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COMPLETIONS IN BIAFFINE SETS

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Abstract. The theory of completion of T_0 objects in categories of affine objects over a given complete category developed by the second author in [12] is extended to the case of T_0 objects in categories of 2affine objects. In the paper the case of the category **Set** and target object the two-point set is studied in detail and an internal characterization of 2affine sets is provided.

1. Introduction

Let $\mathbf{X}$ be a complete category endowed with a proper factorization structure for morphisms and A an $\mathbf{X}$-object. The category $\mathbf{X}(A)$ of affine $\mathbf{X}$-objects over A is defined as follows: the objects are the pairs $(X, \mathcal{U})$ where $X \in \mathbf{X}$ and $\mathcal{U}$ is a subset of the set $\mathbf{X}(X, A) = A^X$ of all $\mathbf{X}$-morphisms from X to A and the morphisms (affine X-morphisms) from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ are $\mathbf{X}$-morphisms $f : X \longrightarrow Y$ such that $v \circ f \in \mathcal{U}$ whenever $v \in \mathcal{V}$.

It is shown in [12] that the category $\mathbf{X}(A)$, which is topological over $\mathbf{X}$, has a natural closure operator, called Zariski closure operator (first considered by Skula [19] in the category of topological spaces), and that the subcategory $\mathbf{T}_0\mathbf{X}(A)$ of all affine $\mathbf{X}$-objects whose diagonal is Zariski closed admits completions (admits a firm reflection in the terminology of [2] and [3]). Moreover Zariski compactness ([4]) implies completeness.

In [10, 11] (see also [5] and [14]) the second author studied in detail the case $\mathbf{X} = \mathbf{Set}$ with A= two-point set, extended the above result to every hereditary coreflective subcategory of $\mathbf{Set}(\{0, 1\})$ (there denoted by **SSet**) and provided an internal characterization of complete (algebraic in [6]) objects, explaining that “completion” coincides with “sobrification” in the familiar examples.

As the theory of topological spaces has a parallel theory of bitopological spaces, the theory of affine objects admits a parallel theory of 2affine objects.

A 2affine object over A is a triple $(X, \mathcal{P}, \mathcal{Q})$, with X a $\mathbf{X}$-object and $\mathcal{P}, \mathcal{Q}$ affine structures over A. An affine $\mathbf{X}$-morphism from $(X, \mathcal{P}, \mathcal{Q})$ to $(X', \mathcal{P}', \mathcal{Q}')$ is an affine $\mathbf{X}$-morphism both from $(X, \mathcal{P})$ to $(X', \mathcal{P}')$ and from $(X, \mathcal{Q})$ to $(X', \mathcal{Q}')$.

The aim of the paper is to study in detail (as in [10, 11]) the case $\mathbf{X} = \mathbf{Set}$, with factorization structure (onto, one to one) and target the two-point set $\{0, 1\}$.

In Section 2 the category **2SSET** of 2affine sets over $\{0, 1\}$ is introduced and some

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elementary properties are listed. In Section 3 and Section 4 the Zariski closure and the separated (T_0) 2affine sets are introduced and studied, with emphasis on epimorphisms and regular monomorphisms.

Section 5 contains the main result: (*Theorem* 5.5), the category $\mathbf{2SSET}_0$ of separated 2affine sets admits completions. That is: $\mathbf{2SSET}_0$ admits a (unique determined) subcategory $\mathbf{C}$ which is embedding reflective and such that every z-dense embedding into an element of $\mathbf{C}$ is a $\mathbf{C}$-reflection map (i.e. $\mathbf{C}$ is a firm reflection in the terminology of [2, 3]). Moreover (*Proposition* 5.7) a separated 2affine set is complete if and only if it is injective with respect to Zariski-dense embeddings, if and only if it is Zariski-closed in every separated 2affine set in which it can be embedded.

In Section 6 the above results are extended to all hereditary coreflective subcategories of $\mathbf{2SSET}$, recovering, among others, the fact that the category of all T_0 bitopological spaces admits completions (cf. [13]).

Section 7 contains a general method to provide hereditary coreflective subcategories of $\mathbf{2SSET}$ and Section 8 contains an internal characterization of complete 2affine sets.

The categorical terminology is that of [1] and [16] and for the categorical theory of closure operators we refer to [8].

2. The category **2SSet** of 2affine sets

A *2affine set* over the two point set $S = \{0, 1\}$ is a triple $(X, \mathcal{P}, \mathcal{Q})$, where X is a set and $\mathcal{P}$ and $\mathcal{Q}$ are subsets of the power set S^X .

A *2affine map* from $(X, \mathcal{P}, \mathcal{Q})$ to $(Y, \mathcal{L}, \mathcal{R})$ is a function $f : X \longrightarrow Y$ such that $l \circ f \in \mathcal{P}$ whenever $l \in \mathcal{L}$ and $r \circ f \in \mathcal{Q}$ whenever $r \in \mathcal{R}$.

2SSet will denote the category of 2affine sets (over S) and 2affine maps.

Since the elements of $\mathcal{P}$ and $\mathcal{Q}$ are characteristic functions of X, we can also describe **2SSet** using subsets: objects are triples $(X, \mathcal{P}, \mathcal{Q})$ where X is a set and $\mathcal{P}$ and $\mathcal{Q}$ are subsets of the power set $\mathcal{P}$(X), and the morphisms from $(X, \mathcal{P}, \mathcal{Q})$ to $(Y, \mathcal{L}, \mathcal{R})$ are functions $f : X \longrightarrow Y$ such that $f^{-1}(L) \in \mathcal{P}$ for every $L \in \mathcal{L}$ and $f^{-1}(R) \in \mathcal{Q}$ for every $R \in \mathcal{R}$. When we use this description we speak about $\mathcal{P}$-open and $\mathcal{Q}$-open sets and 2-continuous maps.

