

DISTRIBUTIVE LAWS FOR PSEUDOMONADS

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ABSTRACT. We define distributive laws between pseudomonads in a *Gray*-category $\mathbf{A}$, as the classical two triangles and the two pentagons but commuting only up to isomorphism. These isomorphisms must satisfy nine coherence conditions. We also define the *Gray*-category $\mathbf{PSM}(\mathbf{A})$ of pseudomonads in $\mathbf{A}$, and define a lifting to be a pseudomonad in $\mathbf{PSM}(\mathbf{A})$. We define what is a pseudomonad with compatible structure with respect to two given pseudomonads. We show how to obtain a pseudomonad with compatible structure from a distributive law, how to get a lifting from a pseudomonad with compatible structure, and how to obtain a distributive law from a lifting. We show that one triangle suffices to define a distributive law in case that one of the pseudomonads is a (co-)KZ-doctrine and the other a KZ-doctrine.

1. Introduction

Distributive laws for monads were introduced by J. Beck in [2]. As pointed out by G. M. Kelly in [7], strict distributive laws for higher dimensional monads are rare. We need then a study of pseudo-distributive laws. The first step in this direction is quite easy: just replace the two commutative triangles, and the two commutative pentagons of [2] by appropriate invertible cells. The problem is to determine what coherence conditions to impose on these invertible cells. We should point out that, in [7], there is a step in this direction, keeping commutativity on the nose on the triangles and one of the pentagons, and asking for commutativity up to isomorphism in the remaining pentagon, plus five coherence conditions. The structure obtained from such a distributive law between two strict 2-monads is not, in general, a strict 2-monad, and since that article deals exclusively with strict 2-monads, what is obtained is a reflection result.

In this paper, instead of working with 2-monads we work with the more general pseudomonads. We will see that the structure obtained from a distributive law between pseudomonads is a pseudomonad. We define a distributive law between pseudomonads as we said above, that is to say, asking for commutativity up to isomorphism of the two triangles and the two pentagons. We propose nine coherence conditions for these isomorphisms. See section 4 below. We observe that the coherence conditions of [7] and the ones proposed in this paper coincide if in our setting we ask for commutativity on the nose of the two triangles and one of the pentagons. Thus, the examples of distributive laws given there are examples here as well.

But why exactly these nine coherence conditions?

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A more conceptual approach to distributive laws for monads is given by R. Street in [11]. It is shown that for a 2-category $\mathbf{C}$, a distributive law is the same thing as a monad in the 2-category $\mathbf{MND}(\mathbf{C})$, whose objects are monads in $\mathbf{C}$. In this paper we introduce the corresponding structure $\mathbf{PSM}(\mathbf{A})$, of pseudomonads for a corresponding three dimensional structure $\mathbf{A}$, see section 7.

In M. Barr and C. Wells' book [1], exercise (DL) asks to prove that, for monads, a distributive law, a lifting of one monad structure to the algebras of the other, and a monad with compatible structure with the two given monads, are essentially the same thing. For pseudomonads we have already mentioned distributive laws. We define a lifting as a pseudomonad in $\mathbf{PSM}(\mathbf{A})$ in section 8. In section 6, we define what a pseudomonad whose structure is compatible with two given pseudomonads is.

We show how to obtain a composite pseudomonad with compatible structure from a distributive law between pseudomonads, how to obtain a lifting from a pseudomonad with compatible structure, and, closing the cycle, how to obtain a distributive law from a lifting.

We see then, that the nine coherence conditions can be shown to hold if we define a distributive law from a lifting. In turn, these coherence conditions allow us to define a lifting from a distributive law between pseudomonads.

The situation for distributive laws between KZ-doctrines and (co-)KZ-doctrines is a lot simpler. We show that either one of the triangles commuting up to isomorphism (satisfying coherence conditions) is enough to obtain a distributive law. One such example is the following. It is well known that adding free (finite) coproducts to categories is a KZ-doctrine over $\mathbf{Cat}$, and adding free (finite) products is a co-KZ-doctrine. There is a more or less obvious distributive law of the co-KZ-doctrine over the KZ-doctrine. Observe however that even if we arrange for these KZ-doctrine and co-KZ-doctrine to produce strict pseudomonads, the distributive law obtained is not strict.

This article is possible thanks to the definition of tricategories given in [6]. It is simplified by the fact that a tricategory is triequivalent to a **Gray**-category, a fact proved in the same paper. We thus work in the framework of **Gray**-categories, as in [6], continuing the development of the formal theory of pseudomonads started in [9].

This paper is organized as follows:

In section 2 we provide a brief description of the framework that we use, namely that of **Gray**-categories. For more details we refer the reader to [6, 5].

In section 3 we recall the definition and some properties of pseudomonads given in [9], the definition uses the definition of pseudomonoid given in [3]. We also define the change of base 2-functors, change of base strong transformations and the change of base modifications that we will need in later sections. Change of base turns out to be a **Gray**-natural transformation.

In section 4 we define distributive laws for pseudomonads by replacing commutativity on the nose by commutativity up to isomorphism. We give here the nine coherence conditions that these isomorphisms should satisfy.

The first step to obtain compatible structures is to define a composite pseudomonad

from a distributive law. This is what we do in section 5.

In section 6 we define what a pseudomonad with compatible structure is with respect to given pseudomonads. Furthermore, we exhibit the structure that makes compatible the composite pseudomonad defined in the previous section.

We introduce the **Gray**-category $\mathbf{PSM}(\mathbf{A})$ in section 7, to define, in section 8, a lifting as a pseudomonad in the **Gray**-category $\mathbf{PSM}(\mathbf{A})$.

In section 9 we show how to construct a pseudomonad in $\mathbf{PSM}(\mathbf{A})$, from given pseudomonads with compatible structure. In the following section, we go from a pseudomonad in $\mathbf{PSM}(\mathbf{A})$ to a distributive law.

Section 11 deals with distributive laws of (co-)KZ-doctrines over KZ-doctrines. We refer the reader to [9] for the definition and properties we use of KZ-doctrines, but see [8] as well. We show that one triangle, plus coherence, is enough to produce a distributive law. Compare with [10], where it is shown that one of the triangles suffices for a distributive law between idempotent monads. The case of KZ-doctrines over KZ-doctrines is formally very similar. In this latter case, we show that the composite pseudomonad is again a KZ-doctrine.

I would like to thank the referee for helping improve the readability of this paper, and for suggesting condition (12), after which all the conditions of section 6 were modeled.

2. Gray-categories

As in [9] we will work with a **Gray**-category $\mathbf{A}$, where **Gray** is the symmetric monoidal closed category whose underlying category is 2-Cat with tensor product as in [6]. A **Gray**-category is a category enriched in the category **Gray** as in [4]. We will briefly spell out what this means, and we refer the reader to [6] and [4] for more details.

A **Gray**-category $\mathbf{A}$ has objects $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$. For every pair of objects $\mathcal{A}, \mathcal{B}$ of $\mathbf{A}$, $\mathbf{A}$ has a 2-category $\mathbf{A}(\mathcal{A}, \mathcal{B})$. Given another object $\mathcal{C}$ in $\mathbf{A}$, $\mathbf{A}$ has a 2-functor $\mathbf{A}(\mathcal{C}, \mathcal{B}) \otimes \mathbf{A}(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{A}(\mathcal{A}, \mathcal{C})$. This 2-functor corresponds to a cubical functor $M : \mathbf{A}(\mathcal{B}, \mathcal{C}) \times \mathbf{A}(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{A}(\mathcal{A}, \mathcal{C})$. We will denote M by juxtaposition, $M(G, F) = GF$ for $F \in \mathbf{A}(\mathcal{A}, \mathcal{B})$ and $G \in \mathbf{A}(\mathcal{B}, \mathcal{C})$. Given $f : F \rightarrow F'$ in $\mathbf{A}(\mathcal{A}, \mathcal{B})$ and $g : G \rightarrow G'$ in $\mathbf{A}(\mathcal{B}, \mathcal{C})$ we will denote the invertible 2-cell $M_{g,f}$ by

$$\begin{array}{ccc} GF & \xrightarrow{gF} & G'F \\ Gf \downarrow & \not\Downarrow_{g_f} & \downarrow G'f \\ GF' & \xrightarrow{gF'} & G'F'. \end{array}$$

What the definition of being cubical means for M is the following: Given $\varphi : f \rightarrow f' : F \rightarrow F'$, and $f'' : F' \rightarrow F''$ in $\mathbf{A}(\mathcal{A}, \mathcal{B})$, and $\gamma : g \rightarrow g' : G \rightarrow G'$, and $g'' : G' \rightarrow G''$ in $\mathbf{A}(\mathcal{B}, \mathcal{C})$, we have that $(-)F : \mathbf{A}(\mathcal{B}, \mathcal{C}) \rightarrow \mathbf{A}(\mathcal{A}, \mathcal{C})$ and $G(-) : \mathbf{A}(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{A}(\mathcal{A}, \mathcal{C})$ are 2-functors, $(-)f : (-)F \rightarrow (-)F'$ and $g(-) : G(-) \rightarrow G'(-)$ are strong transformations, $(-)\varphi : (-)f \rightarrow (-)f'$ and $\gamma(-) : g(-) \rightarrow g'(-)$ are modifications, and the following three

equations are satisfied

$$\begin{array}{ccc}
 \begin{array}{ccc}
 GF & \xrightarrow{gF} & G'F \\
 \downarrow Gf & \Downarrow \gamma^F & \downarrow G'f \\
 GF' & \xrightarrow{g'F'} & G'F'
 \end{array} & = & \begin{array}{ccc}
 GF & \xrightarrow{gF} & G'F \\
 \downarrow Gf & \Downarrow \gamma^F & \downarrow G'f \\
 GF' & \xrightarrow{g'F'} & G'F'
 \end{array} \\
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 GF & \xrightarrow{gF} & G'F \\
 \downarrow Gf & \Downarrow g_f & \downarrow G'f \\
 GF' & \xrightarrow{g'F'} & G'F' \\
 \downarrow Gf'' & \Downarrow g_{f''} & \downarrow G'f'' \\
 GF'' & \xrightarrow{g'F''} & G'F''
 \end{array} & = & \begin{array}{ccc}
 GF & \xrightarrow{gF} & G'F \\
 \downarrow G(f'' \circ f) & \Downarrow g_{f'' \circ f} & \downarrow G'(f'' \circ f) \\
 GF'' & \xrightarrow{g'F''} & G'F''
 \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 GF & \xrightarrow{gF} & G'F & \xrightarrow{g''F} & G''F \\
 \downarrow Gf & \Downarrow g_f & \downarrow G'f & \Downarrow g''_f & \downarrow G''f \\
 GF' & \xrightarrow{g'F'} & G'F' & \xrightarrow{g''F'} & G''F'
 \end{array} & = & \begin{array}{ccc}
 GF & \xrightarrow{(g'' \circ g)F} & G''F \\
 \downarrow Gf & \Downarrow (g'' \circ g)_f & \downarrow G''f \\
 GF' & \xrightarrow{(g'' \circ g)F'} & G''F'
 \end{array}
 \end{array}$$

and if either f or g is an identity, then g_f is an identity 2-cell. Now, for every object $\mathcal{A}$ of $\mathbf{A}$, there is a distinguished object $1_{\mathcal{A}}$. The triangle in the definition of enriched categories means that the action of multiplying by $1_{\mathcal{A}}$ is trivial. Now, the pentagon means that for another object $\mathcal{D}$ in $\mathbf{A}$, and $\kappa : k \rightarrow k' : K \rightarrow K'$ in $\mathbf{A}(\mathcal{C}, \mathcal{D})$ the following equations hold:

$$(KG)F = K(GF),$$

$$(KG)f = K(Gf), \quad (Kg)F = K(gF), \quad (kG)F = k(GF),$$

$$(KG)\varphi = K(G\varphi), \quad (K\gamma)F = K(\gamma F), \quad (\kappa G)F = \kappa(GF),$$

$$(Kg)_f = K(g_f), \quad (kG)_f = k_{Gf}, \quad \text{and} \quad (k_g)F = k_{gF}.$$

We will use these properties freely, without further mention.

3. Pseudomonads

For the convenience of the reader we will recall here the definition of a pseudomonad in a **Gray**-category $\mathbf{A}$, for more details we refer the reader to [9]. We adopt the definition of pseudomonad given in [3].

3.1. DEFINITION. A pseudomonad $\mathbb{D}$ on an object $\mathcal{K}$ of a Gray-category $\mathbf{A}$ is a pseudomonoid in the Gray monoid $\mathbf{A}(\mathcal{K}, \mathcal{K})$.

We give now, in elementary terms, what this means. A pseudomonad $\mathbb{D}$ as above consists of an object D in $\mathbf{A}(\mathcal{K}, \mathcal{K})$, and 1-cells $d : 1_{\mathcal{K}} \rightarrow D$, and $m : DD \rightarrow D$ and invertible 2-cells

$$\begin{array}{ccc} D & \xrightarrow{dD} & DD & \xleftarrow{Dd} & D \\ & \searrow \beta & \downarrow m & \nearrow \eta & \\ Id_D & & D & & Id_D \end{array} \quad \begin{array}{ccc} DDD & \xrightarrow{Dm} & DD \\ mD \downarrow & \searrow \mu & \downarrow m \\ DD & \xrightarrow{m} & D, \end{array}$$

such that the following two equations are satisfied:

$$\begin{array}{ccc} DDDD & \xrightarrow{DDm} & DDD \\ \downarrow mDD & \searrow DmD & \downarrow D\mu \\ DDD & \xrightarrow{Dm} & DD \\ \downarrow mD & \searrow \mu D & \downarrow m \\ DD & \xrightarrow{m} & D \end{array} = \begin{array}{ccc} DDDD & \xrightarrow{DDm} & DDD \\ \downarrow mDD & \searrow m_m^{-1} & \downarrow mD \\ DDD & \xrightarrow{Dm} & DD \\ \downarrow mD & \searrow \mu & \downarrow m \\ DD & \xrightarrow{m} & D, \end{array} \quad (1)$$

$$\begin{array}{ccc} & DD & \\ Dm \nearrow & & \searrow m \\ DD \xrightarrow{DdD} DDD & \xrightarrow{\mu} & D \\ mD \searrow & & \nearrow m \\ & DD & \end{array} = \begin{array}{ccc} & DDD & \\ DdD \nearrow & & \searrow Dm \\ DD \xrightarrow{D\beta} DDD & \xrightarrow{\eta D} & DD \xrightarrow{m} D. \\ DdD \searrow & & \nearrow mD \end{array} \quad (2)$$

It is shown in [9] that the following three equations hold for any pseudomonad $\mathbb{D}$:

$$\begin{array}{ccc} & DD & \\ dD \nearrow & & \searrow m \\ 1_{\mathcal{K}} \xrightarrow{d} D & \xrightarrow{Id_D} & D \\ Dd \searrow & & \nearrow m \\ & DD & \end{array} = \begin{array}{ccc} & D & \\ d \nearrow & & \searrow dD \\ 1_{\mathcal{K}} \xrightarrow{d_d^{-1}} D & \xrightarrow{m} & DD \xrightarrow{m} D, \\ d \searrow & & \nearrow Dd \end{array} \quad (3)$$

$$\begin{array}{ccc} DD & \xrightarrow{dDD} & DDD & \xrightarrow{Dm} & DD \\ \downarrow Id_{DD} & \searrow \beta D & \downarrow mD & \searrow \mu & \downarrow m \\ DD & \xrightarrow{m} & D \end{array} = \begin{array}{ccc} DD & \xrightarrow{dDD} & DDD \\ \downarrow m & \searrow d_m & \downarrow Dm \\ D & \xrightarrow{dD} & DD \\ \downarrow Id_D & \searrow \beta & \downarrow m \\ D & & \end{array} \quad (4)$$

$$\begin{array}{ccc}
\begin{array}{ccc}
DD & & \\
DDd \downarrow & \searrow Id_{DD} & \\
DDD & \xleftarrow{D\eta} & DD \\
mD \downarrow & \xleftarrow{\mu} & \downarrow m \\
DD & \xrightarrow{m} & D
\end{array} & = & \begin{array}{ccc}
DD & \xrightarrow{m} & D \\
DDd \downarrow & \xleftarrow{m_d} & \downarrow Dd \eta \\
DDD & \xrightarrow{mD} & DD \xrightarrow{m} D.
\end{array}
\end{array} \quad (5)$$

We recall as well the 2-categories of algebras for a pseudomonad $\mathbb{D}$. Let $\mathcal{X}$ be another object in $\mathbf{A}$. An object in the 2-category $\mathbb{D}\text{-Alg}_{\mathcal{X}}$ consists of an object X in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, together with a 1-cell $x : DX \rightarrow X$ and invertible 2-cells

$$\begin{array}{ccc}
X & \xrightarrow{dX} & DX \\
\searrow Id_X & \Downarrow \psi & \downarrow x \\
& & X
\end{array}
\quad
\begin{array}{ccc}
DDX & \xrightarrow{Dx} & DX \\
mX \downarrow & \Downarrow \chi & \downarrow x \\
DX & \xrightarrow{x} & X,
\end{array}$$

such that the following two equations are satisfied

$$\begin{array}{ccc}
\begin{array}{ccc}
DDDX & \xrightarrow{DDx} & DDX \\
mDX \downarrow & \searrow DmX & \xleftarrow{D\chi} \\
DDX & \xrightarrow{Dx} & DX \\
\mu X \downarrow & \searrow mX & \xleftarrow{\chi} \\
DX & \xrightarrow{x} & X
\end{array} & = & \begin{array}{ccc}
DDDX & \xrightarrow{DDx} & DDX \\
mDX \downarrow & \searrow mX & \xleftarrow{m_x^{-1}} \\
DDX & \xrightarrow{Dx} & DX \\
mX \downarrow & \searrow mX & \xleftarrow{\chi} \\
DX & \xrightarrow{x} & X
\end{array}
\end{array} \quad (6)$$

$$\begin{array}{ccc}
\begin{array}{ccc}
DX & \xrightarrow{DdX} & DDX \\
\searrow Id & \Downarrow \beta_X & \downarrow mX \\
DX & \xrightarrow{x} & X
\end{array} & = & \begin{array}{ccc}
DX & \xrightarrow{DdX} & DDX \\
\searrow Id & \Downarrow \eta_X & \downarrow mX \\
DX & \xrightarrow{x} & X
\end{array}
\end{array} \quad (7)$$

It is shown in [9] that for every object (ψ, χ) in $\mathbb{D}\text{-Alg}_{\mathcal{X}}$, the following equality holds:

$$\begin{array}{ccc}
\begin{array}{ccc}
DX & \xrightarrow{dDX} & DDX \\
\searrow Id & \Downarrow \beta_X & \downarrow mX \\
DX & \xrightarrow{x} & X
\end{array} & = & \begin{array}{ccc}
DX & \xrightarrow{dDX} & DDX \\
\searrow Id & \Downarrow \psi & \downarrow x \\
DX & \xrightarrow{x} & X
\end{array}
\end{array} \quad (8)$$

A 1-cell $(h, \rho) : (\psi, \chi) \rightarrow (\psi', \chi')$ in $\mathbb{D}\text{-Alg}_{\mathcal{X}}$ consists of a 1-cell $h : X \rightarrow X'$ in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, together with an invertible 2-cell

$$\begin{array}{ccc} DX & \xrightarrow{Dh} & DX' \\ x \downarrow & \Downarrow \rho & \downarrow x' \\ X & \xrightarrow{h} & X', \end{array}$$

that satisfies the following two equations:

$$\begin{array}{ccc} X & \xrightarrow{dX} & DX \xrightarrow{Dh} DX' \\ \searrow \psi & \Downarrow \rho & \downarrow x' \\ Id & \searrow & X \xrightarrow{h} X' \end{array} = \begin{array}{ccc} X & \xrightarrow{dX} & DX \\ h \downarrow & \Downarrow d_h & \downarrow Dh \\ X' & \xrightarrow{dX'} & DX' \\ \searrow \psi' & \Downarrow \rho' & \downarrow x' \\ Id & \searrow & X' \end{array} \quad (9)$$

$$\begin{array}{ccc} DDX & \xrightarrow{DDh} & DDX' \\ mX \downarrow & \Downarrow D\rho & \downarrow mX' \\ DX & \xrightarrow{Dh} & DX' \\ \chi \downarrow & \Downarrow \rho & \downarrow x' \\ X & \xrightarrow{h} & X' \end{array} = \begin{array}{ccc} DDX & \xrightarrow{DDh} & DDX' \\ mX \downarrow & \Downarrow m_h^{-1} & \downarrow mX' \\ DX & \xrightarrow{Dh} & DX' \\ x \downarrow & \Downarrow \rho & \downarrow x' \\ X & \xrightarrow{h} & X' \end{array} \quad (10)$$

A 2-cell $\xi : (h, \rho) \rightarrow (h', \rho') : (\psi, \chi) \rightarrow (\psi', \chi')$ is a 2-cell $\xi : h \rightarrow h'$ such that the following condition is satisfied:

$$\begin{array}{ccc} DX & \xrightarrow{Dh} & DX' \\ x \downarrow & \Downarrow D\xi & \downarrow x' \\ X & \xrightarrow{h'} & X' \end{array} = \begin{array}{ccc} DX & \xrightarrow{Dh} & DX' \\ x \downarrow & \Downarrow \rho & \downarrow x' \\ X & \xrightarrow{h} & X' \end{array} \quad (11)$$

Given another object $\mathcal{Z}$ of $\mathbf{A}$, and $K \in \mathbf{A}(\mathcal{Z}, \mathcal{X})$, we can define a change of base 2-functor $\widehat{K} : \mathbb{D}\text{-Alg}_{\mathcal{X}} \rightarrow \mathbb{D}\text{-Alg}_{\mathcal{Z}}$. If $\xi : (h, \rho) \rightarrow (h', \rho') : (\psi, \chi) \rightarrow (\psi', \chi')$ is in $\mathbb{D}\text{-Alg}_{\mathcal{X}}$, then its image under $\widehat{K}$ is $\xi K : (hK, \rho K) \rightarrow (h'K, \rho'K) : (\psi K, \chi K) \rightarrow (\psi'K, \chi'K)$. If $k : K \rightarrow K'$ then we define the strong transformation $\widehat{k} : \widehat{K} \rightarrow \widehat{K}'$ such that $\widehat{k}_{(\psi, \chi)} = (Xk, x_k^{-1})$ and $\widehat{k}_{(h, \rho)} = h_k^{-1}$. If $\kappa : k \rightarrow k' : K \rightarrow K'$ in $\mathbf{A}(\mathcal{Z}, \mathcal{X})$, then $\widehat{\kappa}_{(\psi, \chi)} = X\kappa$ defines a modification $\widehat{\kappa} : \widehat{k} \rightarrow \widehat{k}'$. We have actually defined a **Gray**-functor $\mathbb{D}\text{-Alg} : \mathbf{A}^{op} \rightarrow \mathbf{Gray}$.

