $A \times R \xRightarrow{\bar{\rho}_\diamond} R$ and $R \times B \xRightarrow{\bar{\rho}^\diamond} R$ by $\langle a, r\rangle\bar{\rho}_\diamond := (x \circ a)r$ and $\langle r, b\rangle\bar{\rho}^\diamond := r(b \cdot y)$. I-modules arising in this fashion might be called *affine*.

For interpolative semi-groups the multiplication is a left and a right action and induces an idempotent endo-module, the identity module. For ordinary semi-groups this can fail. In case of the positive natural numbers $\boldsymbol{N}_+$ under addition we obtain $\boldsymbol{N}_+ \bullet \boldsymbol{N}_+ \cong \boldsymbol{N}_{>1}$, on which $\boldsymbol{N}_+$ acts from both sides.

Just as monoids are categories with one object, m-modules between monoids can be interpreted as special categories with two objects, *cf.* part (4) below.

(1) Replacing $\boldsymbol{set}$ above with $\boldsymbol{ab}$, the category of abelian groups, yields unitary rings as monads. Bimodules and their homomorphisms serve as m-modules and modisms. The composition of bimodules is their tensor product, *cf.* Example 1.03.(2).

For the suspension of $\boldsymbol{ab}$, interpolads will be "rngs", *i.e.*, "rings not necessarily with unit", that satisfy the interpolation property. Since regular epis in $\boldsymbol{ab}$ are cokernels and hence surjective, the same description as in (0) applies.

(2) For $\boldsymbol{Y} = \boldsymbol{rel}$, the bicategory of sets, relations and inclusions, an endo-1-cell on A is just a binary relation $a \subseteq A \times A$. A module from a to $b \subseteq B \times B$ is a

relation $r \subseteq A \times B$ that satisfies $ar \subseteq r \supseteq rb$. Since coequalizers in $\boldsymbol{rel}$ are trivial, modules compose like ordinary relations. Modisms are just inclusions.

The interpolads in $\boldsymbol{rel}$ turn out to be Steve Vickers' *information systems* [12], *i.e.*, sets equipped with an idempotent (= transitive and interpolative) relation. An i-module from $\langle A, < \rangle$ to $\langle B, \sqsubset \rangle$ now is a relation $r \subseteq A \times B$ that satisfies $(\, < r) = r = (r \sqsubset \,)$. Since the relation $<$ on $\boldsymbol{2}$ is transitive but not idempotent, we see again that identity modules may fail to exist in the absence of the interpolation property. The resulting bicategory, which we call $\boldsymbol{kar}$, also has been studied by Rosebrugh and Wood [8]. However, their motivation was not a weakening of the notion of monad, but rather the issue of Cauchy-completeness. We return to this subject in Section 4.

A monad in $\boldsymbol{rel}$ is simply a pre-ordered set *i.e.*, a set equipped with a reflexive and transitive relation (sometimes called *graph*). M-modules from $\langle A, \leq \rangle$ to $\langle B, \sqsubseteq \rangle$ are those relations $r \subseteq A \times B$ that satisfy $(\, \leq r) \subseteq r \supseteq (r \sqsubseteq \,)$ (by reflexivity the other inclusions hold automatically) or, equivalently, for which the characteristic function preserves order from $\langle A, \leq \rangle^{\mathrm{op}} \times \langle B, \sqsubseteq \rangle$ to $\boldsymbol{2}$. Such relations are known as *order-ideals*. Pre-ordered sets, order-ideals and inclusions form a sub-bicategory $\boldsymbol{idl}$ of $\boldsymbol{kar}$. Since $\boldsymbol{kar}$ is locally ordered, this inclusions happens to be full.

(3) Every $\boldsymbol{ord}$-enriched category $\boldsymbol{Y}$ may be viewed as a bicategory – the hom-categories are ordered sets and the composition preserves order. Coequalizers in the hom-categories are identities (or isos, if we do not insist on antisymmetry) and hence are trivially preserved. Endo-1-cells on A are elements of the poset $[A, A]$, and a module from $a \in [A, A]$ to $b \in [B, B]$ is an element $r \in [A, B]$ that satisfies $ar \leq r \geq rb$. In this situation we need not distinguish between i-modules and m-modules: the actions simply have to be identities, *i.e.*, we have $ar = r = rb$. For $\boldsymbol{Y} = \boldsymbol{ord}$ interpolads correspond to idempotent endo-functions, while monads correspond to closure operators. Another special case is the bicategory $\boldsymbol{rel}$, *cf.* part (2). It is even enriched in the category $\boldsymbol{cslat}$ of complete lattices with additive or sup-preserving (= left adjoint) functions, since relations from A to B bijectively correspond to left-adjoint functions from $A\boldsymbol{P}$ to $B\boldsymbol{P}$. Categories enriched in $\boldsymbol{cslat}$ are also known as *quantaloids*, *cf.* Example 5.11.(2).

(4) Just as a relation between sets A and B can be viewed as a 2-valued $A \times B$ matrix, a span $A \overset{r}{\longrightarrow\!\!\triangleright} B$, *i.e.*, a function $R \xrightarrow{r} A \times B$, can be viewed as a $\boldsymbol{set}$-valued $A \times B$ matrix, assigning to each pair $\langle x, y \rangle \in A \times B$ a set of *formal arrows*. The composition with $B \overset{s}{\longrightarrow\!\!\triangleright} C$ now is just the matrix product with cartesian product $\times$ and disjoint union $+$ instead of boolean meet $\wedge$ and join $\vee$, *i.e.*, $\langle x, z \rangle (rs)$ is given by $\sum_{y \in B} \langle x, y \rangle r \times \langle y, z \rangle s$. The identity matrices for this operation have singletons in the diagonal, and the empty set everywhere else. Span-morphisms between spans from A to B are $A \times B$ matrices of functions. Their vertical composition as well as the formation of (co-)limits is performed component-wise. The resulting bicategory is called $\boldsymbol{spn}$.

An endo-span a on A can be interpreted as a directed multigraph with vertex set A. If b is another such with vertex set B, a module $a \stackrel{R}{\Longrightarrow} b$ specifies sets of formal arrows from A-vertices to B-vertices that compose associatively with the a-arrows and the b-arrows. If $b \xrightarrow{S} c$ is another module, the composite RS at each B-vertex simply links incoming and outgoing arrows, while $R \bullet S$ takes the possible 1-step transfers within the directed multigraph b into account, and factors by the appropriate equivalence relation.

Two pairs of arrows $\langle f, \bar{h}\rangle \in \langle x, u\rangle r \times \langle u, z\rangle s$ and $\langle \bar{f}, h\rangle \in \langle x, v\rangle r \times \langle v, z\rangle s$ have a common pre-image wrt. $\rho^{\diamond} s$ and $r\sigma_{\diamond}$ iff $\langle f, \bar{h}\rangle$ "diagonalizes" over $\langle \bar{f}, h\rangle$, *i.e.*, iff $\bar{f} = f \,;\, g$ and $\bar{h} = g \,;\, h$ for some $g \in \langle u, v\rangle b$, or vice versa. Hence $\langle f, \bar{h}\rangle$ and $\langle \bar{f}, h\rangle$ are equivalent iff they can be linked by a finite "zig-zag" of diagonals. The corresponding relation $\leftrightsquigarrow_{\langle x,z\rangle}$ is defined on $\sum_{y \in B} \langle x, y\rangle r \times \langle y, z\rangle s$, the $\langle x, z\rangle$-component of rs. Just as span-morphisms are matrices of functions, here we have a matrix $\leftrightsquigarrow$ of relations, a *span-relation* on rs.

