Theory and Applications of Categories, Vol. 5, No. 5, 1999, pp. 91–147.

DISTRIBUTIVE LAWS FOR PSEUDOMONADS

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Transmitted by Ross Street

ABSTRACT. We define distributive laws between pseudomonads in a Gray-category **A**, as the classical two triangles and the two pentagons but commuting only up to isomorphism. These isomorphisms must satisfy nine coherence conditions. We also define the Gray-category PSM(**A**) of pseudomonads in **A**, and define a lifting to be a pseudomonad in PSM(**A**). We define what is a pseudomonad with compatible structure with respect to two given pseudomonads. We show how to obtain a pseudomonad with compatible structure from a distributive law, how to get a lifting from a pseudomonad with compatible structure, and how to obtain a distributive law from a lifting. We show that one triangle suffices to define a distributive law in case that one of the pseudomonads is a (co-)KZ-doctrine and the other a KZ-doctrine.

1. Introduction

Distributive laws for monads were introduced by J. Beck in [2]. As pointed out by G. M. Kelly in [7], strict distributive laws for higher dimensional monads are rare. We need then a study of pseudo-distributive laws. The first step in this direction is quite easy: just replace the two commutative triangles, and the two commutative pentagons of [2] by appropriate invertible cells. The problem is to determine what coherence conditions to impose on these invertible cells. We should point out that, in [7], there is a step in this direction, keeping commutativity on the nose on the triangles and one of the pentagons, and asking for commutativity up to isomorphism in the remaining pentagon, plus five coherence conditions. The structure obtained from such a distributive law between two strict 2-monads is not, in general, a strict 2-monad, and since that article deals exclusively with strict 2-monads, what is obtained is a reflection result.

In this paper, instead of working with 2-monads we work with the more general pseudomonads. We will see that the structure obtained from a distributive law between pseudomonads is a pseudomonad. We define a distributive law between pseudomonads as we said above, that is to say, asking for commutativity up to isomorphism of the two triangles and the two pentagons. We propose nine coherence conditions for these isomorphisms. See section 4 below. We observe that the coherence conditions of [7] and the ones proposed in this paper coincide if in our setting we ask for commutativity on the nose of the two triangles and one of the pentagons. Thus, the examples of distributive laws given there are examples here as well.

But why exactly these nine coherence conditions?

Received by the editors 1998 June 24 and, in revised form, 1998 November 22.
Published on 1999 March 18.
1991 Mathematics Subject Classification: 18C15, 18D05, 18D20.
Key words and phrases: Pseudomonads, distributive laws, KZ-doctrines, Gray-categories.

A more conceptual approach to distributive laws for monads is given by R. Street in [11]. It is shown that for a 2-category C, a distributive law is the same thing as a monad in the 2-category MND(C), whose objects are monads in C. In this paper we introduce the corresponding structure PSM(**A**), of pseudomonads for a corresponding three dimensional structure **A**, see section 7.

In M. Barr and C. Wells' book [1], exercise (DL) asks to prove that, for monads, a distributive law, a lifting of one monad structure to the algebras of the other, and a monad with compatible structure with the two given monads, are essentially the same thing. For pseudomonads we have already mentioned distributive laws. We define a lifting as a pseudomonad in PSM(**A**) in section 8. In section 6, we define what a pseudomonad whose structure is compatible with two given pseudomonads is.

We show how to obtain a composite pseudomonad with compatible structure from a distributive law between pseudomonads, how to obtain a lifting from a pseudomonad with compatible structure, and, closing the cycle, how to obtain a distributive law from a lifting.

We see then, that the nine coherence conditions can be shown to hold if we define a distributive law from a lifting. In turn, these coherence conditions allow us to define a lifting from a distributive law between pseudomonads.

The situation for distributive laws between KZ-doctrines and (co-)KZ-doctrines is a lot simpler. We show that either one of the triangles commuting up to isomorphism (satisfying coherence conditions) is enough to obtain a distributive law. One such example is the following. It is well known that adding free (finite) coproducts to categories is a KZ-doctrine over Cat, and adding free (finite) products is a co-KZ-doctrine. There is a more or less obvious distributive law of the co-KZ-doctrine over the KZ-doctrine. Observe however that even if we arrange for these KZ-doctrine and co-KZ-doctrine to produce strict pseudomonads, the distributive law obtained is not strict.

This article is possible thanks to the definition of tricategories given in [6]. It is simplified by the fact that a tricategory is triequivalent to a Gray-category, a fact proved in the same paper. We thus work in the framework of Gray-categories, as in [6], continuing the development of the formal theory of pseudomonads started in [9].

This paper is organized as follows:

In section 2 we provide a brief description of the framework that we use, namely that of Gray-categories. For more details we refer the reader to [6, 5].

In section 3 we recall the definition and some properties of pseudomonads given in [9], the definition uses the definition of pseudomonoid given in [3]. We also define the change of base 2-functors, change of base strong transformations and the change of base modifications that we will need in later sections. Change of base turns out to be a Gray-natural transformation.

In section 4 we define distributive laws for pseudomonads by replacing commutativity on the nose by commutativity up to isomorphism. We give here the nine coherence conditions that these isomorphisms should satisfy.

The first step to obtain compatible structures is to define a composite pseudomonad from a distributive law. This is what we do in section 5.

In section 6 we define what a pseudomonad with compatible structure is with respect to given pseudomonads. Furthermore, we exhibit the structure that makes compatible the composite pseudomonad defined in the previous section.

We introduce the Gray-category PSM($\mathbf{A}$) in section 7, to define, in section 8, a lifting as a pseudomonad in the Gray-category PSM($\mathbf{A}$).

In section 9 we show how to construct a pseudomonad in PSM($\mathbf{A}$), from given pseudomonads with compatible structure. In the following section, we go from a pseudomonad in PSM($\mathbf{A}$) to a distributive law.

Section 11 deals with distributive laws of (co-)KZ-doctrines over KZ-doctrines. We refer the reader to [9] for the definition and properties we use of KZ-doctrines, but see [8] as well. We show that one triangle, plus coherence, is enough to produce a distributive law. Compare with [10], where it is shown that one of the triangles suffices for a distributive law between idempotent monads. The case of KZ-doctrines over KZ-doctrines is formally very similar. In this latter case, we show that the composite pseudomonad is again a KZ-doctrine.

I would like to thank the referee for helping improve the readability of this paper, and for suggesting condition (12), after which all the conditions of section 6 were modeled.

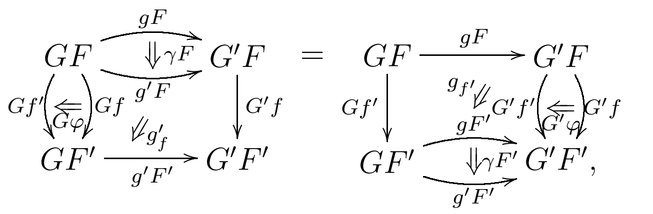
2. Gray-categories

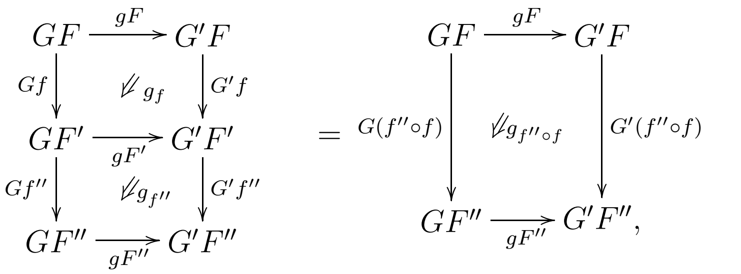
As in [9] we will work with a Gray-category $\mathbf{A}$, where Gray is the symmetric monoidal closed category whose underlying category is 2-Cat with tensor product as in [6]. A Gray-category is a category enriched in the category Gray as in [4]. We will briefly spell out what this means, and we refer the reader to [6] and [4] for more details.

A Gray-category $\mathbf{A}$ has objects $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... For every pair of objects $\mathcal{A}$, $\mathcal{B}$ of $\mathbf{A}$, $\mathbf{A}$ has a 2-category $\mathbf{A}(\mathcal{A},\mathcal{B})$. Given another object $\mathcal{C}$ in $\mathbf{A}$, $\mathbf{A}$ has a 2-functor $\mathbf{A}(\mathcal{C},\mathcal{B}) \otimes \mathbf{A}(\mathcal{A},\mathcal{B}) \to \mathbf{A}(\mathcal{A},\mathcal{C})$. This 2-functor corresponds to a cubical functor $M : \mathbf{A}(\mathcal{B},\mathcal{C}) \times \mathbf{A}(\mathcal{A},\mathcal{B}) \to \mathbf{A}(\mathcal{A},\mathcal{C})$. We will denote M by juxtaposition, $M(G,F) = GF$ for $F \in \mathbf{A}(\mathcal{A},\mathcal{B})$ and $G \in \mathbf{A}(\mathcal{B},\mathcal{C})$. Given $f : F \to F'$ in $\mathbf{A}(\mathcal{A},\mathcal{B})$ and $g : G \to G'$ in $\mathbf{A}(\mathcal{B},\mathcal{C})$ we will denote the invertible 2-cell $M_{g,f}$ by

$$\begin{array}{ccc} GF & \xrightarrow{gF} & G'F \\ {\scriptstyle Gf}\big\downarrow & \swarrow_{g_f} & \big\downarrow{\scriptstyle G'f} \\ GF' & \xrightarrow[gF']{} & G'F'. \end{array}$$

What the definition of being cubical means for M is the following: Given $\varphi : f \to f' : F \to F'$, and $f'' : F' \to F''$ in $\mathbf{A}(\mathcal{A},\mathcal{B})$, and $\gamma : g \to g' : G \to G'$, and $g'' : G' \to G''$ in $\mathbf{A}(\mathcal{B},\mathcal{C})$, we have that $(\_)F : \mathbf{A}(\mathcal{B},\mathcal{C}) \to \mathbf{A}(\mathcal{A},\mathcal{C})$ and $G(\_) : \mathbf{A}(\mathcal{A},\mathcal{B}) \to \mathbf{A}(\mathcal{A},\mathcal{C})$ are 2-functors, $(\_)f : (\_)F \to (\_)F'$ and $g(\_) : G(\_) \to G'(\_)$ are strong transformations, $(\_)\varphi : (\_)f \to (\_)f'$ and $\gamma(\_) : g(\_) \to g'(\_)$ are modifications, and the following three equations are satisfied





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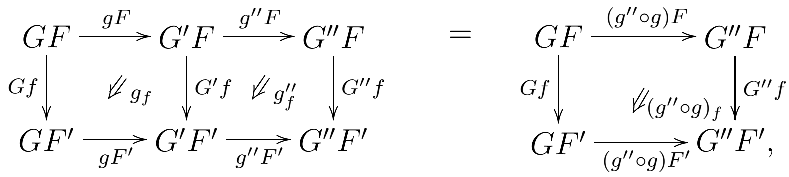


and if either f or g is an identity, then g_f is an identity 2-cell. Now, for every object $\mathcal{A}$ of $\mathbf{A}$, there is a distinguished object $1_{\mathcal{A}}$. The triangle in the definition of enriched categories means that the action of multiplying by $1_{\mathcal{A}}$ is trivial. Now, the pentagon means that for another object $\mathcal{D}$ in $\mathbf{A}$, and $\kappa : k \to k' : K \to K'$ in $\mathbf{A}(\mathcal{C}, \mathcal{D})$ the following equations hold:

$$(KG)F = K(GF),$$

$$(KG)f = K(Gf), \quad (Kg)F = K(gF), \quad (kG)F = k(GF),$$

$$(KG)\varphi = K(G\varphi), \quad (K\gamma)F = K(\gamma F), \quad (\kappa G)F = \kappa(GF),$$

$$(Kg)_f = K(g_f), \quad (kG)_f = k_{Gf}, \text{ and } (k_g)F = k_{gF}.$$

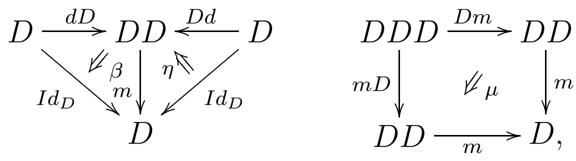
We will use these properties freely, without further mention.

3. Pseudomonads

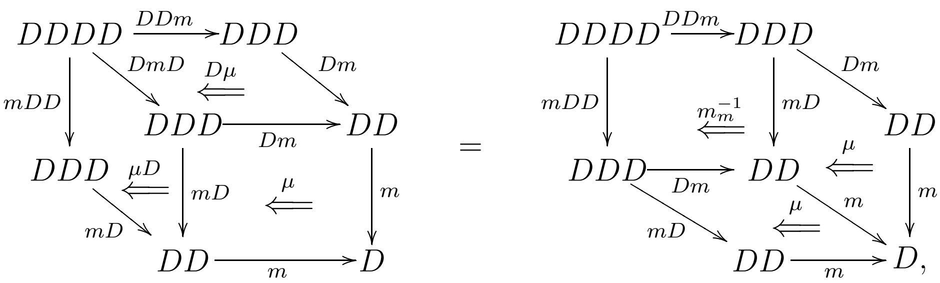
For the convenience of the reader we will recall here the definition of a pseudomonad in a Gray-category $\mathbf{A}$, for more details we refer the reader to [9]. We adopt the definition of pseudomonoid given in [3].

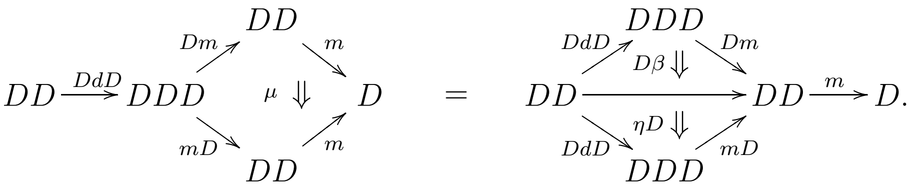
3.1. Definition. *A pseudomonad* $\mathbb{D}$ *on an object* $\mathcal{K}$ *of a* Gray*-category* $\mathbf{A}$ *is a pseudomonoid in the* Gray *monoid* $\mathbf{A}(\mathcal{K},\mathcal{K})$.

We give now, in elementary terms, what this means. A pseudomonad $\mathbb{D}$ as above consists of an object D in $\mathbf{A}(\mathcal{K},\mathcal{K})$, and 1-cells $d : 1_{\mathcal{K}} \to D$, and $m : DD \to D$ and invertible 2-cells

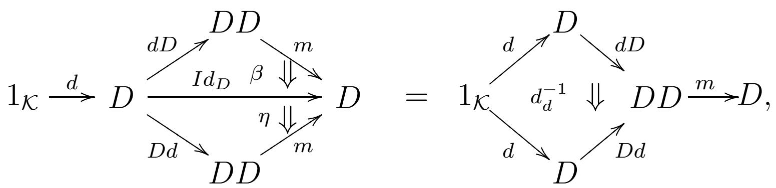


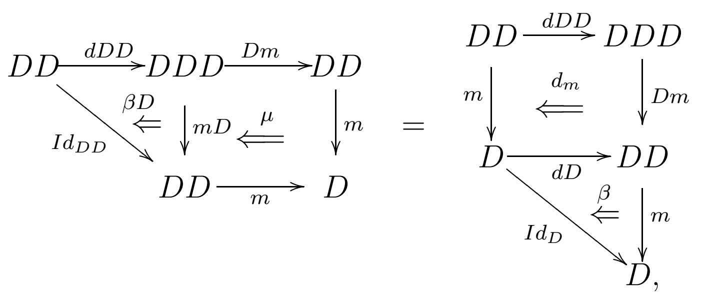
such that the following two equations are satisfied:

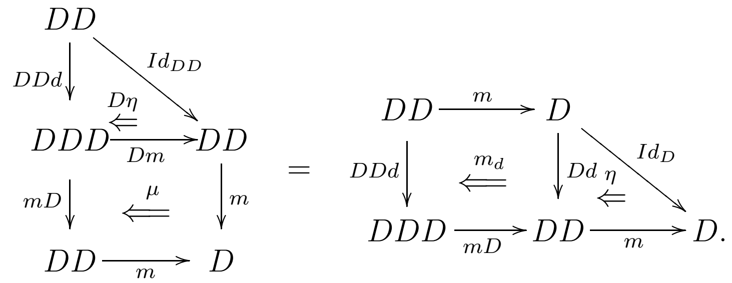

(1)


(2)

It is shown in [9] that the following three equations hold for any pseudomonad $\mathbb{D}$:

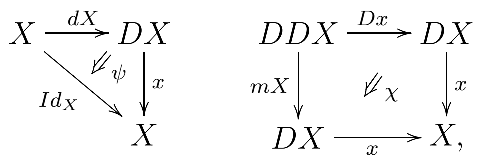

(3)


(4)

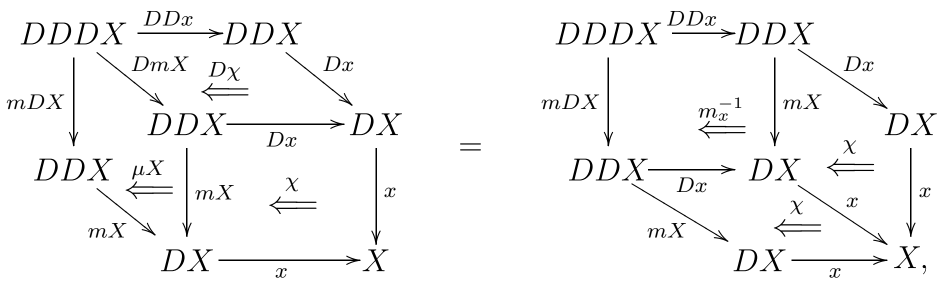


(5)

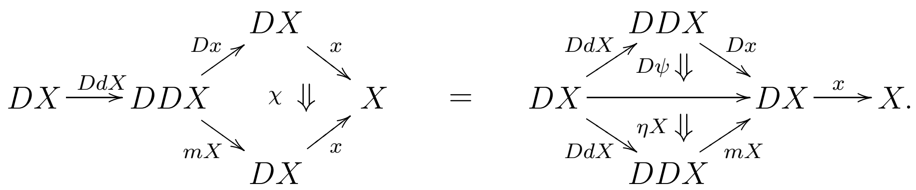
We recall as well the 2-categories of algebras for a pseudomonad $\mathbb{D}$. Let $\mathcal{X}$ be another object in **A**. An object in the 2-category $\mathbb{D}\text{-Alg}_{\mathcal{X}}$ consists of an object X in $\mathbf{A}(\mathcal{X},\mathcal{K})$, together with a 1-cell $x : DX \to X$ and invertible 2-cells



such that the following two equations are satisfied

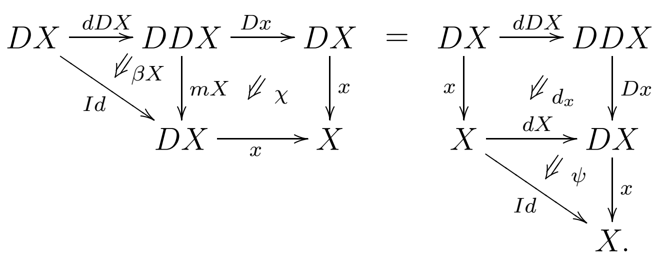


(6)



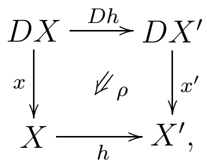
(7)

It is shown in [9] that for every object (ψ, χ) in $\mathbb{D}\text{-Alg}_{\mathcal{X}}$, the following equality holds:

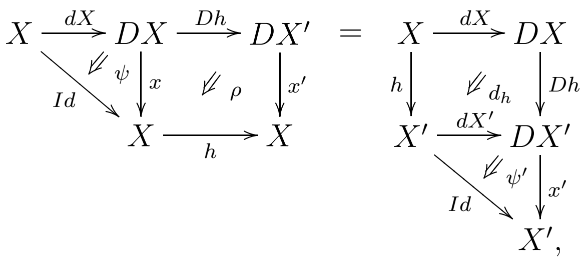


(8)

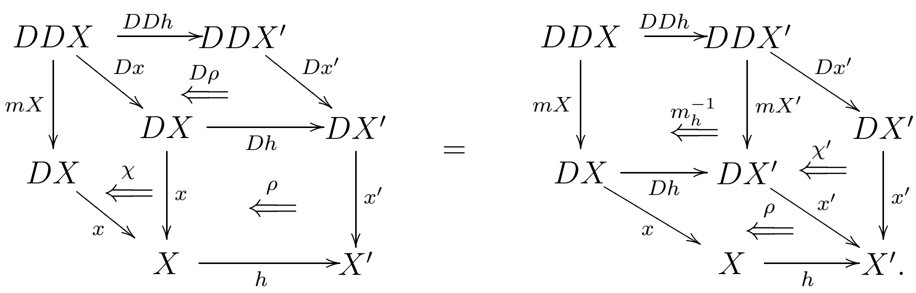
A 1-cell $(h, \rho) : (\psi, \chi) \to (\psi', \chi')$ in $\mathbb{D}$-$\mathrm{Alg}_{\mathcal{X}}$ consists of a 1-cell $h : X \to X'$ in $\mathbf{A}(\mathcal{X}, \mathcal{K})$, together with an invertible 2-cell



that satisfies the following two equations:

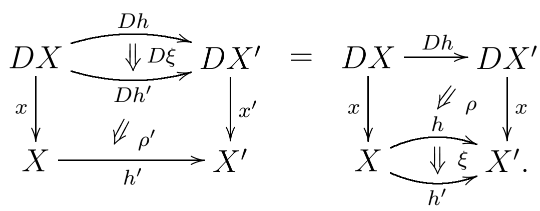


(9)



(10)

A 2-cell $\xi : (h, \rho) \to (h', \rho') : (\psi, \chi) \to (\psi', \chi')$ is a 2-cell $\xi : h \to h'$ such that the following condition is satisfied:



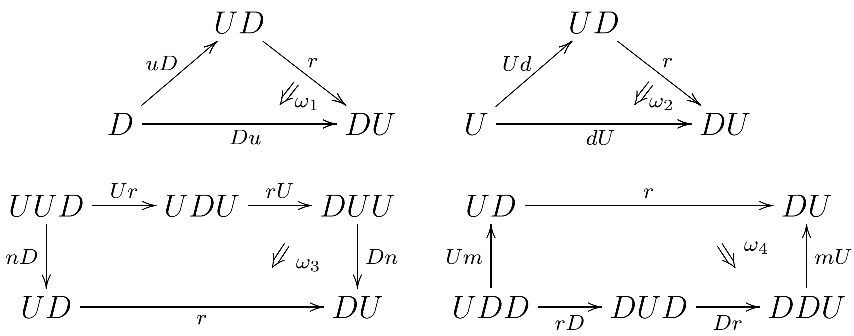
(11)

Given another object $\mathcal{Z}$ of $\mathbf{A}$, and $K \in \mathbf{A}(\mathcal{Z}, \mathcal{X})$, we can define a change of base 2-functor $\widehat{K} : \mathbb{D}\text{-Alg}_{\mathcal{X}} \to \mathbb{D}\text{-Alg}_{\mathcal{Z}}$. If $\xi : (h, \rho) \to (h', \rho') : (\psi, \chi) \to (\psi', \chi')$ is in $\mathbb{D}\text{-Alg}_{\mathcal{X}}$, then its image under $\widehat{K}$ is $\xi K : (hK, \rho K) \to (h'K, \rho' K) : (\psi K, \chi K) \to (\psi' K, \chi' K)$. If $k : K \to K'$ then we define the strong transformation $\widehat{k} : \widehat{K} \to \widehat{K'}$ such that $\widehat{k}_{(\psi,\chi)} = (Xk, x_k^{-1})$ and $\widehat{k}_{(h,\rho)} = h_k^{-1}$. If $\kappa : k \to k' : K \to K'$ in $\mathbf{A}(\mathcal{Z}, \mathcal{X})$, then $\widehat{\kappa}_{(\psi,\chi)} = X\kappa$ defines a modification $\widehat{\kappa} : \widehat{k} \to \widehat{k}'$. We have actually defined a Gray-functor $\mathbb{D}\text{-Alg} : \mathbf{A}^{op} \to \mathsf{Gray}$.

For every object $\mathcal{Z}$, we have an obvious forgetful 2-functor $\mathbb{D}\text{-Alg}_{\mathcal{Z}} \to \mathbf{A}(\mathcal{Z}, \mathcal{K})$. These 2-functors define a forgetful Gray-natural transformation $\Phi : \mathbb{D}\text{-Alg} \to \mathbf{A}(\_, \mathcal{K})$.