Both descriptions of **2SSet** will be used throughout the paper.

We will denote by $U : \mathbf{2SSet} \longrightarrow \mathbf{Set}$ the obvious forgetful functor.

2.1. Proposition. $(\mathbf{2SSet}, U)$ *is a topological category.*

Proof. To show that every U-structured source admits a unique initial lift, let X be a set, $\{(Y_i, \mathcal{L}_i, \mathcal{R}_i) \mid i \in I\}$ a family of 2affine sets and $\mathcal{F} = \{f_i : X \longrightarrow Y_i \mid i \in I\}$ a family of functions. The $(\mathcal{P}, \mathcal{Q})$ structure on X, where $\mathcal{P} = \{f_i^{-1}(L) \mid L \in \mathcal{L}_i, i \in I\}$ and $\mathcal{Q} = \{f_i^{-1}(R) \mid R \in \mathcal{R}_i, i \in I\}$, is the unique initial structure on X determined by $\mathcal{F}$. ∎

Thus a 2affine map $f : (X, \mathcal{P}, \mathcal{Q}) \longrightarrow (Y, \mathcal{L}, \mathcal{R})$ is *initial* (with respect to U) if and only if every $P \in \mathcal{P}$ is of the form $f^{-1}(L)$, $L \in \mathcal{L}$, and every $Q \in \mathcal{Q}$ is of the form $f^{-1}(R)$, $R \in \mathcal{R}$.

As usual an initial monomorphism (= injection) will be called *embedding*.

It is clear that every subset M of the underlying set X of a 2affine set $(X, \mathcal{P}, \mathcal{Q})$ carries as initial structure the families of the restrictions to M of the functions in $\mathcal{P}$ and in $\mathcal{Q}$, or, equivalently, the families of subsets $\mathcal{V} = \{P \cap M \mid P \in \mathcal{P}\}$ and $\mathcal{W} = \{Q \cap M \mid Q \in \mathcal{Q}\}$.

In this case we say that $(M, \mathcal{V}, \mathcal{W})$ is a *2affine subset* of $(X, \mathcal{P}, \mathcal{Q})$.

Some consequences (cf.[1]) of *Proposition* 2.1 are collected in the:

2.2. Corollary.

1. *Every F-structured sink $\{g_i : F(X_i, \mathcal{P}_i, \mathcal{Q}_i) \longrightarrow Y \mid i \in I\}$ admits a unique final lift. A surjective and final 2affine map is called* quotient map.
2. *In* **2SSet** *the epimorphisms are the surjective 2affine maps and the monomorphisms are the injective 2affine maps.*
3. *In* **2SSet** *the embeddings coincide with the regular monomorphisms (= equalizers of two 2affine maps).*
4. *Every 2affine map admits an essentially unique (surjective, embedding)-factorization.*
5. **2SSet** *is complete. Every limit is obtained as the initial lift of the corresponding limit in* **Set**.

   In particular, the product of a family $\{(X_i, \mathcal{P}_i, \mathcal{Q}_i) \mid i \in I\}$ of 2affine sets is the cartesian product X of the family $\{(X_i) \mid i \in I\}$ endowed with the 2affine structure $\mathcal{P} = \{p \circ \pi_i \mid p \in \mathcal{P}_i, i \in I\}$ and $\mathcal{Q} = \{q \circ \pi_i \mid q \in \mathcal{Q}_i, i \in I\}$ (or, equivalently, $\mathcal{P} = \{\pi_i^{-1}(P) \mid P \in \mathcal{P}_i, i \in I\}$ and $\mathcal{Q} = \{\pi_i^{-1}(Q) \mid Q \in \mathcal{Q}_i, i \in I\}$), where $\pi_i : X \longrightarrow X_i$ is the i-th projection.
6. **2SSet** *is cocomplete. Every colimit is obtained as the final lift of the corresponding colimit in* **Set**.
7. *Every class $\mathcal{C}$ of 2affine sets admits an epireflective hull $E(\mathcal{C})$ (given by the 2affine subsets of products of elements of $\mathcal{C}$). If $E(\mathcal{C})$ is stable under refinements then $E(\mathcal{C})$ is quotient reflective.*
8. *A full and isomorphism closed subcategory* **A** *of* **2SSet** *is hereditary coreflective if and only if it is stable under coproducts, quotients and embeddings.*

2.3. Example. Every bitopological space (cf. [13]) is a 2affine set and a function between bitopological spaces is continuous if and only if it is 2affine. Thus the category **2Top** of bitopological spaces is fully embedded in **2SSet**.

2.4. Remark. The category **2SSet** is not well-fibred even though every set admits a set of 2affine structures. In fact the empty set admits four 2affine structures. This defect can be removed by assuming that every 2affine structure contains the empty set and the whole set (using the functional description: every 2affine structure contains the constant functions c_0 and c_1). We shall consider this full subcategory of **2SSet**, denoted by **2SSET**, in the next Section.

Note that **2SSET** is universal in the sense of Marny [17].

3. Separated ($= T_0$) 2affine sets

3.1. Definition. *A 2affine set* $(X, \mathcal{P}, \mathcal{Q})$ *is called* separated*, or* T_0*, if* $\mathcal{P} \cup \mathcal{Q}$ *separates the points of* X *i.e. for every* $x \neq y \in X$ *exists* $\alpha \in \mathcal{P} \cup \mathcal{Q}$ *such that* $\alpha x \neq \alpha y$*.*

Equivalently for every $x \neq y \in X$ *there exists* $A \in \mathcal{P} \cup \mathcal{Q}$ *such that* $x \in A$ *and* $y \notin A$*.*

3.2. Remark. Obviously every refinement of a T_0 2affine structure is T_0. $\mathbf{2SSET}_0$ will denote the full subcategory of separates 2affine sets.