A span morphism $aa \stackrel{\alpha}{\Longrightarrow} a$ equips a directed multigraph a with a composition operation, *i.e.*, a family of functions $\sum_{y \in A} \langle x, y\rangle a \times \langle y, z\rangle a \xrightarrow{\langle x,z\rangle \alpha} \langle x, z\rangle a$. Clearly, associativity of α is equivalent to the associativity of the composition operation. If two vertices are connected by two different directed paths, we may ask if one path can be "deformed" into the other, using the composition operation. Geometrically speaking, this is possible iff the "region" between the paths can be completely "triangulated". Thus we obtain a notion of homotopy equivalence between paths. The pair $\langle a, \alpha\rangle$ now is an interpolad provided that for paths of length 2 with common domain and codomain the homotopy equivalence (the same as $\leftrightsquigarrow$ above) is the kernel of the composition operation.

It is now obvious that a monad $\langle a, \alpha, \alpha'\rangle$ in $\boldsymbol{spn}$ on A turns out to be a small category. The $\langle x, x\rangle$-component $1 \longrightarrow \langle x, x\rangle a$ of α' selects the identity morphism on $x \in A$. M-modules from $\mathcal{A}$ to another monad $\mathcal{B}$ correspond to profunctors, *i.e.*, functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \boldsymbol{set}$. Small categories, profunctors and appropriate natural transformations form a bicategory $\boldsymbol{prof}$. In fact, an m-module from $\mathcal{A}$ to $\mathcal{B}$ may be viewed as a special category on the disjoint union $A + B$ that contains $\mathcal{A}$ and $\mathcal{B}$ as full subcategories and has no arrows from $\mathcal{B}$-objects to $\mathcal{A}$-objects.

Paré and Wood briefly considered the notion of "category without identities" in an attempt to generalize idempotent relations. For their so-called *taxonomies* they just dropped the requirement for identities. But without interpolative composition, identity modules, *i.e.*, generalized hom-functors, need not exist. Hence we propose interpolads in $\boldsymbol{spn}$ as more useful taxonomies. The following graphs on $\{w, x, y, z\}$ illustrate the difference. In the first one the interpolation property fails since both triangles commute, but there is no "zig-zag" linking y and w. The central trapezoid of the second graph does not commute, hence the interpolation property is satisfied.

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr] \arrow[dr] & & z \\ & w \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] \arrow[dd, phantom, "\neq"] & \\ x \arrow[ur] \arrow[dr] \arrow[rr, bend left=60] \arrow[rr, bend right=60] & & z \\ & w \arrow[ur] & \end{tikzcd}$$

Hayashi [5] introduced the notion of *semi-functor* between categories by dropping the requirement for functors that identities have to be preserved, *cf.* also [6]. Any semi-functor between (small) categories that is not a functor induces a module between monads in $\boldsymbol{spn}$ that is an i-module, but not an m-module.

(5) The construction in (4) works for any complete and cocomplete monoidal category $\boldsymbol{V}$. The bicategory $\boldsymbol{V}$-$\boldsymbol{spn}$ has sets as objects and matrices of $\boldsymbol{V}$-objects as 1-cells. Monads in $\boldsymbol{V}$-$\boldsymbol{spn}$ then correspond to small $\boldsymbol{V}$-categories.

Already Benabou [1] introduced a “multi-object” version of the notion of monad in $\boldsymbol{Y}$, which he termed “polyad”. Later this was interpreted as a “category enriched in $\boldsymbol{Y}$”, *cf.*, *e.g.*, [9] and the historical remarks therein. It is straightforward to give a similar generalization of interpolads, resulting in the notion of “taxonomy enriched in a bicategory”. Instead of coequalizers, more complicated colimits are needed. This also applies to the corresponding notion of i-module. We leave it to the reader to spell out the details.

4. Splitting idempotents

Recall that an idempotent morphism f in a category is said to *split*, if it can be factored as a split epi e followed by a split mono m such that $m\,;e$ is an identity. A category is called *Cauchy-complete*, if all idempotents split. A *Cauchy-completion* of a category $\boldsymbol{C}$ can, *e.g.*, be realized within $\boldsymbol{C}/\boldsymbol{C}$ by taking all idempotent morphisms as objects and only those morphisms, where both components coincide. These notions immediately carry over to locally ordered bicategories. The failure of $\boldsymbol{rel}$ to be Cauchy-complete in this sense prompted Rosebrugh and Wood [8] to work with $\boldsymbol{kar}$ instead, which happens to be the Cauchy-completion of $\boldsymbol{rel}$, *cf.* Example 3.07.(2).

In a general bicategory already the notion of idempotent 1-cell becomes more subtle. Requiring $aa = a$ for a 1-cell $A \xrightarrow{a} A$ certainly is too strong, we rather should specify an iso $aa \overset{\alpha}{\Longrightarrow} a$. But in order to be able to replace any power a^n, $n > 0$, by a, which seems to capture the spirit of idempotency, we must ask α to induce only one iso from each such power to a, *i.e.*, to be associative. As an iso, α then automatically is a coequalizer of $a\alpha$ and αa, *i.e.*, we are looking at special interpolads.

4.00. Definition. (0) An *idempolad* $\langle a, \alpha\rangle$ on $A \in \underline{\boldsymbol{Y}}$ consists of a 1-cell $A \xrightarrow{a} A$ and an iso $aa \overset{\alpha}{\Longrightarrow} a$ that satisfies $a\alpha = \alpha a$.

(1) A bicategory $\boldsymbol{Y}$ is called *Cauchy-complete*, if all idempolads in $\boldsymbol{Y}$ split.

(2) The Cauchy-completion $\boldsymbol{Y}$-$\boldsymbol{chy}$ of a bicategory $\boldsymbol{Y}$ is the smallest Cauchy-complete bicategory that contains $\boldsymbol{Y}$ as a full sub-bicategory.

4.01. Examples. (0) The interpretation of $\boldsymbol{Y}$-objects as trivial interpolads does in fact yield idempolads.

(1) If $\boldsymbol{Y}$ is the suspension of a cartesian category $\boldsymbol{C}$, then for every idempolad $\langle X, x\rangle$ in $\boldsymbol{C}\Sigma$ the object X is sub-terminal in $\boldsymbol{C}$: consider $\boldsymbol{C}$-arrows $f, g \in \langle C, X\rangle\boldsymbol{C}$; since the bijections $X \times x$ and $x \times X$ coincide, the projections from $X \times X$ to X agree on $\langle f, f, f\rangle \,;(X \times x)$ and $\langle f, g, f\rangle \,;(x \times X)$. Therefore $\langle f, f, f\rangle = \langle f, g, f\rangle$, and hence $f = g$.

In particular, for $\boldsymbol{C} = \boldsymbol{set}$ we see that although every infinite set X is isomorphic to its square, no isomorphism from $X \times X$ to X can be associative. Hence the only idempolads in the suspension of $\boldsymbol{set}$ are singletons, which trivially split.