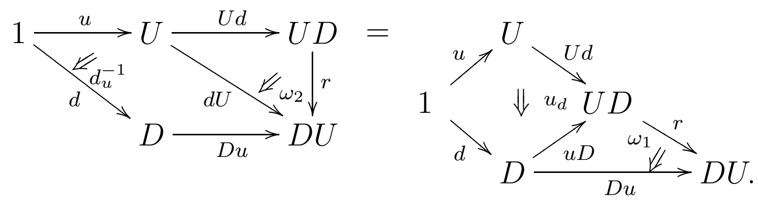
4. Distributive laws

Let $\mathbb{D} = (D, d, m, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \mu_{\mathbb{D}})$ and $\mathbb{U} = (U, u, n, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \mu_{\mathbb{U}})$ be pseudomonads on the same object $\mathcal{K}$ of the Gray-category $\mathbf{A}$. A *distributive law* of $\mathbb{U}$ over $\mathbb{D}$ consists of a 1-cell $r : UD \to DU$ in $\mathbf{A}(\mathcal{K}, \mathcal{K})$, together with invertible 2-cells

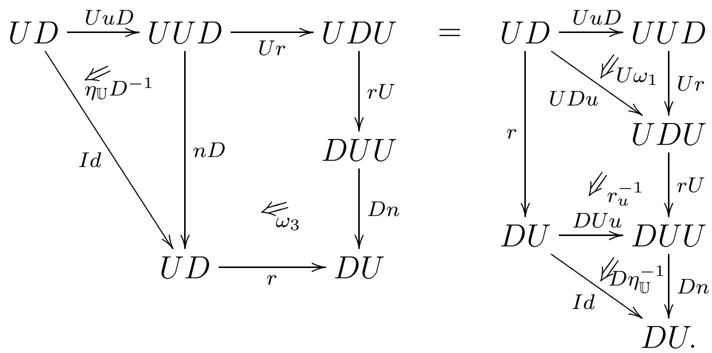


subject to the following coherence conditions:

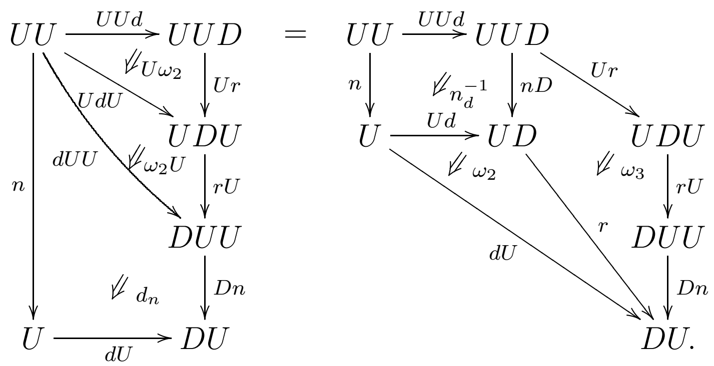
(coh 1)



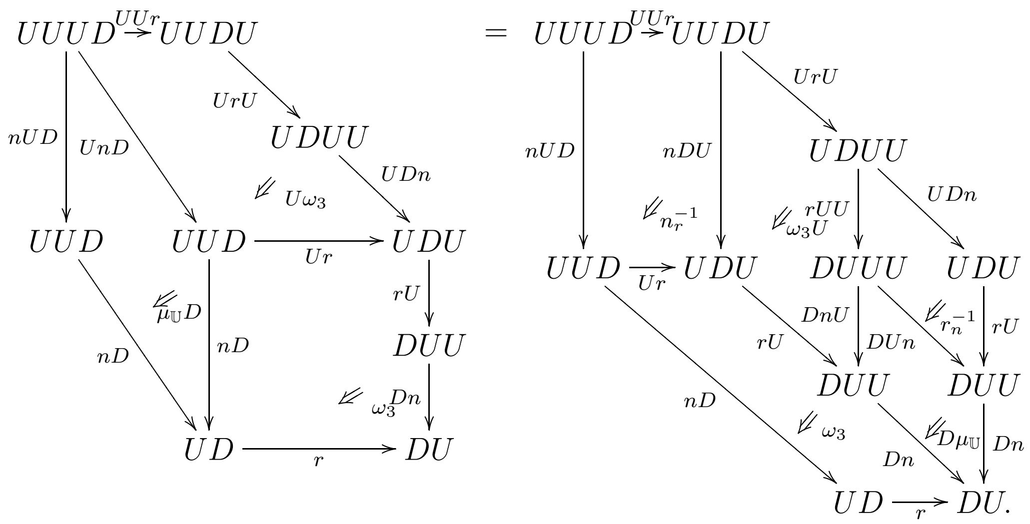
(coh 2)



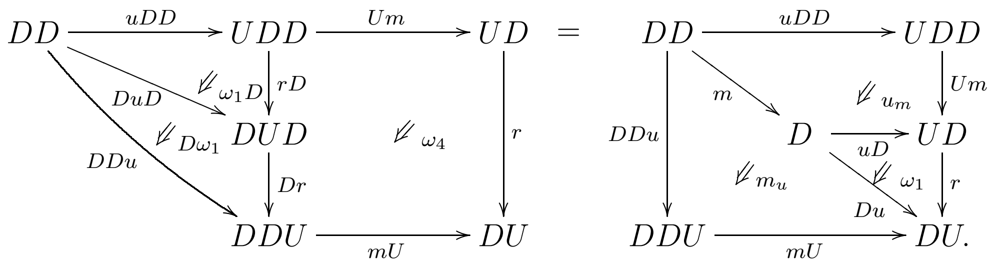
(coh 3)



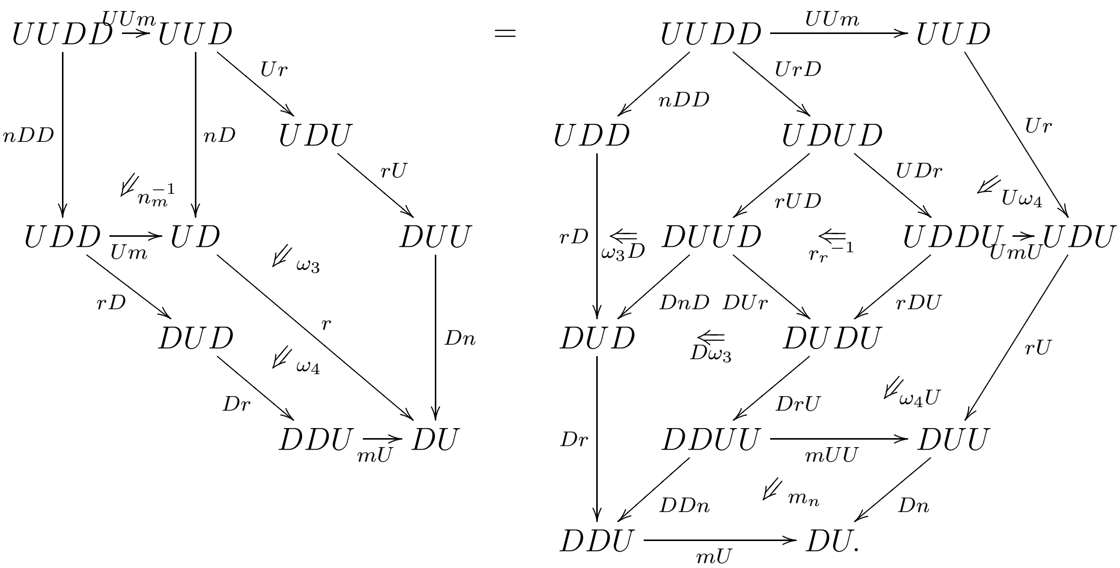
(coh 4)



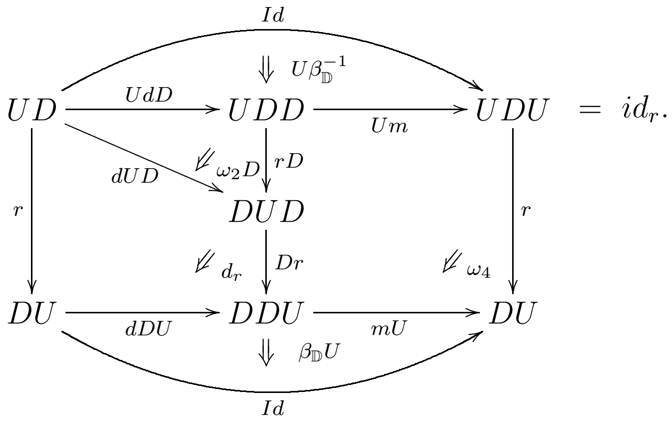
(coh 5)



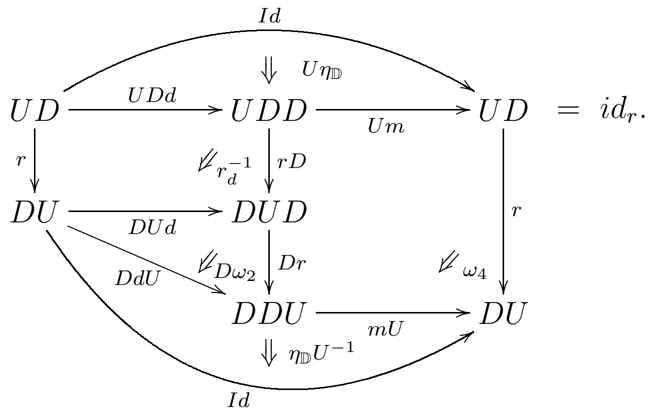
(coh 6)



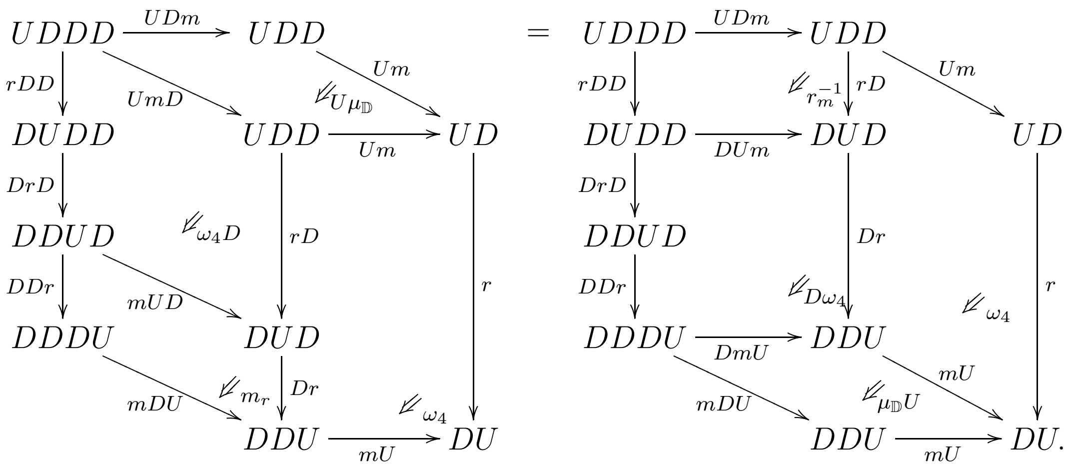
(coh 7)



(coh 8)



(coh 9)



Observe that if the pseudomonads are strict (β, η and μ are identities), and ω_1, ω_2 and ω_3 are identities, then we obtain the coherence conditions of the "mild" extensions of the classical distributive laws given in [7].

5. The composite pseudomonad given by a distributive law

Assume we have a distributive law of $\mathbb{U}$ over $\mathbb{D}$ as in section 4. The first question is how to produce a composite pseudomonad from $\mathbb{U}$, $\mathbb{D}$, and the distributive law. This is what we do in this section.

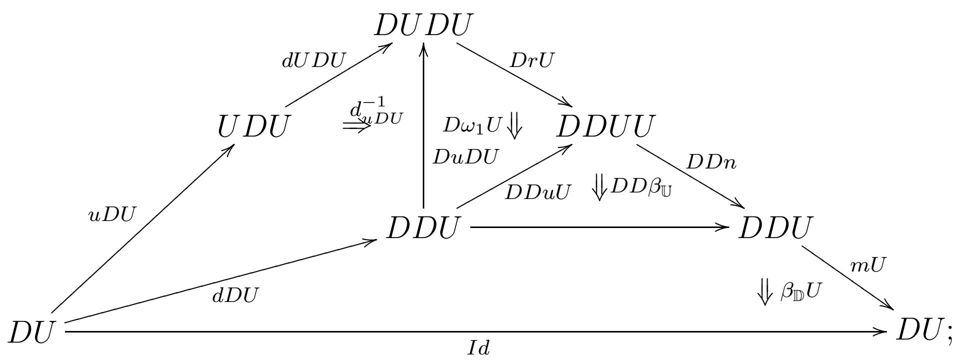
Define $\mathbb{V} = (V, v, p, \beta_{\mathbb{V}}, \eta_{\mathbb{V}}, \mu_{\mathbb{V}})$ as follows: $V = DU \in \mathbf{A}(\mathcal{K}, \mathcal{K})$; v is defined as the composite

$$1_{\mathcal{K}} \xrightarrow{u} U \xrightarrow{dU} DU;$$

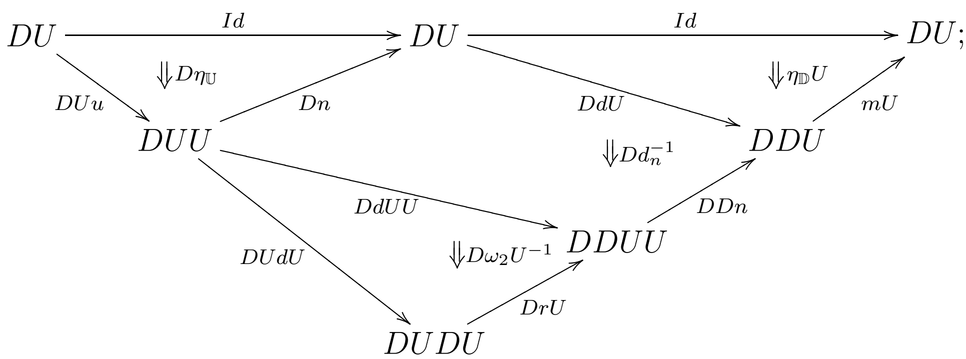
p is the composite

$$DUDU \xrightarrow{DrU} DDUU \xrightarrow{DDn} DDU \xrightarrow{mU} DU;$$

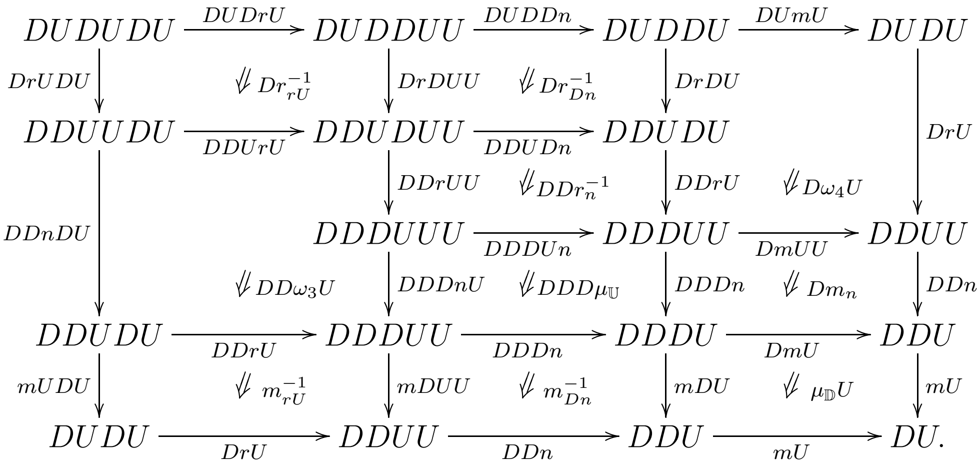
$\beta_{\mathbb{V}}$ is defined to be the pasting



$\eta_{\mathbb{V}}$ as the pasting

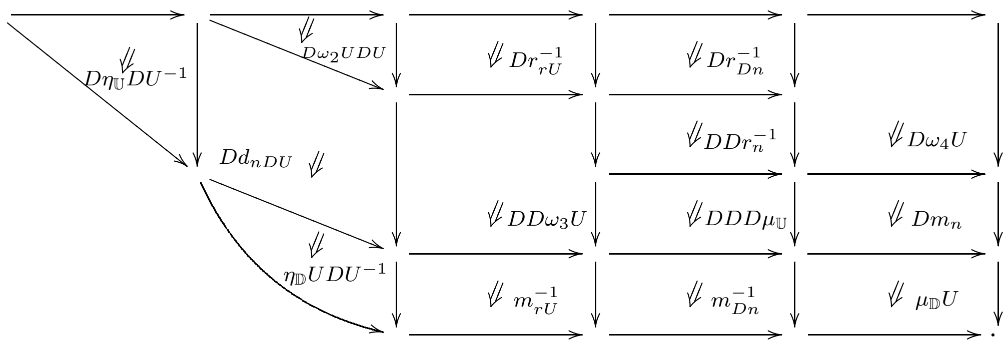


and $\mu_{\mathbb{V}}$ as the pasting

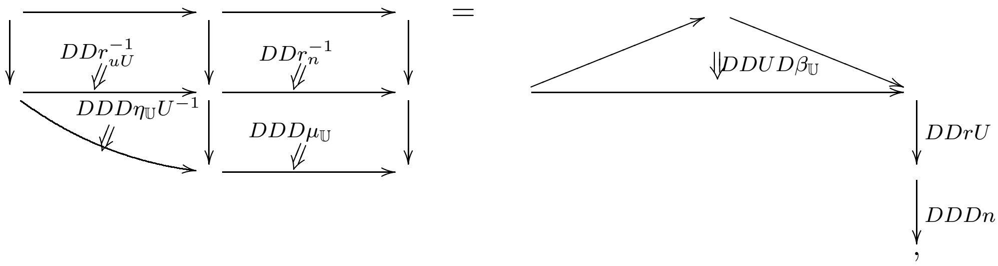


5.1. Theorem. $\mathbb{V} = (V, v, p, \beta_{\mathbb{V}}, \eta_{\mathbb{V}}, \mu_{\mathbb{V}})$, *as defined above, is a pseudomonad on the object* $\mathcal{K}$.

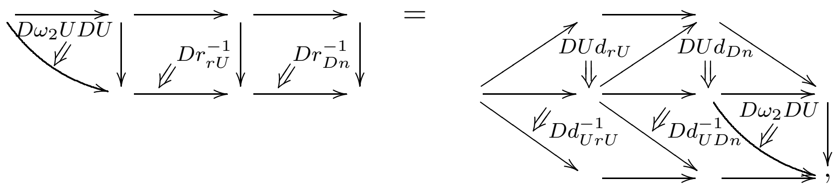
Proof. Observe that the pasting of $\mu_{\mathbb{V}}$ and $\eta_{\mathbb{V}}V^{-1}$ is



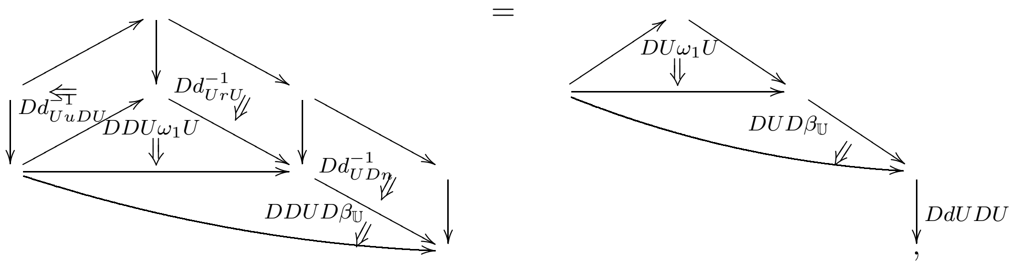
We must show that this pasting equals $p \circ V\beta_{\mathbb{V}}$. Substitute the pasting of Dd_{nDU} and $D\eta_{\mathbb{U}}DU^{-1}$ by the pasting of Dd^{-1}_{UuDU} and $DD\eta_{\mathbb{U}}DU$. Then use (coh 2). With the help of (2) show that



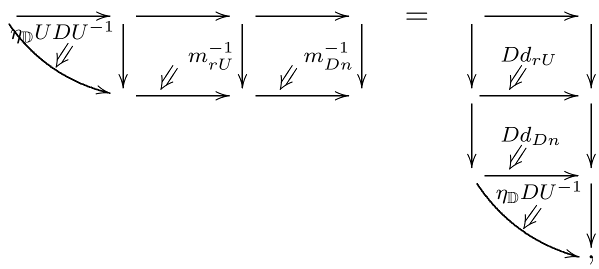
and make the substitution. Make the substitution



followed by the substitutions



and



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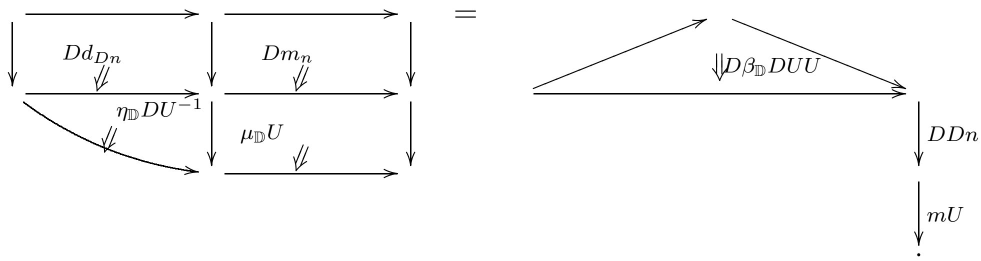
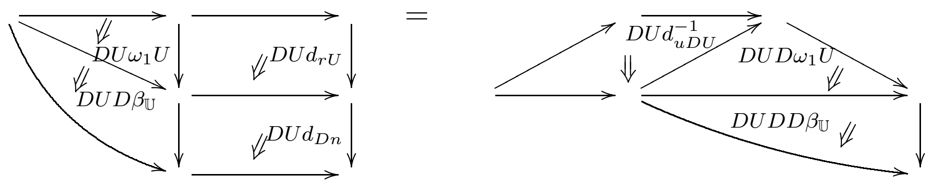
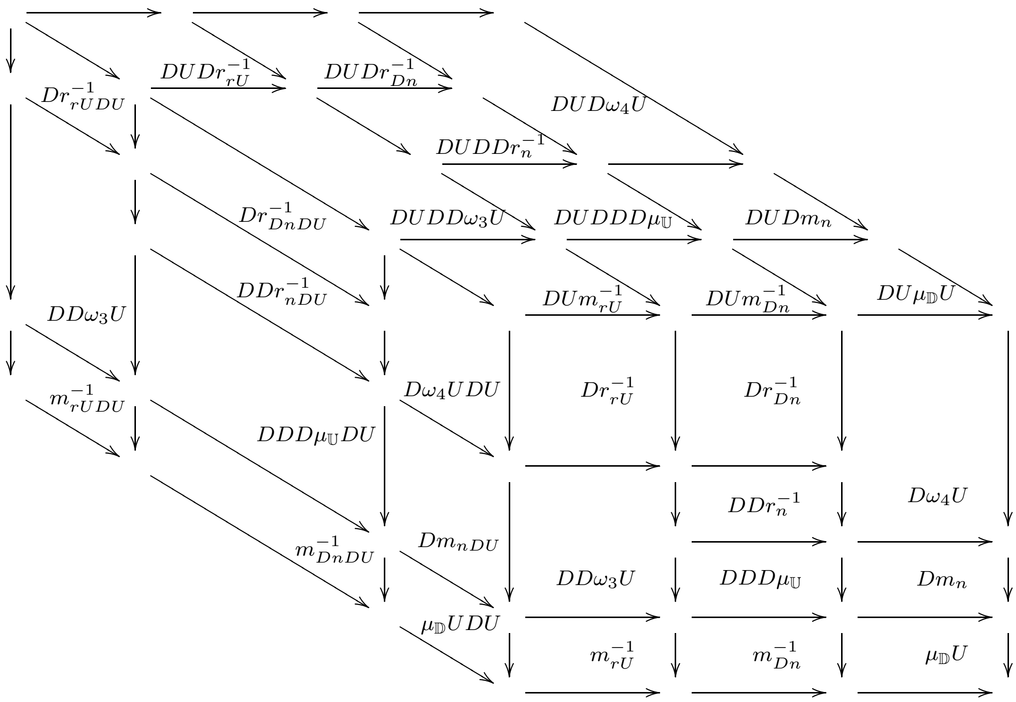


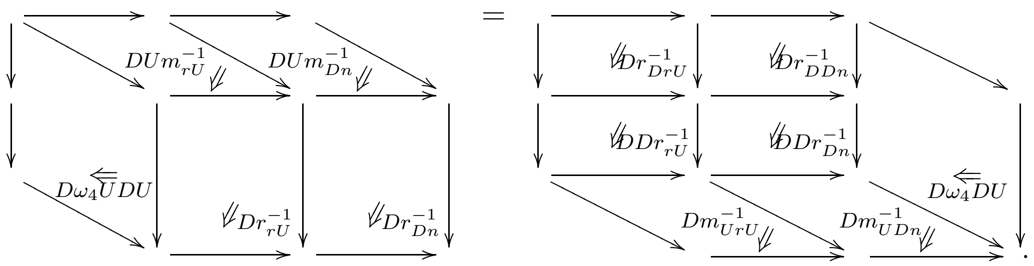
Use (coh 7), and finish with the substitution



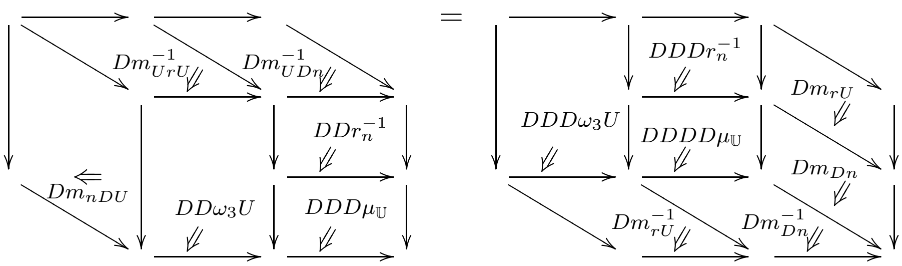
In regards to the other condition, observe that the pasting of $V\mu_{\mathbb{V}}$, $\mu_{\mathbb{V}}V$ and $\mu_{\mathbb{V}}$ is:



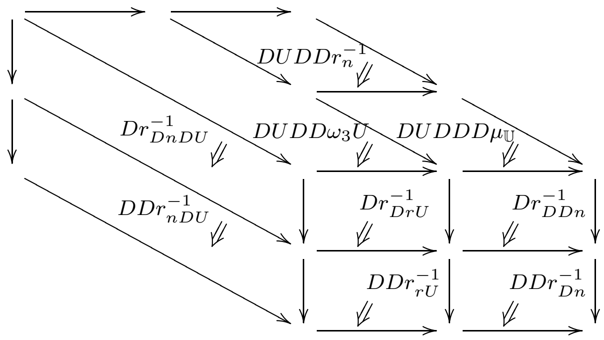
Where we have only put the name of the corresponding 2-cell in each parallelogram. To show that this pasting is equal to the pasting of p_p, $\mu_{\mathbb{V}}$ and $\mu_{\mathbb{V}}$ we do the following. First make the substitution



Then make the substitutions



and



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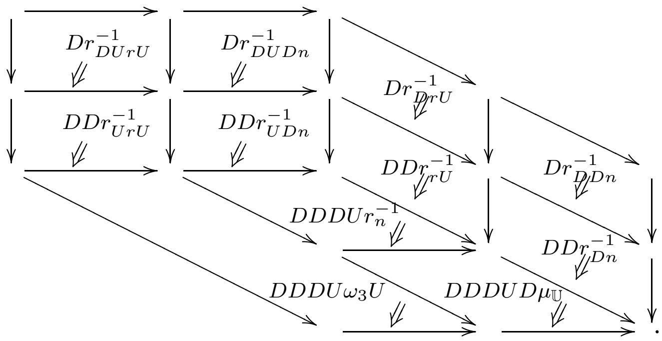
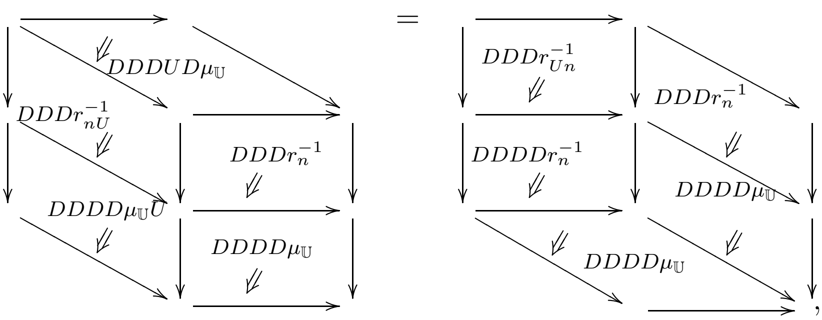


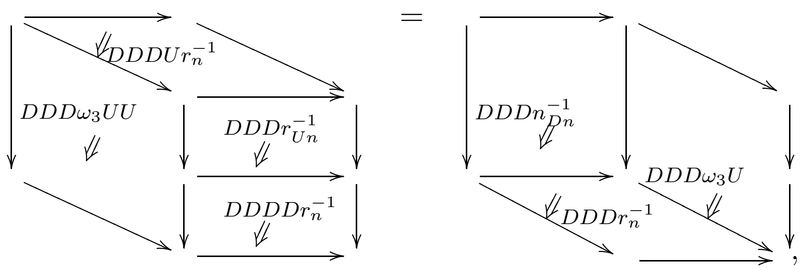
.