For every object $\mathcal{Z}$, we have an obvious forgetful 2-functor $\mathbb{D}\text{-Alg}_{\mathcal{Z}} \rightarrow \mathbf{A}(\mathcal{Z}, \mathcal{K})$. These 2-functors define a forgetful **Gray**-natural transformation $\Phi : \mathbb{D}\text{-Alg} \rightarrow \mathbf{A}(_, \mathcal{K})$.

4. Distributive laws

Let $\mathbb{D} = (D, d, m, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \mu_{\mathbb{D}})$ and $\mathbb{U} = (U, u, n, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \mu_{\mathbb{U}})$ be pseudomonads on the same object $\mathcal{K}$ of the **Gray**-category $\mathbf{A}$. A *distributive law* of $\mathbb{U}$ over $\mathbb{D}$ consists of a 1-cell $r : UD \rightarrow DU$ in $\mathbf{A}(\mathcal{K}, \mathcal{K})$, together with invertible 2-cells

$$\begin{array}{ccc}
 \begin{array}{c} UD \\ \swarrow uD \quad \searrow r \\ D \xrightarrow{Du} DU \end{array} & \Downarrow \omega_1 & \begin{array}{c} UD \\ \swarrow Ud \quad \searrow r \\ U \xrightarrow{dU} DU \end{array} \\
 \begin{array}{ccccc} UUD & \xrightarrow{Ur} & UDU & \xrightarrow{rU} & DUU \\ nD \downarrow & & \Downarrow \omega_3 & & \downarrow Dn \\ UD & \xrightarrow{r} & DU & & \end{array} & & \begin{array}{ccccc} UD & \xrightarrow{r} & DU \\ Um \uparrow & & \Downarrow \omega_4 & & \uparrow mU \\ UDD & \xrightarrow{rD} & DUD & \xrightarrow{Dr} & DDU \end{array}
 \end{array}$$

subject to the following coherence conditions:

(coh 1)

$$\begin{array}{c}
 1 \xrightarrow{u} U \xrightarrow{Ud} UD \\
 \searrow d \quad \swarrow dU^{-1} \quad \searrow dU \quad \swarrow r \\
 D \xrightarrow{Du} DU
 \end{array}
 =
 \begin{array}{c}
 1 \xrightarrow{u} U \xrightarrow{Ud} UD \\
 \searrow d \quad \swarrow Ud \quad \searrow r \\
 D \xrightarrow{Du} DU
 \end{array}$$

(coh 2)

$$\begin{array}{ccc}
 UD \xrightarrow{UuD} UUD \xrightarrow{Ur} UDU & = & UD \xrightarrow{UuD} UUD \\
 \swarrow \eta_U^{-1} \quad \downarrow nD & & \swarrow U\omega_1 \quad \downarrow Ur \\
 UD \xrightarrow{r} DU & & UDU \\
 \swarrow \omega_3 & & \swarrow rU^{-1} \quad \downarrow rU \\
 & & DU \xrightarrow{DUu} DUU \\
 & & \swarrow D\eta_U^{-1} \quad \downarrow Dn \\
 & & DU
 \end{array}$$

(coh 3)

$$\begin{array}{ccc}
 UU \xrightarrow{UUd} UUD & = & UU \xrightarrow{UUd} UUD \\
 \swarrow U\omega_2 \quad \downarrow Ur & & \swarrow n \quad \downarrow nD \\
 UDU & & UDU \\
 \swarrow dUU \quad \swarrow \omega_2 U \quad \downarrow rU & & \swarrow \omega_2 \quad \downarrow rU \\
 DUU & & DUU \\
 \swarrow d_n \quad \downarrow Dn & & \swarrow r \quad \downarrow Dn \\
 U \xrightarrow{dU} DU & & DU
 \end{array}$$

(coh 4)

$$\begin{array}{c}
UUUD \xrightarrow{UU_r} UUUDU \\
\downarrow nUD \quad \searrow UrU \quad \downarrow UrU \\
UUD \quad \quad \quad UDUU \\
\downarrow nUD \quad \searrow U\omega_3 \quad \downarrow UDn \\
UUD \quad \quad \quad UDU \\
\downarrow nD \quad \searrow \mu_U D \quad \downarrow Ur \\
UD \quad \quad \quad DUU \\
\downarrow nD \quad \searrow \omega_3^{Dn} \quad \downarrow rU \\
UD \quad \quad \quad DU
\end{array}
=
\begin{array}{c}
UUUD \xrightarrow{UU_r} UUUDU \\
\downarrow nUD \quad \searrow nDU \quad \downarrow UrU \\
UUD \quad \quad \quad UDUU \\
\downarrow nUD \quad \searrow \omega_3^{-1} \quad \downarrow rUU \quad \downarrow UDn \\
UUD \xrightarrow{Ur} UDU \quad \quad \quad DUUU \quad \quad \quad UDU \\
\downarrow nD \quad \searrow rU \quad \downarrow DnU \quad \searrow \omega_3^{-1} \quad \downarrow rU \\
DUU \quad \quad \quad DUU \quad \quad \quad DUU \\
\downarrow nD \quad \searrow \omega_3 \quad \downarrow Dn \quad \searrow D\mu_U \quad \downarrow Dn \\
UD \xrightarrow{r} DU.
\end{array}$$

(coh 5)

$$\begin{array}{c}
DD \xrightarrow{uDD} UDD \xrightarrow{Um} UD \\
\downarrow DuD \quad \searrow \omega_1 D \quad \downarrow rD \\
DD \quad \quad \quad DUD \\
\downarrow DDu \quad \searrow D\omega_1 \quad \downarrow Dr \\
DDU \xrightarrow{mU} DU
\end{array}
=
\begin{array}{c}
DD \xrightarrow{uDD} UDD \\
\downarrow DDu \quad \searrow m \quad \downarrow D \\
DDU \quad \quad \quad D \xrightarrow{uD} UD \\
\downarrow DDu \quad \searrow m_u \quad \downarrow Du \quad \searrow \omega_1 \quad \downarrow r \\
DDU \xrightarrow{mU} DU.
\end{array}$$

(coh 6)

$$\begin{array}{c}
UUDD \xrightarrow{UUm} UUD \\
\downarrow nDD \quad \searrow Ur \quad \downarrow nD \\
UDD \quad \quad \quad UDU \\
\downarrow UUm \quad \searrow \omega_3 \quad \downarrow rU \\
UD \quad \quad \quad DUU \\
\downarrow rD \quad \searrow \omega_4 \quad \downarrow r \\
DUD \quad \quad \quad DUU \\
\downarrow Dr \quad \searrow mU \quad \downarrow Dn \\
DDU \xrightarrow{mU} DU
\end{array}
=
\begin{array}{c}
UUDD \xrightarrow{UUm} UUD \\
\downarrow nDD \quad \searrow UrD \quad \downarrow Ur \\
UDD \quad \quad \quad UDUD \\
\downarrow rD \quad \searrow \omega_3 D \quad \downarrow rUD \quad \searrow \omega_4 \\
DUU \quad \quad \quad DUU \quad \quad \quad UDDU \xrightarrow{UmU} UDU \\
\downarrow DnD \quad \searrow DUr \quad \downarrow rDU \\
DUD \quad \quad \quad DUDU \\
\downarrow DrU \quad \searrow \omega_4 U \quad \downarrow rU \\
DDU \quad \quad \quad DUU \\
\downarrow DDn \quad \searrow m_n \quad \downarrow Dn \\
DDU \xrightarrow{mU} DU.
\end{array}$$

(coh 7)

$$\begin{array}{ccccc}
& & Id & & \\
& & \Downarrow U\beta_D^{-1} & & \\
UD & \xrightarrow{UdD} & UDD & \xrightarrow{Um} & UDU = id_r. \\
& \searrow dUD & \Downarrow \omega_2 D & \searrow rD & \\
& & DUD & & \\
& & \Downarrow d_r & \Downarrow Dr & \\
DU & \xrightarrow{dDU} & DDU & \xrightarrow{mU} & DU \\
& & \Downarrow \beta_D U & & \\
& & Id & &
\end{array}$$

(coh 8)

$$\begin{array}{ccccc}
& & Id & & \\
& & \Downarrow U\eta_D & & \\
UD & \xrightarrow{UDd} & UDD & \xrightarrow{Um} & UD = id_r. \\
& \searrow & \Downarrow r_d^{-1} & \searrow rD & \\
& & DUD & & \\
& & \Downarrow D\omega_2 & \Downarrow Dr & \\
DU & \xrightarrow{DUd} & DDU & \xrightarrow{mU} & DU \\
& \searrow DdU & \Downarrow \eta_D U^{-1} & & \\
& & Id & &
\end{array}$$

(coh 9)

$$\begin{array}{ccc}
\begin{array}{c}
UDDD \xrightarrow{UDm} UDD \\
\begin{array}{c} rDD \downarrow \\ DUDD \end{array} \xrightarrow{UmD} UDD \\
\begin{array}{c} DrD \downarrow \\ DDUD \end{array} \xrightarrow{\omega_4 D} UDD \\
\begin{array}{c} DDr \downarrow \\ DDDU \end{array} \xrightarrow{mUD} UDD \\
\begin{array}{c} mDU \searrow \\ DDU \end{array} \xrightarrow{mU} UD
\end{array}
&
\begin{array}{c}
UDD \xrightarrow{Um} UD \\
\begin{array}{c} rD \downarrow \\ DUD \end{array} \xrightarrow{Um} UD \\
\begin{array}{c} rD \downarrow \\ DUD \end{array} \xrightarrow{Um} UD \\
\begin{array}{c} Dr \downarrow \\ DUD \end{array} \xrightarrow{Um} UD \\
\begin{array}{c} mU \searrow \\ DU \end{array}
\end{array}
&
= \begin{array}{c}
UDDD \xrightarrow{UDm} UDD \\
\begin{array}{c} rDD \downarrow \\ DUDD \end{array} \xrightarrow{DUm} DUD \\
\begin{array}{c} DrD \downarrow \\ DDUD \end{array} \xrightarrow{DUm} DUD \\
\begin{array}{c} DDr \downarrow \\ DDDU \end{array} \xrightarrow{DmU} DDU \\
\begin{array}{c} mDU \searrow \\ DDU \end{array} \xrightarrow{mU} DU
\end{array}
\end{array}$$

Observe that if the pseudomonads are strict (β , η and μ are identities), and ω_1 , ω_2 and ω_3 are identities, then we obtain the coherence conditions of the “mild” extensions of the classical distributive laws given in [7].

5. The composite pseudomonad given by a distributive law

Assume we have a distributive law of $\mathbb{U}$ over $\mathbb{D}$ as in section 4. The first question is how to produce a composite pseudomonad from $\mathbb{U}$, $\mathbb{D}$, and the distributive law. This is what we do in this section.

Define $\mathbb{V} = (V, v, p, \beta_{\mathbb{V}}, \eta_{\mathbb{V}}, \mu_{\mathbb{V}})$ as follows: $V = DU \in \mathbf{A}(\mathcal{K}, \mathcal{K})$; v is defined as the composite

$$1_{\mathcal{K}} \xrightarrow{u} U \xrightarrow{dU} DU;$$

p is the composite

$$DU DU \xrightarrow{DrU} DDUU \xrightarrow{DDn} DDU \xrightarrow{mU} DU;$$

$\beta_{\mathbb{V}}$ is defined to be the pasting

$$\begin{array}{ccccc}
 & & DUDU & & \\
 & \nearrow^{dUDU} & \uparrow & \searrow^{DrU} & \\
 & UDU & \xrightarrow{d_{uDU}^{-1}} & DDUU & \\
 & \nearrow^{uDU} & \uparrow^{D\omega_1 U \Downarrow} & \nearrow^{DDn} & \\
 DU & \xrightarrow{dDU} & DDU & \xrightarrow{DDuU \Downarrow DD\beta_{\mathbb{U}}} & DDU & \xrightarrow{mU} DU; \\
 & \searrow^{dDU} & \downarrow \beta_{\mathbb{D}} U & & \\
 & & Id & &
 \end{array}$$

$\eta_{\mathbb{V}}$ as the pasting

$$\begin{array}{ccccc}
 DU & \xrightarrow{Id} & DU & \xrightarrow{Id} & DU; \\
 \downarrow D\eta_{\mathbb{U}} & & \downarrow \eta_{\mathbb{D}} U & & \\
 DUU & \xrightarrow{Dn} & DDU & \xrightarrow{mU} & \\
 \downarrow Ddu & \searrow^{DDUU} & \downarrow Dd_n^{-1} & \nearrow^{DDn} & \\
 & DUDU & \xrightarrow{DrU} & DDUU & \\
 & \downarrow D\omega_2 U^{-1} & & &
 \end{array}$$

and $\mu_{\mathbb{V}}$ as the pasting

$$\begin{array}{ccccccc}
DUDUDU & \xrightarrow{DUDrU} & DUDDUU & \xrightarrow{DUDDn} & DUDDU & \xrightarrow{DUmU} & DUDU \\
\downarrow DrUDU & \Downarrow Dr_{rU}^{-1} & \downarrow DrDUU & \Downarrow Dr_{Dn}^{-1} & \downarrow DrDU & & \downarrow DrU \\
DDUUDU & \xrightarrow{DDUrU} & DDUDUU & \xrightarrow{DDUDn} & DDUDU & & \\
\downarrow DDnDU & & \downarrow DDrUU & \Downarrow DDr_n^{-1} & \downarrow DDrU & \Downarrow D\omega_4U & \\
DDUDU & \xrightarrow{DDrU} & DDDUUU & \xrightarrow{DDDUU} & DDDUU & \xrightarrow{DmUU} & DDUU \\
\downarrow mUDU & \Downarrow m_{rU}^{-1} & \downarrow DDDnU & \Downarrow DDD\mu_U & \downarrow DDDn & \Downarrow Dm_n & \downarrow DDn \\
DDUDU & \xrightarrow{DDrU} & DDDUU & \xrightarrow{DDDn} & DDDU & \xrightarrow{DmU} & DDU \\
\downarrow mUDU & \Downarrow m_{rU}^{-1} & \downarrow mDUU & \Downarrow m_{Dn}^{-1} & \downarrow mDU & \Downarrow \mu_{\mathbb{D}}U & \downarrow mU \\
DUDU & \xrightarrow{DrU} & DDUU & \xrightarrow{DDn} & DDU & \xrightarrow{mU} & DU.
\end{array}$$

5.1. THEOREM. $\mathbb{V} = (V, v, p, \beta_{\mathbb{V}}, \eta_{\mathbb{V}}, \mu_{\mathbb{V}})$, as defined above, is a pseudomonad on the object $\mathcal{K}$.

Proof. Observe that the pasting of $\mu_{\mathbb{V}}$ and $\eta_{\mathbb{V}}V^{-1}$ is

$$\begin{array}{ccccccc}
& \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \searrow D\eta_U DU^{-1} & \searrow D\omega_2 U DU & \Downarrow Dr_{rU}^{-1} & \Downarrow Dr_{Dn}^{-1} & & \downarrow \\
& & \searrow Dd_n DU & & \Downarrow DDr_n^{-1} & \searrow D\omega_4 U & \downarrow \\
& & & \Downarrow DD\omega_3 U & \Downarrow DDD\mu_U & \Downarrow Dm_n & \downarrow \\
& & \searrow \eta_{\mathbb{D}} U DU^{-1} & \Downarrow m_{rU}^{-1} & \Downarrow m_{Dn}^{-1} & \Downarrow \mu_{\mathbb{D}} U & \downarrow \\
& & & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad}
\end{array}$$

We must show that this pasting equals $p \circ V\beta_{\mathbb{V}}$. Substitute the pasting of $Dd_n DU$ and $D\eta_U DU^{-1}$ by the pasting of Dd_{UuDU}^{-1} and $DD\eta_U DU$. Then use (coh 2). With the help of (2) show that

$$\begin{array}{ccc}
\begin{array}{ccc}
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
\downarrow & \Downarrow DDr_{uU}^{-1} & \downarrow \\
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \Downarrow DDD\eta_U U^{-1} & \downarrow \\
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \Downarrow DDD\mu_U & \downarrow
\end{array} & = & \begin{array}{ccc}
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \Downarrow DDU D\beta_U & \downarrow \\
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \Downarrow DDrU & \downarrow \\
& \xrightarrow{\quad} & \xrightarrow{\quad} \\
& \Downarrow DDDn & \downarrow
\end{array}
\end{array}$$

and make the substitution. Make the substitution

$$\begin{array}{c} \xrightarrow{D\omega_2 U D U} \\ \searrow \Downarrow \\ \xrightarrow{\quad} \xrightarrow{D r_{rU}^{-1}} \xrightarrow{D r_{Dn}^{-1}} \end{array} = \begin{array}{c} \xrightarrow{D U d_{rU}} \xrightarrow{D U d_{Dn}} \\ \searrow \Downarrow D d_{UrU}^{-1} \quad \searrow \Downarrow D d_{UDn}^{-1} \quad \searrow \Downarrow D\omega_2 D U \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array},$$

followed by the substitutions

$$\begin{array}{c} \xrightarrow{\quad} \xrightarrow{D d_{UrU}^{-1}} \\ \searrow \Downarrow D d_{UuDU}^{-1} \quad \searrow \Downarrow DDU\omega_1 U \quad \searrow \Downarrow D d_{UD}^{-1} \eta \\ \xrightarrow{\quad} \xrightarrow{D d_{UD}^{-1} \eta} \\ \searrow \Downarrow DDU D \beta_U \\ \xrightarrow{\quad} \end{array} = \begin{array}{c} \xrightarrow{D U \omega_1 U} \\ \searrow \Downarrow D U D \beta_U \\ \xrightarrow{\quad} \xrightarrow{D d U D U} \end{array},$$

and

$$\begin{array}{c} \xrightarrow{\eta_D U D U^{-1}} \\ \searrow \Downarrow \\ \xrightarrow{\quad} \xrightarrow{m_{rU}^{-1}} \xrightarrow{m_{Dn}^{-1}} \end{array} = \begin{array}{c} \xrightarrow{D d_{rU}} \\ \searrow \Downarrow \\ \xrightarrow{D d_{Dn}} \\ \searrow \Downarrow \eta_D D U^{-1} \\ \xrightarrow{\quad} \end{array},$$