4.02. Proposition. *$\boldsymbol{Y}$-$\boldsymbol{int}$ is Cauchy-complete.*

Proof. Consider an endo-i-module $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ on an interpolad $\langle a, \alpha\rangle$ and an associative isomorphism $R\bullet R \overset{\mu}{\Longrightarrow} R$. Let φ, φ', and φ'' be the coequalizers used in the construction of $R\bullet R$, $(R\bullet R)\bullet R$, and $R\bullet(R\bullet R)$, respectively, *cf.* Definition 2.08.. Since both $\varphi R \,; \varphi'$ and $R\varphi \,; \varphi''$ are multi-coequalizers of the four 2-cells $\rho^\diamond\rho^\diamond r$, $\rho^\diamond r \rho_\diamond$, $r\rho_\diamond\rho_\diamond$ and $r\rho_\diamond a r \,; r\rho^\diamond r = r a \rho^\diamond r \,; r\rho_\diamond r$ from $RaRaR$ to RRR, (*cf.* the proof of Proposition 2.09., and in particular Diagram (2-04)), we now have

$$\begin{tikzcd}
 & & RR \arrow[dr, Rightarrow, "\varphi"] & & \\
 & (R\bullet R)R \arrow[ur, Rightarrow, "\mu R"] \arrow[dr, Rightarrow, "\varphi'"'] & & R\bullet R \arrow[dr, Rightarrow, "\mu"] & \\
RRR \arrow[ur, Rightarrow, "\varphi R"] \arrow[dr, Rightarrow, "R\varphi"'] & & R\bullet R\bullet R \arrow[ur, Rightarrow, "\mu\bullet R"] \arrow[dr, Rightarrow, "R\bullet\mu"'] & & R \\
 & R(R\bullet R) \arrow[ur, Rightarrow, "\varphi''"'] \arrow[dr, Rightarrow, "R\mu"'] & & R\bullet R \arrow[ur, Rightarrow, "\mu"'] & \\
 & & RR \arrow[ur, Rightarrow, "\varphi"'] & &
\end{tikzcd} \tag{4-00}$$

Therefore $(\varphi \,; \mu)R \,; \varphi = R(\varphi \,; \mu) \,; \varphi$ has the same multiple coequalizer property, and it follows immediately that φ, and hence also $\varphi \,; \mu$ is in fact a coequalizer of $(\varphi \,; \mu)R$ and $R(\varphi \,; \mu)$. Hence $\langle r, \varphi \,; \mu\rangle$ is an interpolad in $\boldsymbol{Y}$.

To see that $U := \langle \rho_\diamond, r, \varphi \,; \mu\rangle$ and $V := \langle \varphi \,; \mu, r, \rho^\diamond\rangle$ are i-modules from $\langle a, \alpha\rangle$ to $\langle r, \varphi \,; \mu\rangle$ and back, observe that in both cases the stacking condition follows directly from $\varphi \,; \mu$ being modism. Axioms (I0) and (I1) are satisfied, since R is an i-module and since $\langle r, \varphi \,; \mu\rangle$ is an interpolad. Finally, it is easy to see that $U\bullet V = R\bullet R \cong R$, while $V\bullet U$ essentially is the identity module on $\langle r, \varphi \,; \mu\rangle$. ∎

We expect the full sub-bicategory $\boldsymbol{Y}$-$\boldsymbol{idm}$ of $\boldsymbol{Y}$-$\boldsymbol{int}$, spanned by the idempolads, to be the Cauchy-completion of $\boldsymbol{Y}$. Since the notion of idempolad is self-dual, in case that $\boldsymbol{Y}$ locally has stable equalizers as well, we would also expect to recover the Cauchy-completion of $\boldsymbol{Y}$ as a full sub-bicategory inside the bicategory of co-interpolads. Hence i-modules between idempolads ought to be i-comodules and their composition must be independent of the local (co-)completeness properties of $\boldsymbol{Y}$.

4.03. Proposition. *I-modules between idempolads have isomorphisms as actions and bijectively correspond to i-comodules. Moreover, if $\mathcal{B} = \langle b, \beta \rangle$ is an idempolad, then any i-modules $\mathcal{A} \xrightarrow{R} \mathcal{B} \xrightarrow{S} \mathcal{C}$ satisfy $R \bullet S = RS$.*

Proof. Set $R = \langle \rho_\diamond, r, \rho^\diamond \rangle$. Since $\rho^\diamond$ is a coequalizer of $\rho^\diamond b$ and $r\beta$, stability implies that $\rho^\diamond b$ is a coequalizer of $\rho^\diamond bb$ and $r\beta b = rb\beta$. In particular $\rho^\diamond bb\,;\rho^\diamond b = rb\beta\,;\rho^\diamond b = \rho^\diamond bb\,;r\beta$. But $\rho^\diamond bb$ is epi, so we obtain $\rho^\diamond b = r\beta$ and therefore $\rho^\diamond$ is an isomorphism. If $\mathcal{A}$ is an idempolad as well, we see that $\rho_\diamond$ is also an isomorphism, and clearly $\langle (\rho_\diamond)^{-1}, r, (\rho^\diamond)^{-1} \rangle$ is an i-comodule.

For the second i-module $S = \langle \sigma_\diamond, s, \sigma^\diamond \rangle$ we get $\beta s = b\sigma_\diamond$, and therefore $\rho^\diamond bs = rb\sigma_\diamond$. Now $\rho^\diamond bs\,; r\sigma_\diamond = rb\sigma_\diamond\,;\rho^\diamond s$ implies $r\sigma_\diamond = \rho^\diamond s$. Hence the coequalizer φ used in the construction of $r \bullet s$ is an isomorphism. ∎

4.04. Proposition. *For every bicategory $\boldsymbol{X}$ the bicategory $\boldsymbol{X}$-**idm** of idempolads, i-modules and modisms is a Cauchy-completion of $\boldsymbol{Y}$.*

Proof. As we just saw in Proposition 4.03., the composition of i-modules between idempolads is well-defined for any bicategory. This establishes the bicategory $\boldsymbol{X}$-**idm**. Its Cauchy-completeness follows just as that of $\boldsymbol{Y}$-**int** in Proposition 4.02. and the minimality is obvious. ∎

What about the Cauchy-completion of $\boldsymbol{Y}$-**mnd**? Under mild hypotheses on the hom-categories every interpolad in $\boldsymbol{Y}$ occurs by splitting an idempolad in $\boldsymbol{Y}$-**mnd**.

4.05. Proposition. *If $\boldsymbol{Y}$ is distributive, for an interpolad $\langle a, \alpha \rangle$ the 1-cell $A \xrightarrow{a+A} A$ carries a monad structure with multiplication $[\alpha, a\mathbf{u}_\diamond, a\mathbf{u}^\diamond] + A\mathbf{u}$ and unit $A \overset{A\iota}{\Longrightarrow} a + A$. Moreover, $\langle \alpha, a, \alpha \rangle$ is an idempotent m-module on this monad, which in $\boldsymbol{Y}$-**int** splits through $\langle a, \alpha \rangle$.*

Proof. By distributivity, $(a + A)(a + A)$ is isomorphic to $aa + aA + Aa + AA$. Now the 2-cells

$$aa \overset{\alpha}{\Longrightarrow} a \quad , \quad Aa \overset{a\mathbf{u}_\diamond}{\Longrightarrow} a \quad , \quad aA \overset{a\mathbf{u}^\diamond}{\Longrightarrow} a \quad \text{and} \quad AA \overset{A\mathbf{u}}{\Longrightarrow} A$$

induce a 2-cell $\beta := [\alpha, a\mathbf{u}_\diamond, a\mathbf{u}^\diamond] + A\mathbf{u}$ from $(a + A)(a + A)$ to $a + A$ that is easily seen to be associative. The coprojection $A \overset{A\iota}{\Longrightarrow} a + A$ serves as a 2-sided unit. Hence $\langle a + A, \beta, A\iota \rangle$ is a monad.