3.3. Example. The 2affine set $\mathbf{S} = \{S \times S; \{\pi_1, c_0, c_1\}, \{\pi_2, c_0, c_1\}\}$ is separated.

$\mathbf{S}$ is called *Sierpinski 2affine set.*

3.4. Lemma. *Let* $(X, \mathcal{P}, \mathcal{Q})$ *be a 2affine set; then* $f : (X, \mathcal{P}, \mathcal{Q}) \longrightarrow \mathbf{S}$ *is 2affine if and only if* $\pi_1 \circ f \in \mathcal{P}$ *and* $\pi_2 \circ f \in \mathcal{Q}$*.*

Proof. Trivial. ■

3.5. Proposition. *Let* $(X, \mathcal{P}, \mathcal{Q})$ *be a 2affine set; then:*

1. *the canonical map, defined by* $\phi(x) = \{(px, qx) \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$,

$$\phi : (X, \mathcal{P}, \mathcal{Q}) \longrightarrow \mathbf{S}^{(\mathcal{P} \times \mathcal{Q})}$$

   (which is 2affine) is an initial map.

2. ϕ *is an embedding if and only if* $(X, \mathcal{P}, \mathcal{Q})$ *is separated.*

Proof.

1. The 2affine maps from $(X, \mathcal{P}, \mathcal{Q})$ to $\mathbf{S}$ correspond to pairs in $(\mathcal{P} \times \mathcal{Q})$, by *Lemma* 3.4. Thus, for every $p \in \mathcal{P}$ (resp. $q \in \mathcal{Q}$) there exists $c_0 \in \mathcal{P}$ (resp. $c_0 \in \mathcal{Q}$) such that $p \times c_0, c_0 \times q \in \mathcal{P} \times \mathcal{Q}$; finally we have $p = \pi_1 \circ \pi_{(p \times c_0)} \circ \phi$ and $q = \pi_2 \circ \pi_{(c_0 \times q)} \circ \phi$.
2. Let $x_1 \neq x_2$ in X and assume, by separation of $(X, \mathcal{P}, \mathcal{Q})$, that $p \in \mathcal{P}$ satisfies $px_1 \neq px_2$. Then $(px_1, 0) \neq (px_2, 0)$ and these two elements are the (p)-components of $\phi(x_1)$ and $\phi(x_2)$ respectively. ■

3.6. Proposition. $\mathbf{2SSET}_0$ *is quotient reflective in* **2SSET***.*

Proof. $\mathbf{2SSET}_0$ is the epireflective hull (see *Corollary* 2.2.7) of the Sierpinski 2affine set $\mathbf{S}$ in virtue of *Proposition* 3.5. Then by *Remark* 3.2 it is quotient reflective. ■

4. Zariski closure and epimorphisms

Recall from [10] that an affine set over the two point set $S = \{0, 1\}$ is a pair $(X, \mathcal{U})$, where X is a set and $\mathcal{U}$ is a subset of $\mathcal{P}(X)$, containing X and $\emptyset$ (equivalently : $\mathcal{U}$ is a family of functions from X to S containing the constants).

An affine map is a function inversely preserving the elements of the structures.

If $\mathcal{U}$ separates the points of X, $(X, \mathcal{U})$ is a T_0 (separated) affine set.

The category of (T_0) affine sets is denoted by ($\mathbf{SSET}_0$) $\mathbf{SSET}$.

A closure operator z (called Zariski closure) is introduced and studied to characterize the epimorphisms and regular monomorphisms of $\mathbf{SSET}_0$ and the main result is that $\mathbf{SSET}_0$ admits a firm reflection (cf. [10], Theorem 5.2). Here we will establish a similar existence theorem.

Given an affine set $(X, \mathcal{U})$ and $M \subset X$, the Zariski closure is defined, functionally, by:

$$z_{(X,\mathcal{U})}M = \bigcap\{Eq(u, v) \mid u, v \in \mathcal{U}\, and\, M \subset Eq(u, v)\}$$

and using "open" sets by:

$$z_{(X,\mathcal{U})}M = \{x \in X \mid (\nexists U, V \in \mathcal{U})(U \cap M = V \cap M, x \in (U \setminus V))\}.$$

We want to introduce here an analogous Zariski closure in $\mathbf{2SSET}$ to obtain epimorphisms and regular monomorphisms of $\mathbf{2SSET}_0$.

4.1. Definition. *Let $(X, \mathcal{P}, \mathcal{Q})$ be a 2affine set, and $M \subset X$. Set*

$$z_{(X,\mathcal{P},\mathcal{Q})}M = \left(\bigcap\{Eq(p_1, p_2) \mid p_1, p_2 \in \mathcal{P}\, and\, M \subset Eq(p_1, p_2)\}\right)$$
$$\cap \left(\bigcap\{Eq(q_1, q_2) \mid q_1, q_2 \in \mathcal{Q}\, and\, M \subset Eq(q_1, q_2)\}\right).$$

Equivalently:

$$z_{(X,\mathcal{P},\mathcal{Q})}M = (\{x \in X \mid (\nexists P_1, P_2 \in \mathcal{P})(P_1 \cap M = P_2 \cap M, x \in (P_1 \setminus P_2))\} \cap$$
$$\bigcap\{x \in X \mid (\nexists Q_1, Q_2 \in \mathcal{Q})(Q_1 \cap M = Q_2 \cap M, x \in (Q_1 \setminus Q_2))\}).$$

Thus, by definition

$$z_{(X,\mathcal{P},\mathcal{Q})}M = z_{(X,\mathcal{P})}M \cap z_{(X,\mathcal{Q})}M$$

so (Cf [10], Theorem 3.2) we obtain:

4.2. Theorem. *The Zariski closure in $\mathbf{2SSET}$ is a grounded (i.e. $z(\emptyset) = \emptyset$), idempotent (i.e $z_{(X,\mathcal{P},\mathcal{Q})}(z_{(X,\mathcal{P},\mathcal{Q})}M) = z_{(X,\mathcal{P},\mathcal{Q})}M$) and hereditary (i.e. for every $M \subset Y \subset X$, $z_{(Y,\mathcal{V},\mathcal{W})}M = (z_{(X,\mathcal{P},\mathcal{Q})}M) \cap Y$, where Y is endowed with the initial structure induced by the inclusion $Y \subset X$) closure operator in the sense of [7].*

Proof. The intersection of closure operators is a closure operator and all the additional properties above are preserved. ∎

Throughout the paper we will use a property, weak heredity, which is implied by idempotency and hereditariness of z: for every $M \subset X$

$$z_{z_{(X,\mathcal{P},\mathcal{Q})}M} M = z_{(X,\mathcal{P},\mathcal{Q})} M.$$

4.3. Remark. Note that z is additive in **SSET** while it is not in **2SSET**. (cf. [13], 3.7)

The terms z-dense map, z-closed embedding (subset) have the usual meaning.

4.4. Corollary. *(z-dense map, z-closed embedding) is a factorization structure of* **2SSET**. *That is: every 2-affine map $f : X \longrightarrow Y$ admits a factorization $f = m \circ e$, where e is a z-dense map and m is a z-closed embedding (factorization property); moreover, for every $g \circ e = m \circ f$, with e z-dense map and m z-closed embedding, there exists a unique d such that $m \circ d = g$ and $d \circ e = f$ (diagonalization property).*

Proof. Follows from a general result on idempotent and weakly hereditary closure operators (cf.[7], *Proposition 3.1, 3.2.*) ■

4.5. Corollary. *The epimorphisms in* $\mathbf{2SSET}_0$ *are the z-dense 2affine maps and the regular monomorphisms (= extremal monomorphism) are the z-closed embeddings.*

Proof. It is clear from *Proposition* 3.5, *Proposition* 3.6 and from the definition of the Zariski closure that z is the regular closure operator induced by $\mathbf{2SSET}_0$ in **2SSET**. (cf. [7], section 5)

Thus the epimorphisms in $\mathbf{2SSET}_0$ are the z-dense 2affine maps and, by weak heredity of z, the regular monomorphisms are the z-closed embeddings. ■

5. Complete 2affine sets

Recall, from [3], that a full, isomorphism closed subcategory $\mathcal{C}$ of $\mathbf{2SSET}_0$ is (embedding-) firm reflective if:

(firm1) $\mathcal{C}$ is embedding reflective in $\mathbf{2SSET}_0$, i.e. it is reflective and the reflections maps $r_X : X \longrightarrow RX$, $RX \in \mathcal{C}$ are embeddings in **2SSET**.

(firm2) If $f : X \longrightarrow Y$ is a z-dense embedding in $\mathbf{2SSET}_0$ with $Y \in \mathcal{C}$ then there is an isomorphism $f' : RX \longrightarrow Y$ such that $f' \circ r_X = f$.

5.1. Remark. Note that every category has at most one firm reflective subcategory (cf. [3]).

5.2. Remark. The category of metric spaces and non-expansive maps admits as firm reflective subcategory the one of all complete metric spaces. This is a motivating example for the abstract notion of firm epireflection and motivates even the name *complete* for the members of the firm epireflective subcategory (cf. [2], [3]).

We will show now that $\mathbf{2SSET}_0$ admits a firm reflection (i.e. $\mathbf{2SSET}_0$ admits completions).

5.3. Definition. *$X \in \mathbf{2SSET}_0$ is called* z-injective *if it is injective with respect to the class of z-dense embeddings. That is: for every z-dense embedding $m : M \longrightarrow Y$ and 2affine map $f : M \longrightarrow X$ there exists an 2affine map $f' : Y \longrightarrow X$ such that $f' \circ m = f$.*

We denote with $Inj(\mathbf{2SSET}_0)$ the full subcategory of all z-injective 2affine sets.

The 2affine set $\mathbf{S}$ is trivially z-injective and the class of all z-injective 2affine sets is stable under products and z-closed embeddings by a standard argument, so that $Inj(\mathbf{2SSET}_0)$ is epireflective in $\mathbf{2SSET}_0$ by *Corollary* 4.5. In addition we have:

5.4. Proposition. *$Inj(\mathbf{2SSET}_0)$ is embedding reflective in $\mathbf{2SSET}_0$.*

Proof. Let $(X, \mathcal{P}, \mathcal{Q})$ be a T_0 2affine set and $r_X : X \longrightarrow RX$ the restriction of the canonical map ϕ of *Proposition* 3.5.2, to $z_{\mathbf{S}^{\mathcal{P}\times\mathcal{Q}}}(\phi X) = RX$. Since z is idempotent RX is z-closed in the z-injective 2affine set $\mathbf{S}^{\mathcal{P}\times\mathcal{Q}}$, so that RX is z-injective.

Let now $f : X \longrightarrow Y$ be a 2affine map with $Y \in Inj(\mathbf{2SSET}_0)$. Then, by z-injectivity of Y, there is a 2affine map $f' : RX \longrightarrow Y$ such that $f' \circ r_X = f$. Now r_X is an embedding, and it is z-dense since z is weakly hereditary. ∎

5.5. Theorem. *$Inj(\mathbf{2SSET}_0)$ is firm reflective in $\mathbf{2SSET}_0$.*

Proof. Thanks to *Proposition* 5.4 we need to show that the uniqueness property **(firm2)** is fulfilled.

Now let $f : X \longrightarrow Y$ be as in **(firm 2)**. Then by the universal property of reflections there exists a (unique) $f' : RX \longrightarrow Y$ such that $f' \circ r_X = f$.

On the other hand RX is z-injective and f is a z-dense embedding so there exists $r' : Y \longrightarrow RX$ with $r' \circ f = r_X$.