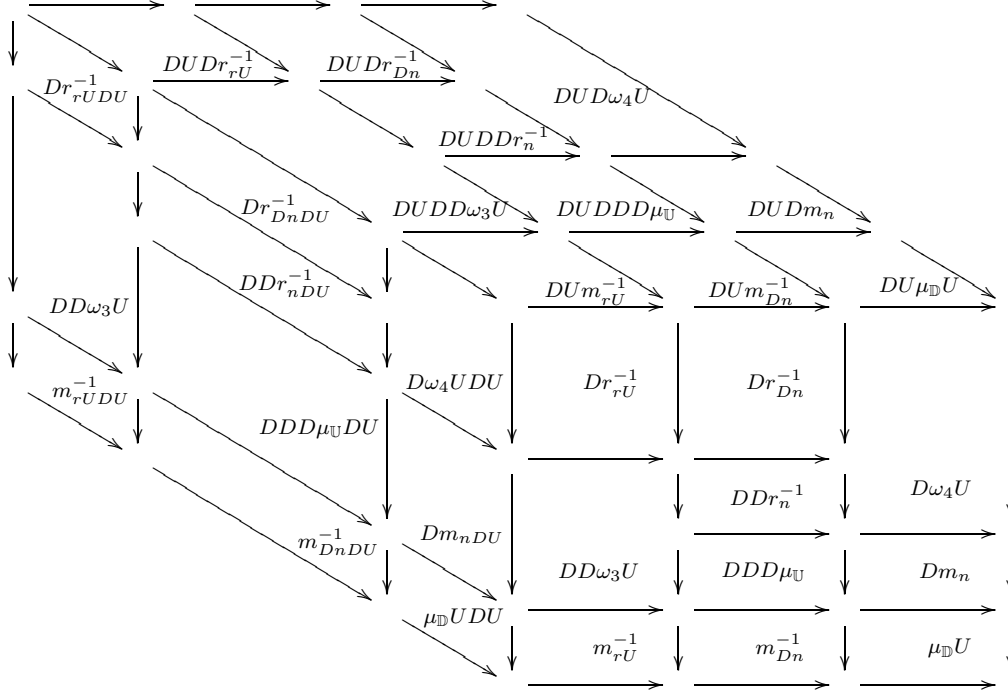
and

$$\begin{array}{c} \xrightarrow{D d_{Dn}} \xrightarrow{D m_n} \\ \searrow \Downarrow \eta_D D U^{-1} \quad \searrow \Downarrow \mu_D U \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array} = \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\Downarrow D \beta_D D U U} \\ \searrow \Downarrow D D n \\ \xrightarrow{\quad} \xrightarrow{m U} \end{array}.$$

Use (coh 7), and finish with the substitution

$$\begin{array}{c} \xrightarrow{D U \omega_1 U} \xrightarrow{D U d_{rU}} \\ \searrow \Downarrow D U D \beta_U \quad \searrow \Downarrow D U d_{Dn} \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array} = \begin{array}{c} \xrightarrow{D U d_{uDU}^{-1}} \xrightarrow{D U D \omega_1 U} \\ \searrow \Downarrow D U D D \beta_U \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array}$$

In regards to the other condition, observe that the pasting of $V\mu_{\mathbb{V}}$, $\mu_{\mathbb{V}}V$ and $\mu_{\mathbb{V}}$ is:



Where we have only put the name of the corresponding 2-cell in each parallelogram. To show that this pasting is equal to the pasting of p_p , $\mu_{\mathbb{V}}$ and $\mu_{\mathbb{V}}$ we do the following. First make the substitution

$$\begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \swarrow & \xrightarrow{DUm_r^{-1}} & \searrow \\
 \downarrow & \xrightarrow{DUm_{Dn}^{-1}} & \downarrow \\
 \swarrow & \xrightarrow{D\omega_4 UDU} & \searrow \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \swarrow & \xrightarrow{\Downarrow Dr_{DrU}^{-1}} & \searrow \\
 \downarrow & \xrightarrow{\Downarrow Dr_{DnU}^{-1}} & \downarrow \\
 \swarrow & \xrightarrow{Dm_{UrU}^{-1}} & \searrow \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}$$

Then make the substitutions

$$\begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \swarrow & \xrightarrow{Dm_{UrU}^{-1}} & \searrow \\
 \downarrow & \xrightarrow{Dm_{DnU}^{-1}} & \downarrow \\
 \swarrow & \xrightarrow{DD\omega_3 U} & \searrow \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \swarrow & \xrightarrow{DDDr_n^{-1}} & \searrow \\
 \downarrow & \xrightarrow{DDDD\mu_{\mathbb{U}}} & \downarrow \\
 \swarrow & \xrightarrow{Dm_{rU}} & \searrow \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}$$

[illegible]

Commutative diagram showing relationships between operators and their inverses:

$$\begin{array}{ccccccc}
 & \xrightarrow{\quad} & \xrightarrow{\quad} & \searrow & & \searrow & \\
 \downarrow & \text{\scriptsize Dr_{DUrU}^{-1}} & \text{\scriptsize Dr_{DUDn}^{-1}} & & \text{\scriptsize Dr_{rU}^{-1}} & \downarrow & \text{\scriptsize Dr_{rDn}^{-1}} \\
 & \Downarrow & \Downarrow & & \Downarrow & & \Downarrow \\
 \downarrow & \text{\scriptsize DDr_{UrU}^{-1}} & \text{\scriptsize DDr_{UDn}^{-1}} & \searrow & \text{\scriptsize DDr_{rU}^{-1}} & \downarrow & \text{\scriptsize DDr_{rDn}^{-1}} \\
 & \Downarrow & \Downarrow & & \Downarrow & & \Downarrow \\
 \downarrow & \xrightarrow{\quad} & \xrightarrow{\quad} & \searrow & \text{\scriptsize $DDDUr_n^{-1}$} & \searrow & \text{\scriptsize $DDDU\mu_U^{-1}$} \\
 & & & \searrow & \Downarrow & \searrow & \Downarrow \\
 & & & \text{\scriptsize $DDDU\omega_3 U$} & & \text{\scriptsize $DDDU\mu_U$} & \\
 & & & \Downarrow & & \Downarrow & \\
 & & & \xrightarrow{\quad} & & \xrightarrow{\quad} &
 \end{array}$$
$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \searrow \text{\scriptsize $DDDUD\mu_U$} \\ \downarrow \quad \searrow \text{\scriptsize $DDDr_n^{-1}$} \\ \downarrow \quad \searrow \text{\scriptsize $DDDD\mu_U$} \\ \downarrow \quad \searrow \text{\scriptsize $DDDD\mu_U$} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \text{\scriptsize $DDDr_n^{-1}$} \\ \downarrow \text{\scriptsize $DDDD\mu_U$} \\ \downarrow \text{\scriptsize $DDDD\mu_U$} \end{array} \\
= & & \begin{array}{ccc} \xrightarrow{\quad} & \downarrow \text{\scriptsize $DDDr_n^{-1}$} & \searrow \\ \downarrow \text{\scriptsize $DDDr_n^{-1}$} & \downarrow \text{\scriptsize $DDDD\mu_U$} & \searrow \\ \downarrow \text{\scriptsize $DDDD\mu_U$} & \downarrow \text{\scriptsize $DDDD\mu_U$} & \searrow \\ \downarrow \text{\scriptsize $DDDD\mu_U$} & \downarrow \text{\scriptsize $DDDD\mu_U$} & \searrow \end{array}
\end{array}$$
$$\begin{array}{ccc}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
\downarrow & \searrow \Downarrow DDDU r_n^{-1} & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD\omega_3 U U & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD r_{U_n}^{-1} & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD D r_n^{-1} & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow
\end{array}
& = &
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD n^{-1} D_n & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD\omega_3 U & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & \searrow \Downarrow DDD r_n^{-1} & \searrow \\
\longrightarrow & \longrightarrow & \longrightarrow
\end{array}
\end{array}$$

and then

$$\begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \mu_{\mathbb{D}} \Upsilon DU & \xleftarrow{\quad} & \mu_{\mathbb{D}} \Upsilon DU \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 Dm_{rU}^{-1} & \xrightarrow{\quad} & Dm_{Dn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{rU}^{-1} & \xrightarrow{\quad} & m_{Dn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{DrU}^{-1} & \xrightarrow{\quad} & m_{DDn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{rU}^{-1} & \xrightarrow{\quad} & m_{Dn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array},$$

followed by the substitution

$$\begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DDDr_n^{-1} & \xrightarrow{\quad} & DDD\omega_3 U \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{DnDU}^{-1} & \xrightarrow{\quad} & m_{DrU}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{DDn}^{-1} & \xrightarrow{\quad} & m_{DDn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{DUrU}^{-1} & \xrightarrow{\quad} & m_{DDn}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DDr_n^{-1} & \xrightarrow{\quad} & DDD\mu_U \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DD\omega_3 U & \xrightarrow{\quad} & DDD\mu_U \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}.$$

Use (coh 9) to show that

$$\begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DU \mu_{\mathbb{D}} U & \xrightarrow{\quad} & DU \mu_{\mathbb{D}} U \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 D\omega_4 DU & \xrightarrow{\quad} & D\omega_4 U \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 Dm_{rU} & \xrightarrow{\quad} & Dm_n \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 \mu_{\mathbb{D}} DU & \xrightarrow{\quad} & \mu_{\mathbb{D}} U \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 Dr_{mU}^{-1} & \xrightarrow{\quad} & Dr_{mU}^{-1} \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DD\omega_4 U & \xrightarrow{\quad} & DD\omega_4 U \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 DDm_n & \xrightarrow{\quad} & DDm_n \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 m_{mU} & \xrightarrow{\quad} & m_{mU} \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}$$

and make the substitution. Next the substitution

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{DU Dr_r^{-1}} & \downarrow & \searrow^{DU Dr_{Dn}^{-1}} & \downarrow \\
 & \xrightarrow{Dr_r^{-1} DU} & \Downarrow & \xrightarrow{Dr_{DUrU}^{-1}} & \Downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{DD\omega_3 U DU} & \Downarrow & \xrightarrow{DDr_{U r U}^{-1}} & \Downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{m_{rDU}^{-1}} & \Downarrow & \xrightarrow{m_{DUrU}^{-1}} & \Downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 & = &
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{Dr_U^{-1} Dr_r U} & \downarrow & \searrow^{Dr_U^{-1} DDn} & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{\quad} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{DDn_{DrU}^{-1}} & \Downarrow & \xrightarrow{DDn_{DDn}^{-1}} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{m_U^{-1} Dr_r U} & \Downarrow & \xrightarrow{m_U^{-1} DDn} & \downarrow \\
 & \searrow^{Dr_r^{-1} U} & \Downarrow & \searrow^{Dr_{Dn}^{-1} U} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 ,
 \end{array}$$

followed by the substitution

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{Dr_r^{-1} DU} & \downarrow & \searrow^{DU D\omega_4 U} & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{\quad} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{Dr_{DrU}^{-1}} & \Downarrow & \xrightarrow{\quad} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{Dr_{DDn}^{-1}} & \Downarrow & \xrightarrow{DDm_n} & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{Dr_{mU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 & = &
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{Dr_U^{-1} m_U} & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{DDU \omega_4 U} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{Dr_r^{-1} U} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{DDU m_n} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 .
 \end{array}$$

Substitute the pasting of $DDUm_n$, $DD\omega_4 U$, DDr_{Dn}^{-1} and $DDDr_n^{-1}$ by the pasting of $DD\omega_4 UU$, DDr_n^{-1} and DDm_{Un} . Then observe that the pasting of DDm_{Un} , DDm_n and $DDDD\mu_U$ equals the pasting of $DDD\mu_U$, DDm_{nU} and DDm_n . Use (coh 6). To finish the proof, make the substitution

$$\begin{array}{c}
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{m_{rDU}^{-1}} & \downarrow & \searrow^{DD\omega_4 U} & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{\quad} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{m_{DrU}^{-1}} & \Downarrow & \xrightarrow{\quad} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{m_{DDn}^{-1}} & \Downarrow & \xrightarrow{DDm_n} & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{m_{mU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 & = &
 \begin{array}{ccccc}
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \downarrow & \searrow^{m_U^{-1} m_U} & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{D\omega_4 U} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{m_r^{-1} U} & \downarrow \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & \xrightarrow{\quad} & \Downarrow & \xrightarrow{m_{Dn}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{Dm_n} & \downarrow \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 \end{array}
 .
 \end{array}$$

■

6. Compatible pseudomonad structures

We consider now the question of when can a pseudomonad be considered as the composite of two pseudomonads.

Let $\mathbb{D}$, $\mathbb{U}$ be pseudomonads on the same object $\mathcal{K}$ of the **Gray**-category **A**. Given another pseudomonad $\mathbb{V} = (V, v, p, \beta_V, \eta_V, \mu_V)$ on the same object $\mathcal{K}$, we say that $\mathbb{V}$ is *compatible* with the pseudomonads $\mathbb{D}$ and $\mathbb{U}$ if $V = DU$ and there are invertible 2-cells:

$$\begin{array}{ccc}
 \begin{array}{c} U \\ \swarrow u \quad \searrow dU \\ 1_{\mathcal{K}} \xrightarrow{v} DU \end{array} & \begin{array}{c} DUDU \\ \swarrow DUdU \quad \searrow p \\ DUU \xrightarrow{Dn} DU \end{array} & \begin{array}{c} DUDU \\ \swarrow DuDU \quad \searrow p \\ DDU \xrightarrow{mU} DU \end{array}
 \end{array}$$

$\theta_1 \Downarrow$ $\theta_2 \Downarrow$ $\theta_3 \Downarrow$

subject to coherence conditions. To describe the coherence conditions introduce the following pastings:

$$\theta_4 = \begin{array}{ccc}
 DUDUU & \xrightarrow{DUDn} & DUDU \\
 \downarrow pU & \searrow DUdU & \downarrow DU\theta_2^{-1} \\
 & DUDUDU & \downarrow DU p \\
 & \downarrow pDU & \downarrow \mu_V \\
 & DUDU & \downarrow \mu_V^{-1} \\
 DUU & \xrightarrow{DUDU} & DU \\
 & \downarrow Dn & \downarrow p
 \end{array}$$

and

$$\theta_5 = \begin{array}{ccc}
 DDU DU & \xrightarrow{mUDU} & DUDU \\
 \downarrow Dp & \searrow DuDU & \downarrow \theta_3 DU^{-1} \\
 & DUDUDU & \downarrow pDU \\
 & \downarrow DU p & \downarrow \mu_V^{-1} \\
 & DUDU & \downarrow \mu_V^{-1} \\
 DDU & \xrightarrow{DuDU} & DU \\
 & \downarrow mU & \downarrow p
 \end{array}$$

The coherence conditions are:

$$\begin{array}{ccc}
 DU & \xrightarrow{Id} & DU \\
 \downarrow DUv & \searrow \eta_V & \downarrow p \\
 & DUDU & \downarrow \theta_2 \\
 DUu \searrow DU\theta_1^{-1} & & DUU \xrightarrow{DUdU} \\
 & & \downarrow Dn
 \end{array}
 = D\eta_U,
 \tag{12}$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & DUU & & \\
 & \nearrow^{DuU} & \searrow^{DUdU} & \nearrow^{Dn} & \\
 DU & \xrightarrow{DdU} & DDU & \xrightarrow{mU} & DU \\
 & \searrow^{Id} & \nearrow^{DUdU} & \searrow^{p} & \\
 & & DUDU & & \\
 & \nearrow^{DuDU} & \searrow^{p} & \nearrow^{p} & \\
 & & DU & &
 \end{array}
 \end{array}
 = D\beta_U,
 \quad (13)$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & DDU & & \\
 & \nearrow^{dDU} & \searrow^{DuDU} & \nearrow^{mU} & \\
 DU & \xrightarrow{uDU} & UDU & \xrightarrow{dUDU} & DU \\
 & \searrow^{Id} & \nearrow^{DUdU} & \searrow^{p} & \\
 & & DUDU & & \\
 & \nearrow^{DuDU} & \searrow^{p} & \nearrow^{p} & \\
 & & DU & &
 \end{array}
 \end{array}
 = \beta_{DU},
 \quad (14)$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & DDUU & & DUU \\
 & \nearrow^{DDn} & \searrow^{DuDUU} & \nearrow^{pU} & \\
 DDUU & \xrightarrow{mUU} & DUDUU & \xrightarrow{pU} & DUU \\
 & \searrow^{DDn} & \nearrow^{DUdUU} & \searrow^{pU} & \\
 & & DUDU & & \\
 & \nearrow^{DuDU} & \searrow^{p} & \nearrow^{p} & \\
 & & DU & &
 \end{array}
 \end{array}
 = m_n
 \quad (15)$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & DUUU & & DUU \\
 & \nearrow^{DUU} & \searrow^{DUdUU} & \nearrow^{pU} & \\
 DUUU & \xrightarrow{DnU} & DUDUU & \xrightarrow{pU} & DUU \\
 & \searrow^{DUU} & \nearrow^{DUdUU} & \searrow^{pU} & \\
 & & DUDU & & \\
 & \nearrow^{DUdU} & \searrow^{p} & \nearrow^{p} & \\
 & & DU & &
 \end{array}
 \end{array}
 = D\mu_U^{-1}
 \quad (16)$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & DDDU & & DDU \\
 & \nearrow^{mDU} & \searrow^{DDuDU} & \nearrow^{Dp} & \\
 DDDU & \xrightarrow{DmU} & DDDU & \xrightarrow{Dp} & DDU \\
 & \searrow^{mDU} & \nearrow^{DDuDU} & \searrow^{Dp} & \\
 & & DDDU & & \\
 & \nearrow^{DuDU} & \searrow^{p} & \nearrow^{p} & \\
 & & DU & &
 \end{array}
 \end{array}
 = \mu_{DU}
 \quad (17)$$

Some other equations we will need later are contained in the following:

6.1. PROPOSITION. Assume we have invertible 2-cells θ_1 , θ_2 , and θ_3 , satisfying the coherence conditions. Then

$$(Dn, \theta_4) : (\beta_{\mathbb{V}}U, \mu_{\mathbb{V}}U) \rightarrow (\beta_{\mathbb{V}}, \mu_{\mathbb{V}}) \quad (18)$$

is a 1-cell in $\mathbb{V}\text{-Alg}_{\mathcal{K}}$, and the following are 2-cells in $\mathbb{V}\text{-Alg}_{\mathcal{K}}$:

$$\begin{array}{ccc} (\beta_{\mathbb{V}}, \mu_{\mathbb{V}}) & \xrightarrow{Id} & (\beta_{\mathbb{V}}, \mu_{\mathbb{V}}) \\ & \searrow (DUu, p_u^{-1}) \quad \Downarrow D\eta_U & \nearrow (Dn, \theta_4) \\ & (\beta_{\mathbb{V}}U, \mu_{\mathbb{V}}U) & \end{array} \quad (19)$$

$$\begin{array}{ccccc} & (DUu, p_u^{-1}) & (\beta_{\mathbb{V}}U, \mu_{\mathbb{V}}U) & (Dn, \theta_4) & \\ & \nearrow & \Downarrow D\mu_U & \searrow & \\ (\beta_{\mathbb{V}}UU, \mu_{\mathbb{V}}UU) & & & & (\beta_{\mathbb{V}}, \mu_{\mathbb{V}}), \\ & (DnU, \theta_4U) & (\beta_{\mathbb{V}}U, \mu_{\mathbb{V}}U) & (Dn, \theta_4) & \end{array} \quad (20)$$

We also have that

$$(p, \theta_5^{-1}) : (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) \rightarrow (\beta_{\mathbb{D}}U, \mu_{\mathbb{D}}U) \quad (21)$$

is a 1-cell in $\mathbb{D}\text{-Alg}_{\mathcal{K}}$, and the following are 2-cells in $\mathbb{D}\text{-Alg}_{\mathcal{K}}$:

$$\begin{array}{ccccc} & (DUu, p_u^{-1}) & (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) & (p, \theta_5^{-1}) & \\ & \nearrow & \Downarrow \mu_{\mathbb{V}} & \searrow & \\ (\beta_{\mathbb{D}}UDUDU, \mu_{\mathbb{D}}UDUDU) & & & & (\beta_{\mathbb{D}}U, \mu_{\mathbb{D}}U), \\ & (pDU, \theta_5DU^{-1}) & (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) & (p, \theta_5^{-1}) & \end{array} \quad (22)$$

$$\begin{array}{ccccc} & & (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) & & \\ & (DUdU, m_{UdU}^{-1}) \nearrow & \searrow (p, \theta_5^{-1}) & & \\ (\beta_{\mathbb{D}}UU, \mu_{\mathbb{D}}UU) & \xrightarrow{(Dn, m_n^{-1})} & & \searrow \theta_2 \Downarrow & (\beta_{\mathbb{D}}U, \mu_{\mathbb{D}}U), \end{array} \quad (23)$$

$$\begin{array}{ccccc} & (DUDn, m_{UDn}^{-1}) & (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) & (p, \theta_5^{-1}) & \\ & \nearrow & \Downarrow \theta_4 & \searrow & \\ (\beta_{\mathbb{D}}UDUU, \mu_{\mathbb{D}}UDUU) & & & & (\beta_{\mathbb{D}}U, \mu_{\mathbb{D}}U), \\ & (pU, \theta_5U^{-1}) & (\beta_{\mathbb{D}}UU, \mu_{\mathbb{D}}UU) & (Dn, m_n^{-1}) & \end{array} \quad (24)$$

Furthermore, if we define σ_1 as,

$$\begin{array}{ccccc} & & UDU & & \\ & \nearrow uDU & \searrow dUDU & & \\ DU & \xrightarrow{vDU} & & \searrow p & DU, \\ & \nearrow \theta_1 DU \Downarrow & & \searrow \beta_{\mathbb{V}} \Downarrow & \\ & & DU DU & & \\ & \xrightarrow{Id} & & & \end{array}$$

σ_2 as

$$\begin{array}{ccccc}
UU DU & \xrightarrow{U d U DU} & U DU DU & \xrightarrow{U p} & U DU \\
\downarrow n DU & \searrow d U U DU & \downarrow d U DU DU & \searrow d U_p^{-1} & \downarrow d U DU \\
& DU U DU & \xrightarrow{DU d U DU} & DU DU DU & \xrightarrow{DU p} & DU DU \\
& \searrow d_n DU & \searrow \theta_2 DU & \downarrow p DU & \searrow \mu_V & \downarrow p \\
& & D n DU & & & DU \\
& & \searrow & & & \\
& & DU DU & \xrightarrow{p} & DU,
\end{array}$$

 γ as

$$\begin{array}{ccc}
UU & \xrightarrow{U d U} & U DU \\
\downarrow n & \searrow d U U & \downarrow d U DU \\
& DU U & \xrightarrow{DU d U} & DU DU \\
& \searrow D n & \searrow \theta_2 & \downarrow p \\
& & d_n & DU \\
& & \searrow & \\
& & U & \xrightarrow{d U} & DU.
\end{array}$$

then (σ_1, σ_2) is an object in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$, and $(dU, \gamma) : (\beta_U, \mu_U) \rightarrow (\sigma_1, \sigma_2)$ is a 1-cell in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$.