Now use (coh 4). Proceed with the following substitutions



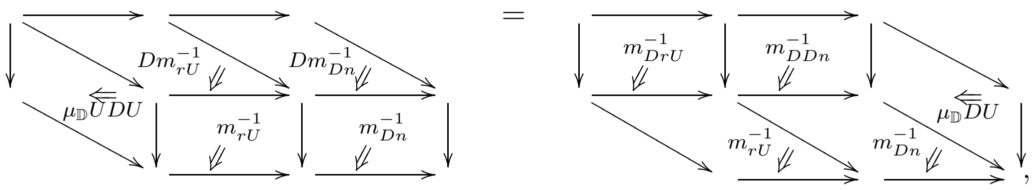
,

followed by

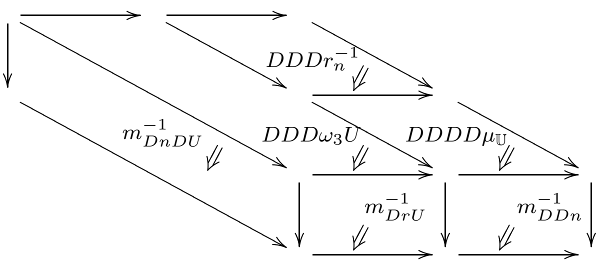


,

and then



followed by the substitution



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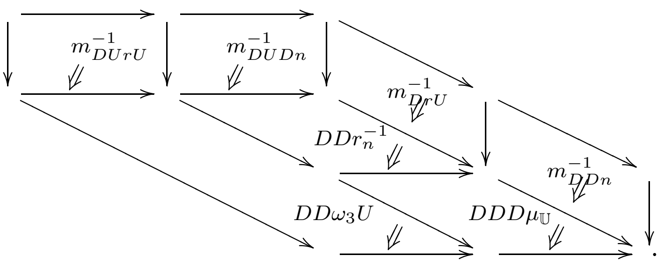
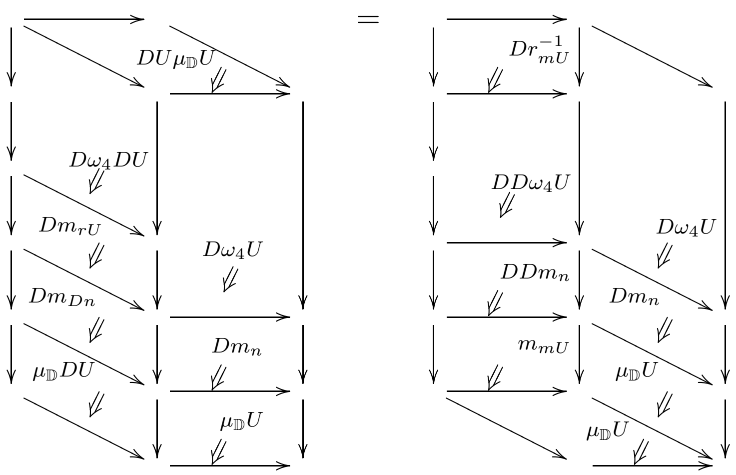
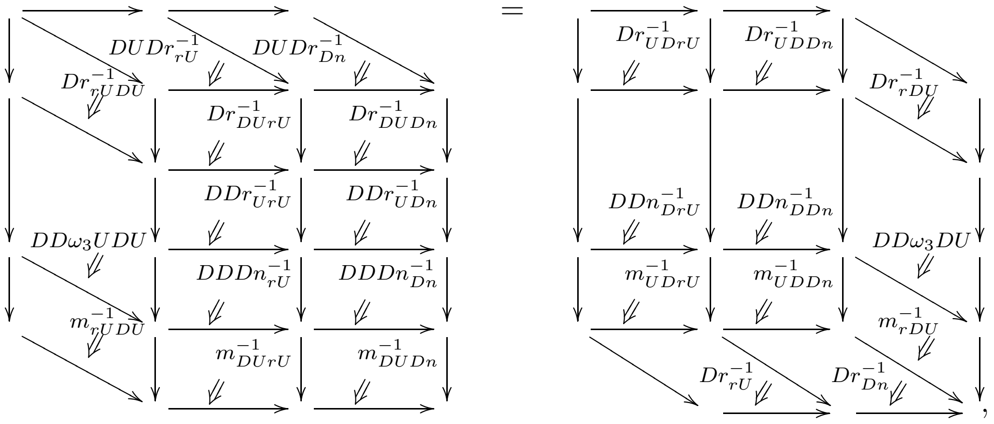


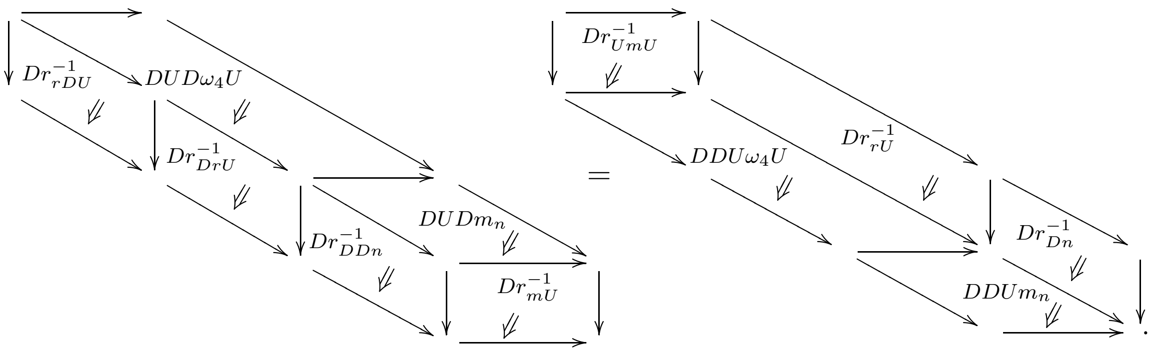
Use (coh 9) to show that



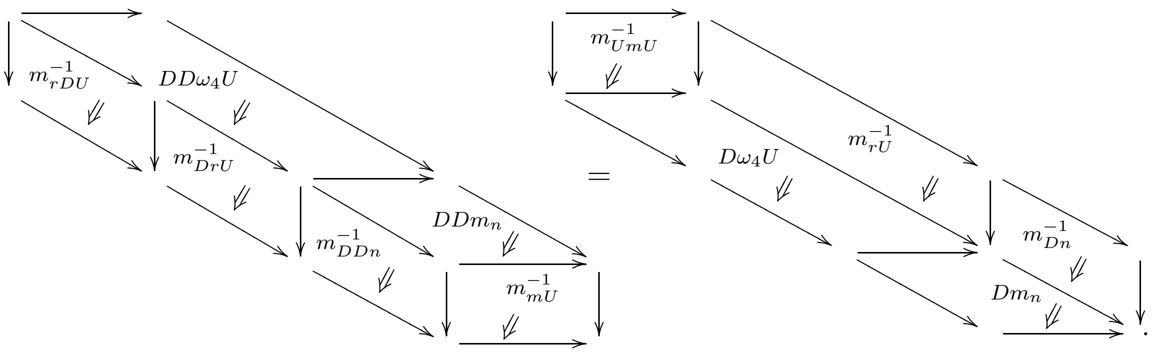
and make the substitution. Next the substitution



followed by the substitution



Substitute the pasting of $DDUm_n$, $DD\omega_4U$, DDr^{-1}_{Dn} and $DDDr^{-1}_n$ by the pasting of $DD\omega_4UU$, DDr^{-1}_n and DDm_{Un}. Then observe that the pasting of DDm_{Un}, DDm_n and $DDDD\mu_{\mathbb{U}}$ equals the pasting of $DDD\mu_{\mathbb{U}}$, DDm_{nU} and DDm_n. Use (coh 6). To finish the proof, make the substitution

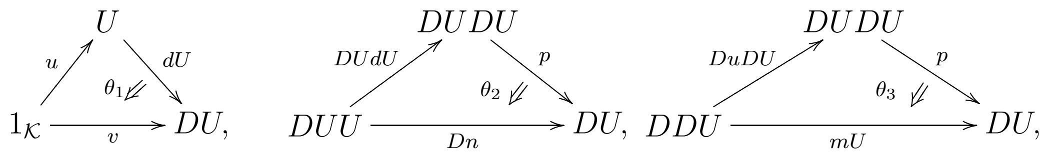


■

6. Compatible pseudomonad structures

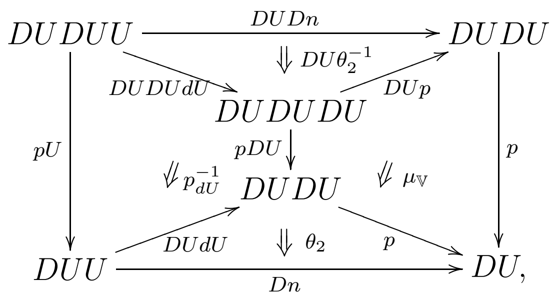
We consider now the question of when can a pseudomonad be considered as the composite of two pseudomonads.

Let $\mathbb{D}$, $\mathbb{U}$ be pseudomonads on the same object $\mathcal{K}$ of the Gray-category $\mathbf{A}$. Given another pseudomonad $\mathbb{V} = (V, v, p, \beta_{\mathbb{V}}, \eta_{\mathbb{V}}, \mu_{\mathbb{V}})$ on the same object $\mathcal{K}$, we say that $\mathbb{V}$ is *compatible* with the pseudomonads $\mathbb{D}$ and $\mathbb{U}$ if $V = DU$ and there are invertible 2-cells:



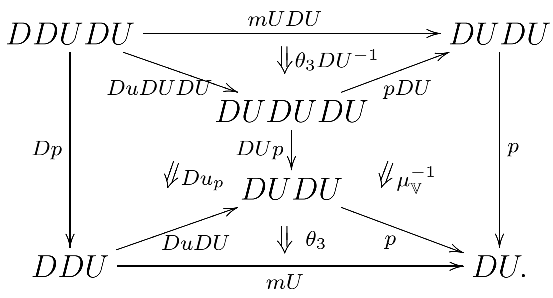
subject to coherence conditions. To describe the coherence conditions introduce the following pastings:

$\theta_4 \quad =$

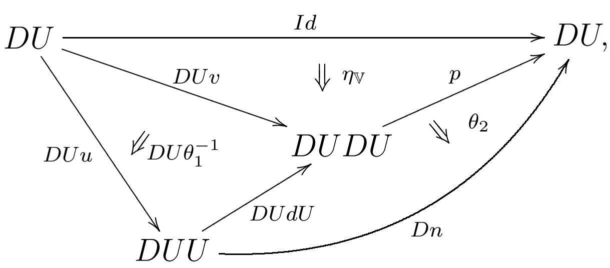


and

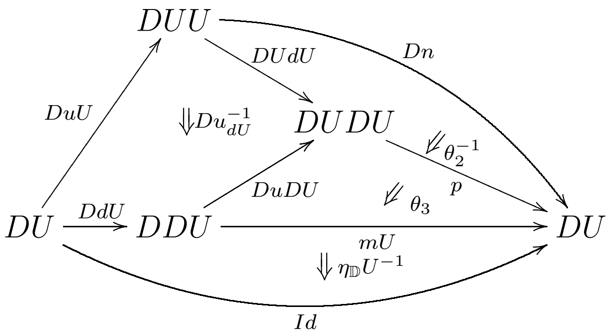
$\theta_5 \quad =$



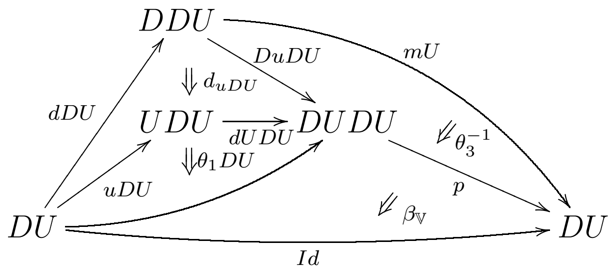
The coherence conditions are:



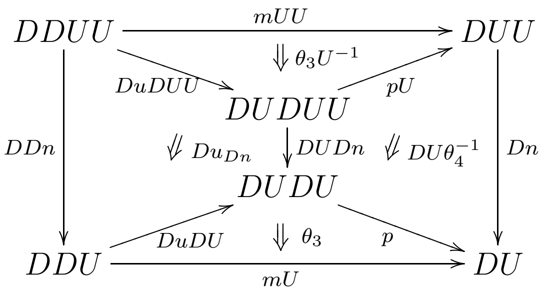
$= \quad D\eta_{\mathbb{U}},$ (12)



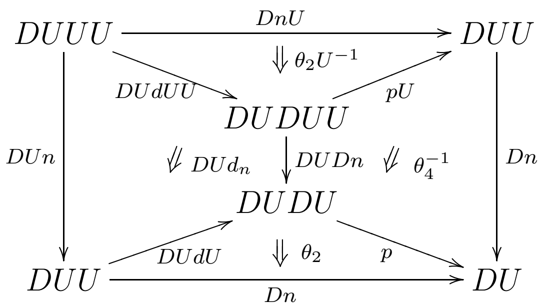
$= \quad D\beta_{\mathbb{U}},$ (13)



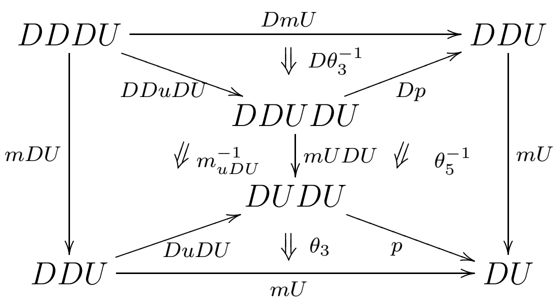
$= \quad \beta_{\mathbb{D}}U,$ (14)



$= \quad m_n$ (15)



$= \quad D\mu_{\mathbb{U}}^{-1}$ (16)



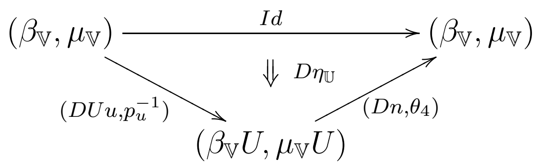
$= \quad \mu_{\mathbb{D}}U$ (17)

Some other equations we will need later are contained in the following:

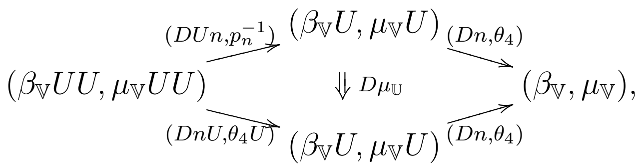
6.1. Proposition. *Assume we have invertible 2-cells θ_1, θ_2, and θ_3, satisfying the coherence conditions. Then*

$$(Dn, \theta_4) : (\beta_{\mathbb{V}}U, \mu_{\mathbb{V}}U) \to (\beta_{\mathbb{V}}, \mu_{\mathbb{V}}) \tag{18}$$

is a 1-cell in $\mathbb{V}$-$Alg_{\mathcal{K}}$, and the following are 2-cells in $\mathbb{V}$-$Alg_{\mathcal{K}}$:



(19)

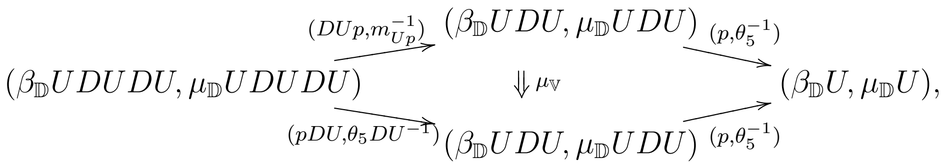


(20)

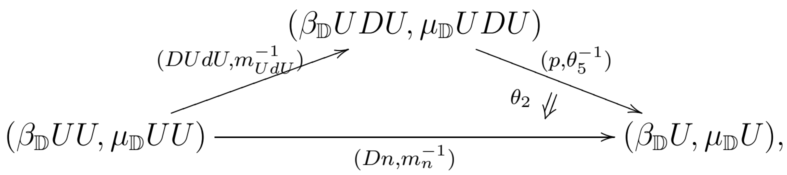
We also have that

$$(p, \theta_5^{-1}) : (\beta_{\mathbb{D}}UDU, \mu_{\mathbb{D}}UDU) \to (\beta_{\mathbb{D}}U, \mu_{\mathbb{D}}U) \tag{21}$$

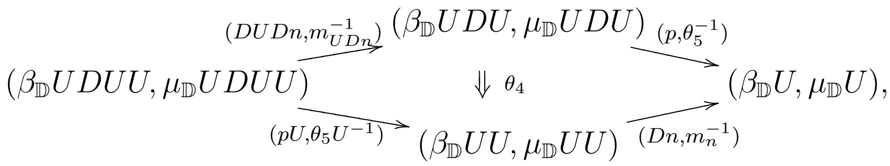
is a 1-cell in $\mathbb{D}$-$Alg_{\mathcal{K}}$, and the following are 2-cells in $\mathbb{D}$-$Alg_{\mathcal{K}}$:



(22)

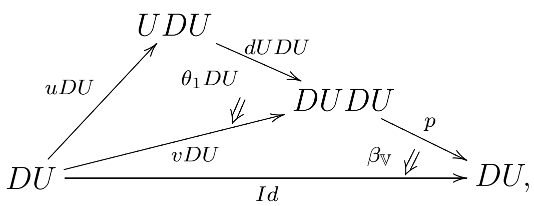


(23)

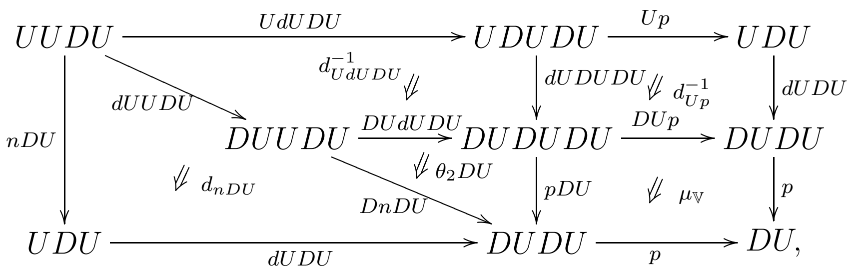


(24)

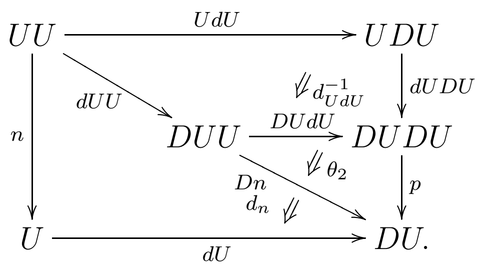
Furthermore, if we define σ_1 as,



σ_2 as

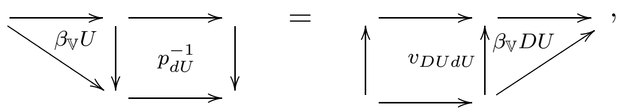


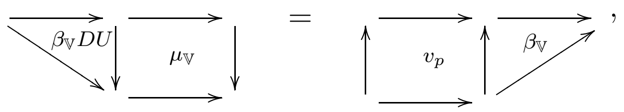
γ as



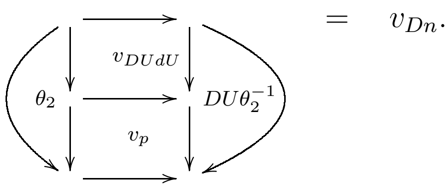
then (σ_1, σ_2) is an object in $\mathbb{U}$-$Alg_{\mathcal{K}}$, and $(dU, \gamma) : (\beta_{\mathbb{U}}, \mu_{\mathbb{U}}) \to (\sigma_1, \sigma_2)$ is a 1-cell in $\mathbb{U}$-$Alg_{\mathcal{K}}$.

Proof. We show first that (Dn, θ_4) is a 1-cell in $\mathbb{V}$-Alg$_{\mathcal{K}}$. To show that (Dn, θ_4) satisfies (9), start on the left hand side and make the substitutions:

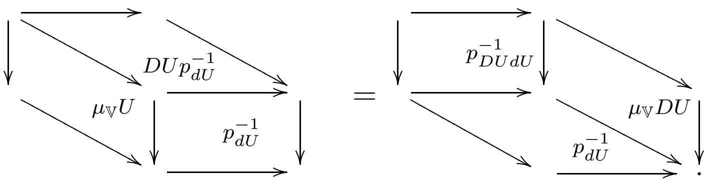




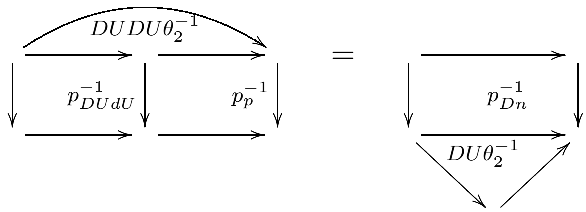
and



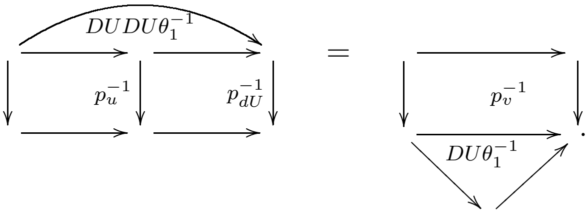
To show that (Dn, θ_4) satisfies (10), start on the left hand side, cancel $DU\theta_2$ and its inverse, make the substitution



Then use (1), and conclude with

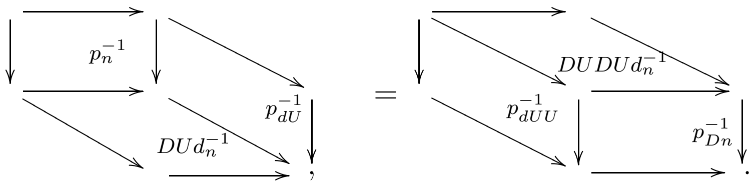


Next we show that $D\eta_{\mathbb{U}}$ in (19) is a 2-cell in $\mathbb{V}$-Alg$_{\mathcal{K}}$. To show that satisfies (11), start on the left hand side of (11). Use (12) to substitute the pasting of $DUD\eta_{\mathbb{U}}$ and $DU\theta_2^{-1}$ for the pasting of $DU\beta_{\mathbb{V}}$ and $DUDU\theta_1^{-1}$. Then make the substitution



Now use (5), and then use (12) once again.

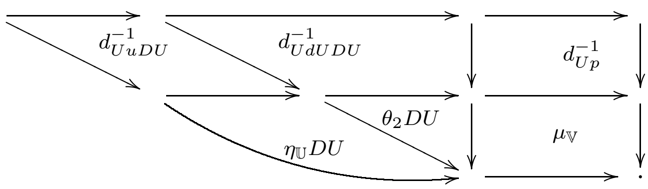
Next we show that $D\mu_{\mathbb{U}}$ in (20) is a 2-cell of $\mathbb{V}$-algebras. Start on the right hand side of (11). Use (16) to substitute the pasting of θ_2 and $D\mu_{\mathbb{U}}$, by the pasting of DUd_n^{-1}, $DU\theta_4$ and $\theta_2 U$. Make the substitution



Since we already know that (Dn, θ_4) is a $\mathbb{V}$-algebra morphism, we can substitute the pasting of $\mu_{\mathbb{V}}$, p_{Dn}^{-1}, and θ_4, by the pasting of $DU\theta_4$, θ_4 and $\mu_{\mathbb{V}}$. By the definition of θ_4, we can substitute the pasting of $\mu_{\mathbb{V}}U$, p_{dUU}^{-1} and $\theta_2 U$ by the pasting of $DU\theta_2 U$ and $\theta_4 U$. Finally, use (16).

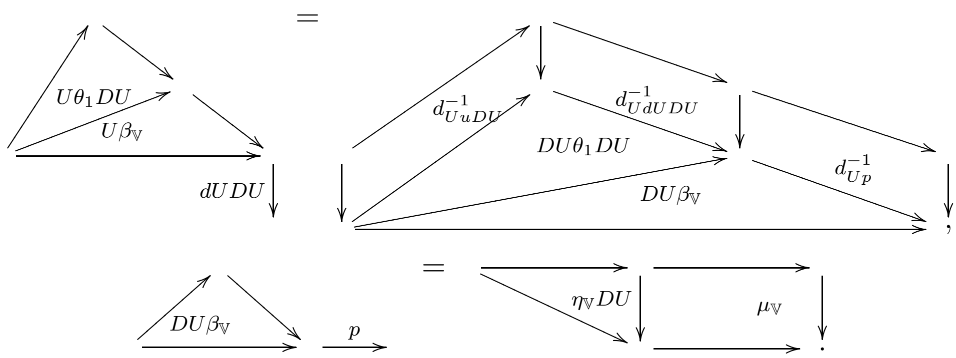
That (p, θ_5^{-1}) is a morphism of $\mathbb{D}$-algebras, and that $\mu_{\mathbb{V}}$, θ_2, and θ_4 are 2-cells of $\mathbb{D}$-algebras are similar and left to the reader.