Now

$$(r' \circ f') \circ r_X =$$
$$r' \circ (f' \circ r_X) =$$
$$r' \circ f = r_X =$$
$$1_{RX} \circ r_X$$

so that $r' \circ f' = 1_{RX}$ by r_X (z-dense, hence) epimorphism.

Furthermore

$$(f' \circ r') \circ f =$$
$$f' \circ (r' \circ f) =$$
$$f' \circ r_X = f =$$
$$1_Y \circ f$$

so that $f' \circ r' = 1_Y$ by f (z-dense, hence) epimorphism.

Thus f' is an isomorphism and the proof is complete. ∎

5.6. DEFINITION. *$X \in \mathbf{2SSET}_0$ is called* absolutely z-closed *(resp.* z-saturated*) if for every embedding (resp. z-dense embedding) $k : X \longrightarrow Y$, $Y \in \mathbf{2SSET}_0$, kX is z-closed in Y (resp. k is an isomorphism).*

5.7. PROPOSITION. *For $X \in \mathbf{2SSET}_0$ equivalent are:*

1. *X is z-injective*
2. *X is absolutely z-closed*
3. *X is z-saturated.*

PROOF. 1 $\Rightarrow$ 2 Let X be z-injective, $f : X \longrightarrow Y$ be an embedding with $Y \in \mathbf{2SSET}_0$ and $f = m \circ e$ a factorization of f, with e z-dense map and m z-closed embedding.

We need to show that e is an isomorphism. Since X is z-injective the identity 1_X admits an extension $\alpha : M \longrightarrow X$, i.e. there exists α such that $\alpha \circ e = 1_X$. Thus e is a section and an embedding so it is an isomorphism.

2 $\Rightarrow$ 3 Trivial.

3 $\Rightarrow$ 1 Let X be z-saturated. Then $r_X : X \longrightarrow RX$ is a z-dense embedding (cf. *Proposition* 5.4) and so it is an isomorphism and $RX \in Inj(\mathbf{2SSET}_0)$. ■

6. Completions in coreflective subcategories

Let $\mathbf{X}$ be a (bi-)coreflective subcategory of $\mathbf{2SSET}$; since $\mathbf{2SSET}$ is topological over $\mathbf{Set}$, $\mathbf{X}$ bicoreflective means that $\mathbf{X}$ is stable under quotients (= final surjective 2affine maps) and coproducts (= disjoint unions with final structures). Furthermore it is not restrictive to assume that the coreflection $c_X : X \longrightarrow CX$ of X in $\mathbf{X}$ is the identity 1_X in the underlying sets.

Then $\mathbf{X}$ is still a topological category over $\mathbf{Set}$ (cf. [1]), thus in particular (surjective map, embedding (in $\mathbf{X}$)) is a factorization structure of $\mathbf{X}$.

Let us assume, in addition, that $\mathbf{X}$ is *hereditary* in $\mathbf{2SSET}$; that is: if $f : X \longrightarrow Y$ is an embedding with $Y \in \mathbf{X}$ then $X \in \mathbf{X}$.

Then the above factorization structure coincides with the restriction to $\mathbf{X}$ of the factorization structure of $\mathbf{2SSET}$.

Again by heredity of $\mathbf{X}$ the Zariski closure in $\mathbf{2SSET}$ can be restricted to $\mathbf{X}$.

Let $\hat{\mathbf{S}}$ be the coreflection of $\mathbf{S}$ in $\mathbf{X}$.

6.1. LEMMA. *Let $(X, \mathcal{P}, \mathcal{Q}) \in \mathbf{X}$ and $f : X \longrightarrow S \times S$ be a function. Then $f : (X, \mathcal{P}, \mathcal{Q}) \longrightarrow \hat{\mathbf{S}}$ is 2affine if and only if $f : (X, \mathcal{P}, \mathcal{Q}) \longrightarrow \mathbf{S}$ is 2affine.*

PROOF. The proof directly follows by the universal property of bicoreflections. ■

Set $\mathbf{X}_0 = \mathbf{X} \cap \mathbf{2SSET}_0$.

6.2. PROPOSITION.

(a) *The (z-dense map, z-closed embedding)-factorization of* $\mathbf{2SSET}_0$ *can be restricted to* $\mathbf{X}_0$.

(b) *The inclusion functor* $K : \mathbf{X}_0 \longrightarrow \mathbf{2SSET}_0$ *preserves epimorphisms and regular (= extremal) monomorphisms.*

PROOF.

(a) Directly follows from the heredity of $\mathbf{X}$.

(b) *Lemma* 3.4 together with *Lemma* 6.1 say that the Zariski closure of $\mathbf{2SSET}$, restricted to $\mathbf{X}$, is the regular closure of $\mathbf{X}$ induced by $\mathbf{X}_0$. So the proof follows from heredity of z.

∎

Now we are ready for the proof of :

6.3. THEOREM. *Let* $\mathbf{X}$ *be a bicoreflective and hereditary subcategory of* $\mathbf{2SSET}$ *and let* $\mathbf{X}_0$ *consist of* T_0 *2affine sets in* $\mathbf{X}$. *Then* $\mathbf{X}_0$ *admits a firm epireflection given by* $Inj(\mathbf{X}_0)$, *i.e. the z-injective 2affine sets of* $\mathbf{X}_0$.

PROOF. Follows the lines of the proof of *Proposition* 5.4 and *Theorem* 5.5, with $Inj(\mathbf{X}_0)$, $\mathbf{X}_0$ and $\hat{\mathrm{S}}$ instead of $Inj(\mathbf{2SSET}_0)$, $\mathbf{2SSET}_0$ and S, taking in account *Proposition* 6.2 and *Lemma* 6.1. ∎

6.4. COROLLARY. *The complete 2affine sets of* $\mathbf{X}_0$ *are precisely the z-closed 2affine subsets of products in* $\mathbf{X}$ *of* $\hat{\mathrm{S}}$.