Proof. We show first that (Dn, θ_4) is a 1-cell in $\mathbb{V}\text{-Alg}_{\mathcal{K}}$. To show that (Dn, θ_4) satisfies (9), start on the left hand side and make the substitutions:

$$\begin{array}{ccc}
\begin{array}{ccc} \rightarrow & \rightarrow & \\ \beta_V U \searrow & \downarrow p_{dU}^{-1} & \downarrow \\ \rightarrow & \rightarrow & \end{array} & = & \begin{array}{ccc} \rightarrow & \rightarrow & \\ \uparrow v_{DU dU} & \uparrow \beta_V DU & \nearrow \\ \rightarrow & \rightarrow & \end{array}, \\
\begin{array}{ccc} \rightarrow & \rightarrow & \\ \beta_V DU \searrow & \downarrow \mu_V & \downarrow \\ \rightarrow & \rightarrow & \end{array} & = & \begin{array}{ccc} \rightarrow & \rightarrow & \\ \uparrow v_p & \uparrow \beta_V & \nearrow \\ \rightarrow & \rightarrow & \end{array},
\end{array}$$

and

$$\begin{array}{ccc}
\begin{array}{ccc} \rightarrow & \rightarrow & \\ \downarrow \theta_2 & \rightarrow & \downarrow DU \theta_2^{-1} \\ \downarrow & \rightarrow & \downarrow \\ \downarrow & \rightarrow & \downarrow \\ \downarrow & \rightarrow & \downarrow \end{array} & = & v_{Dn}.
\end{array}$$

To show that (Dn, θ_4) satisfies (10), start on the left hand side, cancel $DU \theta_2$ and its inverse, make the substitution

$$\begin{array}{ccc}
\begin{array}{ccc} \rightarrow & \rightarrow & \\ \downarrow & \searrow DU p_{dU}^{-1} & \rightarrow \\ \downarrow \mu_V U & \downarrow p_{dU}^{-1} & \downarrow \\ \downarrow & \rightarrow & \downarrow \end{array} & = & \begin{array}{ccc} \rightarrow & \rightarrow & \\ \downarrow & \searrow p_{DU dU}^{-1} & \rightarrow \\ \downarrow & \searrow \mu_V DU & \downarrow \\ \downarrow & \searrow p_{dU}^{-1} & \downarrow \end{array}.
\end{array}$$

Then use (1), and conclude with

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{DUDU\theta_2^{-1}} & \\
 \downarrow & \xrightarrow{p_{DUdU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 & \xrightarrow{p_p^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 & = &
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{p_{Dn}^{-1}} & \downarrow \\
 & \xrightarrow{DU\theta_2^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 \end{array}$$

Next we show that $D\eta_U$ in (19) is a 2-cell in $\mathbb{V}\text{-Alg}_{\mathcal{K}}$. To show that satisfies (11), start on the left hand side of (11). Use (12) to substitute the pasting of $DUD\eta_U$ and $DU\theta_2^{-1}$ for the pasting of $DU\beta_V$ and $DUDU\theta_1^{-1}$. Then make the substitution

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{DUDU\theta_1^{-1}} & \\
 \downarrow & \xrightarrow{p_u^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 & \xrightarrow{p_{dU}^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 & = &
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{p_v^{-1}} & \downarrow \\
 & \xrightarrow{DU\theta_1^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 \end{array}$$

Now use (5), and then use (12) once again.

Next we show that $D\mu_U$ in (20) is a 2-cell of $\mathbb{V}$ -algebras. Start on the right hand side of (11). Use (16) to substitute the pasting of θ_2 and $D\mu_U$, by the pasting of DUd_n^{-1} , $DU\theta_4$ and θ_2U . Make the substitution

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{p_n^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 & \xrightarrow{DUd_n^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 & = &
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{DUd_n^{-1}} & \downarrow \\
 & \xrightarrow{p_{dUU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \downarrow \\
 & \xrightarrow{p_{Dn}^{-1}} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 \end{array}$$

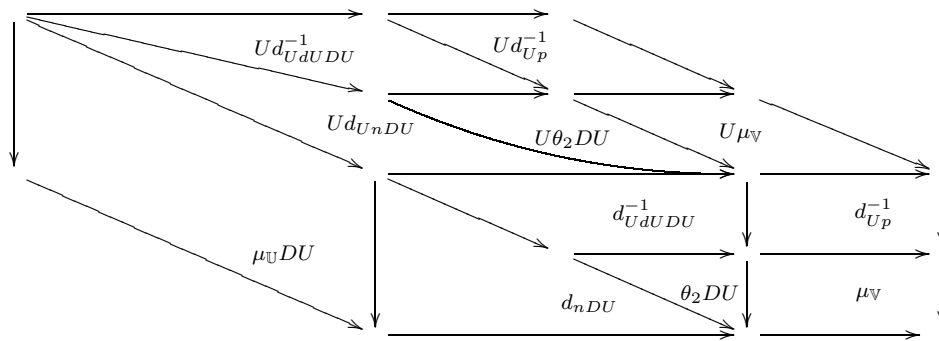
Since we already know that (Dn, θ_4) is a $\mathbb{V}$ -algebra morphism, we can substitute the pasting of μ_V , p_{Dn}^{-1} , and θ_4 , by the pasting of $DU\theta_4$, θ_4 and μ_V . By the definition of θ_4 , we can substitute the pasting of μ_VU , p_{dUU}^{-1} and θ_2U by the pasting of $DU\theta_2U$ and θ_4U . Finally, use (16).

That (p, θ_5^{-1}) is a morphism of $\mathbb{D}$ -algebras, and that μ_V , θ_2 , and θ_4 are 2-cells of $\mathbb{D}$ -algebras are similar and left to the reader.

Now we show that (σ_1, σ_2) is an object in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. Paste $\eta_U DU^{-1}$ onto the left hand side of (7), and make a substitution on it to obtain

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{d_{UuDU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 & \xrightarrow{\eta_U DU} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow
 \end{array}
 & \xrightarrow{\quad} &
 \begin{array}{ccc}
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{d_{UdUDU}^{-1}} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\theta_2 DU} & \downarrow \\
 & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\mu_V} & \downarrow
 \end{array}
 \end{array}
 \quad (25)$$

Now, the left hand side of (6) in this case, is the following pasting:



$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \searrow U d_n DU \\ d_{U n DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U d U DU}^{-1} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow d_{U d U U DU}^{-1} \quad \searrow \\ \xrightarrow{\quad} \quad \downarrow d_{U D n DU}^{-1} \end{array} \\
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \searrow U \theta_2 DU \\ d_{U D n DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U p DU}^{-1} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow d_{U DU d U DU}^{-1} \quad \searrow \\ \xrightarrow{\quad} \quad \downarrow DU \theta_2 DU \end{array}
\end{array}$$

and

$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow U d_{U d U DU} \quad \searrow U d_{U p}^{-1} \\ d_{U d U U DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U DU d U DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U DU p}^{-1} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow d_{U U d U DU}^{-1} \quad \downarrow d_{U U p}^{-1} \quad \searrow \\ \xrightarrow{\quad} \quad \xrightarrow{\quad} \quad \downarrow d_{U d U DU}^{-1} \\ \quad \quad \quad \downarrow DU d_{U d U DU}^{-1} \quad \downarrow DU d_{U p}^{-1} \end{array}
\end{array}$$

On the other hand, the right hand side of (6) in this case is the pasting

$$\begin{array}{c}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \nearrow n_p^{-1} \\ \quad \quad \quad \downarrow n_{d U DU}^{-1} \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow d_n DU \quad \searrow \theta_2 DU \\ d_{U d U DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U p}^{-1} \downarrow \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow d_{U p}^{-1} \quad \searrow \mu_V \\ \quad \quad \quad \downarrow \mu_V \end{array} \\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \nearrow n_p^{-1} \\ \quad \quad \quad \downarrow n_{d U DU}^{-1} \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow d_n DU \quad \searrow \theta_2 DU \\ d_{U d U DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U p}^{-1} \downarrow \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow d_{U p}^{-1} \quad \searrow \mu_V \\ \quad \quad \quad \downarrow \mu_V \end{array}
\end{array}$$

On this pasting make the following substitutions:

$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow n_{d U DU}^{-1} \quad \searrow n_p^{-1} \\ d_{U d U DU}^{-1} \downarrow \quad \xrightarrow{\quad} d_{U p}^{-1} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow d_{U U d U DU}^{-1} \quad \searrow d_{U U p}^{-1} \\ d_n U DU \downarrow \quad \xrightarrow{\quad} D n_{d U DU}^{-1} \downarrow \quad \xrightarrow{\quad} D n_p^{-1} \downarrow \end{array} \\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow D n_{d U DU}^{-1} \quad \searrow D n_p^{-1} \\ \quad \quad \quad \downarrow \theta_2 DU \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \downarrow \quad \searrow \theta_2 U DU \quad \searrow DU d_{U d U DU}^{-1} \\ \quad \quad \quad \downarrow p_{d U DU}^{-1} \quad \downarrow DU d_{U p}^{-1} \end{array}
\end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \quad \downarrow \\ \xrightarrow{p_{dU DU}^{-1}} \\ \downarrow \quad \downarrow \\ \xrightarrow{\quad} \\ \searrow \theta_2 DU \end{array} & = & \begin{array}{c} \xrightarrow{DU \theta_2 DU} \\ \downarrow \quad \searrow \\ \xrightarrow{\quad} \quad \downarrow \mu_{\mathbb{V}} DU^{-1} \\ \searrow \theta_4 DU \quad \downarrow \\ \xrightarrow{\quad} \end{array}
 \end{array} \quad (\text{definition of } \theta_4).$$

Compare both results, and use equation (16).

The claim about (dU, γ) is left to the reader. ■

Next we prove that the composite of two pseudomonads via a distributive law as defined in section 5, is compatible.

6.2. THEOREM. *Assume we have a distributive law as in section 4 and let $\mathbb{V}$ the composite pseudomonad defined in section 5. If we define $\theta_1 = id_{dU \circ u}$, and θ_2 as the pasting*

$$\begin{array}{ccc}
 & DU DU & \\
 DU dU \nearrow & \Downarrow D\omega_2 U & \searrow DrU \\
 DUU & \xrightarrow{DdUU} & DDUU \\
 Dn \downarrow & \not\Downarrow Dd_n & \downarrow DDn \\
 DU & \xrightarrow{DdU} & DDU \\
 & Id \searrow & \not\Downarrow \eta_{\mathbb{D}} U^{-1} \downarrow mU \\
 & & DU,
 \end{array}$$

and θ_3 as the pasting

$$\begin{array}{ccc}
 & DU DU & \\
 Du DU \nearrow & \Downarrow D\omega_1 U & \searrow DrU \\
 DDU & \xrightarrow{DDuU} & DDUU \\
 & \not\Downarrow DD\beta_U & \searrow DDn \\
 & Id \searrow & DDU \xrightarrow{mU} DU,
 \end{array}$$

then the pseudomonad $\mathbb{V}$ is compatible with $\mathbb{U}$ and $\mathbb{D}$.

$$\begin{array}{ccc} \theta_4 & = & \begin{array}{ccc} DUDUU \xrightarrow{DUDn} DUDU & & \\ \text{\scriptsize $DrUU$} \downarrow & \not\Downarrow_{\text{\scriptsize Dr_n^{-1}}} & \downarrow \text{\scriptsize DrU} \\ DDUUU \xrightarrow{DDUn} DDUU & & \\ \text{\scriptsize $DDnU$} \downarrow & \not\Downarrow_{\text{\scriptsize $DD\mu_U$}} & \downarrow \text{\scriptsize DDn} \\ DDUU \xrightarrow{DDn} DDU & & \\ \text{\scriptsize mUU} \downarrow & \not\Downarrow_{\text{\scriptsize m_n^{-1}}} & \downarrow \text{\scriptsize mU} \\ DUU \xrightarrow{Dn} DU, & & \end{array} & \text{and} & \theta_5 = \begin{array}{ccc} DDDUU \xrightarrow{mUDU} DUDU & & \\ \text{\scriptsize $DDrU$} \downarrow & \not\Downarrow_{\text{\scriptsize m_{rU}}} & \downarrow \text{\scriptsize DrU} \\ DDDUU \xrightarrow{mDUU} DDUU & & \\ \text{\scriptsize $DDDn$} \downarrow & \not\Downarrow_{\text{\scriptsize m_{Dn}}} & \downarrow \text{\scriptsize DDn} \\ DDDU \xrightarrow{mDU} DDU & & \\ \text{\scriptsize DmU} \downarrow & \not\Downarrow_{\text{\scriptsize $\mu_{\mathbb{D}}U$}} & \downarrow \text{\scriptsize mU} \\ DDU \xrightarrow{mU} DU. & & \end{array} \end{array}$$
[illegible]
$$\begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} & \begin{array}{c} m_{UdU}^{-1} \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} \\
\begin{array}{c} \downarrow \\ \nearrow \end{array} & \begin{array}{c} D\omega_2 U \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} \\
\begin{array}{c} \downarrow \\ \nearrow \end{array} & \begin{array}{c} Dd_n \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array}
\end{array}
\quad = \quad
\begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} & \begin{array}{c} DD\omega_2 U \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} \\
\begin{array}{c} \downarrow \\ \nearrow \end{array} & \begin{array}{c} m_n^{-1} \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array} \\
\begin{array}{c} \downarrow \\ \nearrow \end{array} & \begin{array}{c} DDd_n \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ \downarrow \\ \searrow \end{array}
\end{array}
\end{array}$$
$$\begin{array}{c}
\text{---} \longrightarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---}
\end{array}
\begin{array}{c}
DDUd_n^{-1} \\
DD\omega_2UU \\
DDr_n^{-1}
\end{array}
=
\begin{array}{c}
\text{---} \longrightarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---} \\
\text{---} \downarrow \text{---} \text{---} \downarrow \text{---}
\end{array}
\begin{array}{c}
DDd_{U_n}^{-1} \\
DD\omega_2U
\end{array}$$

and

The proof for θ_5 is very similar, except that we need the equation:

To prove this last equation, observe that Id_{DU} is isomorphic to $Dn \circ rU \circ uDU$, thus it suffices to show that the equation holds when followed by $Dn \circ rU \circ uDU$. And, since u_r and ω_3 are invertible, we can paste them below on both sides of the equation. This last equation is not hard to prove, using (coh 4) and (coh 2). The rest of the proof is left to the reader. $\blacksquare$

Given pseudomonads $\mathbb{D}$ on $\mathcal{K}$, and $\mathbb{U}$ on $\mathcal{L}$, we denote the 2-category $\text{PSM}(\mathbf{A})(\mathbb{D}, \mathbb{U})$ as $[\mathbb{D}, \mathbb{U}]$, and we define it as follows. The objects of $[\mathbb{D}, \mathbb{U}]$ are pairs (G, G_0) , where $G : \mathbb{U}\text{-Alg} \rightarrow \mathbb{D}\text{-Alg} : \mathbf{A}^{op} \rightarrow \mathbf{Gray}$ is a **Gray**-natural transformation, $G_0 \in \mathbf{A}(\mathcal{L}, \mathcal{K})$, such

that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg} & \xrightarrow{G} & \mathbb{D}\text{-Alg} \\ \Phi \downarrow & & \downarrow \Phi \\ \mathbf{A}(-, \mathcal{L}) & \xrightarrow{G_0(-)} & \mathbf{A}(-, \mathcal{K}) \end{array}$$

commutes, where Φ is the forgetful **Gray**-natural transformation defined in section 3. We can consider $\mathbb{U}\text{-Alg}$ and $\mathbb{D}\text{-Alg}$ as trihomomorphisms and G as a tritransformation. A 1-cell $(G, G_0) \rightarrow (H, H_0)$ in $[\mathbb{D}, \mathbb{U}]$ is a pair (g, g_0) , where $g : G \rightarrow H$ is a strict trimodification, and $g_0 : G_0 \rightarrow H_0$ is in $\mathbf{A}(\mathcal{L}, \mathcal{K})$, such that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg} & \xrightarrow{G} & \mathbb{D}\text{-Alg} \\ \Downarrow g & & \downarrow \Phi \\ \mathbb{U}\text{-Alg} & \xrightarrow{H} & \mathbb{D}\text{-Alg} \\ & & \downarrow \Phi \\ & & \mathbf{A}(-, \mathcal{K}) \end{array} = \begin{array}{ccc} \mathbb{U}\text{-Alg} & & \mathbf{A}(-, \mathcal{L}) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathbf{A}(-, \mathcal{K}) & & \mathbf{A}(-, \mathcal{L}) \end{array} \begin{array}{c} \xrightarrow{G_0(-)} \\ \Downarrow g_0(-) \\ \xrightarrow{H_0(-)} \end{array} \mathbf{A}(-, \mathcal{K}).$$

What we mean by strict trimodification is, a trimodification in which all the invertible modifications required in the definition ([6], pg. 25) are identities. A 2-cell $(g, g_0) \rightarrow (h, h_0)$ in $[\mathbb{D}, \mathbb{U}]$ is a pair (γ, γ_0) , where $\gamma : g \rightarrow h$ is a perturbation, and $\gamma_0 : g_0 \rightarrow h_0$ is in $\mathbf{A}(\mathcal{L}, \mathcal{K})$, such that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{X}} & \xrightarrow{g_{\mathcal{X}}} & \mathbb{D}\text{-Alg}_{\mathcal{X}} \\ \Downarrow \gamma_{\mathcal{X}} & & \downarrow \Phi_{\mathcal{X}} \\ \mathbb{U}\text{-Alg}_{\mathcal{X}} & \xrightarrow{g'_{\mathcal{X}}} & \mathbb{D}\text{-Alg}_{\mathcal{X}} \\ & & \downarrow \Phi_{\mathcal{X}} \\ & & \mathbf{A}(\mathcal{X}, \mathcal{K}) \end{array} = \begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{X}} & & \mathbf{A}(\mathcal{Z}, \mathcal{L}) \\ \downarrow \Phi_{\mathcal{Z}} & & \downarrow \Phi_{\mathcal{Z}} \\ \mathbf{A}(\mathcal{Z}, \mathcal{L}) & & \mathbf{A}(\mathcal{Z}, \mathcal{L}) \end{array} \begin{array}{c} \xrightarrow{g_0(-)} \\ \Downarrow \gamma_0(-) \\ \xrightarrow{h_0(-)} \end{array} \mathbf{A}(\mathcal{X}, \mathcal{K}).$$

Vertical and horizontal compositions are the obvious ones. The rest of the operations are defined as follows:

$$\begin{aligned} (H, H_0)(G, G_0) &= (HG, H_0G_0) \\ (H, H_0)(g, g_0) &= (Hg, H_0g_0) & (h, h_0)(G, G_0) &= (hG, h_0G_0) \\ (H, H_0)(\sigma, \sigma_0) &= (H\sigma, H_0\sigma_0) & (\tau, \tau_0)(G, G_0) &= (\tau G, \tau_0G_0) \end{aligned}$$

8. Liftings

8.1. DEFINITION. A *lifting* is a pseudomonad in the **Gray**-category $\mathbf{PSM}(\mathbf{A})$.

In more detail, observe that a pseudomonad $\tilde{\mathbb{D}}$ in $\mathbf{PSM}(\mathbf{A})$ has to have domain a pseudomonad $\mathbb{U} = (U, u, n, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \mu_{\mathbb{U}})$ in $\mathbf{A}$, with domain some object $\mathcal{K}$ of $\mathbf{A}$. Observe that $\tilde{\mathbb{D}}$ is of the form

$$\tilde{\mathbb{D}} = ((\tilde{D}, D), (\tilde{d}, d), (\tilde{m}, m), (\tilde{\beta}, \beta), (\tilde{\eta}, \eta), (\tilde{\mu}, \mu)).$$

Taking the second entries, we obtain a pseudomonad $\mathbb{D} = (D, d, m, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \mu_{\mathbb{D}})$ in $\mathbf{A}$ on the same object $\mathcal{K}$. Furthermore, given an object $\mathcal{X}$ of $\mathbf{A}$, we can define $D_{\mathcal{X}} = \tilde{D}_{\mathcal{X}}$, and $d_{\mathcal{X}} = \tilde{d}_{\mathcal{X}}$ etcetera, to obtain a pseudomonad $\mathbb{D}_{\mathcal{X}} = (D_{\mathcal{X}}, d_{\mathcal{X}}, m_{\mathcal{X}}, \beta_{\mathcal{X}}, \eta_{\mathcal{X}}, \mu_{\mathcal{X}})$ in $\mathbf{Gray}$, on the 2-category $\mathbb{U}\text{-Alg}_{\mathcal{X}}$. $\mathbb{D}$ and the family $\langle \mathbb{D}_{\mathcal{X}} \rangle_{\mathcal{X}}$ behave well with forgetful 2-functors and change of base.