Now we show that (σ_1, σ_2) is an object in $\mathbb{U}$-Alg$_{\mathcal{K}}$. Paste $\eta_{\mathbb{U}}DU^{-1}$ onto the left hand side of (7), and make a substitution on it to obtain



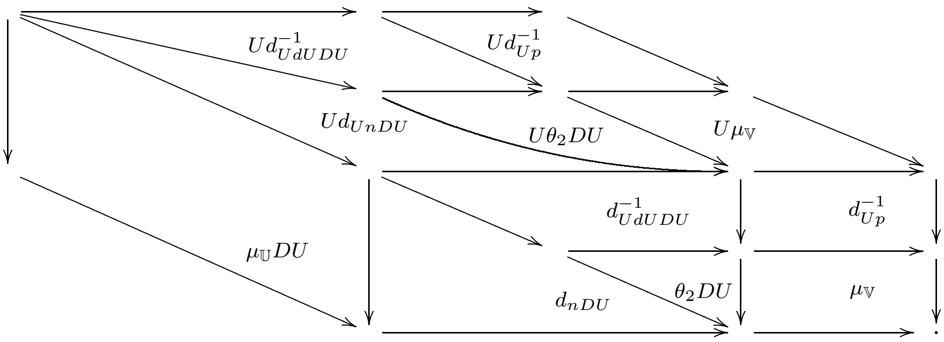
(25)

Also paste $\eta_{\mathbb{U}}DU^{-1}$ on what turns out to be the right hand side of (7), and make the following two substitutions:

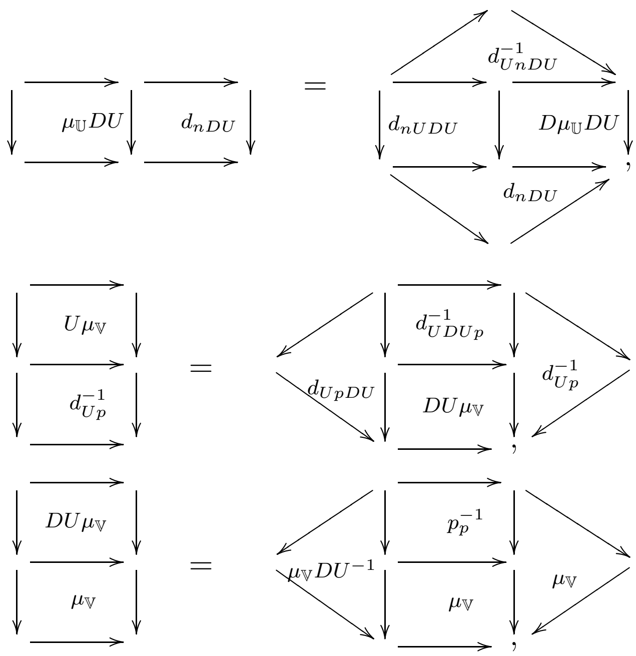


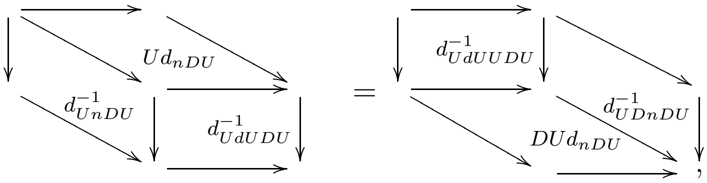
Compare the pasting we arrive at with (25), and use equation (12).

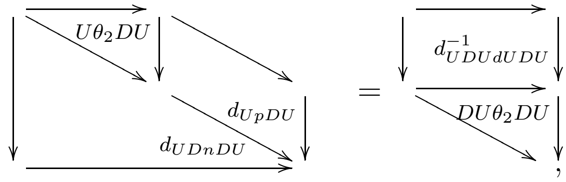
Now, the left hand side of (6) in this case, is the following pasting:



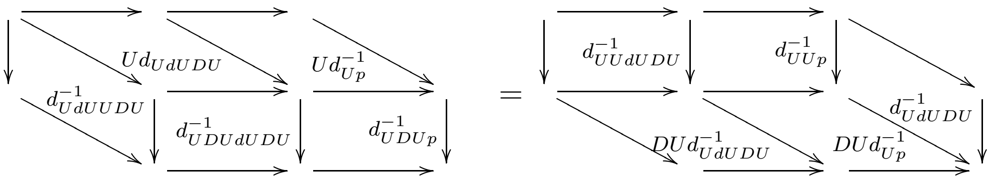
Make the following sequence of substitutions:



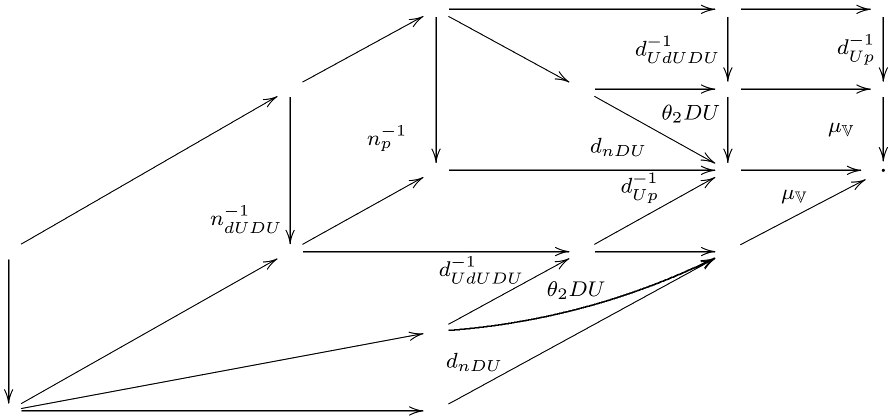




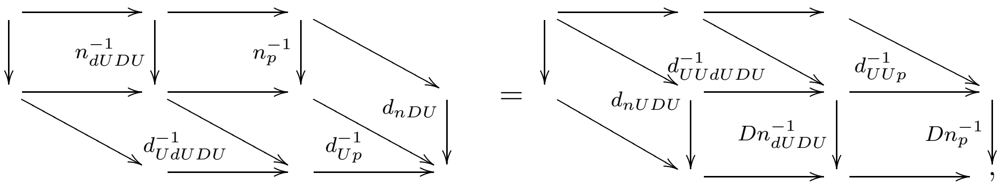
and

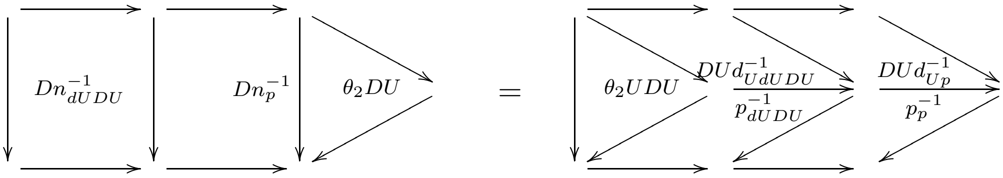


On the other hand, the right hand side of (6) in this case is the pasting

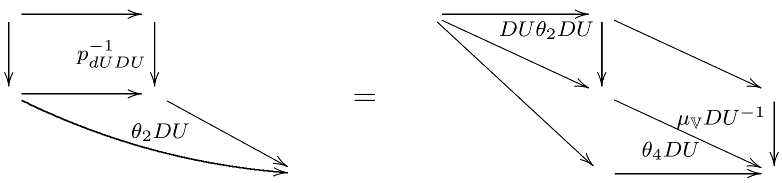


On this pasting make the following substitutions:





and



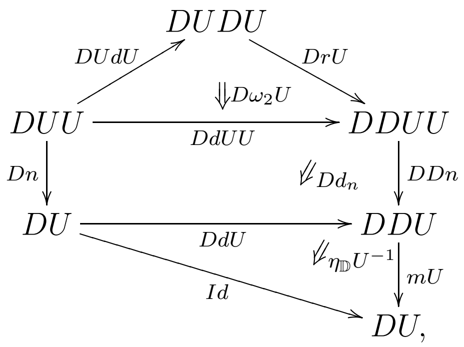
(definition of θ_4).

Compare both results, and use equation (16).

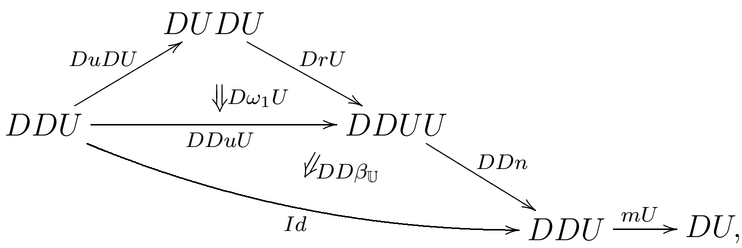
The claim about (dU, γ) is left to the reader. ■

Next we prove that the composite of two pseudomonads via a distributive law as defined in section 5, is compatible.

6.2. Theorem. *Assume we have a distributive law as in section 4 and let $\mathbb{V}$ the composite pseudomonad defined in section 5. If we define $\theta_1 = id_{dU \circ u}$, and θ_2 as the pasting*



and θ_3 as the pasting

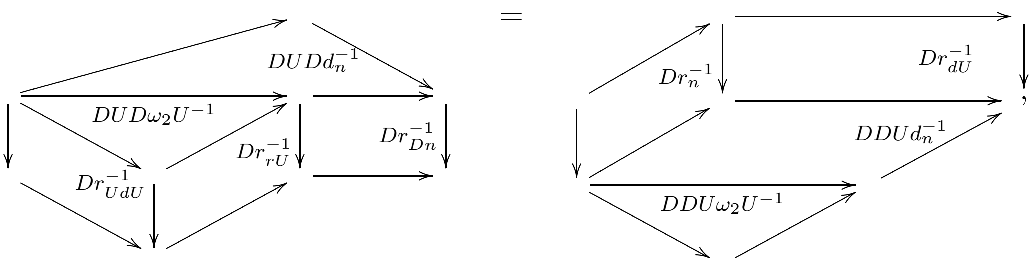


then the pseudomonad $\mathbb{V}$ is compatible with $\mathbb{U}$ and $\mathbb{D}$.

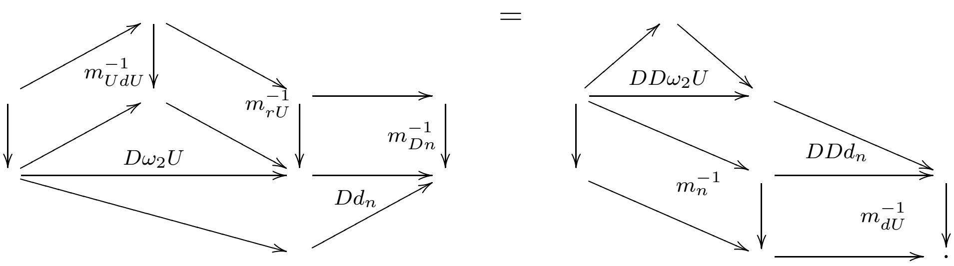
Proof. Conditions (12), (13), and (14) are fairly easy and left to the reader. The remaining ones are also easy once we have shown that

$$\theta_4 \quad = \quad \begin{array}{ccc} DUDUU & \xrightarrow{DUDn} & DUDU \\ {\scriptstyle DrUU}\downarrow & \Downarrow Dr_n^{-1} & \downarrow{\scriptstyle DrU} \\ DDUUU & \xrightarrow[DDUn]{} & DDUU \\ {\scriptstyle DDnU}\downarrow & \Downarrow DD\mu_{\mathbb{U}} & \downarrow{\scriptstyle DDn} \\ DDUU & \xrightarrow[DDn]{} & DDU \\ {\scriptstyle mUU}\downarrow & \Downarrow m_n^{-1} & \downarrow{\scriptstyle mU} \\ DUU & \xrightarrow[Dn]{} & DU, \end{array} \qquad \text{and} \qquad \theta_5 \quad = \quad \begin{array}{ccc} DDUDU & \xrightarrow{mUDU} & DUDU \\ {\scriptstyle DDrU}\downarrow & \Downarrow m_{rU} & \downarrow{\scriptstyle DrU} \\ DDDUU & \xrightarrow[mDUU]{} & DDUU \\ {\scriptstyle DDDn}\downarrow & \Downarrow m_{Dn} & \downarrow{\scriptstyle DDn} \\ DDDU & \xrightarrow[mDU]{} & DDU \\ {\scriptstyle DmU}\downarrow & \Downarrow \mu_{\mathbb{D}}U & \downarrow{\scriptstyle mU} \\ DDU & \xrightarrow[mU]{} & DU. \end{array}$$

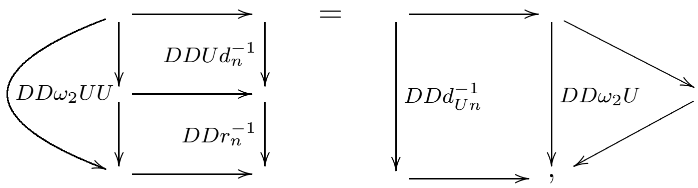
To show the first equality above, start with the definition of θ_4 and make the following substitutions:

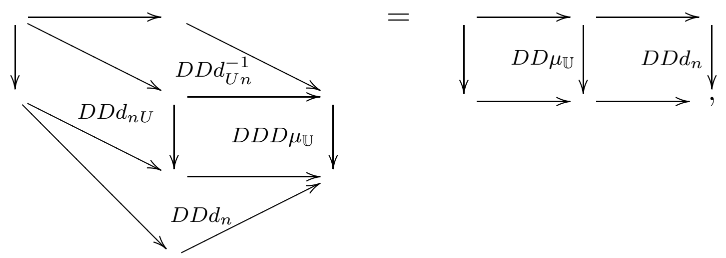


and

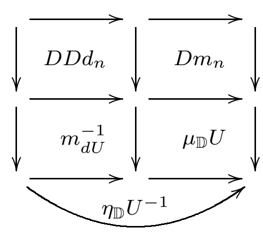


Now use (coh 3) on the pasting of $DDU\omega_2U^{-1}$, $DD\omega_3U$, DDn_{dU}^{-1} and $DD\omega_2U$. Next make the substitutions:





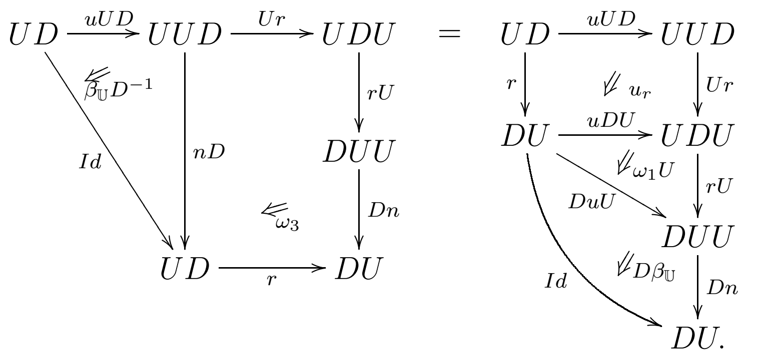
and



$$= mU \circ DDn \circ D\eta_{\mathbb{D}}UU^{-1}.$$

Finally, use (coh 8).

The proof for θ_5 is very similar, except that we need the equation:



To prove this last equation, observe that Id_{DU} is isomorphic to $Dn \circ rU \circ uDU$, thus it suffices to show that the equation holds when followed by $Dn \circ rU \circ uDU$. And, since u_r and ω_3 are invertible, we can paste them below on both sides of the equation. This last equation is not hard to prove, using (coh 4) and (coh 2). The rest of the proof is left to the reader. ■

7. The Gray-category of pseudomonads in a Gray-category

In this section we define the Gray-category PSM($\mathbf{A}$) of pseudomonads on a Gray-category $\mathbf{A}$.

The objects of PSM($\mathbf{A}$) are pseudomonads in $\mathbf{A}$.

Given pseudomonads $\mathbb{D}$ on $\mathcal{K}$, and $\mathbb{U}$ on $\mathcal{L}$, we denote the 2-category PSM($\mathbf{A}$)($\mathbb{D}, \mathbb{U}$) as $[\mathbb{D}, \mathbb{U}]$, and we define it as follows. The objects of $[\mathbb{D}, \mathbb{U}]$ are pairs (G, G_0), where $G : \mathbb{U}\text{-Alg} \to \mathbb{D}\text{-Alg} : \mathbf{A}^{op} \to \mathsf{Gray}$ is a Gray-natural transformation, $G_0 \in \mathbf{A}(\mathcal{L}, \mathcal{K})$, such that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg} & \xrightarrow{G} & \mathbb{D}\text{-Alg} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbf{A}(\_,\mathcal{L}) & \xrightarrow[G_0(\_)]{} & \mathbf{A}(\_,\mathcal{K}) \end{array}$$

commutes, where Φ is the forgetful Gray-natural transformation defined in section 3. We can consider $\mathbb{U}$-Alg and $\mathbb{D}$-Alg as trihomomorphisms and G as a tritransformation. A 1-cell $(G, G_0) \to (H, H_0)$ in $[\mathbb{D}, \mathbb{U}]$ is a pair (g, g_0), where $g : G \to H$ is a strict trimodification, and $g_0 : G_0 \to H_0$ is in $\mathbf{A}(\mathcal{L}, \mathcal{K})$, such that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg} \underset{H}{\overset{G}{\rightrightarrows}} \Downarrow g \ \mathbb{D}\text{-Alg} & = & \mathbb{U}\text{-Alg} \\ \downarrow{\scriptstyle\Phi} & & {\scriptstyle\Phi}\downarrow \\ \mathbf{A}(\_,\mathcal{K}) & & \mathbf{A}(\_,\mathcal{L}) \underset{H_0(\_)}{\overset{G_0(\_)}{\rightrightarrows}} \Downarrow g_0(\_) \ \mathbf{A}(\_,\mathcal{K}). \end{array}$$

What we mean by strict trimodification is, a trimodification in which all the invertible modifications required in the definition ([6],pg. 25) are identities. A 2-cell $(g, g_0) \to (h, h_0)$ in $[\mathbb{D}, \mathbb{U}]$ is a pair (γ, γ_0), where $\gamma : g \to h$ is a perturbation, and $\gamma_0 : g_0 \to h_0$ is in $\mathbf{A}(\mathcal{L}, \mathcal{K})$, such that

$$\begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{X}} \ g_{\mathcal{X}} \Downarrow \xrightarrow{\gamma_{\mathcal{X}}} \Downarrow g'_{\mathcal{X}} \ \mathbb{D}\text{-Alg}_{\mathcal{X}} & & \mathbb{U}\text{-Alg}_{\mathcal{X}} \\ \downarrow{\scriptstyle\Phi_{\mathcal{X}}} & = & \downarrow{\scriptstyle\Phi_{\mathcal{Z}}} \\ \mathbf{A}(\mathcal{X},\mathcal{K}) & & \mathbf{A}(\mathcal{Z},\mathcal{L}) \ g_0(\_) \Downarrow \xrightarrow{\gamma_0(\_)} \Downarrow h_0(\_) \ \mathbf{A}(\mathcal{X},\mathcal{K}). \end{array}$$

Vertical and horizontal compositions are the obvious ones. The rest of the operations are defined as follows:

$$\begin{array}{ll} (H, H_0)(G, G_0) = (HG, H_0G_0) & \\ (H, H_0)(g, g_0) = (Hg, H_0g_0) & (h, h_0)(G, G_0) = (hG, h_0G_0) \\ (H, H_0)(\sigma, \sigma_0) = (H\sigma, H_0\sigma_0) & (\tau, \tau_0)(G, G_0) = (\tau G, \tau_0G_0) \end{array}$$

8. Liftings

8.1. Definition. *A lifting is a pseudomonad in the* Gray*-category* PSM($\mathbf{A}$).

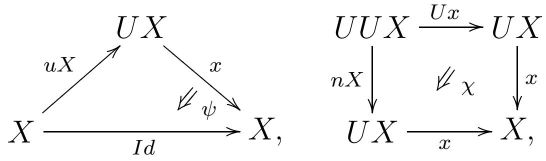
In more detail, observe that a pseudomonad $\widetilde{\mathbb{D}}$ in PSM($\mathbf{A}$) has to have domain a pseudomonad $\mathbb{U} = (U, u, n, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \mu_{\mathbb{U}})$ in $\mathbf{A}$, with domain some object $\mathcal{K}$ of $\mathbf{A}$. Observe that $\widetilde{\mathbb{D}}$ is of the form

$$\widetilde{\mathbb{D}} = ((\widetilde{D}, D), (\widetilde{d}, d), (\widetilde{m}, m), (\widetilde{\beta}, \beta), (\widetilde{\eta}, \eta), (\widetilde{\mu}, \mu)).$$

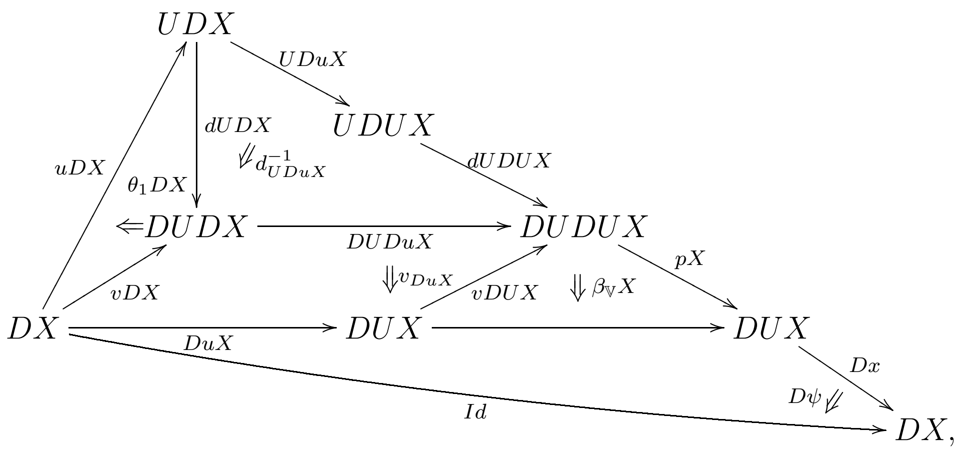
Taking the second entries, we obtain a pseudomonad $\mathbb{D} = (D, d, m, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \mu_{\mathbb{D}})$ in $\mathbf{A}$ on the same object $\mathcal{K}$. Furthermore, given an object $\mathcal{X}$ of $\mathbf{A}$, we can define $D_{\mathcal{X}} = \widetilde{D}_{\mathcal{X}}$, and $d_{\mathcal{X}} = \widetilde{d}_{\mathcal{X}}$ etcetera, to obtain a pseudomonad $\mathbb{D}_{\mathcal{X}} = (D_{\mathcal{X}}, d_{\mathcal{X}}, m_{\mathcal{X}}, \beta_{\mathcal{X}}, \eta_{\mathcal{X}}, \mu_{\mathcal{X}})$ in Gray, on the 2-category $\mathbb{U}\text{-Alg}_{\mathcal{X}}$. $\mathbb{D}$ and the family $\langle \mathbb{D}_{\mathcal{X}} \rangle_{\mathcal{X}}$ behave well with forgetful 2-functors and change of base.

9. From a pseudomonad with compatible structure to a pseudomonad in PSM(A)

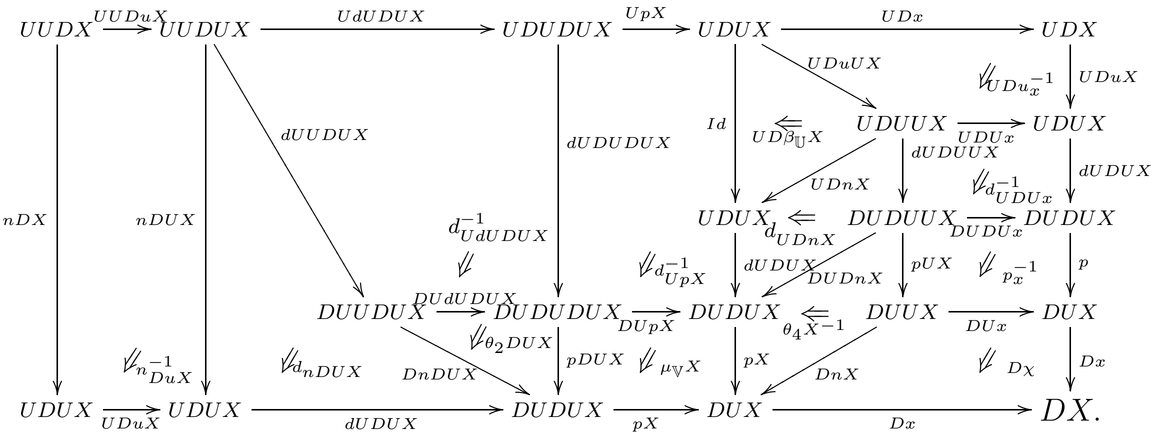
Assume we have a pseudomonad $\mathbb{V}$ that is compatible with pseudomonads $\mathbb{D}$ and $\mathbb{U}$ as in section 6. We want to define a pseudomonad $\widetilde{\mathbb{D}}$ on the object $\mathbb{U}$ of PSM($\mathbf{A}$). Let $\mathcal{X}$ be an object of $\mathbf{A}$. We begin by defining a 2-functor $D_{\mathcal{X}} : \mathbb{U}\text{-Alg}_{\mathcal{X}} \to \mathbb{U}\text{-Alg}_{\mathcal{X}}$. Given an object (ψ, χ) in $\mathbb{U}\text{-Alg}_{\mathcal{X}}$, with



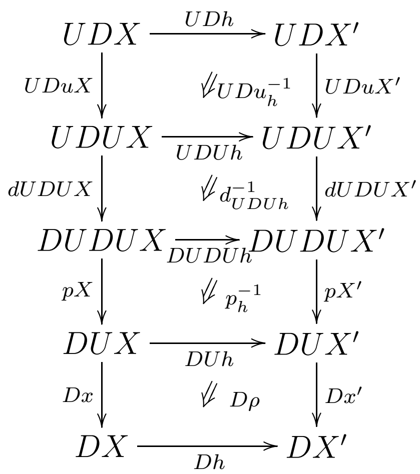
Define the first entry of $\mathbb{D}_{\mathcal{X}}(\psi, \chi)$ as



and the second entry of $\mathbb{D}_{\mathcal{X}}(\psi, \chi)$ as



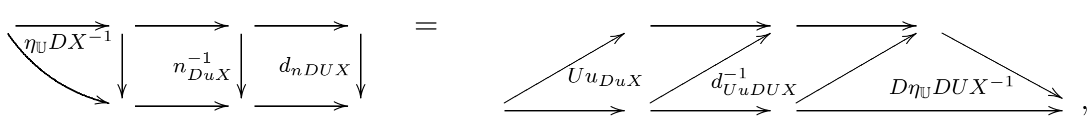
If $(h, \rho) : (\psi, \chi) \to (\psi', \chi')$ in $\mathbb{U}$-Alg$_{\mathcal{X}}$, then define the first entry of $D_{\mathcal{X}}(h, \rho)$ as Dh, and the second as the pasting

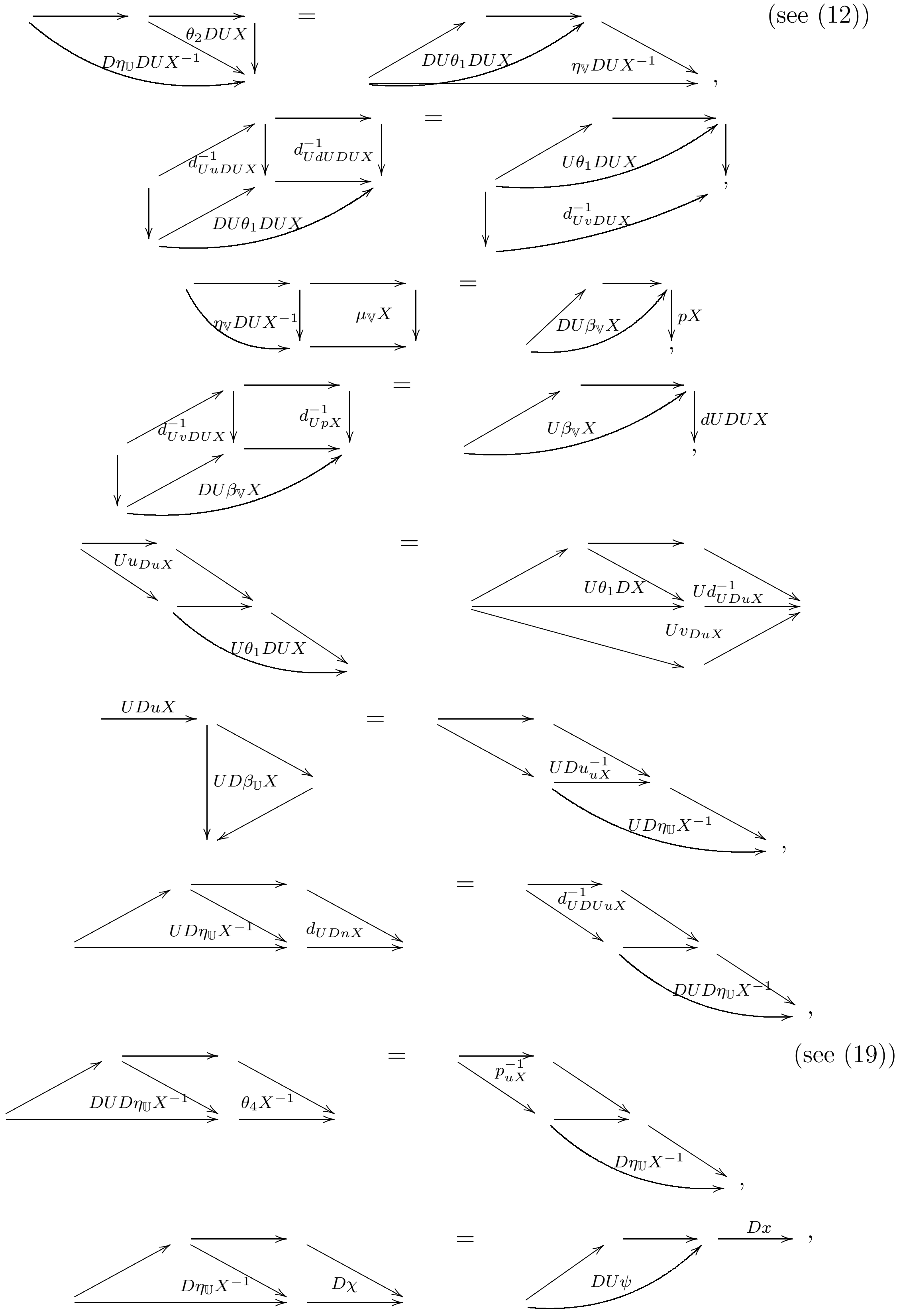


Given $\xi : (h, \rho) \to (h', \rho')$ in $\mathbb{U}$-Alg$_{\mathcal{X}}$, define $D_{\mathcal{X}}(\xi) = D\xi$.