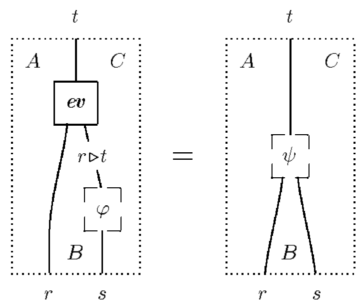
On a we now define a left action $(a + A)a \overset{\rho_\diamond}{\Longrightarrow} a$ via α and $a\mathbf{u}_\diamond$, and a right action $a(a + A) \overset{\rho^\diamond}{\Longrightarrow} a$ via α and $a\mathbf{u}^\diamond$. A straightforward calculation establishes the m-module axioms, and the fact that this m-module in $\boldsymbol{Y}$-**int** splits through the original interpolad $\langle a, \alpha \rangle$. ∎

The Cauchy-completion of $\boldsymbol{Y}$-$\boldsymbol{mnd}$ has idempotent m-modules as objects. These may be viewed as pairs consisting of a monad together with a compatible interpolad. The bifunctor to $\boldsymbol{Y}$-$\boldsymbol{int}$ that forgets the monad is unlikely to be full, unless each interpolad can only occur with essentially one monad. This does happen in the locally ordered case, *e.g.*, for $\boldsymbol{rel}$: an idempotent relation on a set A can be an idempotent m-module only for the preorder obtaind by forming the union with the diagonal on A. In case of $\boldsymbol{spn}$ this fails, since forming disjoint unions with identity modules is not an idempotent operation. Hence in general we cannot realize the Cauchy-completion of $\boldsymbol{Y}$-$\boldsymbol{mnd}$ within $\boldsymbol{Y}$-$\boldsymbol{int}$.

5. The inheritance of closedness

The familiar notion of closedness for a symmetric monoidal category splits into two notions in the absence of symmetry: left-closedness and right-closedness. Their usefulness in connection with m-modules was already observed by Lawvere [7]. These notions make perfect sense in any bicategory. Street and Walters [11] have formalized the corresponding generalizations in terms of the existence of all right extensions respectively all right liftings:

5.00. Definition. For 1-cells $A \xrightarrow{r} B$ and $A \xrightarrow{t} C$, a *right extension along r of t* consists of a 1-cell $B \xrightarrow{r \triangleright t} C$ and a 2-cell $r(r \triangleright t) \xRightarrow{\boldsymbol{ev}} t$ such that for every $s \in [B, C]$ pasting at $r \triangleright t$ of $\boldsymbol{ev}$ is bijective from $\langle s, r \triangleright t\rangle[B, C]$ to $\langle rs, t\rangle[A, C]$, indicated by dashed 2-cells in the following diagrams:



dual notion: *Right liftings* (morally: *left extensions*) in $\boldsymbol{Y}$ are right extensions in $\boldsymbol{Y}^{\mathrm{op}}$; they are denoted by $(t \triangleleft s)s \xRightarrow{\boldsymbol{ve}} s$.

There is an unfortunate name-mismatch between the terms "left-closed" and "right-closed" on one side, and the terms "right extension" and "right lifting" on the other side.

In analogy to the monoidal case, we talk about *(exponential) transposition* and *currying* when referring to the operation of pasting at a right extension or a right lifting and its inverse. These are frequently expressed by means of a notation borrowed from natural deduction:

$$\frac{\dfrac{s \Longrightarrow r \triangleright t}{rs \Longrightarrow t}}{r \Longrightarrow t \triangleleft s}$$

The existence of all right extensions, resp. all right liftings can be expressed equivalently as follows: for any $\boldsymbol{Y}$-objects A, B, and C, there exist adjunctions

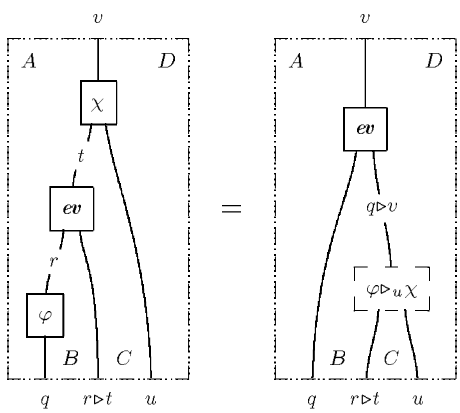
$$[B,C] \underset{[r,C]}{\overset{r\triangleright -}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} [A,C] \quad \text{resp.} \quad [A,B] \underset{[A,s]}{\overset{-\triangleleft s}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} [A,C] \tag{5-00}$$

Consequently, 1-cell composition from the left resp. from the right is left-adjoint and hence preserves colimits, in particular coequalizers. We always write $\boldsymbol{ev}$ for the counit of the first adjunction (reminding us of "evaluation"), and $\boldsymbol{ve}$ for the counit of the second adjunction, regardless of which 2-cells r and s, respectively, are involved. This, we hope, will keep the notation manageable. In particular, $r(r \triangleright t) \overset{\boldsymbol{ev}}{\Longrightarrow} t$ is a transpose of $\boldsymbol{id}_{r\triangleright t}$, and $(t \triangleleft s)s \overset{\boldsymbol{ve}}{\Longrightarrow} t$ is a transpose of $\boldsymbol{id}_{t\triangleleft s}$. Because of the obvious duality, we will restrict our attention to the construction of right extensions. In addition to the existence of stable local coequalizers in $\boldsymbol{Y}$ (*cf.* Assumption 2.06.), for the rest of this section we require

5.01. Assumption. $\boldsymbol{Y}$ *has all right extensions.*

Just as $r\triangleright t$ can be formed whenever r and t have the same domain, the expression $\varphi \triangleright \chi$ for 2-cells φ and χ is meaningful whenever the horizontal domains agree. This generality will greatly simplify computation with right extensions.

5.02. Definition. For 2-cells $q \overset{\varphi}{\Longrightarrow} r$ and $tu \overset{\chi}{\Longrightarrow} v$ with $A \xrightarrow{q,r} B$, $A \xrightarrow{t} C \xrightarrow{u} D$ and $A \xrightarrow{v} D$, we define the 2-cell $\varphi \triangleright_u \chi$ by means of:



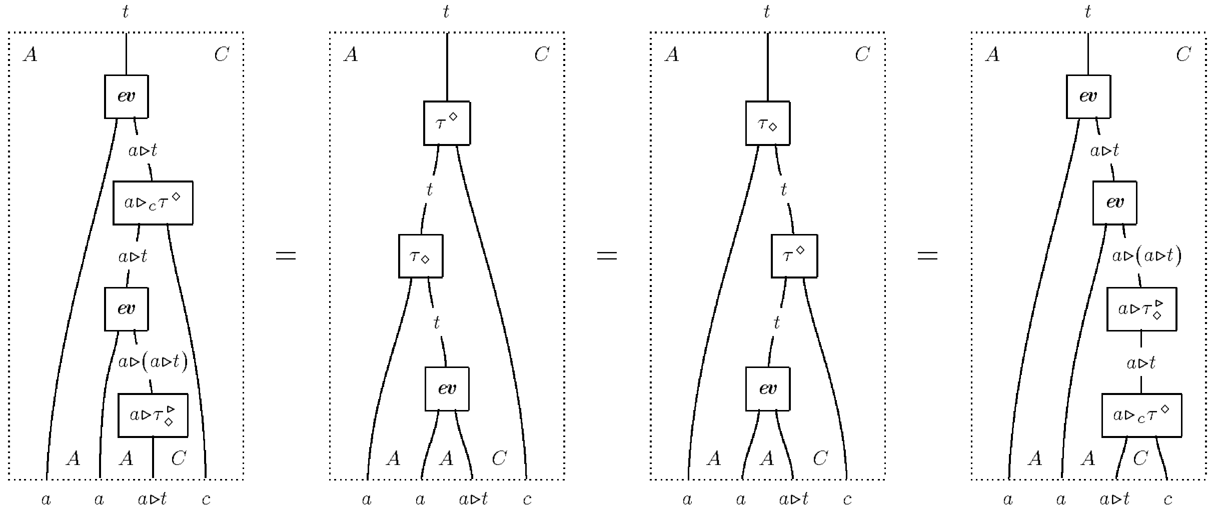
(5-01)

If u happens to be an identity 1-cell, we drop the subscript.