9. From a pseudomonad with compatible structure to a pseudomonad in $\text{PSM}(\mathbf{A})$

Assume we have a pseudomonad $\mathbb{V}$ that is compatible with pseudomonads $\mathbb{D}$ and $\mathbb{U}$ as in section 6. We want to define a pseudomonad $\mathbb{D}$ on the object $\mathbb{U}$ of $\text{PSM}(\mathbf{A})$. Let $\mathcal{X}$ be an object of $\mathbf{A}$. We begin by defining a 2-functor $D_{\mathcal{X}} : \mathbb{U}\text{-Alg}_{\mathcal{X}} \rightarrow \mathbb{U}\text{-Alg}_{\mathcal{X}}$. Given an object (ψ, χ) in $\mathbb{U}\text{-Alg}_{\mathcal{X}}$, with

$$\begin{array}{ccc} & UX & \\ uX \nearrow & & \searrow x \\ X & \xrightarrow{Id} & X, \end{array} \quad \begin{array}{ccc} UUX & \xrightarrow{Ux} & UX \\ nX \downarrow & \not\Downarrow_{\chi} & \downarrow x \\ UX & \xrightarrow{x} & X, \end{array}$$

Define the first entry of $\mathbb{D}_{\mathcal{X}}(\psi, \chi)$ as

$$\begin{array}{ccccccc} & & UDX & & & & \\ & & \uparrow & & & & \\ & & \theta_1 DX \downarrow & & & & \\ & & \Downarrow_{d_{UDuX}^{-1}} & & & & \\ & & UDX & \xrightarrow{UDuX} & UDX & \xrightarrow{dUDUX} & DUDUX \\ & & \Downarrow_{d_{UDuX}^{-1}} & & & & \\ & & DUDUX & \xrightarrow{DUDuX} & DUDUX & \xrightarrow{pX} & DUX \\ & & \Downarrow_{v_{DUX}} & & \Downarrow_{\beta_v X} & & \\ & & DUX & \xrightarrow{v_{DUX}} & DUX & \xrightarrow{Dx} & DX \\ & & \Downarrow_{\beta_v X} & & \Downarrow_{D\psi} & & \\ & & Id & & & & \end{array}$$

[illegible]
$$\begin{array}{ccccc}
UDX & \xrightarrow{UDh} & UDX' \\
UDuX \downarrow & \Downarrow_{UDu_h^{-1}} & UDuX' \\
UDUX & \xrightarrow{UDU_h} & UDUX' \\
dUDUX \downarrow & \Downarrow_{d_{UDU_h}^{-1}} & dUDUX' \\
DUDUX & \xrightarrow{DUDU_h} & DUDUX' \\
pX \downarrow & \Downarrow_{p_h^{-1}} & pX' \\
DUX & \xrightarrow{DU_h} & DUX' \\
Dx \downarrow & \Downarrow_{D\rho} & Dx' \\
DX & \xrightarrow{Dh} & DX'
\end{array}$$

9.1. PROPOSITION. *The above definitions make $D_{\mathcal{X}} : \mathbb{U}\text{-Alg}_{\mathcal{X}} \rightarrow \mathbb{U}\text{-Alg}_{\mathcal{X}}$ a 2-functor. Furthermore, if we define $\tilde{D} : \mathbb{U}\text{-Alg} \rightarrow \mathbb{U}\text{-Alg}$ as $D_{\mathcal{X}}$ at every object $\mathcal{X}$ of $\mathbf{A}$, then $\tilde{D}$ is a Gray-transformation, and $(\tilde{D}, D) \in [\mathbb{U}, \mathbb{U}]$.*

$$\begin{array}{c}
 \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \\
 \eta_U DX^{-1} \downarrow \quad \downarrow n_{D^*UX}^{-1} \quad \downarrow d_n DUX \\
 \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad}
 \end{array}
 =
 \begin{array}{c}
 \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad} \\
 \swarrow U u_{D^*UX} \quad \swarrow d_{U^*DUX}^{-1} \quad \swarrow D \eta_U DUX^{-1} \searrow \\
 \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{\quad}
 \end{array}$$

$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $D\eta_U DUX^{-1}$} \curvearrowright \text{\scriptsize $\theta_2 DUX$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $DU\theta_1 DUX$} \curvearrowright \text{\scriptsize $\eta_V DUX^{-1}$} \downarrow \end{array}, \quad (\text{see (12)}) \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $d_{Uu}^{-1} DUX$} \downarrow \text{\scriptsize $d_{Ud}^{-1} DUX$} \downarrow \\ \downarrow \text{\scriptsize $DU\theta_1 DUX$} \curvearrowright \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $U\theta_1 DUX$} \curvearrowright \text{\scriptsize $d_{Uv}^{-1} DUX$} \downarrow \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $\eta_V DUX^{-1}$} \downarrow \text{\scriptsize $\mu_V X$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $DU\beta_V X$} \downarrow \text{\scriptsize pX} \downarrow \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $d_{Uv}^{-1} DUX$} \downarrow \text{\scriptsize $d_{Up}^{-1} DUX$} \downarrow \\ \downarrow \text{\scriptsize $DU\beta_V X$} \curvearrowright \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $U\beta_V X$} \downarrow \text{\scriptsize $dUDUX$} \downarrow \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $UuDux$} \downarrow \text{\scriptsize $U\theta_1 DUX$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $U\theta_1 DX$} \text{\scriptsize Ud_{UDux}^{-1}} \\ \text{\scriptsize $UvDux$} \curvearrowright \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $UDux$} \downarrow \text{\scriptsize $UD\beta_U X$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize UDu_{uX}^{-1}} \downarrow \text{\scriptsize $UD\eta_U X^{-1}$} \downarrow \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $UD\eta_U X^{-1}$} \downarrow \text{\scriptsize $d_{UDn} X$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $d_{UDU}^{-1} DUX$} \downarrow \text{\scriptsize $DUD\eta_U X^{-1}$} \downarrow \end{array}, \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $DUD\eta_U X^{-1}$} \downarrow \text{\scriptsize $\theta_4 X^{-1}$} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize p_{uX}^{-1}} \downarrow \text{\scriptsize $D\eta_U X^{-1}$} \downarrow \end{array}, \quad (\text{see (19)}) \\
\\
\begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $D\eta_U X^{-1}$} \downarrow \text{\scriptsize DX} \downarrow \end{array} & = & \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \text{\scriptsize $DU\psi$} \downarrow \text{\scriptsize Dx} \downarrow \end{array},
\end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 UD u_{uX}^{-1} & UD u_x^{-1} & \\
 \downarrow & \downarrow & \\
 d_{UD uX}^{-1} & d_{UD x}^{-1} & \\
 \downarrow & \downarrow & \\
 p_{uX}^{-1} & p_x^{-1} & \\
 \downarrow & \downarrow & \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \text{DU}\psi & &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \text{UD}\psi & & \\
 \downarrow & \downarrow & \\
 UD uX & & \\
 \downarrow & \downarrow & \\
 dUDUX & & \\
 \downarrow & \downarrow & \\
 pX & & \\
 \downarrow & \downarrow &
 \end{array}
 \end{array}$$

To get from the left hand side of (6) to the right hand side, make the following sequence of substitutions:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{Un_{DuX}^{-1}} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{n_{DuX}^{-1}} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{\mu_{\mathbb{U}}DX} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{n_{DuX}^{-1}} &
 \end{array}, \\
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{\quad} & d_n DUX \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{\quad} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{d_n^{-1} DUX} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{D\mu_{\mathbb{U}}DX} & \\
 \downarrow & \downarrow & \\
 \mu_{\mathbb{U}}DX & \xrightarrow{d_n DUX} &
 \end{array}, \\
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 D\mu_{\mathbb{U}}DX & \xrightarrow{\quad} & \theta_2 DUX \\
 \downarrow & \downarrow & \\
 D\mu_{\mathbb{U}}DX & \xrightarrow{\quad} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 \theta_2 U DUX & \xrightarrow{DU d_n^{-1} DUX} & \\
 \downarrow & \downarrow & \\
 \theta_2 U DUX & \xrightarrow{\theta_4 DUX} &
 \end{array}, \\
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 U d_n DUX & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 d_{U n DUX}^{-1} & \xrightarrow{\quad} & d_{U d U DUX}^{-1} \\
 \downarrow & \downarrow & \\
 DU d_n^{-1} DUX & \xrightarrow{\quad} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 d_{U d U DUX}^{-1} & \xrightarrow{\quad} & d_{U D n DUX}^{-1} \\
 \downarrow & \downarrow & \\
 d_{U d U DUX}^{-1} & \xrightarrow{\quad} &
 \end{array}, \\
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 U \theta_2 DUX & \xrightarrow{\quad} & U \mu_{\mathbb{V}} X \\
 \downarrow & \downarrow & \\
 d_{U D n DUX}^{-1} & \xrightarrow{\quad} & d_{U p X}^{-1} \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} & DU \mu_{\mathbb{V}} X \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 d_{U D U d U DUX}^{-1} & \xrightarrow{\quad} & d_{U D U p X}^{-1} \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} & DU \mu_{\mathbb{V}} X \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} &
 \end{array}
 \quad \text{(definition of } \theta_4 \text{)} \\
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} & \theta_4 DUX \\
 \downarrow & \downarrow & \\
 DU \theta_2 DUX & \xrightarrow{\quad} &
 \end{array}
 & = &
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow & \downarrow & \\
 p_{d U DUX}^{-1} & \xrightarrow{\quad} & \mu_{\mathbb{V}} DUX \\
 \downarrow & \downarrow & \\
 p_{d U DUX}^{-1} & \xrightarrow{\quad} & \theta_2 DUX \\
 \downarrow & \downarrow & \\
 p_{d U DUX}^{-1} & \xrightarrow{\quad} &
 \end{array}
 \end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& \mu_{\nabla} DUX & \xrightarrow{\mu_{\nabla} X} \\
& \downarrow & \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& p_{pX}^{-1} & \searrow \\
& \mu_{\nabla} X & \downarrow \\
& \mu_{\nabla} X &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{UdUdUX}^{-1} & \xrightarrow{Ud_{UdUX}^{-1}} \\
& \downarrow & \downarrow \\
& d_{UDUdUX}^{-1} & \xrightarrow{d_{UpX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{UUdUX}^{-1} & \xrightarrow{d_{UpX}^{-1}} \\
& \downarrow & \downarrow \\
& DUd_{UdUX}^{-1} & \xrightarrow{DUd_{UpX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& \theta_2 U D U X & \xrightarrow{DUd_{UdUX}^{-1}} \\
& \downarrow & \downarrow \\
& p_{dUX}^{-1} & \xrightarrow{p_{pX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& Dn_{dUX}^{-1} & \xrightarrow{Dn_{pX}^{-1}} \\
& \downarrow & \downarrow \\
& \theta_2 D U X &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{nUX} & \xrightarrow{d_{UdUX}^{-1}} \\
& \downarrow & \downarrow \\
& Dn_{dUX}^{-1} & \xrightarrow{Dn_{pX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& n_{dUX}^{-1} & \xrightarrow{n_{pX}^{-1}} \\
& \downarrow & \downarrow \\
& d_{UdUX}^{-1} & \xrightarrow{d_{UpX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& UUD\beta_{\nabla} X & \xrightarrow{Ud_{UDnX}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& UUD\beta_{\nabla} X & \xrightarrow{Ud_{UDnX}} \\
& \downarrow & \downarrow \\
& &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& UDUD\beta_{\nabla} X & \xrightarrow{dUDUDUX} \\
& \downarrow & \downarrow \\
& pDUX &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{UDUDuX}^{-1} & \xrightarrow{d_{UDUDnX}^{-1}} \\
& \downarrow & \downarrow \\
& p_{DuX}^{-1} & \xrightarrow{p_{DnX}^{-1}} \\
& \downarrow & \downarrow \\
& DUD\beta_{\nabla} X &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{UdUX}^{-1} & \xrightarrow{Ud_{UDuX}^{-1}} \\
& \downarrow & \downarrow \\
& d_{UDUDuX}^{-1} & \xrightarrow{d_{UDUDuX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& d_{UUduX}^{-1} & \xrightarrow{d_{UdUX}^{-1}} \\
& \downarrow & \downarrow \\
& DUd_{UDuX}^{-1} & \xrightarrow{DUd_{UDuX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array},
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& \theta_2 D U X & \xrightarrow{DUd_{UDuX}^{-1}} \\
& \downarrow & \downarrow \\
& p_{DuX}^{-1} & \xrightarrow{p_{DuX}^{-1}} \\
& \downarrow & \downarrow \\
& &
\end{array}
=
\begin{array}{ccc}
& \searrow & \\
\downarrow & & \searrow \\
& Dn_{DuX}^{-1} & \xrightarrow{\theta_2 DUUX} \\
& \downarrow & \downarrow \\
& &
\end{array},
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \searrow & \downarrow \\
& d_{UDuUX}^{-1} & \\
& \downarrow & \downarrow \\
& d_{nDUX} & Dn_{DuUX}^{-1} \\
& \downarrow & \downarrow \\
& & DUD\beta_U X
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{n_{DuUX}^{-1}} & \downarrow \\
& \searrow & \searrow \\
& UD\beta_U X & d_{UDnX} \\
& \downarrow & \downarrow
\end{array},
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \searrow & \downarrow \\
& U\theta_4 X^{-1} & \\
& \downarrow & \downarrow \\
& d_{UDUDnX}^{-1} & d_{UpX}^{-1} \\
& \downarrow & \downarrow \\
& & DU\theta_4 X^{-1}
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{d_{UpUX}^{-1}} & \downarrow \\
& \searrow & \searrow \\
& & d_{UDnX}^{-1} \\
& \downarrow & \downarrow
\end{array},
\end{array}$$

(see (18))

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \searrow & \downarrow \\
& DU\theta_4 X^{-1} & \\
& \downarrow & \downarrow \\
& p_{DnX}^{-1} & \mu_V X \\
& \downarrow & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{\mu_V UX} & \downarrow \\
& \searrow & \searrow \\
& & \theta_4 X \\
& \downarrow & \downarrow
\end{array},
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \xrightarrow{UDnX} & \downarrow \\
& \searrow & \searrow \\
& & UD\beta_U X \\
& \downarrow & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{UD\beta_U UX} & \downarrow \\
& \searrow & \searrow \\
& & UD\mu_U X \\
& \downarrow & \downarrow
\end{array},
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \xrightarrow{UD\mu_U X} & \downarrow \\
& \searrow & \searrow \\
& & d_{UDnX}^{-1} \\
& \downarrow & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{d_{UDUDnX}^{-1}} & \downarrow \\
& \searrow & \searrow \\
& & DUD\mu_U X \\
& \downarrow & \downarrow
\end{array}
\end{array}$$

(see (20))

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \xrightarrow{DU\mu_U X} & \downarrow \\
& \searrow & \searrow \\
& & \theta_4 X^{-1} \\
& \downarrow & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{p_{nX}^{-1}} & \downarrow \\
& \searrow & \searrow \\
& & D\mu_U X \\
& \downarrow & \downarrow
\end{array}
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc}
\downarrow & \xrightarrow{UD\chi} & \downarrow \\
& \searrow & \searrow \\
& & UD\mu_U X \\
& \downarrow & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\downarrow & \xrightarrow{UD\mu_U X} & \downarrow \\
& \searrow & \searrow \\
& & UD\mu_U X \\
& \downarrow & \downarrow
\end{array}
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ & DU\chi & \\ \downarrow & \swarrow & \downarrow \\ D\mu_{\mathbb{U}}X & \xrightarrow{\quad} & D\chi \end{array} & = & \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ Dn_x^{-1} & \downarrow & \downarrow \\ \downarrow & \searrow & \downarrow \\ & D\chi & \end{array}, \\
\\
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ UD\beta_{\mathbb{U}}UX & \xrightarrow{UDu_{\mathbb{U}}^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ d_{UDnUX} & \xrightarrow{d_{UDU}^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ \theta_4 UX^{-1} & \xrightarrow{p_{Ux}^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ & Dn_x^{-1} & \end{array} & = & \begin{array}{ccc} \downarrow & \searrow & \downarrow \\ UD\beta_{\mathbb{U}}X & \xrightarrow{\quad} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ d_{UDnX} & \xrightarrow{d_{UDU}^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ \theta_4 X^{-1} & \xrightarrow{p_x^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ & p_x^{-1} & \end{array} \\
\\
\begin{array}{ccc} \nearrow & \xrightarrow{Ud_{UDU}^{-1}} & \nearrow \\ \downarrow & \xrightarrow{d_{UdUDU}^{-1}} & \downarrow \\ \downarrow & \xrightarrow{d_{UpUX}^{-1}} & \downarrow \\ \downarrow & \xrightarrow{d_{UDU}^{-1}} & \downarrow \end{array} & = & \begin{array}{ccc} \nearrow & \xrightarrow{d_{UdUDU}^{-1}} & \nearrow \\ \downarrow & \xrightarrow{DUd_{UDU}^{-1}} & \downarrow \\ \downarrow & \xrightarrow{DUd_{UDU}^{-1}} & \downarrow \\ \downarrow & \xrightarrow{DUd_{UDU}^{-1}} & \downarrow \end{array} \\
\\
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ & DU p_x^{-1} & \\ \downarrow & \swarrow & \downarrow \\ \mu_{\mathbb{V}}UX & \xrightarrow{\quad} & p_x^{-1} \end{array} & = & \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ p_{DUx}^{-1} & \downarrow & \downarrow \\ \downarrow & \searrow & \downarrow \\ & \mu_{\mathbb{V}}X & \\ \downarrow & \swarrow & \downarrow \\ & p_x^{-1} & \end{array}, \\
\\
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ \theta_2 DUUX & \xrightarrow{DUd_{UDU}^{-1}} & \downarrow \\ \downarrow & \swarrow & \downarrow \\ & p_{DUx}^{-1} & \end{array} & = & \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ Dn_{DUx}^{-1} & \downarrow & \downarrow \\ \downarrow & \searrow & \downarrow \\ & \theta_2 DUX & \end{array}, \\
\\
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ & d_{UU}^{-1} & \\ \downarrow & \swarrow & \downarrow \\ d_n DUUX & \xrightarrow{\quad} & Dn_{DUx}^{-1} \end{array} & = & \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ n_{DUx}^{-1} & \downarrow & \downarrow \\ \downarrow & \searrow & \downarrow \\ & d_n DUX & \\ \downarrow & \swarrow & \downarrow \\ & d_{UU}^{-1} & \end{array}, \\
\\
\begin{array}{ccc} \downarrow & \searrow & \downarrow \\ & UUdu_x^{-1} & \\ \downarrow & \swarrow & \downarrow \\ n_{DuUX}^{-1} & \xrightarrow{\quad} & n_{DUx}^{-1} \end{array} & = & \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ n_{Dx}^{-1} & \downarrow & \downarrow \\ \downarrow & \searrow & \downarrow \\ & n_{DuX}^{-1} & \\ \downarrow & \swarrow & \downarrow \\ & UUdu_x^{-1} & \end{array}.
\end{array}$$

and

■

We are now ready to define $d_{\mathcal{X}} : 1 \rightarrow D_{\mathcal{X}}$, and $m_{\mathcal{X}} : D_{\mathcal{X}}D_{\mathcal{X}} \rightarrow D_{\mathcal{X}}$. The first entry of $(d_{\mathcal{X}})_{(\psi, \chi)}$ is dX , while the second is the pasting

$$\begin{array}{c}
 \begin{array}{ccccc}
 UX & \xrightarrow{UdX} & UDX & & \\
 \downarrow dUX & \searrow UuX & \downarrow \Downarrow Ud_uX & \downarrow Ud_uX & \\
 & UUX & \xrightarrow{UdUX} & UDUX & \\
 & \downarrow \Downarrow d_{UuX}^{-1} & \downarrow dUUX & \downarrow \Downarrow d_{UdUX}^{-1} & \downarrow dUDUX \\
 & DUX & \xrightarrow{DUuX} & DUUX & \xrightarrow{DUdUX} & DUDUX \\
 & \searrow D\eta_U X^{-1} \Downarrow & \searrow \Downarrow \theta_2 X & \downarrow pX & \\
 & & Id & DUX & \downarrow Dx \\
 & & & \downarrow \Downarrow d_x & \\
 X & \xrightarrow{dX} & DX, & &
 \end{array}
 \end{array}$$

and $(d_{\mathcal{X}})_{(h, \rho)} = d_h$.