9.1. Proposition. *The above definitions make $D_{\mathcal{X}} : \mathbb{U}\text{-}Alg_{\mathcal{X}} \to \mathbb{U}\text{-}Alg_{\mathcal{X}}$ a 2-functor. Furthermore, if we define $\widetilde{D} : \mathbb{U}\text{-}Alg \to \mathbb{U}\text{-}Alg$ as $D_{\mathcal{X}}$ at every object $\mathcal{X}$ of $\mathbf{A}$, then $\widetilde{D}$ is a* Gray*-transformation, and $(\widetilde{D}, D) \in [\mathbb{U}, \mathbb{U}]$.*

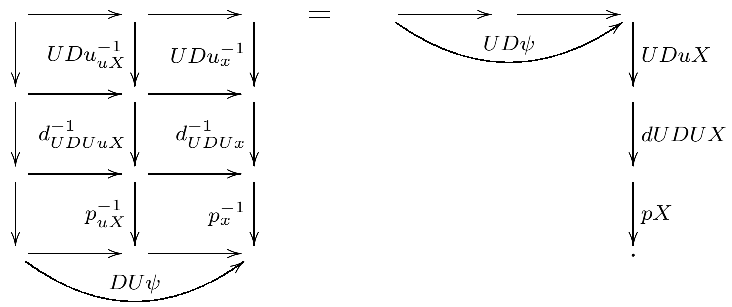
Proof. The hardest part of the proof is to show that $D_{\mathcal{X}}(\psi, \chi)$ is indeed an object in $\mathbb{U}$-Alg$_{\mathcal{X}}$. This is what we do, and we leave the rest to the reader. We must show that (6) and (7) are satisfied. To show (7), pass $\eta_{\mathbb{U}} DX$ to the left hand side, and perform the following sequence of substitutions:



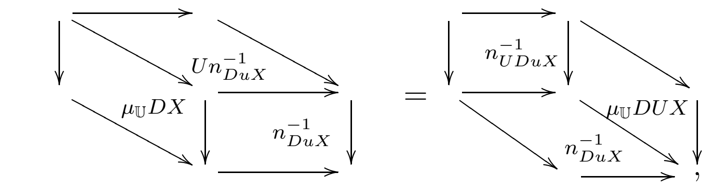


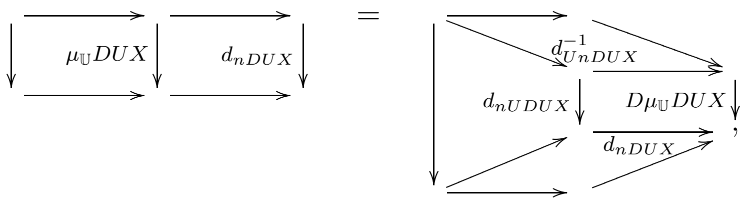
$\theta_2 DUX$
$D\eta_{\mathbb{U}} DUX^{-1}$
=
$DU\theta_1 DUX$
$\eta_{\mathbb{V}} DUX^{-1}$
(see (12))
d^{-1}_{UuDUX}
d^{-1}_{UdUDUX}
$DU\theta_1 DUX$
=
$U\theta_1 DUX$
d^{-1}_{UvDUX}
$\eta_{\mathbb{V}} DUX^{-1}$
$\mu_{\mathbb{V}} X$
=
$DU\beta_{\mathbb{V}} X$
pX
d^{-1}_{UvDUX}
d^{-1}_{UpX}
$DU\beta_{\mathbb{V}} X$
=
$U\beta_{\mathbb{V}} X$
$dUDUX$
Uu_{DuX}
$U\theta_1 DUX$
=
$U\theta_1 DX$
Ud^{-1}_{UDuX}
Uv_{DuX}
$UDuX$
$UD\beta_{\mathbb{U}} X$
=
UDu^{-1}_{uX}
$UD\eta_{\mathbb{U}} X^{-1}$
$UD\eta_{\mathbb{U}} X^{-1}$
d_{UDnX}
=
d^{-1}_{UDUuX}
$DUD\eta_{\mathbb{U}} X^{-1}$
$DUD\eta_{\mathbb{U}} X^{-1}$
$\theta_4 X^{-1}$
=
p^{-1}_{uX}
$D\eta_{\mathbb{U}} X^{-1}$
(see (19))
$D\eta_{\mathbb{U}} X^{-1}$
$D\chi$
=
$DU\psi$
Dx

and

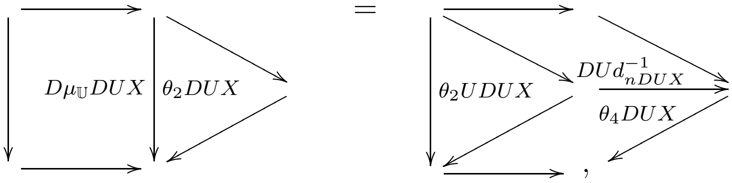


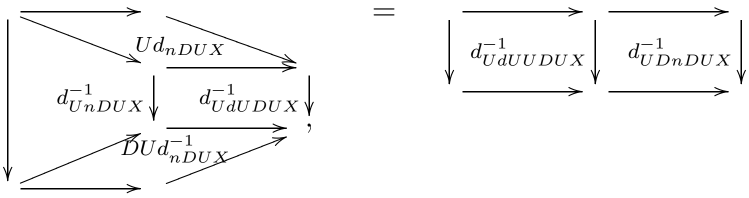
To get from the left hand side of (6) to the right hand side, make the following sequence of substitutions:

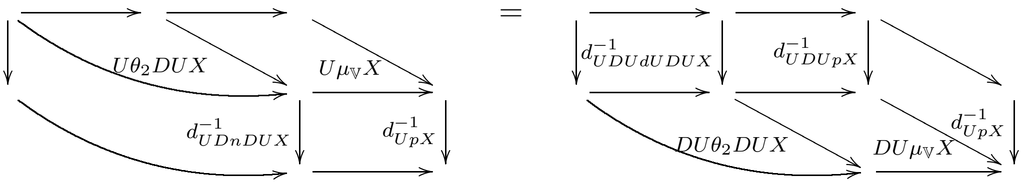




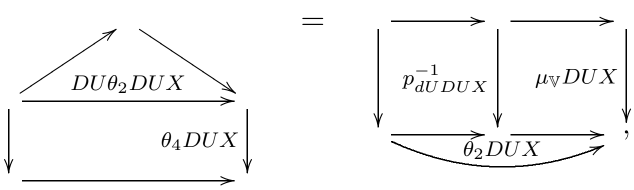
(see (16))

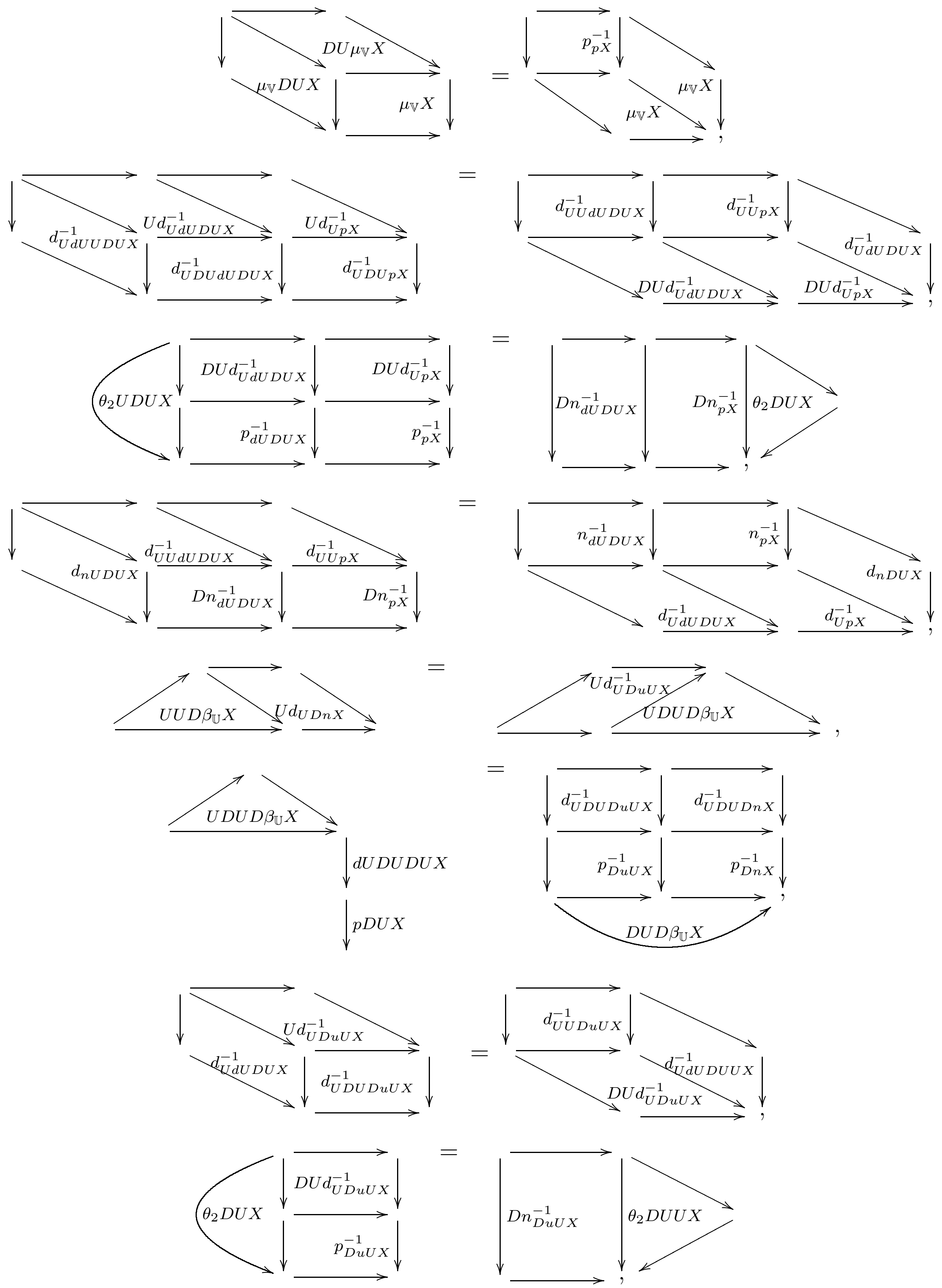




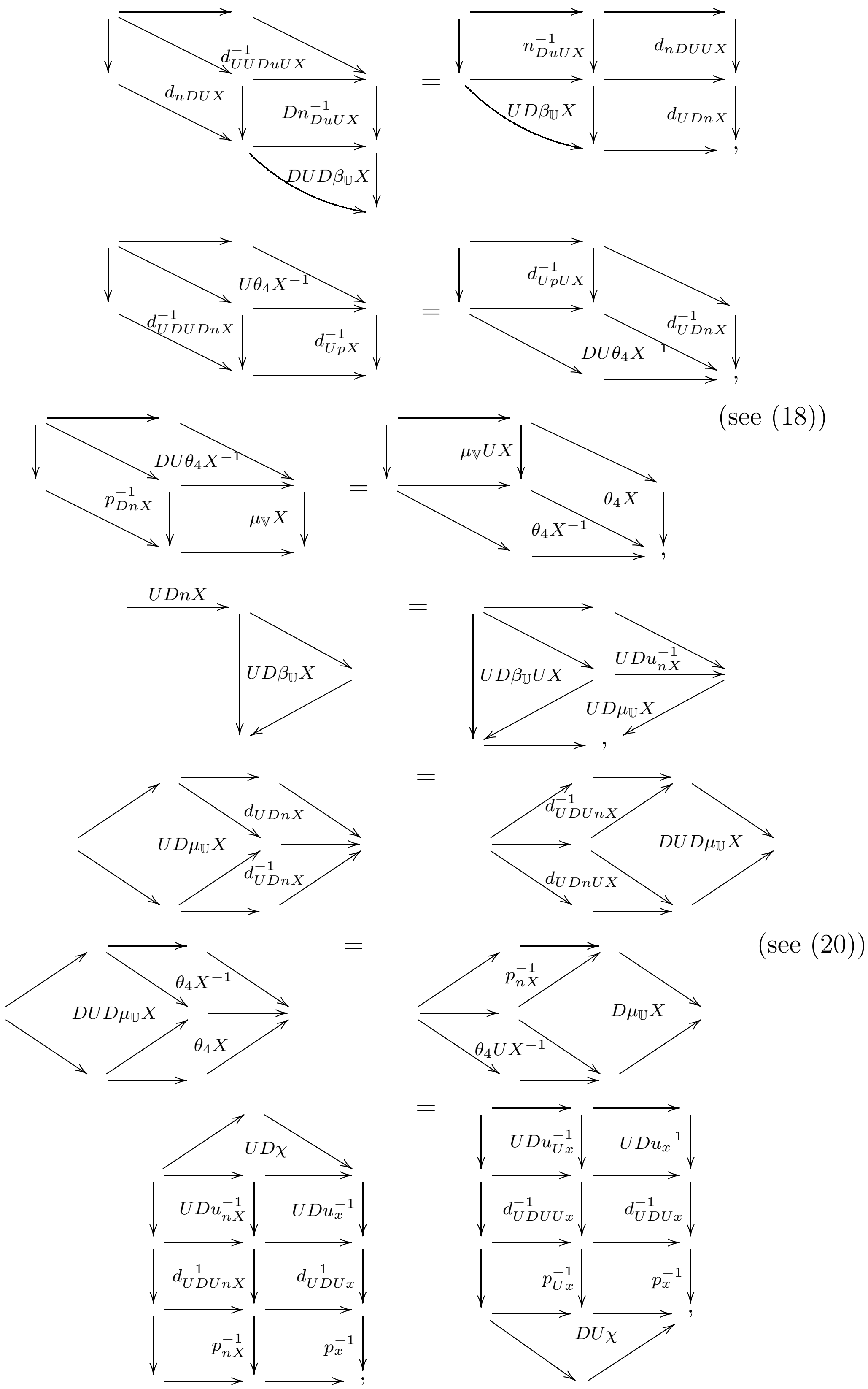


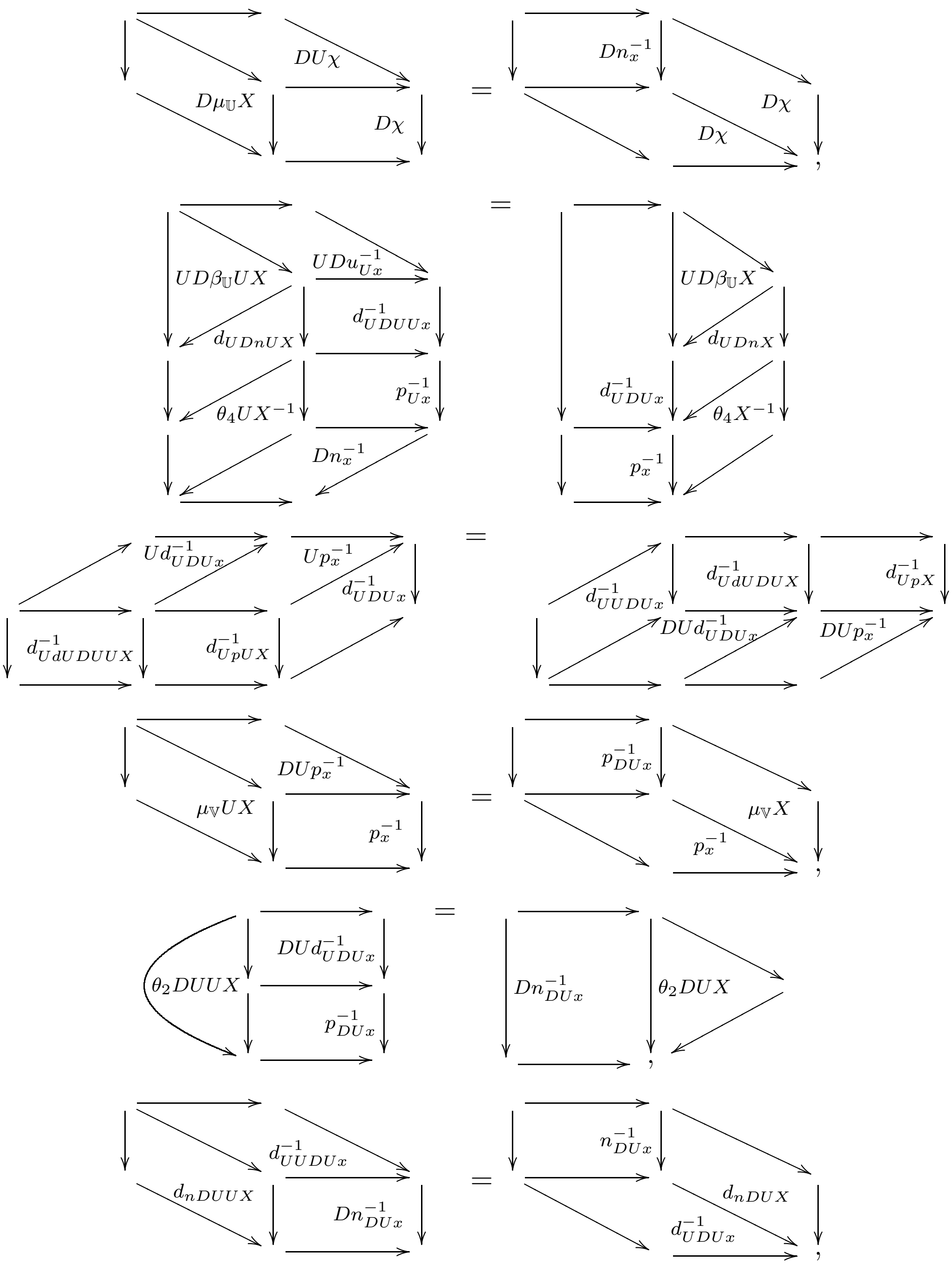
(definition of θ_4)





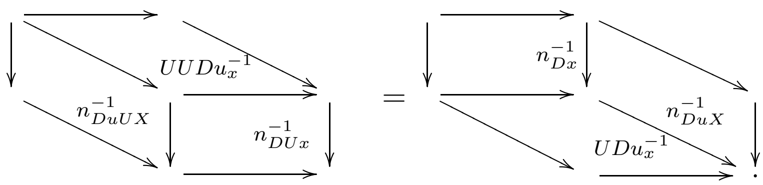
DUμ𝕍X
μ𝕍DUX
μ𝕍X
=
p⁻¹pX
μ𝕍X
μ𝕍X
d⁻¹UdUUDUX
Ud⁻¹UUdUDUX
Ud⁻¹UpX
d⁻¹UDUdUDUX
d⁻¹UDUpX
d⁻¹UUdUDUX
d⁻¹UUpX
d⁻¹UdUDUX
DUd⁻¹UdUDUX
DUd⁻¹UpX
θ2UDUX
DUd⁻¹UdUDUX
DUd⁻¹UpX
p⁻¹dUDUX
p⁻¹pX
Dn⁻¹dUDUX
Dn⁻¹pX
θ2DUX
dnUDUX
d⁻¹UUdUDUX
d⁻¹UUpX
Dn⁻¹dUDUX
Dn⁻¹pX
n⁻¹dUDUX
n⁻¹pX
dnDUX
d⁻¹UdUDUX
d⁻¹UpX
UUDβ𝕌X
UdUDnX
Ud⁻¹UDuUX
UDUDβ𝕌X
UDUDβ𝕌X
dUDUDUX
pDUX
d⁻¹UDUDuUX
d⁻¹UDUDnX
p⁻¹DuUX
p⁻¹DnX
DUDβ𝕌X
Ud⁻¹UDuUX
d⁻¹UdUDUX
d⁻¹UDUDuUX
d⁻¹UUDuUX
d⁻¹UdUDUUX
DUd⁻¹UDuUX
θ2DUX
DUd⁻¹UDuUX
p⁻¹DuUX
Dn⁻¹DuUX
θ2DUUX


d^{-1}_{UUDuUX}
d_{nDUX}
Dn^{-1}_{DuUX}
$DUD\beta_{\mathbb{U}}X$
=
n^{-1}_{DuUX}
d_{nDUUX}
$UD\beta_{\mathbb{U}}X$
d_{UDnX}
,
$U\theta_4X^{-1}$
d^{-1}_{UDUDnX}
d^{-1}_{UpX}
=
d^{-1}_{UpUX}
d^{-1}_{UDnX}
$DU\theta_4X^{-1}$
,
(see (18))
$DU\theta_4X^{-1}$
p^{-1}_{DnX}
$\mu_{\mathbb{V}}X$
=
$\mu_{\mathbb{V}}UX$
θ_4X
θ_4X^{-1}
,
$UDnX$
$UD\beta_{\mathbb{U}}X$
=
$UD\beta_{\mathbb{U}}UX$
UDu^{-1}_{nX}
$UD\mu_{\mathbb{U}}X$
,
d_{UDnX}
$UD\mu_{\mathbb{U}}X$
d^{-1}_{UDnX}
=
d^{-1}_{UDUnX}
$DUD\mu_{\mathbb{U}}X$
d_{UDnUX}
θ_4X^{-1}
$DUD\mu_{\mathbb{U}}X$
θ_4X
=
p^{-1}_{nX}
$D\mu_{\mathbb{U}}X$
θ_4UX^{-1}
(see (20))
$UD\chi$
UDu^{-1}_{nX}
UDu^{-1}_x
d^{-1}_{UDUnX}
d^{-1}_{UDUx}
p^{-1}_{nX}
p^{-1}_x
,
=
UDu^{-1}_{Ux}
UDu^{-1}_x
d^{-1}_{UDUUx}
d^{-1}_{UDUx}
p^{-1}_{Ux}
p^{-1}_x
,
$DU\chi$



$DU\chi$
$D\mu_{\mathbb{U}}X$
$D\chi$
=
Dn_x^{-1}
$D\chi$
$D\chi$
,
=
UDu_{Ux}^{-1}
$UD\beta_{\mathbb{U}}UX$
d_{UDUUx}^{-1}
d_{UDnUX}
p_{Ux}^{-1}
$\theta_4 UX^{-1}$
Dn_x^{-1}
$UD\beta_{\mathbb{U}}X$
d_{UDnX}
d_{UDUx}^{-1}
$\theta_4 X^{-1}$
p_x^{-1}
=
Ud_{UDUx}^{-1}
Up_x^{-1}
d_{UDUx}^{-1}
$d_{UdUDUUX}^{-1}$
d_{UpUX}^{-1}
d_{UdUDUX}^{-1}
d_{UpX}^{-1}
d_{UUDUx}^{-1}
DUd_{UDUx}^{-1}
DUp_x^{-1}
DUp_x^{-1}
$\mu_{\mathbb{V}}UX$
p_x^{-1}
=
p_{DUx}^{-1}
$\mu_{\mathbb{V}}X$
p_x^{-1}
,
=
DUd_{UDUx}^{-1}
$\theta_2 DUUX$
p_{DUx}^{-1}
Dn_{DUx}^{-1}
$\theta_2 DUX$
,
d_{UUDUx}^{-1}
d_{nDUUX}
Dn_{DUx}^{-1}
=
n_{DUx}^{-1}
d_{nDUX}
d_{UDUx}^{-1}
,

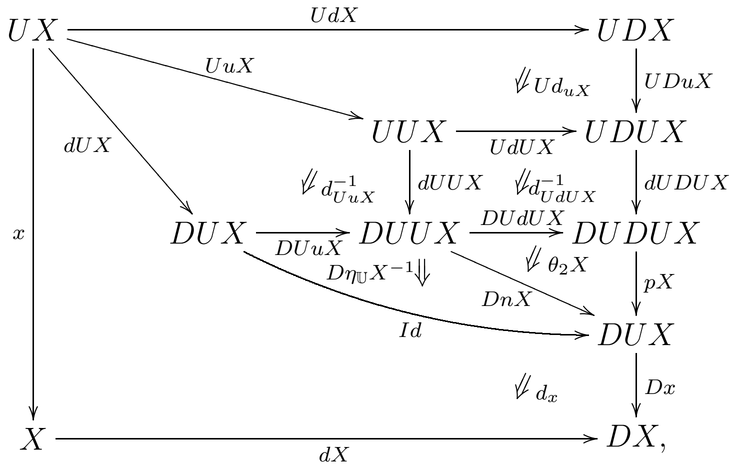
and



$UUDu_x^{-1}$
n_{DuUX}^{-1}
n_{DUx}^{-1}
=
n_{Dx}^{-1}
n_{DuX}^{-1}
UDu_x^{-1}
.