If all right liftings exist, $\varphi \triangleright_u \chi$ decomposes as $(\varphi \triangleright \chi^{\triangleleft})u\ ;\ \boldsymbol{ve}$, where $\chi^{\triangleleft}$ is a curried form of χ. But the following arguments do not need right liftings.

5.03. Proposition. *For a module $T = \langle \tau_\diamond, t, \tau^\diamond \rangle$ from a to c let $\tau_\diamond^\triangleright$ from t to $a \triangleright t$ be the curried form of $\tau_\diamond$. The 1-cell $a \xrightarrow{a\triangleright t} c$ together with the transpose of $a \triangleright \tau_\diamond^\triangleright$ as left action and $a \triangleright_c \tau^\diamond$ as right action constitutes a module $a \triangleright T$ from a to c.*

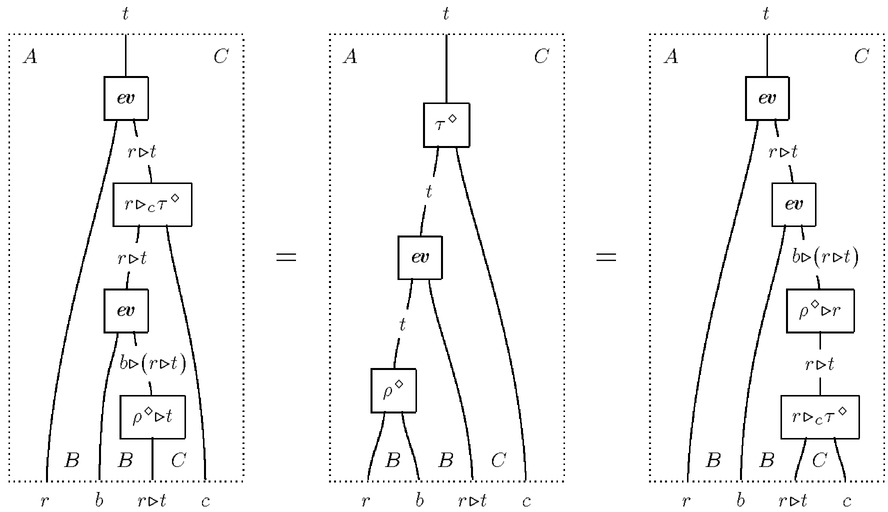
PROOF. The stacking condition for the specified actions follows from



since the right branches under $a(a \triangleright t) \xRightarrow{\boldsymbol{ev}} t$ in the outer diagrams have to agree. ∎

5.04. PROPOSITION. *If a, b and c are endo-1-cells on objects A, B and C, respectively, the functor $[A,B]^{\mathrm{op}} \times [A,C] \xrightarrow{\triangleright} [B,C]$ restricts to a functor from $(\langle a,b\rangle \boldsymbol{mod})^{\mathrm{op}} \times \langle a,c\rangle \boldsymbol{mod}$ to $\langle b,c\rangle \boldsymbol{mod}$ that maps $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ and $T = \langle \tau_\diamond, t, \tau^\diamond\rangle$ to the module $R \triangleright T$ with carrier $r \triangleright t$, left action the transpose of $\rho^\diamond \triangleright t$, and right action $r \triangleright_c \tau^\diamond$. Furthermore, if $\tau^\triangleright_\diamond$ denotes the curried version of $\tau_\diamond$, the 2-cells $r \triangleright t \xRightarrow{\rho_\diamond \triangleright t} ar \triangleright t$ and $r \triangleright t \xRightarrow{r \triangleright \tau^\triangleright_\diamond} r \triangleright (a \triangleright t) = ar \triangleright t$ are modisms from $R \triangleright T$ to $aR \triangleright T = R \triangleright (a \triangleright T)$ that constitute the $\langle R, T\rangle$-components of two natural transformations with domain $\triangleright$.*

PROOF. The stacking condition for $b(\rho_\diamond \triangleright t); \boldsymbol{ev}$ and $r \triangleright \tau^\triangleright_\diamond$ follows from



since the right branches under $r(t \triangleright t) \xRightarrow{\boldsymbol{ev}} t$ in the outer diagrams have to agree.

The stacking conditions for R and T, and the independence of $\rho_\diamond$ and $\tau^\diamond$ as well as $\rho^\diamond$ and $\tau_\diamond$ imply that $\rho^\diamond s$ and $r\sigma_\diamond$ are modisms from $R \triangleright T$ to $aR \triangleright T$. Again, naturality turns out to be a direct consequence of the modism axioms. ■

Just as in Section 2 coequalizers were necessary to define a meaningful composition of modules, we now expect equalizers to play the same role for right extensions of modules.

5.05. Assumption. *$\boldsymbol{Y}$ locally has equalizers.*

Again, for technical reasons we assume that a not necessarily canonical choice of equalizers has been made.

5.06. Proposition. *Let b and c be endo-1-cells on B and C, respectively.*

(0) *The category $\langle b, c\rangle \boldsymbol{mod}$ has equalizers.*

(1) *If $\mathcal{B}$ and $\mathcal{C}$ are monads with carriers b and c respectively, then $\langle \mathcal{B}, \mathcal{C}\rangle\,(\boldsymbol{Y}\text{-}\boldsymbol{mnd})$ is closed under equalizers in $\langle b, c\rangle \boldsymbol{mod}$.*

Proof. (0) For modules $S = \langle \sigma_\diamond, s, \sigma^\diamond\rangle$ and $V = \langle \nu_\diamond, v, \nu^\diamond\rangle$ from b to c consider modisms ψ and χ from S to V. Let $\langle p, \xi\rangle$ be the equalizer of ψ and χ in $[B, C]$. This induces actions $bp \overset{\pi_\diamond}{\Longrightarrow} p$ and $pc \overset{\pi^\diamond}{\Longrightarrow} p$ via

$$\begin{tikzcd}
pc \arrow[r, Rightarrow, "\xi c"] \arrow[d, dotted, "\pi^\diamond"'] & sc \arrow[r, Rightarrow, shift left, "\psi c"] \arrow[r, Rightarrow, shift right, "\chi c"'] \arrow[d, Rightarrow, "\sigma^\diamond"'] & vc \arrow[d, Rightarrow, "\nu^\diamond"] \\
p \arrow[r, Rightarrow, "\xi"] & s \arrow[r, Rightarrow, shift left, "\psi"] \arrow[r, Rightarrow, shift right, "\chi"'] & v \\
bp \arrow[r, Rightarrow, "b\xi"'] \arrow[u, dotted, "\pi_\diamond"] & bs \arrow[r, Rightarrow, shift left, "b\psi"] \arrow[r, Rightarrow, shift right, "b\chi"'] \arrow[u, Rightarrow, "\sigma_\diamond"'] & bv \arrow[u, Rightarrow, "\nu_\diamond"']
\end{tikzcd}$$