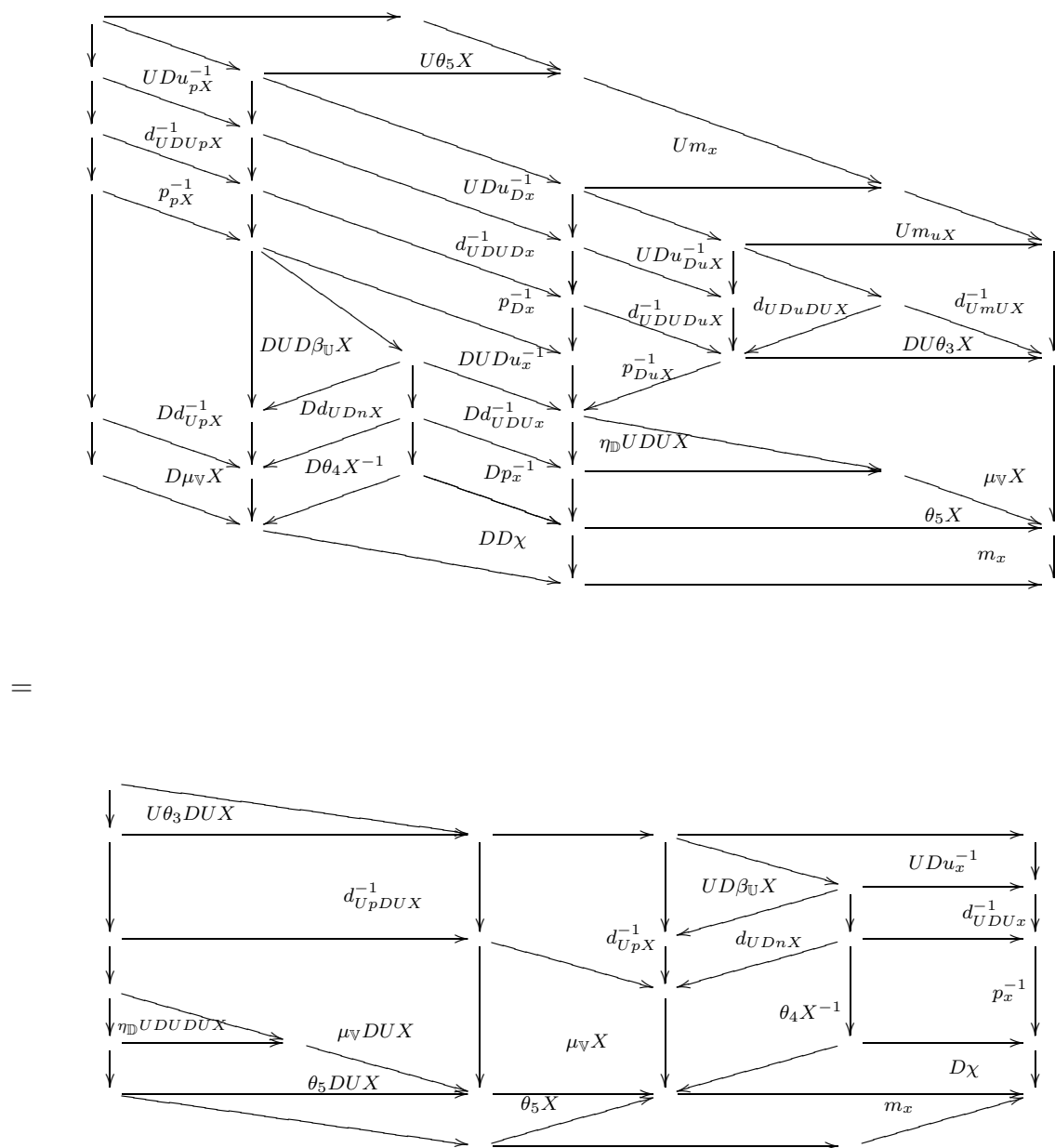
Define the first entry of $(m_{\mathcal{X}})_{(\psi, \chi)}$ as mX , and the second as the pasting

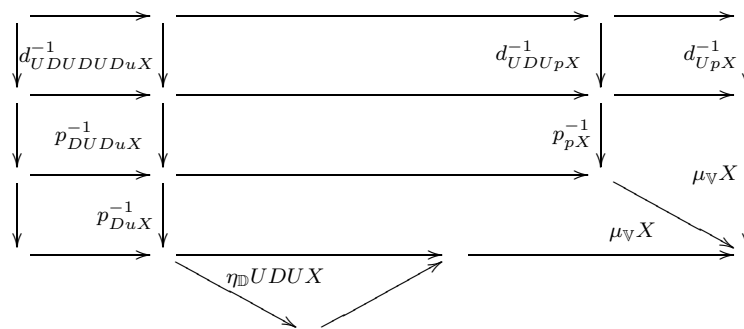
$$\begin{array}{c}
 \begin{array}{ccccc}
 UDDX & \xrightarrow{UmX} & UDU & & \\
 \downarrow Ud_uDX & \searrow UDDuX & \downarrow \Downarrow Um_uX & \downarrow Ud_uX & \\
 UDUDX & \xrightarrow{UDDUX} & UDDUX & \xrightarrow{UmUX} & UDUX \\
 \downarrow dUDUDX & \searrow \Downarrow Ud_u^{-1} & \downarrow Ud_uDUX & \downarrow \Downarrow d_{UmUX}^{-1} & \downarrow dUDUX \\
 DUDUDX & \xrightarrow{UDUDuX} & UDUDUX & \xrightarrow{dUDUDUX} & DUDDUX \\
 \downarrow pDX & \searrow \Downarrow d_{UDUDuX}^{-1} & \downarrow dUDUDUX & \downarrow \Downarrow DU\theta_3 X & \downarrow DUmUX \\
 DUDX & \xrightarrow{DUUDuX} & DUUDUX & \xrightarrow{DU\theta_3 X} & DUDUX \\
 \downarrow DUDuX & \searrow \Downarrow p_{DUX} & \downarrow Id & \downarrow \Downarrow \mu_V X & \downarrow pX \\
 DUDUX & \xrightarrow{DUUDUX} & DUDUX & \xrightarrow{pX} & DUX \\
 \downarrow DdUDUX & \searrow \Downarrow \eta_D UDUX & \downarrow \Downarrow \theta_5 X & \downarrow \Downarrow m_x & \downarrow Dx \\
 DDUDX & \xrightarrow{mUDUX} & DUDUX & \xrightarrow{pX} & DUX \\
 \downarrow DpX & \searrow \Downarrow \theta_5 X & \downarrow \Downarrow m_x & \downarrow \Downarrow m_x & \downarrow Dx \\
 DDUX & \xrightarrow{mUX} & DUX & \xrightarrow{Dx} & DX \\
 \downarrow DDx & \searrow \Downarrow m_x & \downarrow \Downarrow m_x & \downarrow \Downarrow m_x & \\
 DDX & \xrightarrow{mX} & DX. & &
 \end{array}
 \end{array}$$

Define $(m_{\mathcal{X}})_{(h, \rho)} = m_h$.

9.2. PROPOSITION. *With the above definitions, $d_{\mathcal{X}} : 1 \rightarrow D_{\mathcal{X}}$ and $m_{\mathcal{X}} : D_{\mathcal{X}}D_{\mathcal{X}} \rightarrow D_{\mathcal{X}}$ are strong transformations. Furthermore, defining d as $d_{\mathcal{X}}$, and $\tilde{m}$ as $m_{\mathcal{X}}$ at every object*

Proof. The size of the diagrams obtained is what makes it hard to prove that $(m_{\mathcal{X}})_{(\psi, \chi)}$ satisfies (10). To get from the left hand side of (10) to the right hand side make the following three substitutions:



[illegible]

=

The fact that the three equalities above are indeed satisfied and the rest of the proof are left to the reader. $\blacksquare$

It is easily seen that, defining $\tilde{\eta}$, $\tilde{\beta}$, and $\tilde{\mu}$ at every $\mathcal{X}$ as $(\eta_{\mathcal{X}})_{(\psi, \chi)} = \eta_{\mathbb{D}X}$, $(\beta_{\mathcal{X}})_{(\psi, \chi)} = \beta_{\mathbb{D}X}$, and $(\mu_{\mathcal{X}})_{(\psi, \chi)} = \mu_{\mathbb{D}X}$, we obtain 2-cells $(\tilde{\eta}, \eta)$, $(\tilde{\beta}, \beta)$, and $(\tilde{\mu}, \mu)$ in $[\mathbb{U}, \mathbb{U}]$. We thus have:

9.3. THEOREM. $\tilde{\mathbb{D}} = ((\tilde{D}, D), (\tilde{d}, d), (\tilde{m}, m), (\tilde{\beta}, \beta), (\tilde{\eta}, \eta), (\tilde{\mu}, \mu))$ is a pseudomonad in $\text{PSM}(\mathbf{A})$, on the object $\mathbb{U}$. $\blacksquare$

10. From a pseudomonad in $\text{PSM}(\mathbf{A})$ to a distributive law

Assume that we have a pseudomonad in $\text{PSM}(\mathbf{A})$ as in section 8. We produce a distributive law as follows. Consider $D_{\mathcal{K}} : \mathbb{U}\text{-Alg}_{\mathcal{K}} \rightarrow \mathbb{U}\text{-Alg}_{\mathcal{K}}$. Since the diagram

$$\begin{array}{ccc}
 \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} \\
 \Phi_{\mathcal{K}} \downarrow & & \downarrow \Phi_{\mathcal{K}} \\
 \mathbf{A}(\mathcal{K}, \mathcal{K}) & \xrightarrow{D(-)} & \mathbf{A}(\mathcal{K}, \mathcal{K})
 \end{array}$$

commutes, we have that $D_{\mathcal{K}}(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})$ is of the form (σ_1, σ_2) with

$$\begin{array}{ccc}
 & UDU & \\
 uDU \nearrow & & \searrow s \\
 DU & \xrightarrow{Id} & DU, \\
 & \Downarrow \sigma_1 &
 \end{array}
 \quad
 \begin{array}{ccc}
 UUDU & \xrightarrow{Us} & UDU \\
 nDU \downarrow & \Downarrow \sigma_2 & \downarrow s \\
 UDU & \xrightarrow{s} & DU.
 \end{array}$$

Furthermore, since the diagram

$$\begin{array}{ccc}
 \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} \\
 \downarrow \hat{U} & & \downarrow \hat{U} \\
 \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}}
 \end{array}$$

commutes, where $\widehat{U}$ is the change of base 2-functor defined in section 3, then $D_{\mathcal{K}}(n, \mu_{\mathbb{U}}) : D_{\mathcal{K}}(\beta_{\mathbb{U}}U, \mu_{\mathbb{U}}U) \rightarrow D_{\mathcal{K}}(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})$ is of the form $(Dn, \sigma_3) : (\sigma_1U, \sigma_2U) \rightarrow (\sigma_1, \sigma_2)$ with σ_3 of the form

$$\begin{array}{ccc} UDUU & \xrightarrow{UDn} & UDU \\ sU \downarrow & \not\Downarrow \sigma_3 & \downarrow s \\ DUU & \xrightarrow{Dn} & DU. \end{array}$$

Similarly, it can be shown that $(d_{\mathcal{K}})_{(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})}$ is of the form (dU, σ_4) , with

$$\begin{array}{ccc} UU & \xrightarrow{UdU} & UDU \\ n \downarrow & \not\Downarrow \sigma_4 & \downarrow s \\ U & \xrightarrow{dU} & DU. \end{array}$$

If we apply $D_{\mathcal{K}}$ to (σ_1, σ_2) we obtain another $\mathbb{U}$ -algebra (σ'_1, σ'_2) , with

$$\begin{array}{ccc} & UDDU & \\ uDDU \nearrow & & \searrow s' \\ DDU & \xrightarrow{\quad} & DDU, \end{array} \quad \begin{array}{ccc} UUDDU & \xrightarrow{Us'} & UDDU \\ nDDU \downarrow & \not\Downarrow \sigma'_2 & \downarrow s' \\ UDDU & \xrightarrow{s'} & DU. \end{array}$$

Applying $D_{\mathcal{K}}$ to $(s, \sigma_2) : (\beta_{\mathbb{U}}DU, \mu_{\mathbb{U}}DU) \rightarrow (\sigma_1, \sigma_2)$, we obtain a morphism (Ds, τ_1) with

$$\begin{array}{ccc} UDU DU & \xrightarrow{UDs} & UDDU \\ sDU \downarrow & \not\Downarrow \tau_1 & \downarrow s' \\ DUDU & \xrightarrow{Ds} & DDU. \end{array}$$

We also have that $(m_{\mathcal{K}})_{(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})}$ is of the form (mU, γ_1) , with

$$\begin{array}{ccc} UDDU & \xrightarrow{UmU} & UDU \\ s' \downarrow & \not\Downarrow \gamma_1 & \downarrow s \\ DDU & \xrightarrow{mU} & DU. \end{array}$$

Define σ_5 as the pasting

$$\begin{array}{ccc} UDDU & \xrightarrow{UmU} & UDU \\ \downarrow UDuDU & \searrow Id & \downarrow s \\ UDU DU & \xrightarrow{UD\sigma_1^{-1}} & UDDU \\ \downarrow sDU & \searrow UD s & \downarrow s' \\ DUDU & \xrightarrow{Ds} & DDU \\ & \searrow Ds & \downarrow s' \\ & & DDU \xrightarrow{mU} DU. \end{array}$$

$\not\Downarrow \tau_1$ $\not\Downarrow \gamma_1$

It is with the help of σ_1 to σ_5 that we define the distributive law as follows:
Define $r = s \circ UDu$, ω_1 as the pasting

$$\begin{array}{ccccc}
 & & UD & & \\
 & \nearrow^{uD} & & \searrow^{UDu} & \\
 D & & & & UDU \\
 & \searrow_{Du} & & \nearrow_{uD} & \\
 & & DU & &
 \end{array}
 \xrightarrow{Id} DU,$$

$\Downarrow^{uDu} \quad \Downarrow^{\sigma_1}$

ω_2 as the pasting

$$\begin{array}{ccccc}
 & & UD & & \\
 & \nearrow^{Ud} & & \searrow^{UDu} & \\
 U & & & & UDU \\
 & \searrow_{Uu} & & \nearrow_{UdU} & \\
 & & UU & &
 \end{array}
 \xrightarrow{Id} U$$

$\Downarrow^{Ud_u} \quad \Downarrow^{\sigma_4} \quad \Downarrow^{\eta_U^{-1}}$

ω_3 as the pasting

$$\begin{array}{ccccccc}
 UUD & \xrightarrow{UUDu} & UUDU & \xrightarrow{Us} & UDU & \xrightarrow{UDuU} & UDUU \xrightarrow{sU} DUU \\
 \downarrow nD & & \downarrow nDU & & \searrow Id & \downarrow UD\beta_U & \downarrow UDn \\
 & & & & & UDU & \\
 UD & \xrightarrow{UDu} & UDU & \xrightarrow{s} & DU, & &
 \end{array}$$

$\Downarrow^{nD} \quad \Downarrow^{nDU} \quad \Downarrow^{UD\beta_U} \quad \Downarrow^{\sigma_2} \quad \Downarrow^{s^{-1}_{\sigma_3}} \quad \Downarrow^{Dn}$

and ω_4 as the pasting

$$\begin{array}{ccccccc}
 UD & \xrightarrow{UDu} & UDU & \xrightarrow{s} & DU \\
 \uparrow Um & & \uparrow UmU & & \uparrow mU \\
 & & UDDU & \xrightarrow{UPuDU} & UDUDU \\
 & \nearrow^{UDDu} & & \searrow^{UDUDu} & \\
 UDD & \xrightarrow{UDuD} & UDUD & \xrightarrow{sD} & DUD & \xrightarrow{DUDu} & DUDU \xrightarrow{Ds} DDU
 \end{array}$$

$\Downarrow^{Um_u} \quad \Downarrow^{UmU} \quad \Downarrow^{UDu_{DU}} \quad \Downarrow^{s^{-1}_{DU}} \quad \Downarrow^{sDU} \quad \Downarrow^{\sigma_5}$

The proof that we obtain a distributive law in this way is based on the behavior of the 2-cells σ_1 - σ_5 . Aside from the obvious conditions that come from the facts that (σ_1, σ_2) is a $\mathbb{U}$ -algebra, (Dn, σ_3) and (dU, σ_4) are homomorphisms of $\mathbb{U}$ -algebras, we have the following conditions.