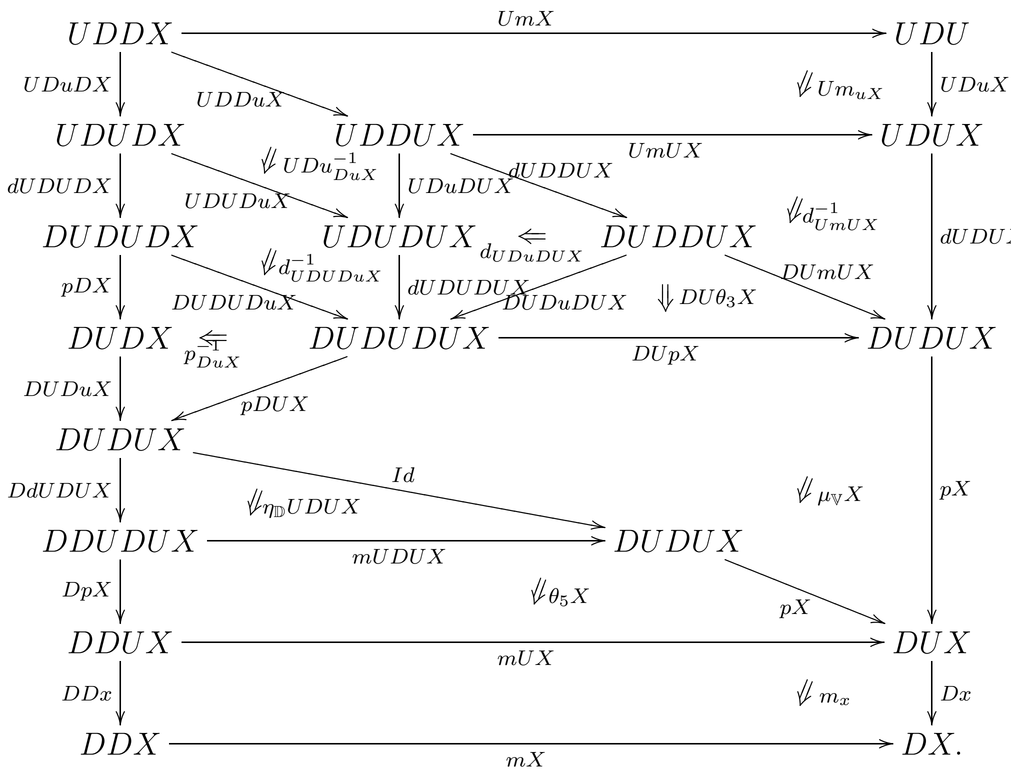
■

We are now ready to define $d_{\mathcal{X}} : 1 \to D_{\mathcal{X}}$, and $m_{\mathcal{X}} : D_{\mathcal{X}}D_{\mathcal{X}} \to D_{\mathcal{X}}$. The first entry of $(d_{\mathcal{X}})_{(\psi,\chi)}$ is dX, while the second is the pasting



and $(d_{\mathcal{X}})_{(h,\rho)} = d_h$.

Define the first entry of $(m_{\mathcal{X}})_{(\psi,\chi)}$ as mX, and the second as the pasting

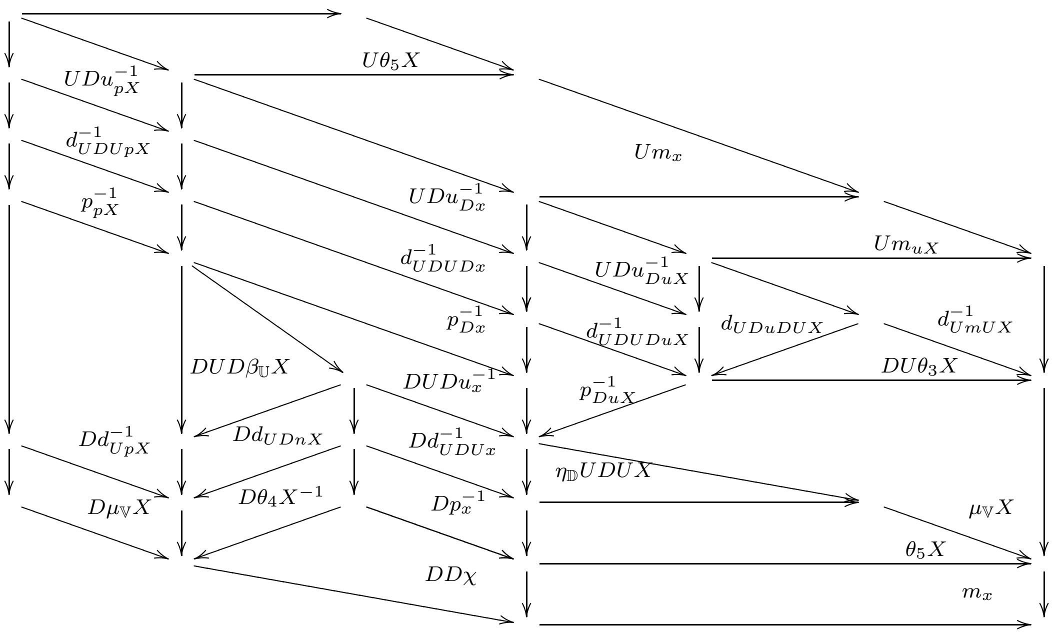


Define $(m_{\mathcal{X}})_{(h,\rho)} = m_h$.

9.2. Proposition. *With the above definitions, $d_{\mathcal{X}} : 1 \to D_{\mathcal{X}}$ and $m_{\mathcal{X}} : D_{\mathcal{X}}D_{\mathcal{X}} \to D_{\mathcal{X}}$ are strong transformations. Furthermore, defining $\widetilde{d}$ as $d_{\mathcal{X}}$, and $\widetilde{m}$ as $m_{\mathcal{X}}$ at every object*

*$\mathcal{X}$ of **A**, we have that, $(\widetilde{d}, d) : 1 \to (\widetilde{D}, D)$ and $(\widetilde{m}, m) : (\widetilde{D}, D)(\widetilde{D}, D) \to (\widetilde{D}, D)$ are 1-cells in the 2-category $[\mathbb{U}, \mathbb{U}]$.*

Proof. The size of the diagrams obtained is what makes it hard to prove that $(m_{\mathcal{X}})_{(\psi,\chi)}$ satisfies (10). To get from the left hand side of (10) to the right hand side make the following three substitutions:



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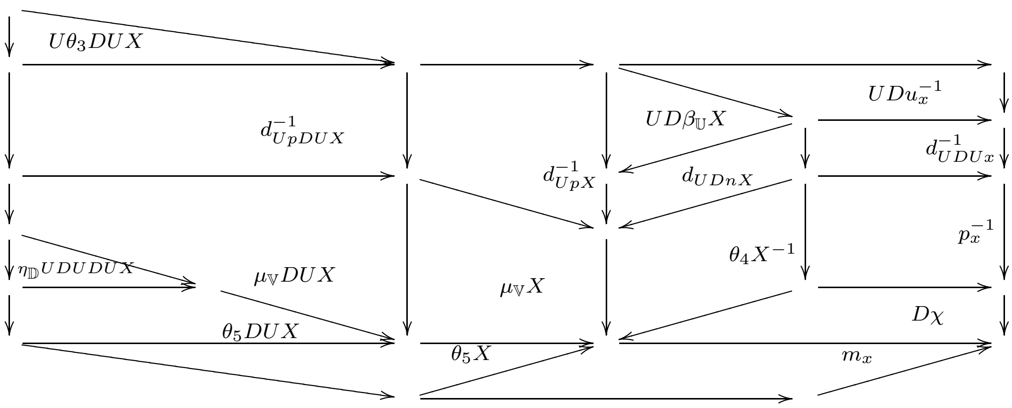
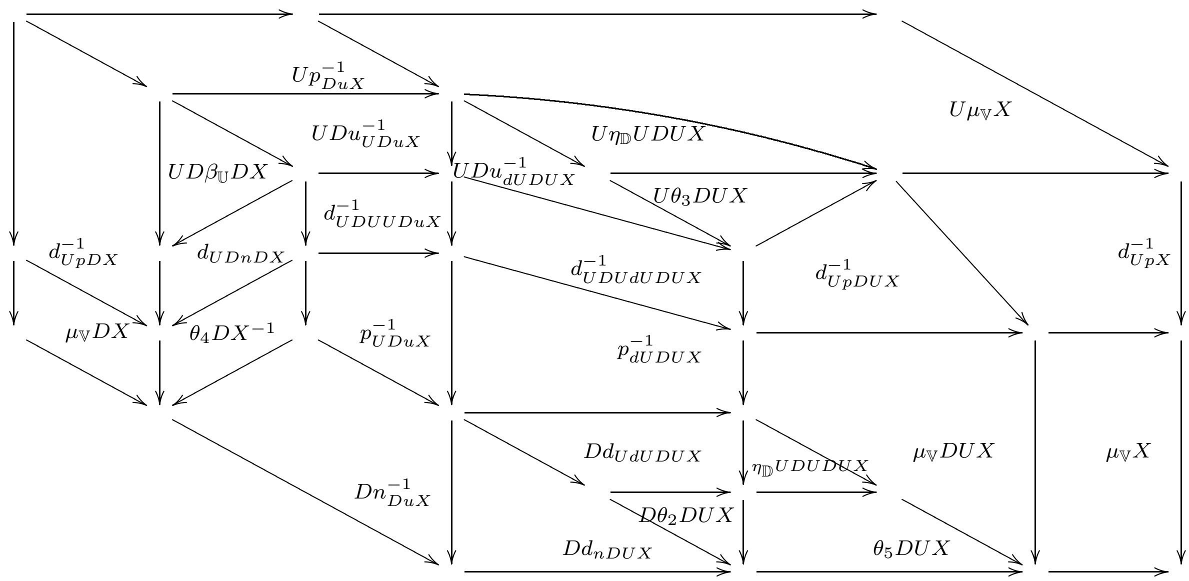
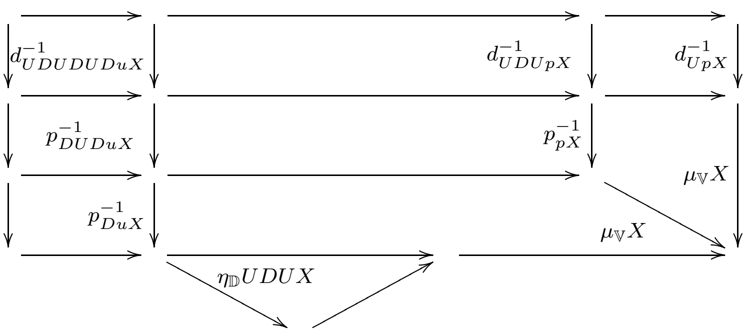
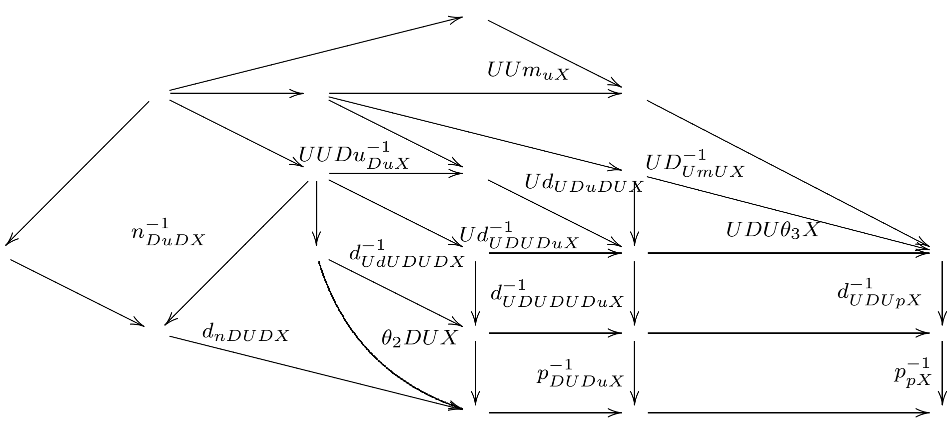



Up^{-1}_{DuX}
$U\mu_{\mathbb{V}}X$
UDu^{-1}_{UDuX}
$U\eta_{\mathbb{D}}UDUX$
$UD\beta_{\mathbb{U}}DX$
UDu^{-1}_{dUDUX}
$U\theta_3DUX$
$d^{-1}_{UDUUDuX}$
d^{-1}_{UpDX}
d_{UDnDX}
$d^{-1}_{UDUdUDUX}$
d^{-1}_{UpDUX}
d^{-1}_{UpX}
$\mu_{\mathbb{V}}DX$
θ_4DX^{-1}
p^{-1}_{UDuX}
p^{-1}_{dUDUX}
Dd_{UdUDUX}
$\eta_{\mathbb{D}}UDUDUX$
$\mu_{\mathbb{V}}DUX$
$\mu_{\mathbb{V}}X$
Dn^{-1}_{DuX}
$D\theta_2DUX$
Dd_{nDUX}
θ_5DUX

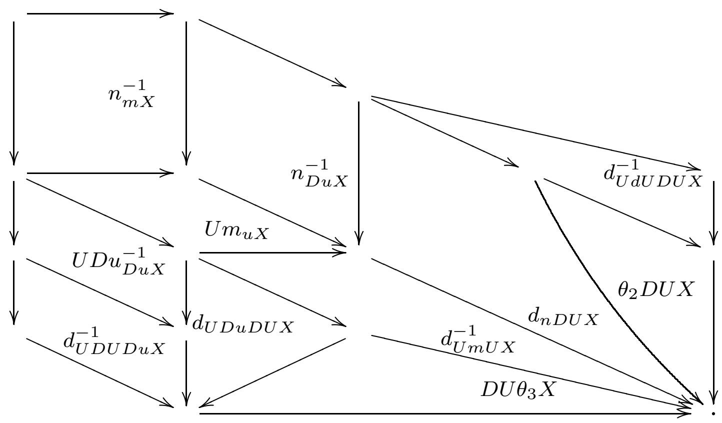
=


$d^{-1}_{UDUDUDuX}$
d^{-1}_{UDUpX}
d^{-1}_{UpX}
p^{-1}_{DUDuX}
p^{-1}_{pX}
$\mu_{\mathbb{V}}X$
p^{-1}_{DuX}
$\mu_{\mathbb{V}}X$
$\eta_{\mathbb{D}}UDUX$

and


UUm_{uX}
$UUDu^{-1}_{DuX}$
UD^{-1}_{UmUX}
Ud_{UDuDUX}
n^{-1}_{DuDX}
Ud^{-1}_{UDUDuX}
$UDU\theta_3X$
$d^{-1}_{UdUDUDX}$
$d^{-1}_{UDUDUDuX}$
d^{-1}_{UDUpX}
d_{nDUDX}
θ_2DUX
p^{-1}_{DUDuX}
p^{-1}_{pX}

=



The fact that the three equalities above are indeed satisfied and the rest of the proof are left to the reader. ∎

It is easily seen that, defining $\widetilde{\eta}$, $\widetilde{\beta}$, and $\widetilde{\mu}$ at every $\mathcal{X}$ as $(\eta_{\mathcal{X}})_{(\psi,\chi)} = \eta_{\mathbb{D}}X$, $(\beta_{\mathcal{X}})_{(\psi,\chi)} = \beta_{\mathbb{D}}X$, and $(\mu_{\mathcal{X}})_{(\psi,\chi)} = \mu_{\mathbb{D}}X$, we obtain 2-cells $(\widetilde{\eta}, \eta)$, $(\widetilde{\beta}, \beta)$, and $(\widetilde{\mu}, \mu)$ in $[\mathbb{U}, \mathbb{U}]$. We thus have:

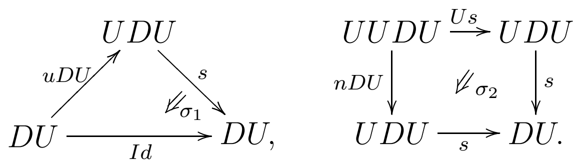
9.3. THEOREM. $\widetilde{\mathbb{D}} = ((\widetilde{D}, D), (\widetilde{d}, d), (\widetilde{m}, m), (\widetilde{\beta}, \beta), (\widetilde{\eta}, \eta), (\widetilde{\mu}, \mu))$ *is a pseudomonad in* PSM(**A**)*, on the object* $\mathbb{U}$. ∎

10. From a pseudomonad in PSM(**A**) to a distributive law

Assume that we have a pseudomonad in PSM(**A**) as in section 8. We produce a distributive law as follows. Consider $D_{\mathcal{K}} : \mathbb{U}\text{-Alg}_{\mathcal{K}} \to \mathbb{U}\text{-Alg}_{\mathcal{K}}$. Since the diagram

$$\begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} \\ {\scriptstyle \Phi_{\mathcal{K}}}\downarrow & & \downarrow{\scriptstyle \Phi_{\mathcal{K}}} \\ \mathbf{A}(\mathcal{K},\mathcal{K}) & \xrightarrow[D(\_)]{} & \mathbf{A}(\mathcal{K},\mathcal{K}) \end{array}$$

commutes, we have that $D_{\mathcal{K}}(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})$ is of the form (σ_1, σ_2) with



Furthermore, since the diagram

$$\begin{array}{ccc} \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} \\ \downarrow{\scriptstyle \widehat{U}} & & \downarrow{\scriptstyle \widehat{U}} \\ \mathbb{U}\text{-Alg}_{\mathcal{K}} & \xrightarrow{D_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} \end{array}$$

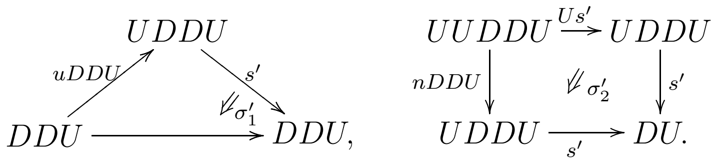
commutes, where $\widehat{U}$ is the change of base 2-functor defined in section 3, then $D_{\mathcal{K}}(n, \mu_{\mathbb{U}}) : D_{\mathcal{K}}(\beta_{\mathbb{U}}U, \mu_{\mathbb{U}}U) \to D_{\mathcal{K}}(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})$ is of the form $(Dn, \sigma_3) : (\sigma_1 U, \sigma_2 U) \to (\sigma_1, \sigma_2)$ with σ_3 of the form

$$\begin{array}{ccc} UDUU & \xrightarrow{UDn} & UDU \\ {\scriptstyle sU}\downarrow & \not\Downarrow \sigma_3 & \downarrow{\scriptstyle s} \\ DUU & \xrightarrow[Dn]{} & DU. \end{array}$$

Similarly, it can be shown that $(d_{\mathcal{K}})_{(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})}$ is of the form (dU, σ_4), with

$$\begin{array}{ccc} UU & \xrightarrow{UdU} & UDU \\ {\scriptstyle n}\downarrow & \not\Downarrow \sigma_4 & \downarrow{\scriptstyle s} \\ U & \xrightarrow[dU]{} & DU. \end{array}$$

If we apply $D_{\mathcal{K}}$ to (σ_1, σ_2) we obtain another $\mathbb{U}$-algebra (σ_1', σ_2'), with



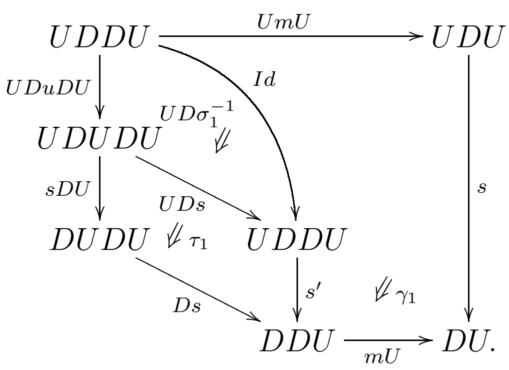
Applying $D_{\mathcal{K}}$ to $(s, \sigma_2) : (\beta_{\mathbb{U}}DU, \mu_{\mathbb{U}}DU) \to (\sigma_1, \sigma_2)$, we obtain a morphism (Ds, τ_1) with

$$\begin{array}{ccc} UDUDU & \xrightarrow{UDs} & UDDU \\ {\scriptstyle sDU}\downarrow & \not\Downarrow \tau_1 & \downarrow{\scriptstyle s'} \\ DUDU & \xrightarrow[Ds]{} & DDU. \end{array}$$

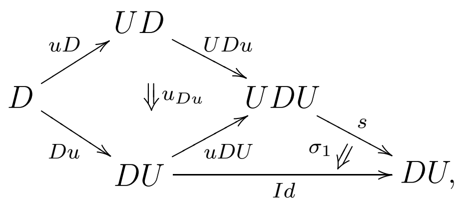
We also have that $(m_{\mathcal{K}})_{(\beta_{\mathbb{U}}, \mu_{\mathbb{U}})}$ is of the form (mU, γ_1), with

$$\begin{array}{ccc} UDDU & \xrightarrow{UmU} & UDU \\ {\scriptstyle s'}\downarrow & \not\Downarrow \gamma_1 & \downarrow{\scriptstyle s} \\ DDU & \xrightarrow[mU]{} & DU. \end{array}$$

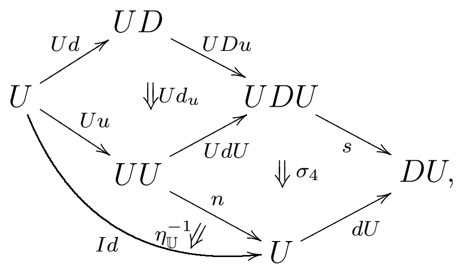
Define σ_5 as the pasting



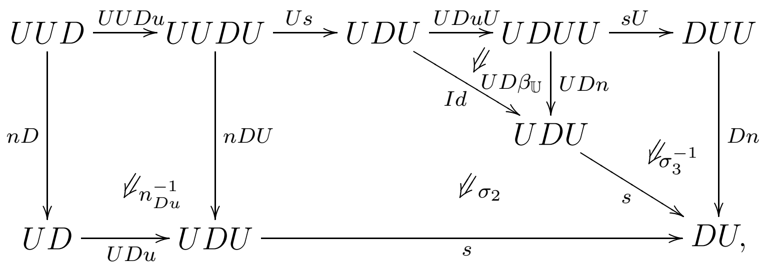
It is with the help of σ_1 to σ_5 that we define the distributive law as follows: Define $r = s \circ UDu$, ω_1 as the pasting



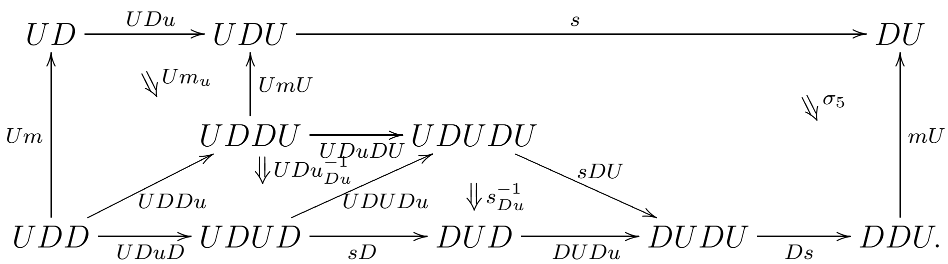
ω_2 as the pasting



ω_3 as the pasting



and ω_4 as the pasting



The proof that we obtain a distributive law in this way is based on the behavior of the 2-cells σ_1-σ_5. Aside from the obvious conditions that come from the facts that (σ_1, σ_2) is a $\mathbb{U}$-algebra, (Dn, σ_3) and (dU, σ_4) are homomorphisms of $\mathbb{U}$-algebras, we have the following conditions.