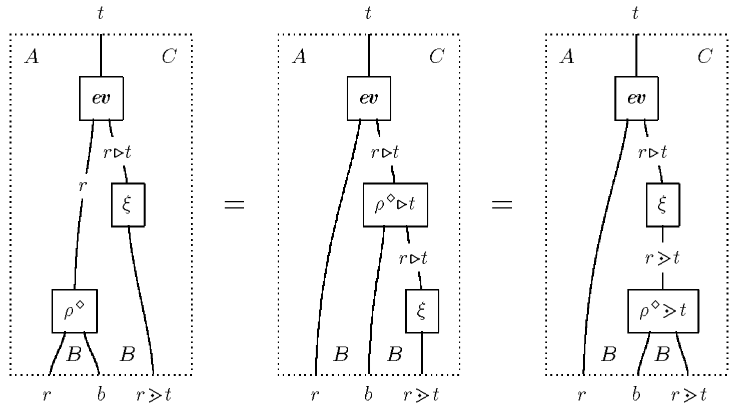
Since ξ is mono, we can derive the stacking condition for $\pi_\diamond$ and $\pi^\diamond$ directly from the stacking condition for $\sigma_\diamond$ and $\sigma^\diamond$. Hence $P = \langle \pi_\diamond, p, \pi^\diamond\rangle$ together with ξ is the equalizer of ψ and χ in $\langle b, c\rangle \boldsymbol{mod}$.

(1) The fact that ξ is mono implies $\langle \pi_\diamond, p, \pi^\diamond\rangle$ is compatible with the monad structures of $\mathcal{A}$ and $\mathcal{B}$ as specified in Proposition 3.04.(b). ■

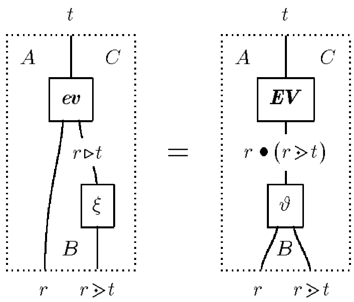
5.07. Definition. For modules $R = \langle \rho_\diamond, r, \rho^\diamond\rangle$ from a to b and $T = \langle \tau_\diamond, t, \tau^\diamond\rangle$ from a to c define $\langle r \succ t, \xi\rangle$ to be the equalizer of $\rho_\diamond \triangleright t$ and $r \triangleright \tau_\diamond^\triangleright$ in $\langle b, c\rangle \boldsymbol{mod}$. We denote the carrier of the module $R \succ T$ by $r \succ t$, and the left and right actions by $\rho^\diamond \succ t$ and $r \succ \tau^\diamond$, respectively.

5.08. Theorem. *All right extensions exist for modules between endo-1-cells in $\boldsymbol{Y}$, and they are constructed by means of $\succ$.*

PROOF. Consider modules $a \xrightarrow{R} b$ and $a \xrightarrow{T} c$. Recall that $R \bullet (R \geqslant T)$ is defined in terms of the coequalizer $r(r \geqslant t) \xRightarrow{\vartheta} r \bullet (r \geqslant t)$ of the 2-cells $\rho^\diamond(r \geqslant t)$ and $r(\rho^\diamond \geqslant t)$ from $rb(r \geqslant t)$ to $r(r \geqslant t)$. By definition of $\rho^\diamond \triangleright t$ and $\rho^\diamond \geqslant t$ we now have

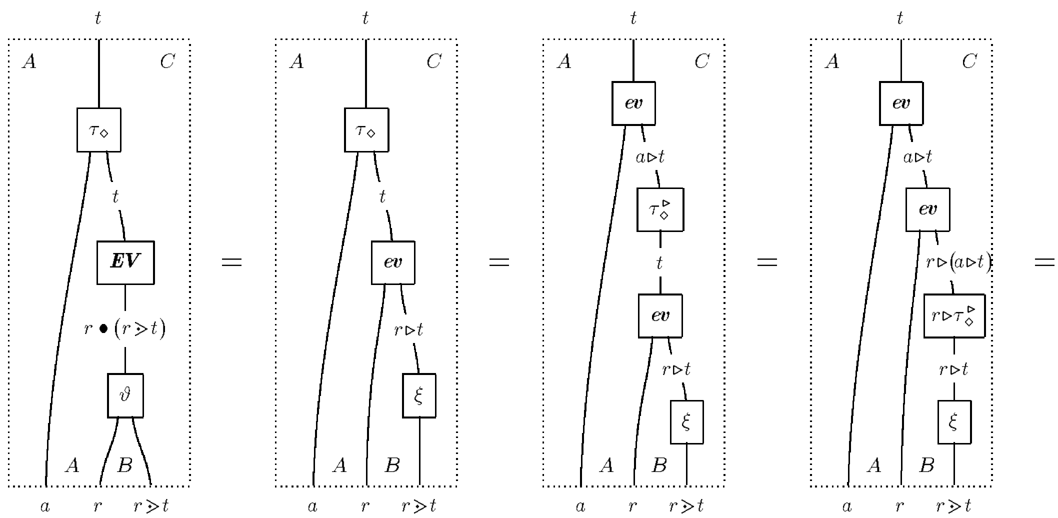


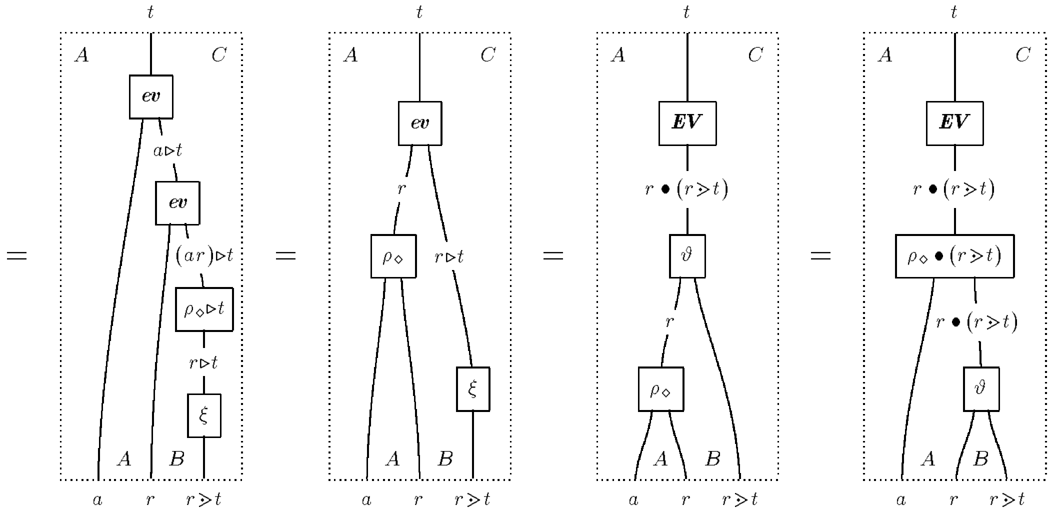
Hence the transpose of $r \geqslant t \xRightarrow{\xi} r \triangleright t$, *i.e.*, the restriction of $\boldsymbol{ev}$ by ξ, coequalizes $\rho^\diamond(r \geqslant t)$ and $r(\rho^\diamond \geqslant t)$. Consequently, there exists a unique 2-cell $r \bullet (r \geqslant t) \xRightarrow{\boldsymbol{EV}} t$ that satisfies