6.5. REMARK. Products in $\mathbf{X}$ are the coreflections of products in $\mathbf{2SSET}$.

6.6. COROLLARY. *For* $X \in X_0$ *equivalent are:*

1. X *is z-injective in* $\mathbf{X}_0$
2. X *is absolutely z-closed in* $\mathbf{X}_0$
3. X *is z-saturated in* $\mathbf{X}_0$.

PROOF. As in *Proposition* 5.7. ∎

7. Hereditary coreflections defined by algebra structures

In this section we give a general method of constructing hereditary coreflections.

Recall that an *algebra structure* in the set S is a family of functions

$$\Omega = \{\omega_T : S^T \longrightarrow S\}$$

where T runs in a given class of sets.

Then, for every set X, the power set S^X carries a Ω-algebra structure defined, for every operation ω_T on S and family of functions $(f_t : X \longrightarrow S)_{t \in T}$, by

$$\omega_T^X(f_t) = \omega_T \circ (< f_t >).$$

If Ω_1 and Ω_2 are algebra structures in S, we denote by $\mathbf{2SSET}(\Omega_1, \Omega_2)$ the subcategory of $\mathbf{2SSET}$ consisting of those 2affine sets $(X, \mathcal{P}, \mathcal{Q})$ for which $\mathcal{P}$ and $\mathcal{Q}$ are respectively Ω_1- and Ω_2-subalgebras of the function algebra S^X.

7.1. PROPOSITION. *$\mathbf{2SSET}(\Omega_1, \Omega_2)$ is a hereditary coreflective subcategory of $\mathbf{2SSET}$.*

PROOF. For every $(X, \mathcal{P}, \mathcal{Q}) \in \mathbf{2SSET}$ the object $(X, \Omega_1(\mathcal{P}), \Omega_2(\mathcal{Q}))$, where $\Omega_1(\mathcal{P})$ is the Ω_1-subalgebra generated by $\mathcal{P}$ (i.e. the intersection of all Ω_1-subalgebras containing $\mathcal{P}$) and $\Omega_2(\mathcal{Q})$, is the Ω_2-subalgebra generated by $\mathcal{Q}$ is in $\mathbf{2SSET}(\Omega_1, \Omega_2)$.

$$1_X : (X, \Omega_1(\mathcal{P}), \Omega_2(\mathcal{Q})) \longrightarrow (X, \mathcal{P}, \mathcal{Q})$$

is the $\mathbf{2SSET}(\Omega_1, \Omega_2)$-coreflection of $(X, \mathcal{P}, \mathcal{Q})$.

Moreover the structure in a 2affine subset is the restriction to the subset of the two structures in the set, so that $\mathbf{2SSET}(\Omega_1, \Omega_2)$ is also hereditary. ∎

7.2. EXAMPLE.

1. Let us consider the algebra structures

$$\Omega_1 = \Omega_2 = \{(\bigvee_T : S^T \longrightarrow S \mid \text{T any set}) \bigcup (c_1 : S \longrightarrow S)\} = \Omega.$$

   Then $\mathbf{2SSET}(\Omega_1, \Omega_2)$ is the category of *2closure spaces*, in fact the objects are 2affine sets $(X, \mathcal{P}, \mathcal{Q})$, with $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ closure spaces and the morphisms are functions continuous with respect to both the structures. We will denote the category of 2closure spaces by $\mathbf{2CS}$. (Recall that a closure space $(X, \mathcal{C})$ is a pair, where X is a set and $\mathcal{C}$ is a subset of the power set $\mathcal{P}(X)$ satisfying the condition that X belongs to $\mathcal{C}$ and that $\mathcal{C}$ is stable under arbitrary unions, and that a morphism between closure spaces is an ordinary continuous map (cf. [5]).)

2. Take

$$\Omega_1 = \Omega_2 = \{(\bigvee_T : S^T \longrightarrow S \mid \text{T any set}), (\bigwedge_F : S^F \longrightarrow S \mid \text{F any finite set})\} = \Omega'.$$

2SSET(Ω_1, Ω_2) is the category of *bitopological spaces*, usually denoted by **2Top** (cf. [13], [15]).

The fact that $\mathbf{2Top}_0$ has a firm epireflection was firstly shown in [13].

3. Let us consider the algebra structures

$$\Omega_1 = \Omega_2 = \{(\bigvee_F : S^F \longrightarrow S \mid \text{F any finite set}) \cup (\bigwedge_F : S^F \longrightarrow S \mid \text{F any finite set}) \cup$$
$$\cup (\tau : S \longrightarrow S \mid \tau(0) = 1 \text{ e } \tau(1) = 0)\} = \Omega''.$$

This is a Boolean algebra structure; **2SSET**(Ω_1, Ω_2) is the category of *2fields of sets*.

4. Every combination of two of the previous Ω-structures gives a new coreflective subcategory. For example **2SSET**(Ω', Ω'') is the category of 2affine sets $(X, \mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ is a topology and $\mathcal{Q}$ is a field of subsets of X.

Moreover there is a functor

$$F : \mathbf{Top} \longrightarrow \mathbf{2SSET}(\Omega_1, \Omega_2)$$

such that $F(X, \mathcal{P}) = (X, \mathcal{P}, \mathcal{Q})$, where $\mathcal{Q}$ is the family of clopen sets in $(X, \mathcal{P})$. We can also define the functor

$$F' : \mathbf{2SSET}(\Omega_1, \Omega_2) \longrightarrow \mathbf{Top}$$

such that $F(X, \mathcal{P}, \mathcal{Q}) = (X, \mathcal{U})$ where $\mathcal{U}$ is the topology generated by $\mathcal{P} \cup \mathcal{Q}$. As $F \circ F' = 1_{\mathbf{Top}}$ F is a retraction.