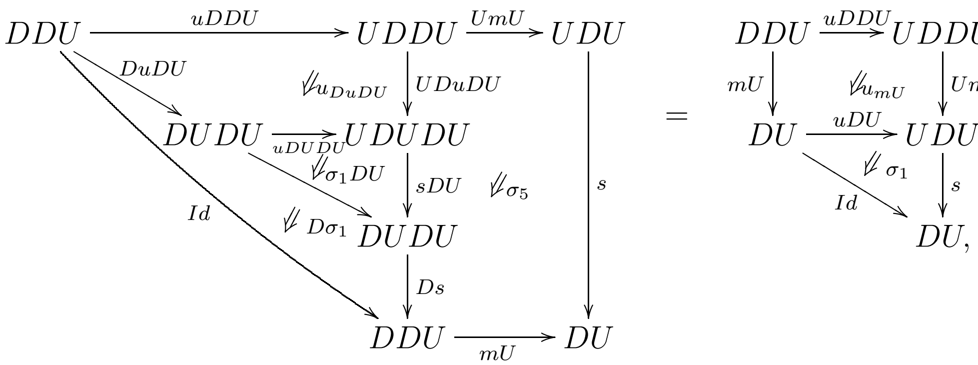
10.1. Proposition. *The following are 2-cells in* $\mathbb{U}$-$Alg_{\mathcal{K}}$*:*

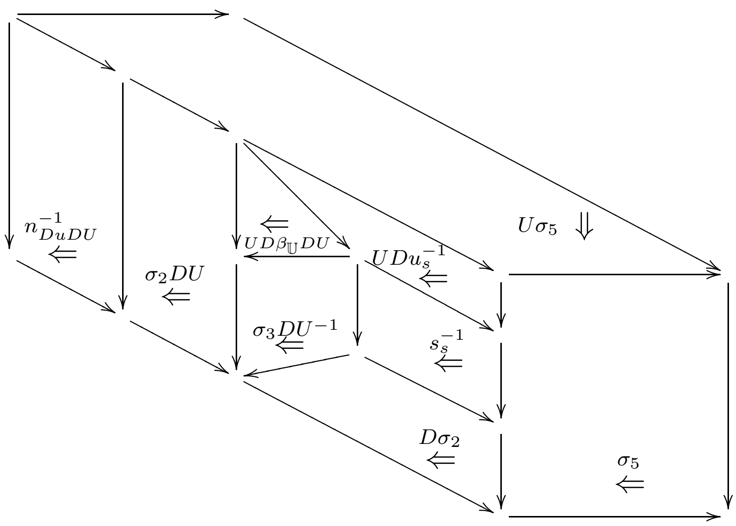
$$D\eta_{\mathbb{U}} : Id \to (Dn, \sigma_3) \circ (DUu, s_u^{-1}). \tag{26}$$

$$D\mu_{\mathbb{U}} : (Dn, \sigma_3) \circ (DUn, s_n^{-1}) \to (Dn, \sigma_3) \circ (DnU, \sigma_3 U).$$

$$d_n : (Dn, \sigma_3) \circ (dUU, \sigma_4 U) \to (dU, \sigma_4) \circ (n, \mu_{\mathbb{U}}). \qquad (27)$$

Furthermore, the following equations are satisfied:





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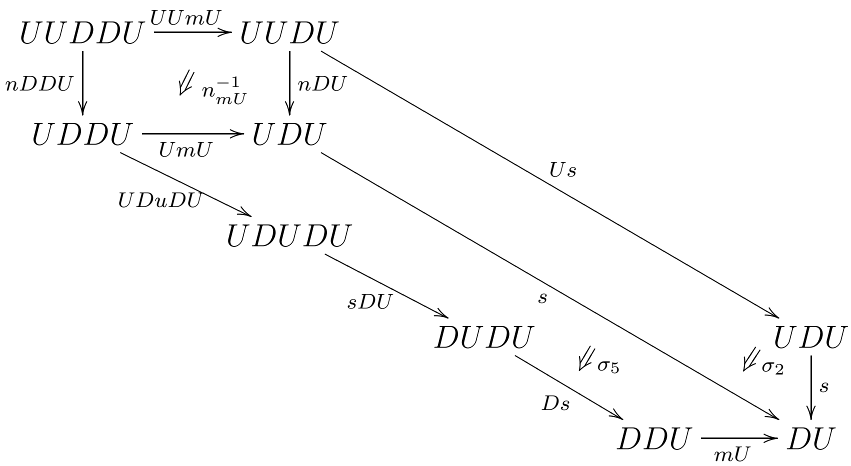
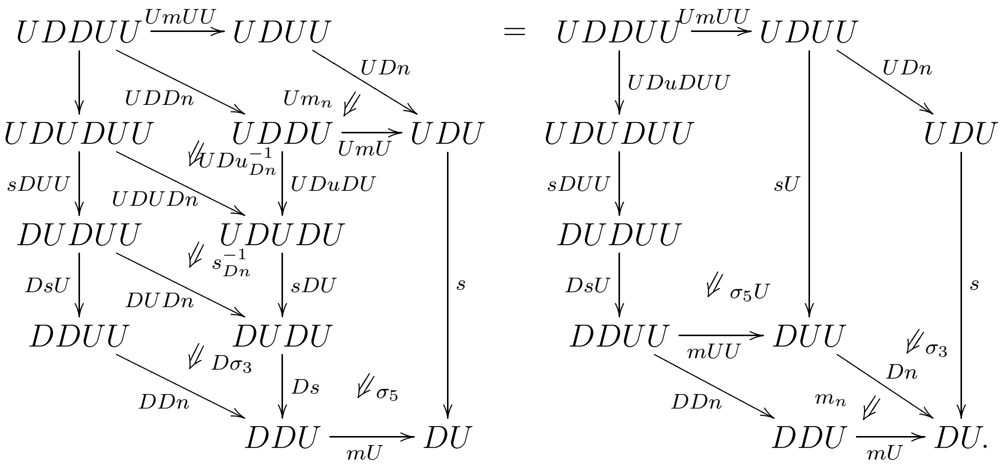


(28)


UDDUU
UmUU
UDUU
=
UDDUU
UmUU
UDUU
UDn
UDDn
Umn
UDUDUU
UDDU
UmU
UDU
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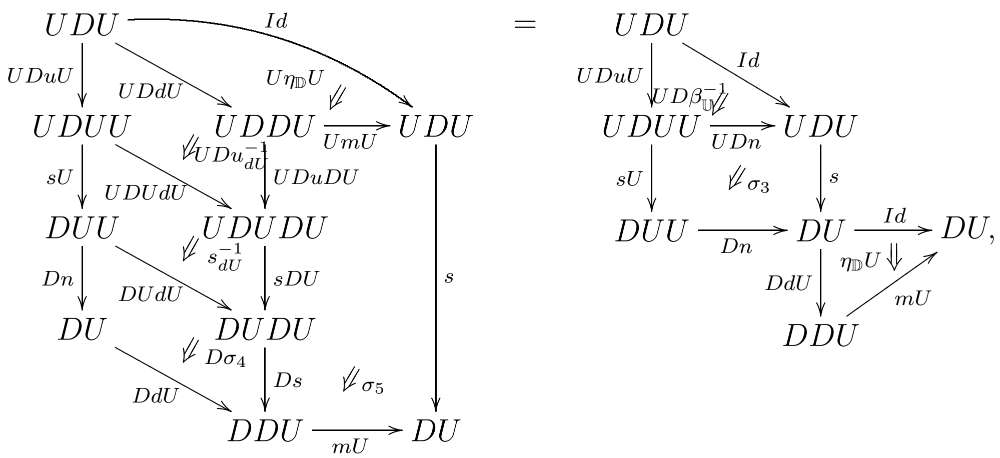
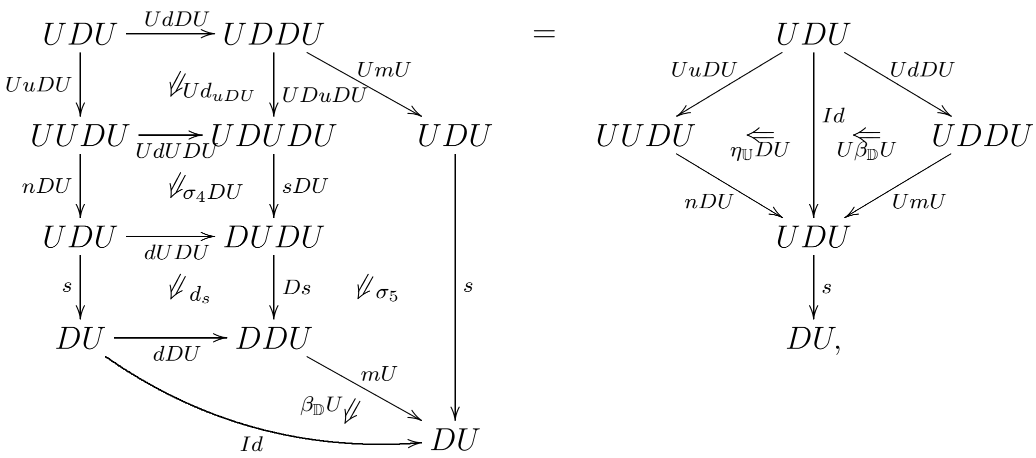
UDUDUU
UDU
UDu−1Dn
UDuDU
sDUU
UDUDn
sDUU
sU
DUDUU
UDUDU
DUDUU
s−1Dn
DsU
DUDn
sDU
s
DsU
σ5U
s
DDUU
DUDU
DDUU
mUU
DUU
σ3
Dσ3
Ds
σ5
Dn
DDn
DDn
mn
DDU
mU
DU
DDU
mU
DU.
(29)


UDU
Id
=
UDU
UDuU
UDdU
UηDU
UDuU
Id
UDβ−1U
UDUU
UDDU
UmU
UDU
UDUU
UDn
UDU
UDu−1dU
UDuDU
sU
UDUdU
sU
σ3
s
DUU
UDUDU
DUU
Dn
DU
Id
DU,
s−1dU
ηDU
Dn
DUdU
sDU
s
DdU
mU
DU
DUDU
DDU
Dσ4
Ds
σ5
DdU
DDU
mU
DU
(30)


UDU
UdDU
UDDU
=
UDU
UmU
UuDU
UdDU
UuDU
UduDU
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UUDU
UdUDU
UDUDU
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UUDU
ηUDU
Id
UβDU
UDDU
nDU
σ4DU
sDU
nDU
UmU
UDU
dUDU
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s
ds
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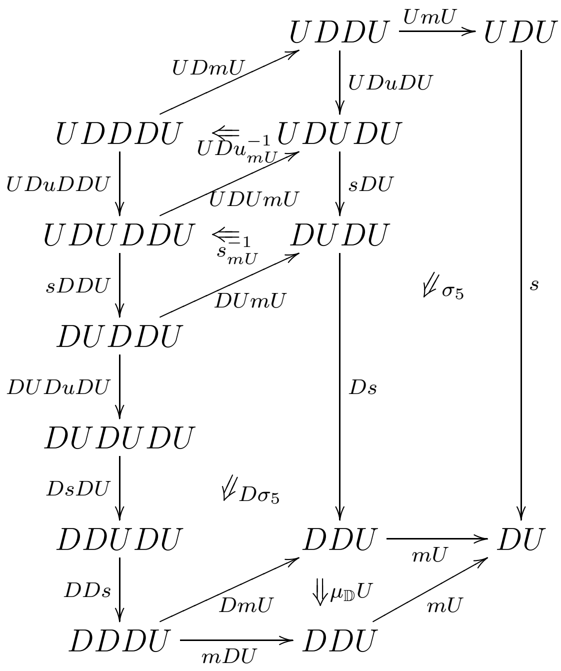
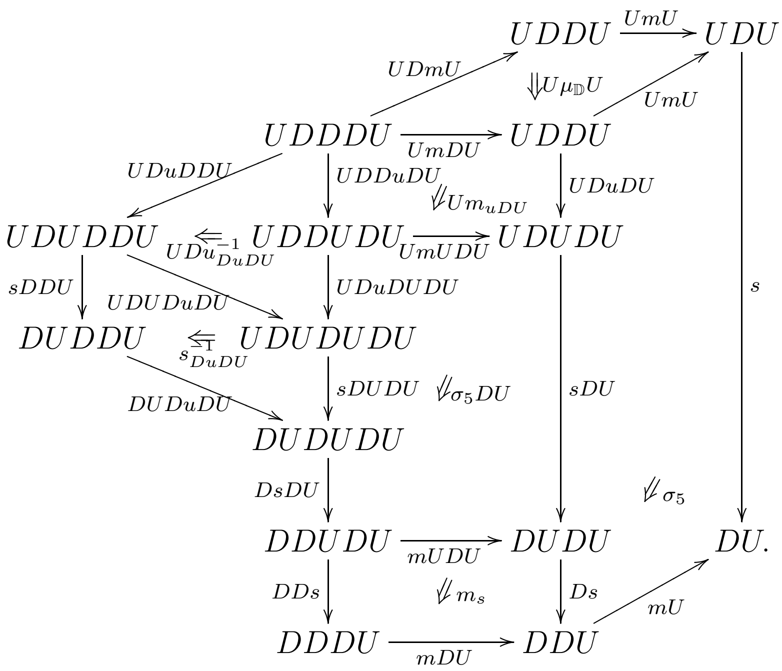
σ5
s
s
DU
dDU
DDU
DU,
mU
βDU
Id
DU



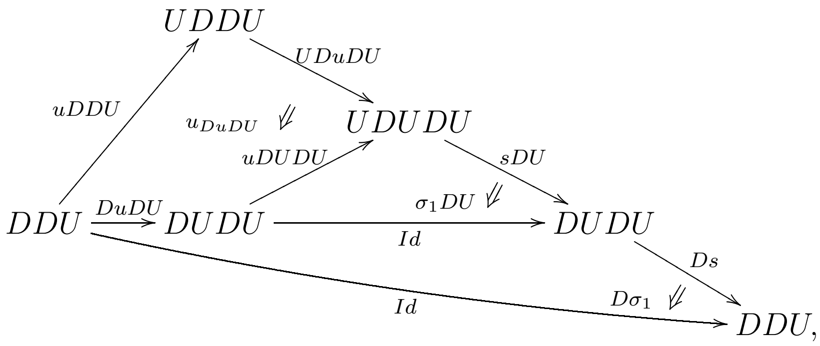
=



(31)

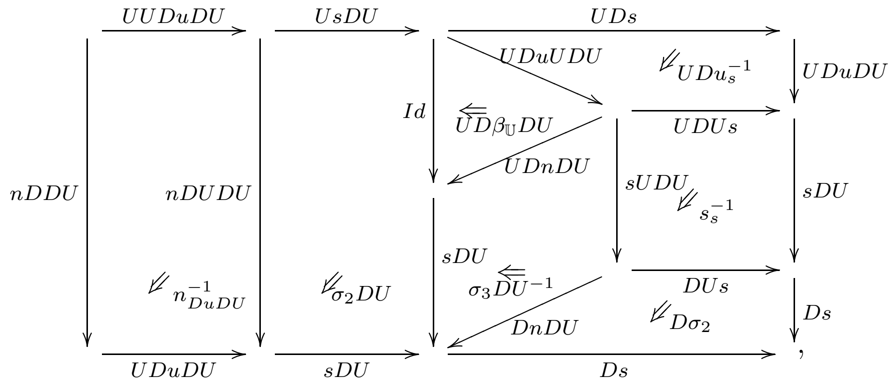
Observation: The equations that appear in the proposition can be written as conditions in $\mathbb{U}$-Alg$_{\mathcal{K}}$. For instance, it can be shown that the pair of pastings:

$\varphi \quad =$



and

$\vartheta \quad =$



define an object in $\mathbb{U}$-Alg$_{\mathcal{K}}$. Thus, the first two equations say that $(mU, \sigma_5) : (\varphi, \vartheta) \to (\sigma_1, \sigma_2)$ is a $\mathbb{U}$-algebra morphism. Similarly the rest of the equations. We leave the details for the interested reader.

Proof. The fact that $D\eta_{\mathbb{U}}$ and $D\mu_{\mathbb{U}}$ are 2-cells in $\mathbb{U}$-Alg$_{\mathcal{K}}$ follow from applying $D_{\mathcal{K}}$ to $\eta_{\mathbb{U}}$ and $\mu_{\mathbb{U}}$, respectively. The statement about d_n follows from pseudonaturality of $d_{\mathcal{K}}$. As for the equations, the proof of the last equation is fairly typical, and it is the only one we do. We begin by introducing some notation and deducing some equations we will need. The notation is

$$
\begin{aligned}
D_{\mathcal{K}}(\,(\beta_{\mathbb{U}}DDU, \mu_{\mathbb{U}}DDU) \xrightarrow{(s',\sigma_2')} (\sigma_1', \sigma_2')) &= (\sigma_1 DDU, \sigma_2 DDU) \xrightarrow{(Ds',\tau_2)} (\sigma_1'', \sigma_2''),\\
D_{\mathcal{K}}\,((\sigma_1 DU, \sigma_2 DU) \xrightarrow{(Ds,\tau_1)} (\sigma_1', \sigma_2')) &= (\sigma_1' DU, \sigma_2' DU) \xrightarrow{(DDs,\tau_3)} (\sigma_1'', \sigma_2''),\\
D_{\mathcal{K}}\,((\sigma_1', \sigma_2') \xrightarrow{(mU,\gamma_1)} (\sigma_1, \sigma_2)) &= (\sigma_1'', \sigma_2'') \xrightarrow{(DmU,\gamma_2)} (\sigma_1', \sigma_2'),
\end{aligned}
$$

and $(m_{\mathcal{K}})_{(\sigma_1,\sigma_2)} = (mDU, \gamma_3) : (\sigma_1'', \sigma_2'') \to (\sigma_1', \sigma_2')$.

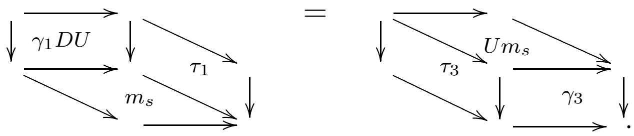
Now the equations. Since

$$
\begin{array}{ccccc}
\mathbb{U}\text{-Alg}_{\mathcal{K}} \underset{D_{\mathcal{K}}}{\overset{D_{\mathcal{K}}D_{\mathcal{K}}}{\Longrightarrow}}\!\!{\scriptstyle \Downarrow m_{\mathcal{K}}} & \mathbb{U}\text{-Alg}_{\mathcal{K}} & = & \mathbb{U}\text{-Alg}_{\mathcal{K}} & \\
& \downarrow{\scriptstyle \Phi_{\mathcal{K}}} & & {\scriptstyle \Phi_{\mathcal{K}}}\downarrow & \\
& \mathbf{A}(\mathcal{K},\mathcal{K}) & & \mathbf{A}(\mathcal{K},\mathcal{K}) \underset{D(\_)}{\overset{DD(\_)}{\Longrightarrow}}\!\!{\scriptstyle \Downarrow m(\_)} & \mathbf{A}(\mathcal{K},\mathcal{K}),
\end{array}
$$

we have that $m_s = (m_{\mathcal{K}})_{(s,\sigma_2)}$ is a 2-cell

$$
\begin{array}{ccc}
(\sigma_1' DU, \sigma_2' DU) & \xrightarrow{(mUDU,\gamma_1 DU)} & (\sigma_1 DU, \sigma_2 DU) \\
{\scriptstyle (DDs,\tau_3)}\downarrow & \Downarrow m_s & \downarrow{\scriptstyle (Ds,\tau_1)} \\
(\sigma_1'', \sigma_2'') & \xrightarrow[(mDU,\gamma_3)]{} & (\sigma_1', \sigma_2')
\end{array}
$$

in $\mathbb{U}\text{-Alg}_{\mathcal{K}}$. From this we obtain the equation

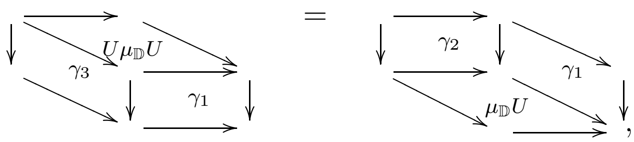


(32)

Similarly, $(\mu_{\mathcal{K}})_{(\beta_{\mathbb{U}},\mu_{\mathbb{U}})}$ is a 2-cell

$$
\begin{array}{ccc}
(\sigma_1'', \sigma_2'') & \xrightarrow{(DmU,\gamma_2)} & (\sigma_1', \sigma_2') \\
{\scriptstyle (mDU,\gamma_3)}\downarrow & \Downarrow \mu_{\mathbb{D}} U & \downarrow{\scriptstyle (mU,\gamma_1)} \\
(\sigma_1', \sigma_2') & \xrightarrow[(mU,\gamma_1)]{} & (\sigma_1, \sigma_2).
\end{array}
$$

We thus obtain the equation



(33)

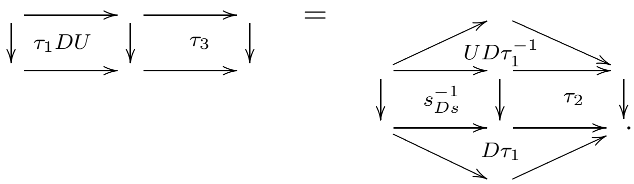
Observe that

$$
\begin{array}{ccc}
(\beta_{\mathbb{U}} DUDU, \mu_{\mathbb{U}} DUDU) & \xrightarrow{(UDs,n_{Ds}^{-1})} & (\beta_{\mathbb{U}} DDU, \mu_{\mathbb{U}} DDU) \\
{\scriptstyle (sDU,\sigma_2 DU)}\downarrow & \Downarrow \tau_1 & \downarrow{\scriptstyle (s',\sigma_2')} \\
(\sigma_1 DU, \sigma_2 DU) & \xrightarrow[(Ds,\tau_1)]{} & (\sigma_1', \sigma_2')
\end{array}
$$

is a 2-cell in $\mathbb{U}$-Alg$_{\mathcal{K}}$. Since $D_{\mathcal{K}}$ behaves well with change of base, then $D_{\mathcal{K}}(UDs, n_{Ds}^{-1}) = (DUDs, s_{Ds}^{-1})$. Thus, applying $D_{\mathcal{K}}$, we have that

$$\begin{array}{ccc}(\sigma_1 DUDU, \sigma_2 DUDU) & \xrightarrow{(DUDs, s_{Ds}^{-1})} & (\sigma_1 DDU, \sigma_2 DDU) \\ {\scriptstyle (DsDU, \tau_1)}\downarrow & \Downarrow D\tau_1 & \downarrow{\scriptstyle (Ds', \tau_2)} \\ (\sigma_1' DU, \sigma_2' DU) & \xrightarrow[(DDs, \tau_3)]{} & (\sigma_1'', \sigma_2'')\end{array}$$

is a 2-cell in $\mathbb{U}$-Alg$_{\mathcal{K}}$. From this we obtain the equation

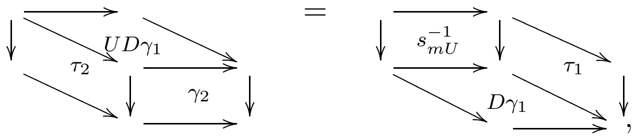


(34)

We also have that

$$\begin{array}{ccc}(\beta_{\mathbb{U}} DDU, \mu_{\mathbb{U}} DDU) & \xrightarrow{(UmU, n_{mU}^{-1})} & (\beta_{\mathbb{U}} DU, \mu_{\mathbb{U}} DU) \\ {\scriptstyle (s', \sigma_2')}\downarrow & \Downarrow \gamma_1 & \downarrow{\scriptstyle (s, \sigma_2)} \\ (\sigma_1', \sigma_2') & \xrightarrow[(mU, \gamma_1)]{} & (\sigma_1, \sigma_2)\end{array}$$

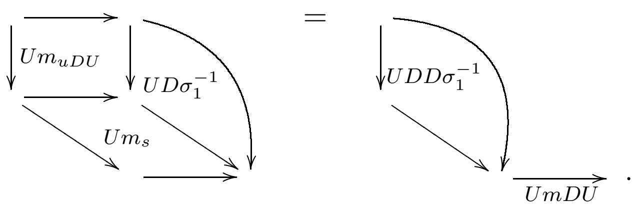
is a 2-cell in $\mathbb{U}$-Alg$_{\mathcal{K}}$. Applying $D_{\mathcal{K}}$ to it we obtain that the equation



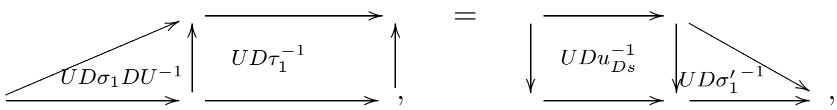
(35)

is satisfied.

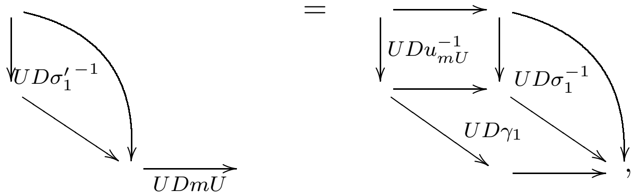
We are ready to show (31). Start with the second member of the equation, and use (32). Next make the substitution



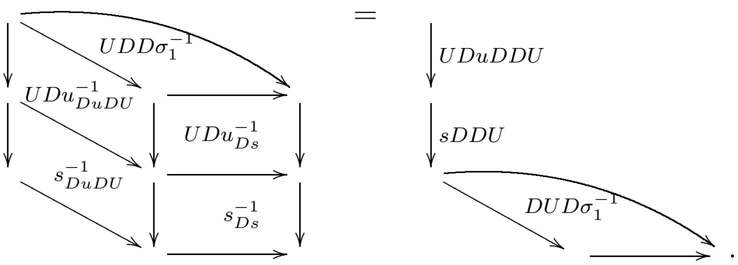
Use now (33) and (34). Make the substitutions



and



using that $(Ds, \tau_1) : (\sigma_1 DU, \sigma_2 DU) \to (\sigma_1', \sigma_2')$ and $(mU, \gamma_1) : (\sigma_1', \sigma_2') \to (\sigma_1, \sigma_2)$ are 1-cells in $\mathbb{U}$-$\mathrm{Alg}_{\mathcal{K}}$. Then the substitution

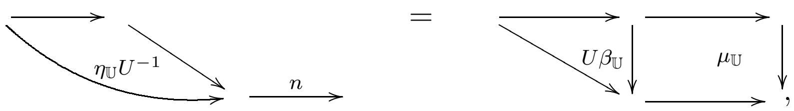


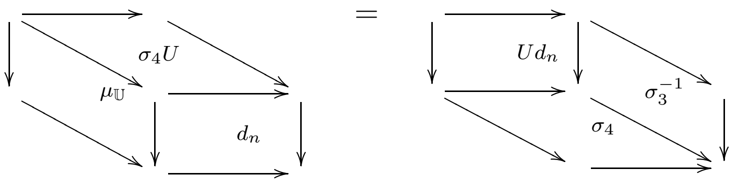
Finally, use (35). ∎

We show now that we do have a distributive law.