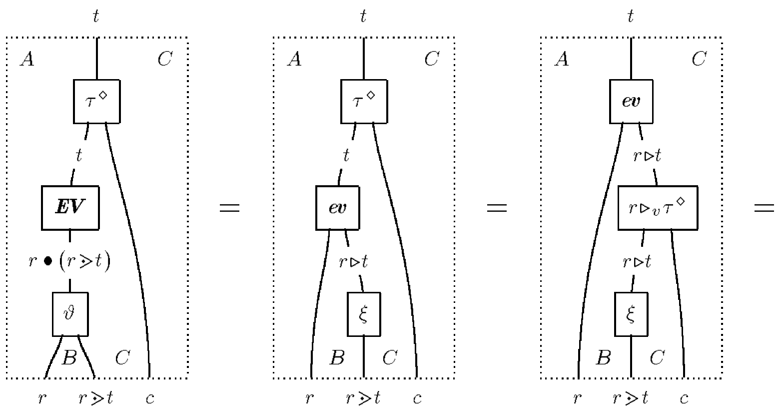
(5-02)

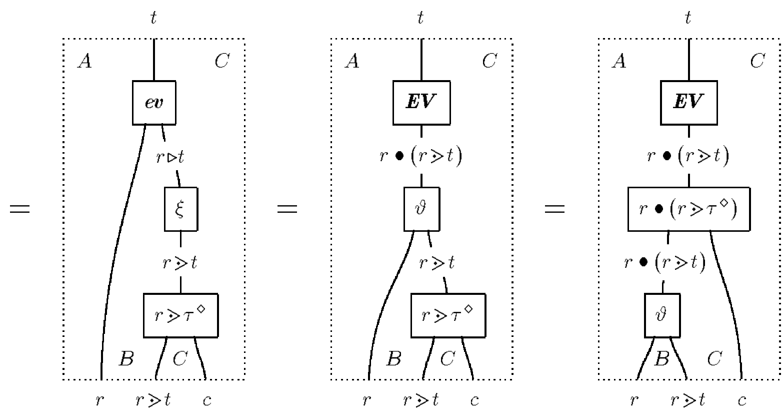
We wish to show that $\boldsymbol{EV}$ is a modism from $R \bullet (R \geqslant T)$ to T. Since $a\vartheta$ is epi, $\boldsymbol{EV}$ commutes with the left actions:





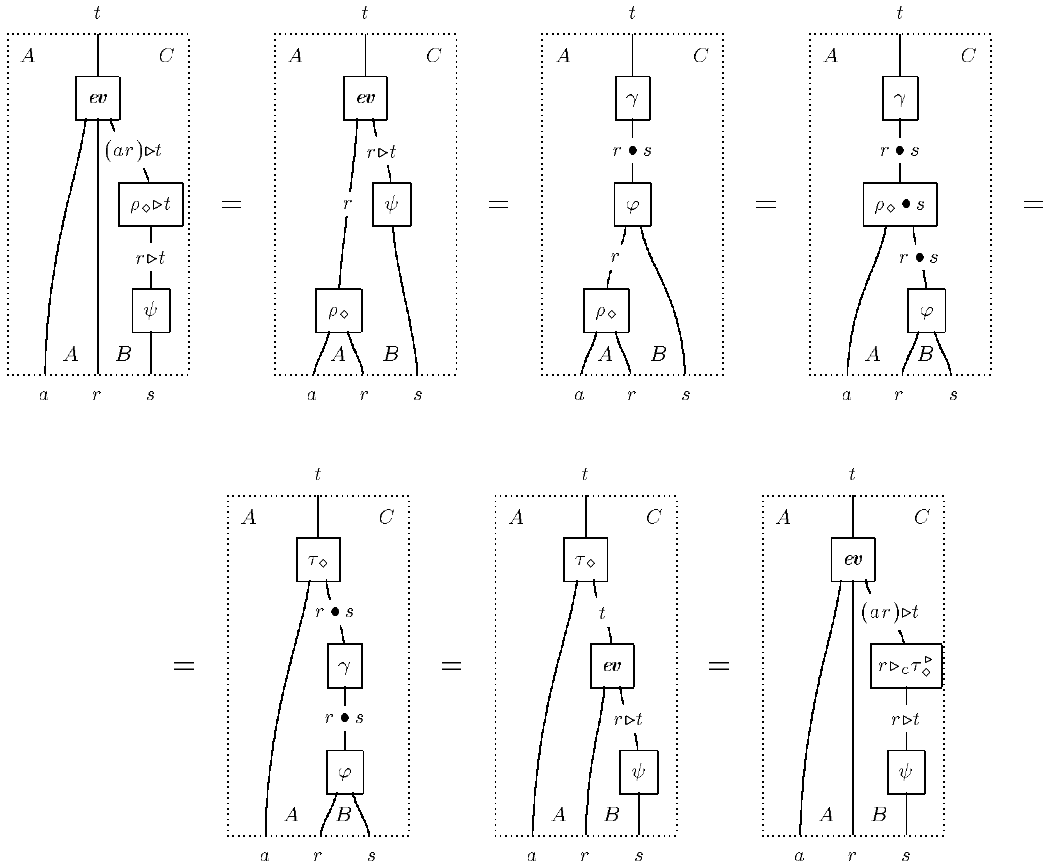
And since ϑc is epi, $\boldsymbol{EV}$ commutes with the right actions as well:





Pasting of modisms at $R \gtrdot T$ of $\boldsymbol{EV}$ corresponds to pasting of 2-cells at $r \gtrdot t$ of $r\xi\,;\,\boldsymbol{ev}$ in $\boldsymbol{Y}$, *cf.* Diagram (5-02), and hence is injective. To establish the surjectivity, consider a modism $R \bullet S \xRightarrow{\gamma} T$. Recall that $R \bullet S$ is defined in terms of the coequalizer

φ of the modisms $\rho^\diamond s$ and $r\sigma_\diamond$ from RbS to RS. Let $s \overset{\psi}{\Longrightarrow} r \triangleright t$ be the transpose of $\varphi\,;\gamma$. It suffices to show that ψ equalizes $\rho_\diamond \triangleright t$ and $r \triangleright \tau_\diamond^\triangleright$ in $[B, C]$. This follows from



since the right branches of $ar(ar \triangleright t) \overset{ev}{\Longrightarrow} t$ in the outer diagrams agree. ∎

5.09. Remark. If one drops the requirement of a tensor unit from the definition of a (not necessarily symmetric) left- (and right-) closed monoidal category, one arrives at the notion of a *Lambek calculus.* What we have here is a 2-dimensional analogue: if the base $\boldsymbol{Y}$ has all right extensions (and right liftings), the endo-1-cells, modules, and modisms nearly form a bicategory with the same property, except for the missing identity modules.

5.10. Theorem. *The bicategories* $\boldsymbol{Y}$*-mnd and* $\boldsymbol{Y}$*-int have all right extensions. In* $\boldsymbol{Y}$*-mnd they are formed via* $>$*, just as for general modules, while in* $\boldsymbol{Y}$*-int pre- and post-composition with the appropriate identitiy i-modules is additionally required.*

Proof. For $\boldsymbol{Y}$-*mnd* this is an immediate consequence of Proposition 5.06.(1).