Question: We don't know any non trivial (i.e. $\mathcal{P} = P(X)$ or $\mathcal{Q} = P(X)$) example of hereditary coreflective subcategory $\mathbf{X}_0$ of $\mathbf{2SSET}_0$ which is not of the form **2SSET**(Ω_1, Ω_2).

8. The structure of complete T_0 2affine sets

8.1. Definition. *Let $(X, \mathcal{P}, \mathcal{Q}) \in$ **2SSET** and $a \subset \mathcal{P}$, $b \subset \mathcal{Q}$ both containing the whole subset X and not the empty set.*

The one-point extension of $(X, \mathcal{P}, \mathcal{Q})$ defined by the pair (a, b) is the 2affine set denoted $(X_\infty, \mathcal{P}_a, \mathcal{Q}_b)$ with $X_\infty = X \cup \infty$, $\infty \notin X$, and $\mathcal{P}_a, \mathcal{Q}_b$ respectively defined by

$$\mathcal{P}_a = \{A \cup \infty \mid A \in a\} \cup (\mathcal{P} \setminus a)$$
$$\mathcal{Q}_b = \{B \cup \infty \mid B \in b\} \cup (\mathcal{Q} \setminus b).$$

Since a and b contain X and not $\emptyset$, $\mathcal{P}_a$ and $\mathcal{Q}_b$ contain $X \cup \{\infty\}$ and the empty set so $(X_\infty, \mathcal{P}_a, \mathcal{Q}_b) \in$ **2SSET** and $(X, \mathcal{P}, \mathcal{Q})$ is clearly a 2affine subset of $(X_\infty, \mathcal{P}_a, \mathcal{Q}_b) \in$ **2SSET**.

Moreover $\infty \in zX$ since, for every $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ never both P and $P \cup \{\infty\}$ or both Q and $Q \cup \{\infty\}$ belong respectively to $\mathcal{P}_a$ or to $\mathcal{Q}_b$.

Let $(X, \mathcal{P}, \mathcal{Q})$ be a 2affine set and $x \in X$. We will denote by $\mathcal{P}_x$ and $\mathcal{Q}_x$ respectively the set of $\mathcal{P}$-open sets containing x and of $\mathcal{Q}$-open sets containing x.

Assume now that $(X, \mathcal{P}, \mathcal{Q})$ is separated. Then the one point extension $(X_\infty, \mathcal{P}_a, \mathcal{Q}_b)$, defined by (a, b) is separated if and only if for all $x \in X$ $(a, b) \neq (\mathcal{P}_x, \mathcal{Q}_x)$.

8.2. Theorem. *$(X,\mathcal{P},\mathcal{Q}) \in \mathbf{2SSET}_0$ is complete if and only if for every pair (a,b) as above there exists (unique) $x \in X$ such that $(a,b) = (\mathcal{P}_x, \mathcal{Q}_x)$.*

Proof. $(\Rightarrow)$

Assume there exists (a,b) such that $(a,b) \neq (\mathcal{P}_x, \mathcal{Q}_x)$ for every $x \in X$ and consider the inclusion map:

$$k : (X,\mathcal{P},\mathcal{Q}) \longrightarrow (X_\infty, \mathcal{P}_a, \mathcal{Q}_b).$$

¿From the above observations we know that k is a z-dense embedding in a separated 2affine set, so, by *Theorem* 5.5 and *Proposition* 5.7, $(X,\mathcal{P},\mathcal{Q})$ is not complete.

$(\Leftarrow)$

Assume that $(X,\mathcal{P},\mathcal{Q})$ is not complete. Since the Zariski closure is weakly hereditary, we may assume that there exists a separated 2affine set $(X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$ such that the inclusion map:

$$k : (X,\mathcal{P},\mathcal{Q}) \longrightarrow (X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$$

is a z-dense embedding. Set

$$a = \{P \in \mathcal{P} \mid P \notin \mathcal{V}\}$$

and

$$b = \{Q \in \mathcal{Q} \mid Q \notin \mathcal{W}\}.$$

Since $(X,\mathcal{P},\mathcal{Q})$ is a 2affine subset of $(X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$ we have $P \cup \{\infty\} \in \mathcal{V}$ whenever $P \in a$ and $Q \cup \{\infty\} \in \mathcal{W}$ whenever $Q \in b$, i.e. $a \subset \{P \in \mathcal{P} \mid P \cup \{\infty\} \in \mathcal{V}\}$ and $b \subset \{Q \in \mathcal{Q} \mid Q \cup \{\infty\} \in \mathcal{W}\}$.

On the other hand, by X z-dense in $(X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$ we have $P \in a$ whenever $P \cup \{\infty\} \in \mathcal{V}$ and $Q \in b$ whenever $Q \cup \{\infty\} \in \mathcal{W}$, i.e. $\{P \in \mathcal{P} \mid P \cup \{\infty\} \in \mathcal{V}\} \subset a$ and $\{Q \in \mathcal{Q} \mid Q \cup \{\infty\} \in \mathcal{W}\} \subset b$.

This means that $(X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$ is the one-point extension of $(X,\mathcal{P},\mathcal{Q})$ defined by the pair (a,b). As $(X \cup \{\infty\}, \mathcal{V}, \mathcal{W})$ is separated we have that $(a,b) \neq (\mathcal{P}_x, \mathcal{Q}_x)$ for every $x \in X$. ∎

8.3. Definition. *Let $\mathbf{X}$ be a hereditary coreflective subcategory of $\mathbf{2SSET}$, and set $\mathbf{X}_0 = \mathbf{X} \cap \mathbf{2SSET}_0$ as done before.*

*Let $(X,\mathcal{P},\mathcal{Q}) \in \mathbf{X}$ and $a \subset \mathcal{P}$, $b \subset \mathcal{Q}$, both containing the whole set X and not the empty set. We say that (a,b) is a $\mathbf{X}$-*compatible pair *if $(X_\infty, \mathcal{P}_a, \mathcal{Q}_b) \in \mathbf{X}$.*

Using the same argument as in *Theorem* 8.2 we obtain:

8.4. Theorem. *Let $\mathbf{X}$ be a hereditary coreflective subcategory of $\mathbf{2SSET}_0$, and $\mathbf{X}_0 = \mathbf{X} \cap \mathbf{2SSET}_0$.*

$(X,\mathcal{P},\mathcal{Q}) \in \mathbf{X}_0$ is (X_0-)complete if and only if for every $\mathbf{X}$-compatible pair (a,b) there exists a unique $x \in X$ such that $(a,b) = (\mathcal{P}_x, \mathcal{Q}_x)$.