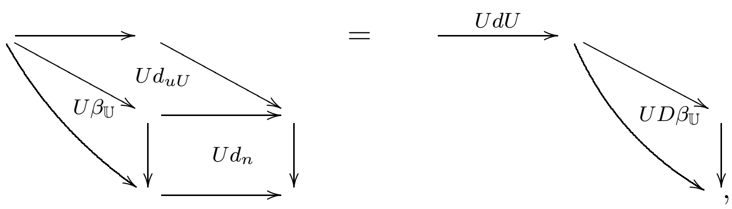
10.2. Theorem. *With the above definitions, $r : UD \to DU$ and the invertible 2-cells $\omega_1, \omega_2, \omega_3$, and ω_4, define a distributive law of $\mathbb{U}$ over $\mathbb{D}$.*

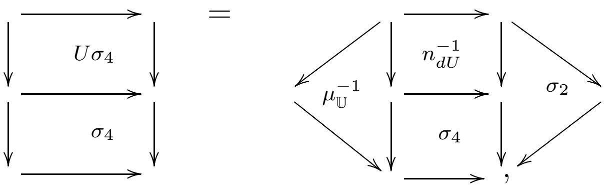
Proof. We show (coh 3, 6 and 8), leaving the rest to the reader. Start on the left hand side of (coh 3) and make the following substitutions:



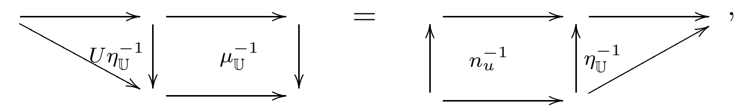


using (27),

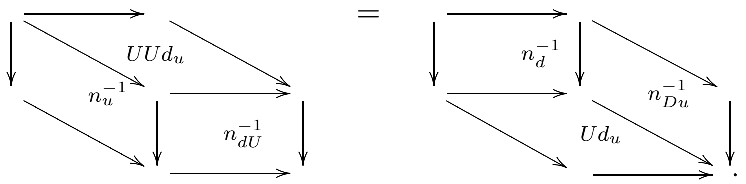




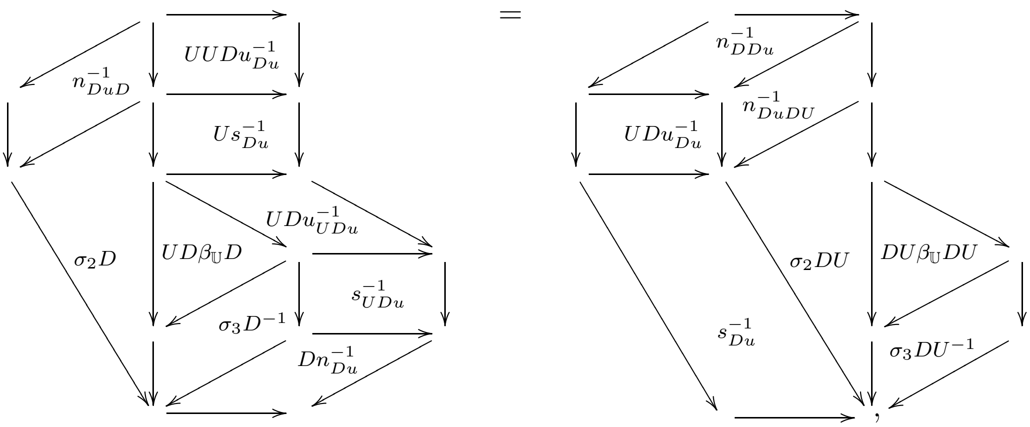
using that $(dU, \sigma_4) : (\beta_{\mathbb{U}}, \mu_{\mathbb{U}}) \to (\sigma_1, \sigma_2)$ is a 1-cell in $\mathbb{U}$-Alg$_{\mathcal{K}}$;

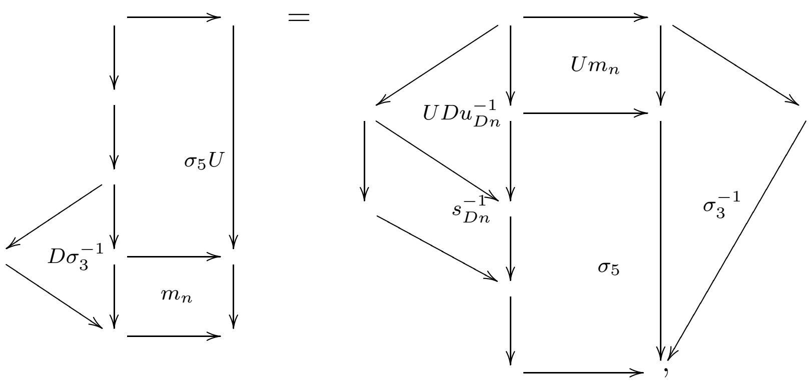


and

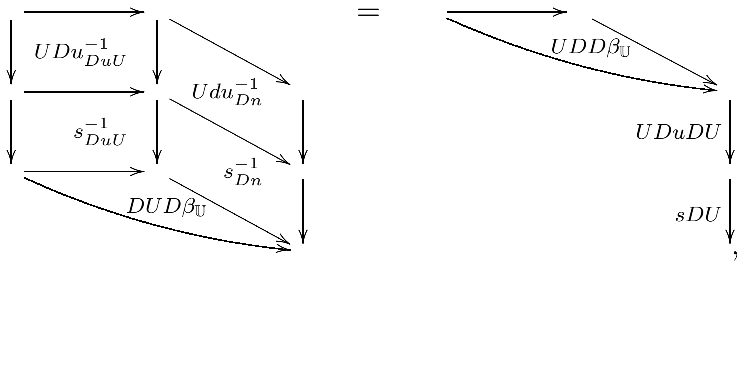


To prove (coh 6), start on the right hand side, and make the following substitutions:





using (29)



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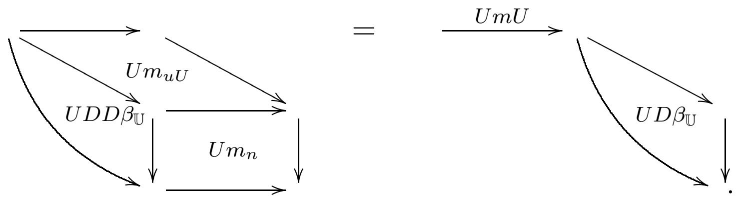
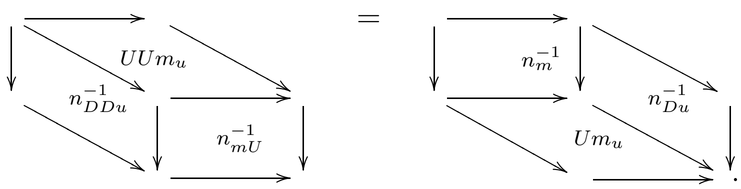
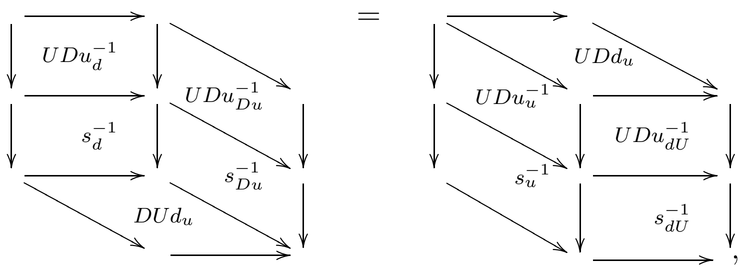


Now use (28). Conclude with the substitution



To prove (coh 8), start on the left hand side, and make the substitutions:



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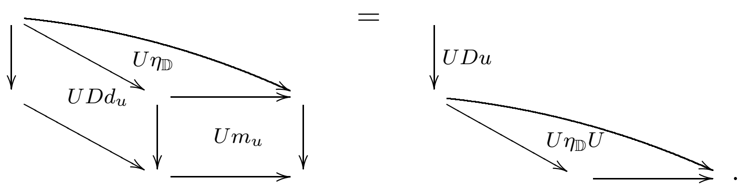


Now, use (30), then the equation obtained from (26). It is easily seen that what remains equals the identity. ∎

11. (Co-)KZ-doctrines over KZ-doctrines

Distributive laws for KZ-doctrines over KZ-doctrines are formally very similar to distributive laws of co-KZ-doctrines over KZ-doctrines. We start this section by recalling the definition and some aspects of KZ-doctrines [9] (see [8] as well). Then we define distributive laws of co-KZ-doctrines over KZ-doctrines, and we show that such a distributive law induces a distributive law between the induced pseudomonads, as defined in section 4. Finally, we show that the composite pseudomonad given by a distributive law of a KZ-doctrine over a KZ-doctrine is again a KZ-doctrine.

11.1. We are still working in a Gray-category $\mathbf{A}$, and $\mathcal{K}$ is an object of $\mathbf{A}$.

11.2. Definition. *A* KZ-doctrine D *on* $\mathcal{K}$ *consists of an object* D*, 1-cells* $d : 1_{\mathcal{K}} \to D$*, and* $m : DD \to D$ *in* $\mathbf{A}(\mathcal{K},\mathcal{K})$ *and a fully-faithful adjoint string* $\eta, \epsilon : Dd \dashv m$*; and* $\alpha, \beta : m \dashv dD : D \to DD$ *such that*

$$\xymatrix{ & & DD \ar[dr]^{m} \ar@{}[d]|{\beta\Downarrow} & \\ 1_{\mathcal{K}} \ar[r]^{d} & D \ar[ur]^{dD} \ar[rr]^{Id_D} \ar[dr]_{Dd} & \ar@{}[d]|{\eta\Downarrow} & D \\ & & DD \ar[ur]_{m} & } \quad = \quad \xymatrix{ & D \ar[dr]^{dD} & & \\ 1_{\mathcal{K}} \ar[ur]^{d} \ar[dr]_{d} & (d_d)^{-1}\Downarrow & DD \ar[r]^{m} & D. \\ & D \ar[ur]_{Dd} & & } \tag{36}$$

The 2-cell $\delta : Dd \to dD$ is defined as the pasting

$$\xymatrix{ D \ar[rr]^{Id_D} \ar[dr]_{dD} & \beta^{-1}\Downarrow & D \ar[dr]^{Dd} & \\ & DD \ar[ur]^{m} \ar[rr]_{Id_{DD}} & \epsilon\Downarrow & DD, }$$

and it satisfies the equations $\delta \circ d = d_d$ and $m \circ \delta = \beta^{-1} \cdot \eta^{-1}$.

In a co-KZ-doctrine U, with $u : 1 \to U$ and $n : UU \to U$, the fully-faithful adjoint string is of the form $uU \dashv n \dashv Uu$.

Proposition 10.1 of [9] says that D induces a pseudomonad $\mathbb{D} = (D, d, m, \beta, \eta, \mu)$ if μ is defined as the pasting

$$\xymatrix{ DDD \ar[rr]^{Id_{DDD}} \ar[dr]_{mD} & \alpha D\Downarrow & DDD \ar[rr]^{Dm} & d_m\Downarrow & DD \ar[dr]^{m} & \\ & DD \ar[ur]_{dDD} \ar[rr]_{m} & & D \ar[ur]_{dD} \ar[rr]_{Id_D} & \beta\Downarrow & D }$$

11.3. Assume we have a KZ-doctrine

$$\mathsf{D} = (D, d, m, \alpha_{\mathsf{D}}, \beta_{\mathsf{D}}, \eta_{\mathsf{D}}, \epsilon_{\mathsf{D}}),$$

and a co-KZ-doctrine $\mathsf{U} = (U, u, n, \alpha_{\mathsf{U}}, \beta_{\mathsf{U}}, \eta_{\mathsf{U}}, \epsilon_{\mathsf{U}})$, with

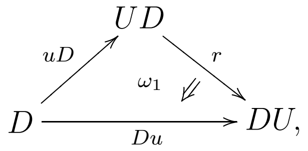
$$\xymatrix{ DD \ar[rr]^{Id} \ar[dr]_{m} & \Downarrow\alpha_{\mathsf{D}} & DD, \\ & D \ar[ur]_{dD} & } \quad \xymatrix{ & DD \ar[dr]^{m} & \\ D \ar[ur]^{dD} \ar[rr]_{Id} & \Downarrow\beta_{\mathsf{D}} & D, } \quad \xymatrix{ D \ar[rr]^{Id} \ar[dr]_{Dd} & \Downarrow\eta_{\mathsf{D}} & D, \\ & DD \ar[ur]_{m} & } \quad \xymatrix{ & D \ar[dr]^{Dd} & \\ DD \ar[ur]^{m} \ar[rr]_{Id} & \Downarrow\epsilon_{\mathsf{D}} & DD, }$$

and

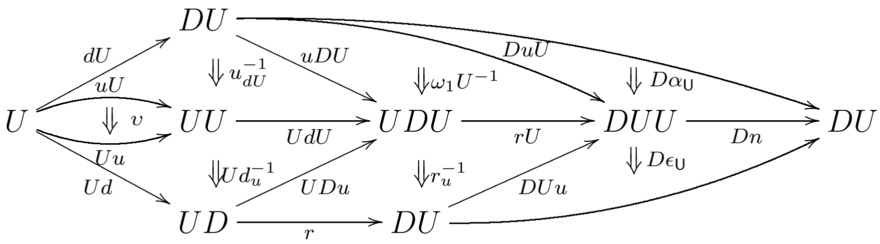
$$\xymatrix{ U \ar[rr]^{Id} \ar[dr]_{uU} & \Downarrow\alpha_{\mathsf{U}} & U, \\ & UU \ar[ur]_{n} & } \quad \xymatrix{ & U \ar[dr]^{uU} & \\ UU \ar[ur]^{n} \ar[rr]_{Id} & \Downarrow\beta_{\mathsf{U}} & UU, } \quad \xymatrix{ UU \ar[rr]^{Id} \ar[dr]_{n} & \Downarrow\eta_{\mathsf{U}} & UU, \\ & U \ar[ur]_{Uu} & } \quad \xymatrix{ & UU \ar[dr]^{n} & \\ U \ar[ur]^{Uu} \ar[rr]_{Id} & \Downarrow\epsilon_{\mathsf{U}} & U. }$$

Since we are assuming D to be a KZ-doctrine, and U to be a co-KZ-doctrine, then β_{D} and η_{D}, and α_{U}, and ϵ_{U} are invertible. Let $\delta : Dd \to dD$ and $\upsilon : uU \to Uu$ be the corresponding induced 2-cells.

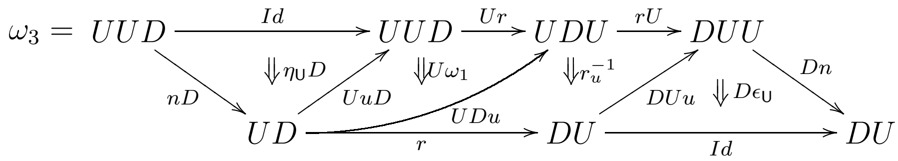
11.4. Definition. *A distributive law of* U *over* D *consists of a 1-cell* $r : UD \to DU$ *in* $\mathbf{A}(\mathcal{K},\mathcal{K})$*, together with an invertible 2-cell*



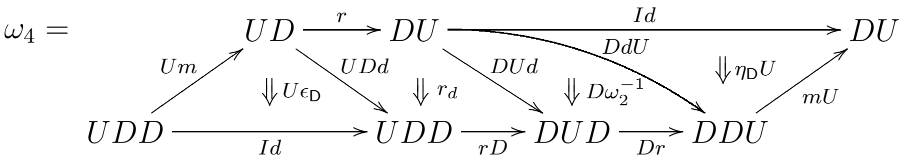
such that the pasting



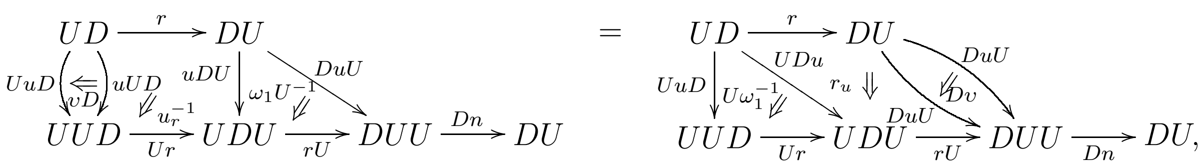
is invertible. We denote the inverse by ω_2*. We further require the pasting*



to be invertible, and the pasting



to be invertible. We also require the following two coherence conditions:



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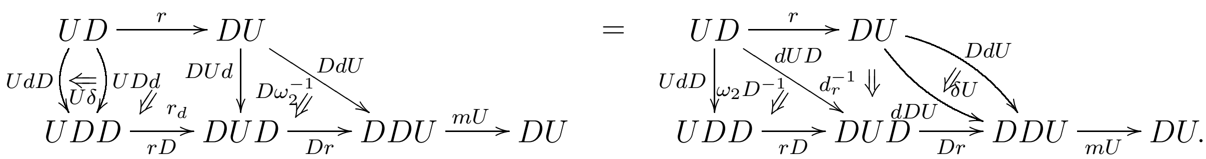
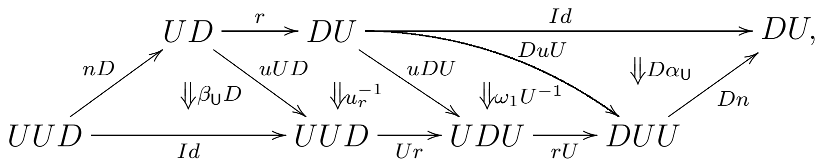
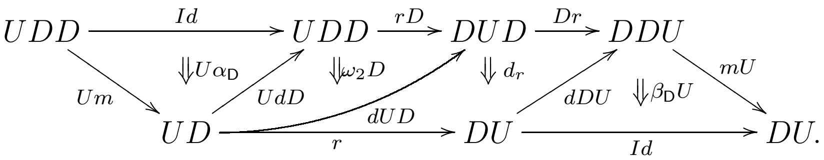


Using the equations $\upsilon \circ n = \eta_{\mathsf{U}} \cdot \beta_{\mathsf{U}}$ and $\delta \circ m = \alpha_{\mathsf{D}} \cdot \epsilon_{\mathsf{D}}$, and the coherence conditions, it is not hard to see that the inverse of ω_3 is



and the inverse of ω_4 is



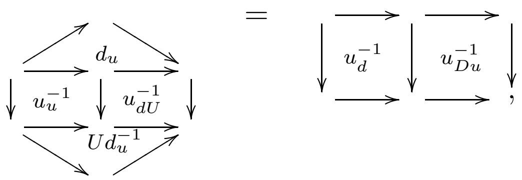
Looking at the definition it seems possible to define a distributive law in terms of ω_2, defining ω_1 with a similar pasting as ω_2 was defined above. It can be shown that if we start with ω_1, define ω_2 as above, and then use a similar pasting to define a new ω_1' we have that $\omega_1 = \omega_1'$. And similarly if we start with ω_2.

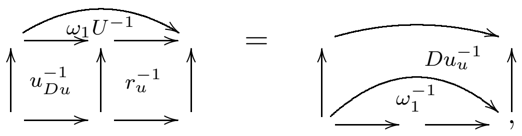
11.5. Theorem. *If $\mathbb{D}$ and $\mathbb{U}$ are the pseudomonads determined by the KZ-doctrine D and the co-KZ-doctrine U respectively, then r together with ω_1, ω_2, ω_3 and ω_4 define a distributive law of $\mathbb{U}$ over $\mathbb{D}$.*

Proof. Since U is a co-KZ-doctrine, we have that $\beta_{\mathbb{U}} = \alpha_{\mathsf{U}}^{-1}$ and $\eta_{\mathbb{U}} = \epsilon_{\mathsf{U}}^{-1}$. With this in mind, we must show that the coherence conditions (coh 1) to (coh 9) of section 4 are satisfied. We show that (coh 1), (coh 3) and (coh 4) are satisfied and leave the others to the reader.

Consider the inverse of the left hand side of (coh 1). To get the inverse of the right hand side make the following four substitutions

$$\upsilon \circ u = u_u^{-1},$$





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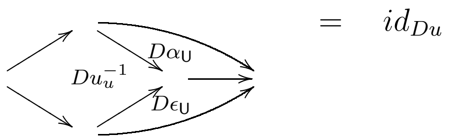
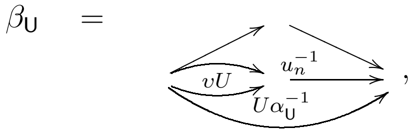
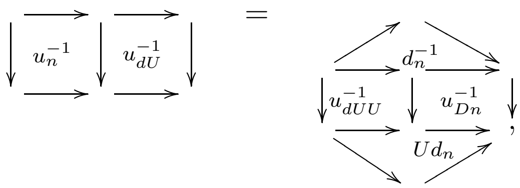
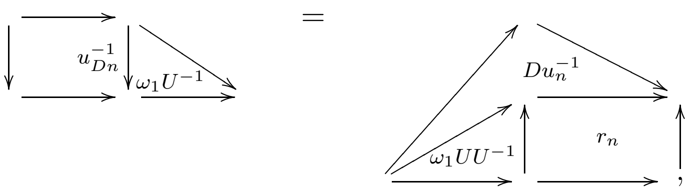


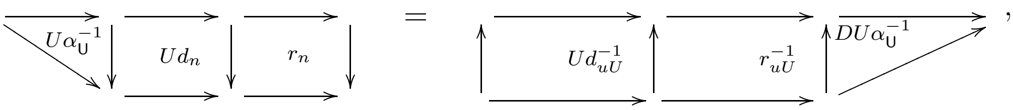
To prove (coh 3) start with the inverse of the right hand side and make the following substitutions:

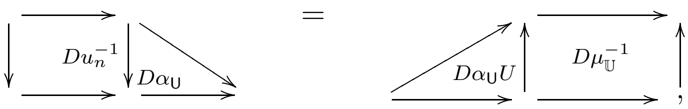
$$v \circ n = \eta_{\mathsf{U}} \cdot \beta_{\mathsf{U}},$$











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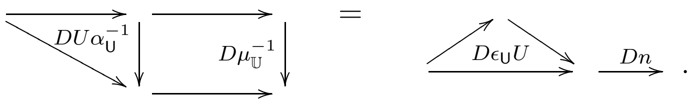
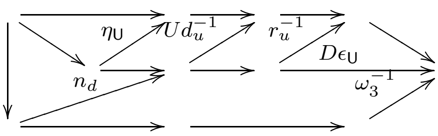


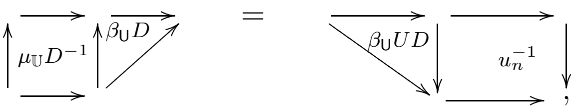
Comparing the pasting obtained with the left hand side of (coh 3), we see that we must show that

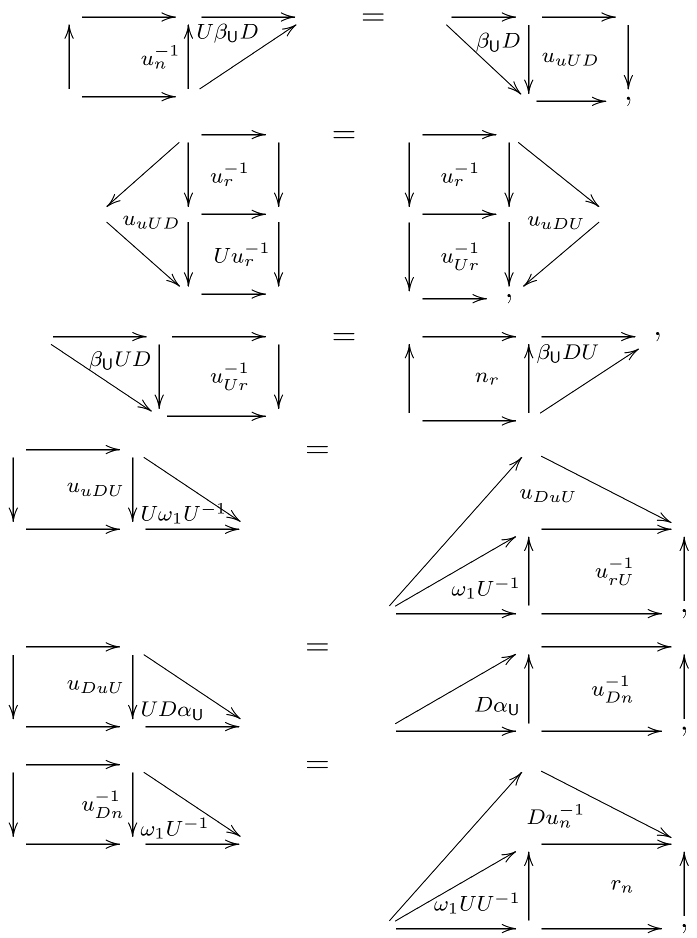


$$= \quad Dn \circ rU \circ U\omega_2^{-1}.$$

Now, pass every 2-cell of the left hand side to the right hand side, except for η_{U}. It is not hard to see, with the help of (coh 1), that the resulting equation is satisfied.

Start with the inverse of the left hand side of (coh 4), and make the following sequence of substitutions:





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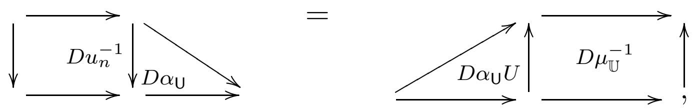

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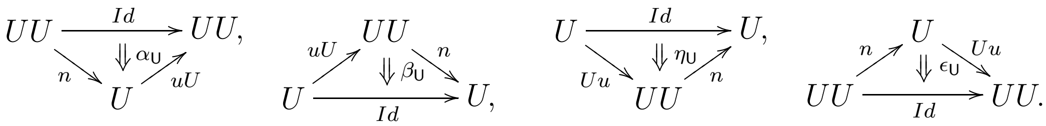
11.6. Assume now that D is a KZ-doctrine as before, but that U is also a KZ-doctrine. The data for D is unchanged, and the data for U is now:

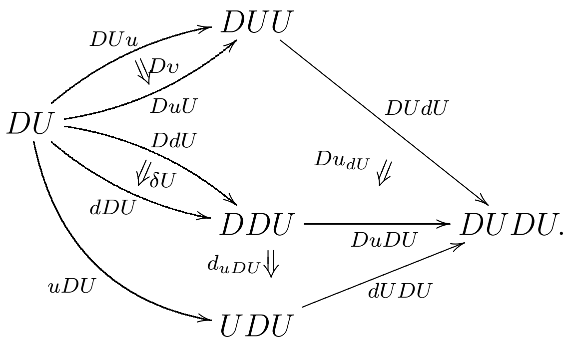


We have $\upsilon : Uu \to uU$.

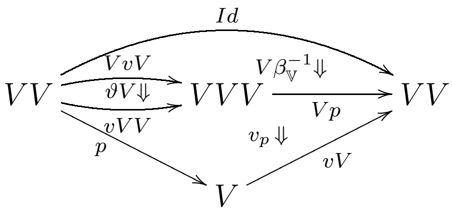
Copying almost exactly the definition of a distributive law of a co-KZ-doctrine over a KZ-doctrine, we obtain the concept of a distributive law of the KZ-doctrine U over the KZ-doctrine D. If $\mathbb{U}$ and $\mathbb{D}$ are the pseudomonads induced by U and D respectively, we obtain, in a very similar way, a distributive law between the pseudomonads $\mathbb{U}$ and $\mathbb{D}$. Let $\mathbb{V}$ be the composite pseudomonad obtained from this last distributive law, as in section 5.

11.7. Theorem. *The composite pseudomonad obtained from a distributive law between KZ-doctrines is again a KZ-doctrine.*

Proof. We follow the notation introduced before the statement of the theorem, and the notation for $\mathbb{V}$ is the same as in section 5. According to theorem 11.1 of [9], it suffices to show that $p \dashv vV$ with counit $\beta_{\mathbb{V}}$. Define $\vartheta : Vv \to vV$ as the pasting



It is not hard to see that $\vartheta \circ v = p_p$ and that $p \circ \vartheta = \beta_{\mathbb{V}}^{-1} \cdot \eta_{\mathbb{V}}^{-1}$. Using these two equations, an easy calculation shows that $p \dashv vV$ with unit



and counit $\beta_{\mathbb{V}}$. ■

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