10.1. PROPOSITION. *The following are 2-cells in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$:*

$$D\eta_{\mathbb{U}} : Id \rightarrow (Dn, \sigma_3) \circ (DUu, s_u^{-1}). \quad (26)$$

Furthermore, the following equations are satisfied:

(28)

$$\begin{array}{ccc}
 UDDUU & \xrightarrow{UmUU} & UDUU \\
 \downarrow & \searrow UDDn & \searrow Um_n \Downarrow \\
 UDUDUU & & UDDU \xrightarrow{UmU} UDU \\
 \downarrow sDUU & \searrow UDUDn & \downarrow UDUDU \\
 DUDUU & & UDUDU \\
 \downarrow DsU & \searrow DUDn & \downarrow sDU \\
 DDUU & & DUDU \\
 \downarrow DDn & \searrow D\sigma_3 & \downarrow Ds \Downarrow \sigma_5 \\
 & DDU \xrightarrow{mU} DU
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 UDDUU & \xrightarrow{UmUU} & UDUU \\
 \downarrow UDUDU & & \downarrow UDn \\
 UDUDUU & & UDU \\
 \downarrow sDUU & & \downarrow sU \\
 DUDUU & & DUDU \\
 \downarrow DsU & \searrow \sigma_5 U & \downarrow s \\
 DDUU & \xrightarrow{mUU} & DUU \\
 \downarrow DDn & \searrow m_n \Downarrow & \downarrow Dn \Downarrow \sigma_3 \\
 & DDU \xrightarrow{mU} DU
 \end{array}
 \quad (29)$$

$$\begin{array}{ccc}
 UDU & \xrightarrow{Id} & UDU \\
 \downarrow UD_uU & \searrow UDDU & \searrow U\eta_D U \Downarrow \\
 UDUU & & UDDU \xrightarrow{UmU} UDU \\
 \downarrow sU & \searrow UDUDU & \downarrow UDUDU \\
 DUU & & UDUDU \\
 \downarrow Dn & \searrow DUdU & \downarrow sDU \\
 DU & & DUDU \\
 \downarrow DdU & \searrow D\sigma_4 & \downarrow Ds \Downarrow \sigma_5 \\
 & DDU \xrightarrow{mU} DU
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 UDU & \xrightarrow{Id} & UDU \\
 \downarrow UD_uU & \searrow UD\beta_U^{-1} & \downarrow UDn \\
 UDUU & & UDU \\
 \downarrow sU & \searrow \sigma_3 & \downarrow s \\
 DUU & \xrightarrow{Dn} & DU \xrightarrow{Id} DU, \\
 \downarrow DdU & \searrow \eta_D U \Downarrow & \downarrow mU \\
 & DDU
 \end{array}
 \quad (30)$$

$$\begin{array}{ccc}
 UDU & \xrightarrow{UdDU} & UDDU \\
 \downarrow UuDU & \searrow Ud_uDU & \downarrow UmU \\
 UUUDU & \xrightarrow{UdUDU} & UDUDU \xrightarrow{sDU} UDU \\
 \downarrow nDU & \searrow \sigma_4 DU & \downarrow sDU \\
 UDU & \xrightarrow{dUDU} & DUDU \\
 \downarrow s & \searrow d_s & \downarrow Ds \Downarrow \sigma_5 \\
 DU & \xrightarrow{dDU} & DDU \xrightarrow{mU} DU \\
 \downarrow Id & & \downarrow \beta_D U \Downarrow
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & UDU & \\
 UuDU \swarrow & \downarrow Id & \searrow UdDU \\
 UUUDU & \eta_D \Leftarrow DU & UDDU \\
 \downarrow nDU & & \downarrow UmU \\
 & UDU & \\
 & \downarrow s & \\
 & DU, &
 \end{array}$$

$$\begin{array}{ccccc}
& & UDDU & \xrightarrow{UmU} & UDU \\
& \nearrow U D m U & \downarrow U D u D U & & \downarrow s \\
& UDDDU & \xleftarrow{U \bar{D} u_m^{-1} U} & UDUDU & \\
& \downarrow U D u D D U & \nearrow U D U m U & \downarrow s D U & \\
& UDUDDU & \xleftarrow{s_m^{-1}} & DUDU & \\
& \downarrow s D D U & \nearrow D U m U & \Downarrow \sigma_5 & \\
& DUDDU & & & \\
& \downarrow D U D u D U & & \downarrow D s & \\
& DUDUDU & & \Downarrow D \sigma_5 & \\
& \downarrow D s D U & & & \\
& DDUDU & \nearrow D m U & DDU & \xrightarrow{mU} & DU \\
& \downarrow D D s & \nearrow D m U & \Downarrow \mu_D U & \nearrow m U & \\
& DDDU & \xrightarrow{m D U} & DDU & &
\end{array}$$

=

$$\begin{array}{ccccc}
& & UDDU & \xrightarrow{UmU} & UDU \\
& \nearrow U D m U & \Downarrow U \mu_D U & \nearrow U m U & \\
& UDDDU & \xrightarrow{UmDU} & UDDU & \\
& \downarrow U D u D D U & \Downarrow U m_u D U & \downarrow U D u D U & \\
& UDUDDU & \xleftarrow{U \bar{D} u_{DuDU}^{-1}} & UDDUDU & \xrightarrow{UmUDU} & UDUDU \\
& \downarrow s D D U & \nearrow U D U D u D U & \downarrow U D u D U D U & & \\
& DUDDU & \xleftarrow{s_{DuDU}^{-1}} & UDUDUDU & & \\
& \downarrow D U D u D U & \nearrow s D U D U & \Downarrow \sigma_5 D U & \downarrow s D U & \\
& DUDUDU & & & \Downarrow \sigma_5 & \\
& \downarrow D s D U & & & & \\
& DDUDU & \xrightarrow{mUDU} & DUDU & & \\
& \downarrow D D s & \Downarrow m_s & \downarrow D s & \nearrow m U & \\
& DDDU & \xrightarrow{m D U} & DDU & &
\end{array}$$

(31)

Observation: The equations that appear in the proposition can be written as condi-

tions in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. For instance, it can be shown that the pair of pastings:

$$\varphi =$$

$DDU,$

and

$$\vartheta =$$

$,$

define an object in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. Thus, the first two equations say that $(mU, \sigma_5) : (\varphi, \vartheta) \rightarrow (\sigma_1, \sigma_2)$ is a $\mathbb{U}$ -algebra morphism. Similarly the rest of the equations. We leave the details for the interested reader.

Proof. The fact that $D\eta_{\mathbb{U}}$ and $D\mu_{\mathbb{U}}$ are 2-cells in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$ follow from applying $D_{\mathcal{K}}$ to $\eta_{\mathbb{U}}$ and $\mu_{\mathbb{U}}$, respectively. The statement about d_n follows from pseudonaturality of $d_{\mathcal{K}}$. As for the equations, the proof of the last equation is fairly typical, and it is the only one we do. We begin by introducing some notation and deducing some equations we will need. The notation is

$$\begin{aligned} D_{\mathcal{K}}((\beta_{\mathbb{U}}DDU, \mu_{\mathbb{U}}DDU) \xrightarrow{(s', \sigma'_2)} (\sigma'_1, \sigma'_2)) &= (\sigma_1DDU, \sigma_2DDU) \xrightarrow{(Ds', \tau_2)} (\sigma''_1, \sigma''_2), \\ D_{\mathcal{K}}((\sigma_1DU, \sigma_2DU) \xrightarrow{(Ds, \tau_1)} (\sigma'_1, \sigma'_2)) &= (\sigma'_1DU, \sigma'_2DU) \xrightarrow{(DDs, \tau_3)} (\sigma''_1, \sigma''_2), \\ D_{\mathcal{K}}((\sigma'_1, \sigma'_2) \xrightarrow{(mU, \gamma_1)} (\sigma_1, \sigma_2)) &= (\sigma''_1, \sigma''_2) \xrightarrow{(DmU, \gamma_2)} (\sigma'_1, \sigma'_2), \end{aligned}$$

and $(m_{\mathcal{K}})_{(\sigma_1, \sigma_2)} = (mDU, \gamma_3) : (\sigma''_1, \sigma''_2) \rightarrow (\sigma'_1, \sigma'_2)$.

Now the equations. Since

$$\begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{K}} & \begin{array}{c} \xrightarrow{D_{\mathcal{K}}D_{\mathcal{K}}} \\ \Downarrow m_{\mathcal{K}} \\ \xrightarrow{D_{\mathcal{K}}} \end{array} & \mathbb{U}\text{-Alg}_{\mathcal{K}} = \mathbb{U}\text{-Alg}_{\mathcal{K}} \\ & \downarrow \Phi_{\mathcal{K}} & \downarrow \Phi_{\mathcal{K}} \\ & \mathbf{A}(\mathcal{K}, \mathcal{K}) & \mathbf{A}(\mathcal{K}, \mathcal{K}) \end{array} \quad \begin{array}{c} \xrightarrow{DD(-)} \\ \Downarrow m(-) \\ \xrightarrow{D(-)} \end{array} \quad \mathbf{A}(\mathcal{K}, \mathcal{K}),$$

we have that $m_s = (m_{\mathcal{K}})_{(s, \sigma_2)}$ is a 2-cell

$$\begin{array}{ccc} (\sigma'_1 DU, \sigma'_2 DU) & \xrightarrow{(mU DU, \gamma_1 DU)} & (\sigma_1 DU, \sigma_2 DU) \\ (DDs, \tau_3) \downarrow & \Downarrow m_s & \downarrow (Ds, \tau_1) \\ (\sigma''_1, \sigma''_2) & \xrightarrow{(mDU, \gamma_3)} & (\sigma'_1, \sigma'_2) \end{array}$$

in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. From this we obtain the equation

$$\begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ \gamma_1 DU \downarrow & \searrow \tau_1 & \downarrow \\ \xrightarrow{\quad} & \searrow m_s & \downarrow \\ & \xrightarrow{\quad} & \downarrow \end{array} = \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ \tau_3 \downarrow & \searrow U m_s & \downarrow \\ \xrightarrow{\quad} & \searrow \gamma_3 & \downarrow \end{array}. \quad (32)$$

Similarly, $(\mu_{\mathcal{K}})_{(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})}$ is a 2-cell

$$\begin{array}{ccc} (\sigma''_1, \sigma''_2) & \xrightarrow{(DmU, \gamma_2)} & (\sigma'_1, \sigma'_2) \\ (mDU, \gamma_3) \downarrow & \Downarrow \mu_{\mathbb{D}} U & \downarrow (mU, \gamma_1) \\ (\sigma'_1, \sigma'_2) & \xrightarrow{(mU, \gamma_1)} & (\sigma_1, \sigma_2). \end{array}$$

We thus obtain the equation

$$\begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ \gamma_3 \downarrow & \searrow U \mu_{\mathbb{D}} U & \downarrow \\ \xrightarrow{\quad} & \searrow \gamma_1 & \downarrow \\ & \xrightarrow{\quad} & \downarrow \end{array} = \begin{array}{ccc} \downarrow & \xrightarrow{\quad} & \downarrow \\ \gamma_2 \downarrow & \searrow \mu_{\mathbb{D}} U & \downarrow \\ \xrightarrow{\quad} & \searrow \gamma_1 & \downarrow \end{array}, \quad (33)$$

Observe that

$$\begin{array}{ccc} (\beta_{\mathbb{U}} DU DU, \mu_{\mathbb{U}} DU DU) & \xrightarrow{(UDs, n_{Ds}^{-1})} & (\beta_{\mathbb{U}} DDU, \mu_{\mathbb{U}} DDU) \\ (sDU, \sigma_2 DU) \downarrow & \Downarrow \tau_1 & \downarrow (s', \sigma'_2) \\ (\sigma_1 DU, \sigma_2 DU) & \xrightarrow{(Ds, \tau_1)} & (\sigma'_1, \sigma'_2) \end{array}$$

is a 2-cell in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. Since $D_{\mathcal{K}}$ behaves well with change of base, then $D_{\mathcal{K}}(UDs, n_{Ds}^{-1}) = (DUDs, s_{Ds}^{-1})$. Thus, applying $D_{\mathcal{K}}$, we have that

$$\begin{array}{ccc} (\sigma_1 DUDU, \sigma_2 DUDU) & \xrightarrow{(DUDs, s_{Ds}^{-1})} & (\sigma_1 DDU, \sigma_2 DDU) \\ \downarrow (DsDU, \tau_1) & \Downarrow D\tau_1 & \downarrow (Ds', \tau_2) \\ (\sigma'_1 DU, \sigma'_2 DU) & \xrightarrow{(DDs, \tau_3)} & (\sigma''_1, \sigma''_2) \end{array}$$

is a 2-cell in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. From this we obtain the equation

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow \tau_1 DU & \downarrow \tau_3 & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \end{array} = \begin{array}{ccc} & \nearrow UD\tau_1^{-1} & \\ \downarrow & \xrightarrow{s_{Ds}^{-1}} & \downarrow \tau_2 \\ & \searrow D\tau_1 & \downarrow \end{array} . \quad (34)$$

We also have that

$$\begin{array}{ccc} (\beta_{\mathbb{U}} DDU, \mu_{\mathbb{U}} DDU) & \xrightarrow{(UmU, n_{mU}^{-1})} & (\beta_{\mathbb{U}} DU, \mu_{\mathbb{U}} DU) \\ \downarrow (s', \sigma'_2) & \Downarrow \gamma_1 & \downarrow (s, \sigma_2) \\ (\sigma'_1, \sigma'_2) & \xrightarrow{(mU, \gamma_1)} & (\sigma_1, \sigma_2) \end{array}$$

is a 2-cell in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. Applying $D_{\mathcal{K}}$ to it we obtain that the equation

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow \tau_2 & \xrightarrow{UD\gamma_1} & \downarrow \gamma_2 \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \end{array} = \begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow s_{mU}^{-1} & \xrightarrow{\quad} & \downarrow \tau_1 \\ \xrightarrow{\quad} & \xrightarrow{D\gamma_1} & \downarrow \end{array} , \quad (35)$$

is satisfied.

We are ready to show (31). Start with the second member of the equation, and use (32). Next make the substitution

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow Um_u DU & \downarrow UD\sigma_1^{-1} & \searrow UD\sigma_1^{-1} \\ \xrightarrow{\quad} & \xrightarrow{Um_s} & \xrightarrow{\quad} \end{array} = \begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow UDD\sigma_1^{-1} & \xrightarrow{\quad} & \downarrow UD\sigma_1^{-1} \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{UmDU} \end{array} .$$

Use now (33) and (34). Make the substitutions

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow UD\sigma_1 DU^{-1} & \downarrow UD\tau_1^{-1} & \downarrow UD\tau_1^{-1} \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \end{array} = \begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow UDu_{Ds}^{-1} & \downarrow UD\sigma_1'^{-1} & \downarrow UD\sigma_1'^{-1} \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} \end{array} ,$$

and

$$\begin{array}{c}
 \downarrow \text{UD}\sigma_1'^{-1} \searrow \\
 \text{UDmU}
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{\quad} & \downarrow \text{UD}u_{mU}^{-1} & \downarrow \text{UD}\sigma_1'^{-1} \\
 \searrow & \xrightarrow{\quad} & \searrow \\
 & \text{UD}\gamma_1 &
 \end{array},$$

using that $(Ds, \tau_1) : (\sigma_1 DU, \sigma_2 DU) \rightarrow (\sigma_1', \sigma_2')$ and $(mU, \gamma_1) : (\sigma_1', \sigma_2') \rightarrow (\sigma_1, \sigma_2)$ are 1-cells in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. Then the substitution

$$\begin{array}{ccc}
 \downarrow \text{UDD}\sigma_1'^{-1} \searrow & & \\
 \text{UD}u_{DuDU}^{-1} \searrow & \xrightarrow{\quad} & \text{UD}u_{Ds}^{-1} \searrow \\
 \downarrow s_{DuDU}^{-1} & \xrightarrow{\quad} & \downarrow s_{Ds}^{-1} \\
 & \text{UD}u_{DuDU}^{-1} & \text{UD}u_{Ds}^{-1} \\
 & \downarrow & \downarrow
 \end{array}
 =
 \begin{array}{ccc}
 \downarrow \text{UD}u_{DDU} & & \\
 \downarrow s_{DDU} & & \\
 & \searrow \text{DUD}\sigma_1'^{-1} &
 \end{array}.$$

Finally, use (35). ■

We show now that we do have a distributive law.

10.2. THEOREM. *With the above definitions, $r : UD \rightarrow DU$ and the invertible 2-cells $\omega_1, \omega_2, \omega_3$, and ω_4 , define a distributive law of $\mathbb{U}$ over $\mathbb{D}$.*

Proof. We show (coh 3, 6 and 8), leaving the rest to the reader. Start on the left hand side of (coh 3) and make the following substitutions:

$$\begin{array}{ccc}
 \xrightarrow{\quad} \searrow \eta_{\mathbb{U}}U^{-1} \xrightarrow{\quad} n & = & \xrightarrow{\quad} \downarrow U\beta_{\mathbb{U}} \downarrow \mu_{\mathbb{U}} \downarrow, \\
 \downarrow \sigma_4 U \searrow \mu_{\mathbb{U}} \downarrow d_n \downarrow & = & \downarrow U d_n \downarrow \sigma_3^{-1} \downarrow \sigma_4 \downarrow \\
 \downarrow U d_u U \searrow U\beta_{\mathbb{U}} \downarrow U d_n \downarrow & = & \xrightarrow{\quad} U d U \searrow U D \beta_{\mathbb{U}} \downarrow, \\
 \downarrow U \sigma_4 \downarrow \sigma_4 \downarrow & = & \downarrow \mu_{\mathbb{U}}^{-1} \downarrow n_{dU}^{-1} \downarrow \sigma_2 \downarrow \sigma_4 \downarrow
 \end{array}
 \quad \text{using (27),}$$

using that $(dU, \sigma_4) : (\beta_{\mathbb{U}}, \mu_{\mathbb{U}}) \rightarrow (\sigma_1, \sigma_2)$ is a 1-cell in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$;

$$\begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \searrow U\eta_{\mathbb{U}}^{-1} & \downarrow \mu_{\mathbb{U}}^{-1} & \downarrow \\
 & \xrightarrow{\quad} &
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \uparrow n_u^{-1} & \uparrow \eta_{\mathbb{U}}^{-1} & \nearrow \\
 & &
 \end{array},$$

and

$$\begin{array}{ccc}
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
\downarrow & \searrow & \downarrow \\
& UUd_u & \\
& \nearrow & \longrightarrow \\
& n_u^{-1} & \downarrow \\
& & n_{dU}^{-1} \\
& & \downarrow
\end{array}
& = &
\begin{array}{ccc}
\longrightarrow & & \longrightarrow \\
\downarrow & \searrow & \downarrow \\
& n_d^{-1} & \downarrow \\
& \nearrow & \searrow \\
& & n_{Du}^{-1} \\
& & \downarrow \\
& & Ud_u \\
& & \downarrow
\end{array}
\end{array}$$

To prove (coh 6), start on the right hand side, and make the following substitutions:

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \swarrow n_{DuD}^{-1} \\ \downarrow \\ \searrow \sigma_2 D \\ \downarrow \\ \swarrow \sigma_3 D^{-1} \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{UUDu_{Du}^{-1}} \\ \xrightarrow{Us_{Du}^{-1}} \\ \downarrow UD\beta_U D \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{UDu_{UDu}^{-1}} \\ \xrightarrow{s_{UDu}^{-1}} \\ \xrightarrow{Dn_{Du}^{-1}} \\ \downarrow \end{array} \\
 & = & \begin{array}{c} \xrightarrow{\quad} \\ \swarrow n_{DDu}^{-1} \\ \downarrow \\ \searrow \sigma_2 DU \\ \downarrow \\ \swarrow s_{Du}^{-1} \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{UDu_{Du}^{-1}} \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{n_{DuDU}^{-1}} \\ \downarrow DU\beta_U DU \\ \downarrow \\ \downarrow \end{array} \\
 & & \text{,}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \swarrow D\sigma_3^{-1} \\ \downarrow \\ \swarrow \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \xrightarrow{m_n} \\ \downarrow \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & = & \begin{array}{c} \xrightarrow{\quad} \\ \swarrow UD u_{Dn}^{-1} \\ \downarrow \\ \swarrow s_{Dn}^{-1} \\ \downarrow \\ \swarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{Um_n} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & & \text{,}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{UDu_{DuU}^{-1}} \\ \xrightarrow{s_{DuU}^{-1}} \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & = & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{Udu_{Dn}^{-1}} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & & \text{,}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & = & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{UDD\beta_U} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \\
 & & \text{,}
 \end{array}$$

using (29)

and

$$\begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{UmU} \\
 \searrow & \xrightarrow{Um_uU} & \searrow \\
 \text{curved arrow} & \xrightarrow{Um_n} & \text{curved arrow} \\
 \swarrow & \xrightarrow{UDD\beta_U} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{UmU} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{UD\beta_U} & \searrow \\
 \text{curved arrow} & & \text{curved arrow} \\
 \swarrow & & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}$$

Now use (28). Conclude with the substitution

$$\begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{UUm_u} & \searrow \\
 \text{curved arrow} & \xrightarrow{n_m^{-1}} & \text{curved arrow} \\
 \swarrow & \xrightarrow{n_{mU}^{-1}} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{n_m^{-1}} & \searrow \\
 \text{curved arrow} & \xrightarrow{n_{Du}^{-1}} & \text{curved arrow} \\
 \swarrow & \xrightarrow{Um_u} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}$$

To prove (coh 8), start on the left hand side, and make the substitutions:

$$\begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{UDu_d^{-1}} & \searrow \\
 \text{curved arrow} & \xrightarrow{UDu_{Du}^{-1}} & \text{curved arrow} \\
 \swarrow & \xrightarrow{s_d^{-1}} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{UDd_u} & \searrow \\
 \text{curved arrow} & \xrightarrow{UDu_u^{-1}} & \text{curved arrow} \\
 \swarrow & \xrightarrow{s_u^{-1}} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}$$

and

$$\begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{U\eta_D} & \searrow \\
 \text{curved arrow} & \xrightarrow{UDd_u} & \text{curved arrow} \\
 \swarrow & \xrightarrow{Um_u} & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}
 =
 \begin{array}{ccc}
 \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 \searrow & \xrightarrow{UDu} & \searrow \\
 \text{curved arrow} & \xrightarrow{U\eta_D U} & \text{curved arrow} \\
 \swarrow & & \swarrow \\
 \xrightarrow{\quad} & & \xrightarrow{\quad}
 \end{array}$$

Now, use (30), then the equation obtained from (26). It is easily seen that what remains equals the identity. $\blacksquare$

11. (Co-)KZ-doctrines over KZ-doctrines

Distributive laws for KZ-doctrines over KZ-doctrines are formally very similar to distributive laws of co-KZ-doctrines over KZ-doctrines. We start this section by recalling the definition and some aspects of KZ-doctrines [9] (see [8] as well). Then we define distributive laws of co-KZ-doctrines over KZ-doctrines, and we show that such a distributive law induces a distributive law between the induced pseudomonads, as defined in section 4. Finally, we show that the composite pseudomonad given by a distributive law of a KZ-doctrine over a KZ-doctrine is again a KZ-doctrine.