If, on the other hand, $\mathcal{A} \xrightarrow{R} \mathcal{B}$ and $\mathcal{T} \xrightarrow{T} \mathcal{C}$ are i-modules between interpolads, the actions $\rho^\diamond > t$ and $r > \tau^\diamond$ of $R > T$ may fail to have the coequalizer property required in axioms (I0) and (I1), although the corresponding diagrams (*cf.* (3-00)) are guaranteed to commute, since ξ is mono.

Let $b(r \gtrdot t) \xRightarrow{\varphi} b \bullet (r \gtrdot t)$ be the chosen coequalizer of $\beta(t \gtrdot t)$ and $b(\rho^\diamond \gtrdot t)$. There exists a unique modism $b \bullet (r \gtrdot t) \xRightarrow{\mu} r \gtrdot t$ that satisfies $\varphi\,;\mu = \rho^\diamond \gtrdot t$. Similarly, for the coequalizer $(r \gtrdot t)c \xRightarrow{\psi} (r \gtrdot t) \bullet c$ of $(r \gtrdot \tau^\diamond)c$ and $(r \gtrdot t)\zeta$ we find a uniqe modism $(r \gtrdot t) \bullet c \xRightarrow{\nu} r \gtrdot t$ with $\psi\,;\nu = r \gtrdot \tau^\diamond$. These clearly satisfy $(b \bullet \nu)\,;\mu = (\mu \bullet c)\,;\nu$. Let η be the resulting modism from the i-module $\mathcal{B} \bullet (R \gtrdot T) \bullet \mathcal{C}$ to $R \gtrdot T$, *cf.*, Proposition 3.03.. We see that $\mathcal{B} \bullet (R \gtrdot T) \bullet \mathcal{C}$ together with $(r \bullet \eta)\,;\boldsymbol{EV}$ constitutes the desired right extension in $\boldsymbol{Y}$-$\boldsymbol{int}$. ■

5.11. Examples. (0) Consider modules $\boldsymbol{R} = \langle \rho_\diamond, R, \rho^\diamond \rangle$ from A to B and $\boldsymbol{T} = \langle \tau_\diamond, T, \tau^\diamond \rangle$ from A to C in the suspension of $\boldsymbol{set}$ (*cf.* Example 3.07.(0)). The set $R \gtrdot T \subseteq R \rhd T = T^R$ consists of *left-homomorphisms*, *i.e.*, functions $R \xrightarrow{f} T$ that satisfy $\rho_\diamond\,; f = (A \times f)\,;\tau_\diamond$. The left action $\rho^\diamond \gtrdot T$ maps $\langle b, f\rangle \in R \times (R \gtrdot T)$ to the left-homomorphism $R \xrightarrow{f_b} T$ defined by $rf_b = (rb)f$, and the right action $R \gtrdot \tau^\diamond$ maps $\langle f, c\rangle \in (R \gtrdot T) \times C$ to the left-homomorphism $R \xrightarrow{f^c} T$ defined by $rf^c = ((rf))\,c$. The restriction of the ordinary evaluation function $R \times (R \rhd T) \xrightarrow{\boldsymbol{ev}} T$ to $R \times (R \gtrdot T)$ coequalizes $\rho^\diamond \times (R \gtrdot t)$ and $R \times (\rho^\diamond \gtrdot T)$, which induces the function $\boldsymbol{R} \bullet (\boldsymbol{R} \gtrdot \boldsymbol{T}) \xRightarrow{\boldsymbol{EV}} \boldsymbol{T}$.

(1) Since for $\boldsymbol{Y} = \boldsymbol{rel}$ local equalizers are trivial, the operators $\rhd$ and $\gtrdot$ agree. Given modules $a \xrightarrow{r} b$ and $a \xrightarrow{t} c$, their right extension is given by

$$r \rhd t = \{\, \langle y, z\rangle \in B \times C \mid \forall x \in A.\, \langle x, y\rangle \in r \Rightarrow \langle x, z\rangle \in t \,\} \tag{5-03}$$

and the 2-cell $\boldsymbol{ev}$ is the inclusion $r(r \rhd t) \subseteq t$. Observe that $(rb) \subseteq r$ implies $b(r \rhd t) \subseteq r \rhd t$, and that $tc \subseteq t$ implies $(r \rhd t)c \subseteq r \rhd t$. If r and t are i-modules between idempotent relations, $r \gtrdot t$ may fail to be idempotent. Then the relation product $b(r \gtrdot t)c$ together with the inclusion $rb(r \rhd t)c = r(r \rhd t)c \subseteq r(r \rhd t) \subseteq t$ serves as the right extension, *cf.*, Rosebrugh and Wood [8].

(2) As we indicated in Example 3.07.(3), the case of $\boldsymbol{rel}$ is subsumed by that of quantaloids, *i.e.*, categories enriched in $\boldsymbol{cslat}$. Given modules $a \xrightarrow{r} b$ and $a \xrightarrow{t} c$, (the underlying 1-cell of) their right extension according to the first adjunction ind (5-00) is given by

$$r \rhd t = \sup\{\, s \in [B, C] : rs \le t \,\} \tag{5-04}$$

and this coincides with $r \gtrdot t$. Equation (5-03) above expresses the same fact at the level of elements of the sets B and C. The description of the right extension of i-modules is as in part (1) above.

If we view $\boldsymbol{cslat}$ as self-enriched, interpolads in this context are additive idempotent endo-functions, while monads correspond to additive closure operators.

(3) In the bicategory $\boldsymbol{spn}$ (*cf.* Example 3.07.(4)) the right extension along $A \xrightarrow{r} B$ of $A \xrightarrow{t} C$ consists of the matrix $r \rhd t$ with $\langle y, z\rangle$-component $\prod_{x \in A} \langle x, y\rangle r \rhd \langle x, z\rangle t$ and the evaluation

$$\sum_{y \in B} \Big(\langle x, y\rangle r \times \prod_{u \in A} \langle u, y\rangle r \rhd \langle u, z\rangle t \Big) \xrightarrow{\boldsymbol{ev}} \langle x, z\rangle t$$

that feeds $h \in \langle x,y\rangle r$ into the x-component of an A-tuple of functions. The $\langle y,z\rangle$-component of the right extension along a module $A \xrightarrow{R} B$ of $A \xrightarrow{T} C$ is the equalizer of the two canonical functions from $\prod_{x\in A} \langle x,y\rangle r \triangleright \langle x,z\rangle t$ to

$$\prod_{x\in A}\Big(\sum_{u\in A} \langle x,u\rangle a \times \langle u,y\rangle r\Big) \triangleright \langle x,z\rangle t \cong \prod_{u\in A} \langle u,y\rangle r \triangleright \Big(\prod_{x\in A} \langle x,u\rangle a \triangleright \langle x,z\rangle t\Big)$$

Concretely this means that the formal arrows $B \ni y \xrightarrow{s} z \in C$ are A-indexed families of functions $\langle x,y\rangle r \xrightarrow{s_x} \langle x,z\rangle t$ that satisfy the following compatibility condition: if in the following diagram of formal arrows the left triangle commutes, so does the right one:

$$\begin{tikzcd} & x \arrow[dl, "h"'] \arrow[dd, "f"] \arrow[dr, "hs_x"] & \\ y & & z \\ & u \arrow[ul, "g"] \arrow[ur, "gs_u"'] & \end{tikzcd}$$

This also describes right extensions in ***prof***, while in ***tax*** $=$ ***spn-int*** we still need to pre- and post-compose with the appropriate identity i-modules.

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