11.1. We are still working in a **Gray**-category **A**, and $\mathcal{K}$ is an object of **A**.

11.2. **DEFINITION.** A KZ-doctrine **D** on $\mathcal{K}$ consists of an object D , 1-cells $d : 1_{\mathcal{K}} \rightarrow D$, and $m : DD \rightarrow D$ in **A**($\mathcal{K}, \mathcal{K}$) and a fully-faithful adjoint string $\eta, \epsilon : Dd \dashv m$; and $\alpha, \beta : m \dashv dD : D \rightarrow DD$ such that

$$1_{\mathcal{K}} \xrightarrow{d} D \begin{array}{c} \nearrow dD \\ \xrightarrow{Id_D} DD \\ \searrow Dd \end{array} \begin{array}{c} \xrightarrow{m} D \\ \Downarrow \beta \\ \xrightarrow{\eta} DD \\ \xrightarrow{m} D \end{array} = 1_{\mathcal{K}} \begin{array}{c} \nearrow d \\ \xrightarrow{(d_d)^{-1}} D \\ \searrow d \end{array} \begin{array}{c} \xrightarrow{dD} DD \\ \Downarrow (d_d)^{-1} \\ \xrightarrow{Dd} DD \end{array} \xrightarrow{m} D. \quad (36)$$

The 2-cell $\delta : Dd \rightarrow dD$ is defined as the pasting

$$D \begin{array}{c} \xrightarrow{Id_D} D \\ \searrow dD \end{array} \begin{array}{c} \xrightarrow{m} DD \\ \Downarrow \beta^{-1} \\ \xrightarrow{\epsilon} DD \end{array} \xrightarrow{Id_{DD}} DD,$$

and it satisfies the equations $\delta \circ d = d_d$ and $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$.

In a co-KZ-doctrine **U**, with $u : 1 \rightarrow U$ and $n : UU \rightarrow U$, the fully-faithful adjoint string is of the form $uU \dashv n \dashv Uu$.

Proposition 10.1 of [9] says that **D** induces a pseudomonad $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$ if μ is defined as the pasting

$$DDD \xrightarrow{Id_{DDD}} DDD \xrightarrow{Dm} DD \begin{array}{c} \xrightarrow{m} D \\ \Downarrow \beta \end{array} D \\ \begin{array}{c} \searrow mD \\ \xrightarrow{dDD} DD \end{array} \xrightarrow{m} D \xrightarrow{Id_D} D \\ \begin{array}{c} \xrightarrow{mD} DD \\ \xrightarrow{m} D \end{array}$$

11.3. Assume we have a KZ-doctrine

$$\mathbf{D} = (D, d, m, \alpha_D, \beta_D, \eta_D, \epsilon_D),$$

and a co-KZ-doctrine $\mathbf{U} = (U, u, n, \alpha_U, \beta_U, \eta_U, \epsilon_U)$, with

$$DD \xrightarrow{Id} DD, \quad \begin{array}{c} DD \\ \searrow m \\ D \end{array} \begin{array}{c} \xrightarrow{Id} DD \\ \Downarrow \alpha_D \\ \xrightarrow{dD} D \end{array}, \quad \begin{array}{c} D \\ \searrow Dd \\ DD \end{array} \begin{array}{c} \xrightarrow{Id} D \\ \Downarrow \beta_D \\ \xrightarrow{m} D \end{array}, \quad \begin{array}{c} D \\ \searrow Dd \\ DD \end{array} \begin{array}{c} \xrightarrow{Id} D \\ \Downarrow \eta_D \\ \xrightarrow{m} D \end{array}, \quad \begin{array}{c} D \\ \searrow Dd \\ DD \end{array} \begin{array}{c} \xrightarrow{Id} D \\ \Downarrow \epsilon_D \\ \xrightarrow{m} D \end{array},$$

and

$$U \xrightarrow{Id} U, \quad \begin{array}{c} U \\ \searrow uU \\ UU \end{array} \begin{array}{c} \xrightarrow{Id} U \\ \Downarrow \alpha_U \\ \xrightarrow{n} UU \end{array}, \quad \begin{array}{c} UU \\ \searrow n \\ U \end{array} \begin{array}{c} \xrightarrow{Id} UU \\ \Downarrow \beta_U \\ \xrightarrow{uU} U \end{array}, \quad \begin{array}{c} UU \\ \searrow n \\ U \end{array} \begin{array}{c} \xrightarrow{Id} UU \\ \Downarrow \eta_U \\ \xrightarrow{Uu} U \end{array}, \quad \begin{array}{c} UU \\ \searrow n \\ U \end{array} \begin{array}{c} \xrightarrow{Id} UU \\ \Downarrow \epsilon_U \\ \xrightarrow{Uu} U \end{array}.$$

Since we are assuming **D** to be a KZ-doctrine, and **U** to be a co-KZ-doctrine, then β_D and η_D , and α_U , and ϵ_U are invertible. Let $\delta : Dd \rightarrow dD$ and $v : uU \rightarrow Uu$ be the corresponding induced 2-cells.

11.4. DEFINITION. A distributive law of $\mathbf{U}$ over $\mathbf{D}$ consists of a 1-cell $r : UD \rightarrow DU$ in $\mathbf{A}(\mathcal{K}, \mathcal{K})$, together with an invertible 2-cell

$$\begin{array}{ccc} & UD & \\ uD \nearrow & & \searrow r \\ D & \xrightarrow{Du} & DU, \end{array} \quad \omega_1 \Downarrow$$

such that the pasting

$$\begin{array}{ccccccc} & & DU & & & & \\ & dU \nearrow & & \searrow uDU & & & \\ U & \xrightarrow{uU} & UU & \xrightarrow{UdU} & UDU & \xrightarrow{rU} & DUU & \xrightarrow{Dn} & DU \\ & \Downarrow v & & & \Downarrow \omega_1 U^{-1} & & \Downarrow D\alpha_U & & \\ & Uu \nearrow & & \searrow UD u & & & & & \\ & Ud \searrow & & \nearrow UD u & & & & & \\ & & UD & \xrightarrow{r} & DU & & & & \end{array}$$

is invertible. We denote the inverse by ω_2 . We further require the pasting

$$\omega_3 = \begin{array}{ccccccc} UUD & \xrightarrow{Id} & UUD & \xrightarrow{Ur} & UDU & \xrightarrow{rU} & DUU \\ & \searrow nD & \Downarrow \eta_U D & \nearrow UuD & \Downarrow U\omega_1 & \Downarrow r_u^{-1} & \nearrow DUu & \Downarrow D\epsilon_U & \searrow Dn \\ & & UD & \xrightarrow{r} & DU & \xrightarrow{Id} & DU \end{array}$$

to be invertible, and the pasting

$$\omega_4 = \begin{array}{ccccccc} & UD & \xrightarrow{r} & DU & \xrightarrow{Id} & DU \\ Um \nearrow & & \searrow UDd & \searrow DUd & \searrow DdU & \searrow \eta_D U & \\ UDD & \xrightarrow{Id} & UDD & \xrightarrow{rD} & DUD & \xrightarrow{Dr} & DDU \end{array}$$

to be invertible. We also require the following two coherence conditions:

$$\begin{array}{ccc} UD \xrightarrow{r} DU & & \\ UuD \left(\begin{array}{c} \Downarrow v \\ \Downarrow u_r^{-1} \end{array} \right) \begin{array}{c} uUD \xrightarrow{u_r} UUD \\ UdU \xrightarrow{rU} DUU \end{array} & = & UD \xrightarrow{r} DU \\ & & UuD \left(\begin{array}{c} \Downarrow U\omega_1^{-1} \\ \Downarrow d_r^{-1} \end{array} \right) \begin{array}{c} UDd \xrightarrow{d_r} DUD \\ DUd \xrightarrow{Dr} DDU \end{array} \end{array}$$

and

$$\begin{array}{ccc} UD \xrightarrow{r} DU & & \\ UdD \left(\begin{array}{c} \Downarrow \delta_D \\ \Downarrow r_d \end{array} \right) \begin{array}{c} UDd \xrightarrow{rD} DUD \\ DUD \xrightarrow{Dr} DDU \end{array} & = & UD \xrightarrow{r} DU \\ & & UdD \left(\begin{array}{c} \Downarrow \omega_2 D^{-1} \\ \Downarrow d_r^{-1} \end{array} \right) \begin{array}{c} UDd \xrightarrow{d_r} DUD \\ DUd \xrightarrow{Dr} DDU \end{array} \end{array}$$

Using the equations $v \circ n = \eta_U \cdot \beta_U$ and $\delta \circ m = \alpha_D \cdot \epsilon_D$, and the coherence conditions, it is not hard to see that the inverse of ω_3 is

$$\begin{array}{ccccc}
 & UD & \xrightarrow{r} & DU & \xrightarrow{Id} & DU, \\
 & \nwarrow nD & \searrow uUD & \searrow uDU & \searrow DuU & \nearrow Dn \\
 & \Downarrow \beta_U D & \Downarrow u_r^{-1} & \Downarrow \omega_1 U^{-1} & \Downarrow D\alpha_U & \\
 UUD & \xrightarrow{Id} & UUD & \xrightarrow{Ur} & UDU & \xrightarrow{rU} & DUU
 \end{array}$$

and the inverse of ω_4 is

$$\begin{array}{ccccccc}
 UDD & \xrightarrow{Id} & UDD & \xrightarrow{rD} & DUD & \xrightarrow{Dr} & DDU \\
 & \searrow Um & \Downarrow U\alpha_D & \Downarrow \omega_2 D & \Downarrow d_r & \Downarrow dDU & \Downarrow \beta_D U \\
 & & UD & \xrightarrow{r} & DU & \xrightarrow{Id} & DU.
 \end{array}$$

Looking at the definition it seems possible to define a distributive law in terms of ω_2 , defining ω_1 with a similar pasting as ω_2 was defined above. It can be shown that if we start with ω_1 , define ω_2 as above, and then use a similar pasting to define a new ω'_1 we have that $\omega_1 = \omega'_1$. And similarly if we start with ω_2 .

11.5. THEOREM. *If $\mathbb{D}$ and $\mathbb{U}$ are the pseudomonads determined by the KZ-doctrine $\mathbb{D}$ and the co-KZ-doctrine $\mathbb{U}$ respectively, then r together with ω_1 , ω_2 , ω_3 and ω_4 define a distributive law of $\mathbb{U}$ over $\mathbb{D}$.*

Proof. Since $\mathbb{U}$ is a co-KZ-doctrine, we have that $\beta_U = \alpha_U^{-1}$ and $\eta_U = \epsilon_U^{-1}$. With this in mind, we must show that the coherence conditions (coh 1) to (coh 9) of section 4 are satisfied. We show that (coh 1), (coh 3) and (coh 4) are satisfied and leave the others to the reader.

Consider the inverse of the left hand side of (coh 1). To get the inverse of the right hand side make the following four substitutions

$$\begin{array}{ccc}
 v \circ u = u_u^{-1}, & & \\
 \begin{array}{ccc}
 \nearrow & d_u & \searrow \\
 \downarrow u_u^{-1} & \downarrow u_{dU}^{-1} & \downarrow \\
 \xrightarrow{Ud_u^{-1}} & &
 \end{array} & = & \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \downarrow u_d^{-1} & \downarrow u_{Du}^{-1} & \downarrow \\
 \xrightarrow{\quad} & \xrightarrow{\quad} &
 \end{array}, \\
 \begin{array}{ccc}
 \xrightarrow{\omega_1 U^{-1}} & & \\
 \uparrow u_{Du}^{-1} & \uparrow r_u^{-1} & \uparrow \\
 \xrightarrow{\quad} & \xrightarrow{\quad} &
 \end{array} & = & \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{Du_u^{-1}} & \\
 \uparrow & \uparrow \omega_1^{-1} & \uparrow \\
 \xrightarrow{\quad} & \xrightarrow{\quad} &
 \end{array}, \\
 \begin{array}{ccc}
 \nearrow & & \searrow \\
 \downarrow & Du_u^{-1} & \downarrow \\
 \nearrow & D\alpha_U & \searrow \\
 \downarrow & D\epsilon_U & \downarrow
 \end{array} & = & id_{Du}
 \end{array}$$

and

To prove (coh 3) start with the inverse of the right hand side and make the following substitutions:

$$v \circ n = \eta_U \cdot \beta_U,$$

$$\begin{aligned} \beta_U &= \begin{array}{ccc} & \nearrow & \\ vU & \xrightarrow{u_n^{-1}} & \\ & \searrow & \\ & U\alpha_U^{-1} & \end{array}, \\ &= \begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & \\ \downarrow u_n^{-1} & \downarrow u_{dU}^{-1} & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \end{array} = \begin{array}{ccc} & \nearrow & \\ & d_n^{-1} & \\ \downarrow u_{dUU}^{-1} & \downarrow u_{Dn}^{-1} & \downarrow \\ \xrightarrow{\quad} & U d_n & \end{array}, \\ &= \begin{array}{ccc} \xrightarrow{\quad} & \downarrow u_{Dn}^{-1} & \\ \downarrow & \omega_1 U^{-1} & \end{array} = \begin{array}{ccc} & \nearrow & \\ & D u_n^{-1} & \\ \omega_1 U U^{-1} & \uparrow & r_n \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \end{array}, \\ &= \begin{array}{ccc} \xrightarrow{U\alpha_U^{-1}} & \xrightarrow{U d_n} & \xrightarrow{r_n} \\ \downarrow & \downarrow & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{U d_u^{-1}} & \xrightarrow{r_u^{-1}} \\ \downarrow & \downarrow & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{D U \alpha_U^{-1}} & \end{array}, \\ &= \begin{array}{ccc} \xrightarrow{\quad} & \downarrow D u_n^{-1} & \\ \downarrow & D \alpha_U & \end{array} = \begin{array}{ccc} \xrightarrow{\quad} & \downarrow D \alpha_U U & \\ \downarrow & D \mu_U^{-1} & \end{array}, \end{aligned}$$

and

$$\begin{array}{ccc} \xrightarrow{\quad} & \downarrow D U \alpha_U^{-1} & \\ \downarrow & D \mu_U^{-1} & \end{array} = \begin{array}{ccc} \xrightarrow{\quad} & \downarrow D \epsilon_U U & \\ \downarrow & D n & \end{array}.$$

Comparing the pasting obtained with the left hand side of (coh 3), we see that we must show that

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\eta_U} & \xrightarrow{U d_u^{-1}} & \xrightarrow{r_u^{-1}} & \xrightarrow{\quad} \\ \downarrow & \searrow n_d & \downarrow & \downarrow D \epsilon_U & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\omega_3^{-1}} & \end{array} = D n \circ r U \circ U \omega_2^{-1}.$$

Now, pass every 2-cell of the left hand side to the right hand side, except for η_U . It is not hard to see, with the help of (coh 1), that the resulting equation is satisfied.

Start with the inverse of the left hand side of (coh 4), and make the following sequence of substitutions:

$$\begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\beta_U D} & \\ \uparrow \mu_U D^{-1} & \uparrow & \downarrow \\ \xrightarrow{\quad} & \xrightarrow{\quad} & \downarrow u_n^{-1} \end{array} = \begin{array}{ccc} \xrightarrow{\beta_U U D} & \xrightarrow{\quad} & \\ \downarrow & \downarrow & \\ \xrightarrow{\quad} & \xrightarrow{u_n^{-1}} & \end{array},$$

$$\begin{array}{c}
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \uparrow \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{U\beta_U D} \\ \uparrow u_n^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \searrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \searrow \beta_U D \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_u U D \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \swarrow \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_r^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_r^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{U r}^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \searrow \beta_U U D \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{U r}^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow n_r \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \beta_U D U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_u D U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow U \omega_1 U^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \swarrow u_{D u} U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \omega_1 U^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{r U}^{-1} \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{D u} U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow U D \alpha_U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \swarrow D \alpha_U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{D n}^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array} \\
\\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow u_{D n}^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \omega_1 U^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \swarrow \omega_1 U U^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow D u_n^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow r_n \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array}
\end{array}$$

and

$$\begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \downarrow D u_n^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow D \alpha_U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& = & \begin{array}{ccc}
\begin{array}{c} \xrightarrow{\quad} \\ \swarrow D \alpha_U U \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow D \mu_U^{-1} \\ \xrightarrow{\quad} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \xrightarrow{\quad} \end{array} \\
& , & &
\end{array}$$

■

11.6. Assume now that D is a KZ-doctrine as before, but that U is also a KZ-doctrine. The data for D is unchanged, and the data for U is now:

$$\begin{array}{c}
UU \xrightarrow{Id} UU, \quad \begin{array}{ccc} uU & UU & n \\ \swarrow & \downarrow \beta_U & \searrow \\ U & & U \end{array} \xrightarrow{Id} U, \quad \begin{array}{ccc} U & \xrightarrow{Id} & U \\ \searrow & \downarrow \eta_U & \swarrow \\ Uu & UU & n \end{array} \\
UU \xrightarrow{n} U \xrightarrow{uU} UU, \quad UU \xrightarrow{n} U \xrightarrow{Uu} UU, \quad UU \xrightarrow{n} U \xrightarrow{Uu} UU, \quad UU \xrightarrow{Id} UU.
\end{array}$$

We have $v : Uu \rightarrow uU$.

Copying almost exactly the definition of a distributive law of a co-KZ-doctrine over a KZ-doctrine, we obtain the concept of a distributive law of the KZ-doctrine U over the KZ-doctrine D . If $\mathbb{U}$ and $\mathbb{D}$ are the pseudomonads induced by U and D respectively, we obtain, in a very similar way, a distributive law between the pseudomonads $\mathbb{U}$ and $\mathbb{D}$. Let $\mathbb{V}$ be the composite pseudomonad obtained from this last distributive law, as in section 5.

11.7. THEOREM. *The composite pseudomonad obtained from a distributive law between KZ-doctrines is again a KZ-doctrine.*

Proof. We follow the notation introduced before the statement of the theorem, and the notation for $\mathbb{V}$ is the same as in section 5. According to theorem 11.1 of [9], it suffices to show that $p \dashv vV$ with counit $\beta_{\mathbb{V}}$. Define $\vartheta : Vv \rightarrow vV$ as the pasting

$$\begin{array}{ccccc}
 & & & DUU & \\
 & \nearrow^{DUu} & & \searrow^{DUdU} & \\
 DU & & & & \\
 & \searrow^{DuU} & & \nearrow^{DuU} & \\
 & \searrow^{DdU} & & \nearrow^{DuU} & \\
 & \searrow^{dDU} & & \nearrow^{DuU} & \\
 & \searrow^{uDU} & & \nearrow^{DuU} & \\
 & & DDU & \xrightarrow{DuDU} & DUDU \\
 & & \nearrow^{d_uDU} & & \nearrow^{dUDU} \\
 & & UDU & &
 \end{array}$$

It is not hard to see that $\vartheta \circ v = p_p$ and that $p \circ \vartheta = \beta_{\mathbb{V}}^{-1} \cdot \eta_{\mathbb{V}}^{-1}$. Using these two equations, an easy calculation shows that $p \dashv vV$ with unit

$$\begin{array}{ccccc}
 & & Id & & \\
 & \nearrow^{VvV} & & \searrow^{V\beta_{\mathbb{V}}^{-1}} & \\
 VV & & & & VV \\
 & \searrow^{\vartheta V} & & \nearrow^{Vp} & \\
 & \searrow^{vVV} & & \nearrow^{v_p} & \\
 & \searrow^p & & \nearrow^{vV} & \\
 & & V & &
 \end{array}$$

and counit $\beta_{\mathbb{V}}$. ■

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