9. Comparison with completions in the category of affine sets.

Recall from [11] that in the one-dimensional case, $(X, \mathcal{P}) \in T_0$ is complete if and only if for every $a \subset \mathcal{P}$ there is a (unique) $x \in X$ such that $a = \mathcal{P}_x$.

Moreover we can define complete objects in **SSET** in the following way:

9.1. Definition. *$(X, \mathcal{P}) \in$ **SSET** is* complete *if for every $a \subset \mathcal{P}$ there is an $x \in X$ such that $a = \mathcal{P}_x$.*

*$(X, \mathcal{P}, \mathcal{Q}) \in$ **2SSET** is* complete *if for every $a \subset \mathcal{P}$ and $b \subset \mathcal{Q}$ there is an $x \in X$ such that $a = \mathcal{P}_x$ and $b = \mathcal{Q}_x$.*

Let T_0 : **SSET** $\longrightarrow$ **SSET**$_0$ be the reflector to **SSET**$_0$ and $2T_0$: **2SSET** $\longrightarrow$ **2SSET**$_0$ be the reflector to **2SSET**$_0$.

9.2. Lemma. *$(X, \mathcal{P}) \in$ **SSET** is complete if and only if $T_0(X, \mathcal{P})$ is complete and $(X, \mathcal{P}, \mathcal{Q}) \in$ **2SSET** is complete if and only if $2T_0(X, \mathcal{P}, \mathcal{Q})$ is complete.*

For $(X, \mathcal{P}, \mathcal{Q}) \in$ **2SSET**, let us consider the three functors F_1, F_2, F_3 : **2SSET** $\longrightarrow$ **SSET** defined by:

$$F_1 : ((X, \mathcal{P}, \mathcal{Q})) = (X, \mathcal{P})$$

$$F_2 : ((X, \mathcal{P}, \mathcal{Q})) = (X, \mathcal{Q})$$

$$F_3 : ((X, \mathcal{P}, \mathcal{Q})) = (X, \mathcal{P} \cup \mathcal{Q}).$$

9.3. Theorem. *F_1, F_2, and F_3 preserve completeness.*

Proof. Assume that $(X, \mathcal{P}, \mathcal{Q})$ is complete. We want to show that also $(X, \mathcal{P})$ is complete, i.e., by *Definition* 9.1 that for every $a \subset \mathcal{P}$ there is an $x \in X$ such that $a = \mathcal{P}_x$. Let $a \subset \mathcal{P}$ be fixed and $b = \{X\}$. As $(X, \mathcal{P}, \mathcal{Q})$ is complete, there is $x \in X$ such that $b = \mathcal{Q}_x$ and $a = \mathcal{P}_x$. So F_1 preserves completeness.

The proof for F_2 is similar.

In the case of F_3, take $d \subset \mathcal{P} \cup \mathcal{Q}$ and assume $d \cap \mathcal{P} = a$ and $d \cap \mathcal{Q} = b$; as $(X, \mathcal{P}, \mathcal{Q})$ is complete, there is $x \in X$ such that $a = \mathcal{P}_x$ and $b = \mathcal{Q}_x$, so $d = \mathcal{D}_x = \mathcal{P}_x \cup \mathcal{Q}_x$ and $(X, \mathcal{P} \cup \mathcal{Q})$ is complete. ∎

9.4. Lemma. *If $(X, \mathcal{P}, \mathcal{Q}) \in$ **2SSET** is complete then for every $x \neq x' \in X$ there is $z \in X$ such that $\mathcal{P}_x = \mathcal{P}_z$ and $\mathcal{Q}_{x'} = \mathcal{Q}_z$.*

Proof. Take $x, x' \in X$, $x \neq x'$ and set $a = \mathcal{P}_x$ and $b = \mathcal{Q}_{x'}$. Note that $X \in a, b$ and $\emptyset \notin a, b$; so, as $(X, \mathcal{P}, \mathcal{Q})$ is complete, there is a unique $z \in X$ such that $a = \mathcal{P}_x = \mathcal{P}_z$ and $b = \mathcal{Q}_{x'} = \mathcal{Q}_z$. ∎

9.5. Theorem. *A separate 2affine set* $(X, \mathcal{P}, \mathcal{Q})$ *is complete if and only if:*

1. $(X, \mathcal{P})$ *is complete;*
2. $(X, \mathcal{Q})$ *is complete;*
3. *for every* $x \neq x' \in X$ *there is* $z \in X$ *such that* $\mathcal{P}_x = \mathcal{P}_z$ *and* $\mathcal{Q}_{x'} = \mathcal{Q}_z$*.*

Proof. ($\Rightarrow$)

If $(X, \mathcal{P}, \mathcal{Q})$ is complete we have *1* and *2* by *Theorem* 9.3 and *3* by *Lemma* 9.4.

($\Leftarrow$)

Assume *1,2,3,* hold and take $a \subset \mathcal{P}$ and $b \subset \mathcal{Q}$; by *1* and *2* there are x, $x' \in X$ such that $a = \mathcal{P}_x$ and $b = \mathcal{Q}_{x'}$. Finally, by *3*, there is $x \in X$ such that $\mathcal{P}_x = \mathcal{P}_z$ and $\mathcal{Q}_{x'} = \mathcal{Q}_z$, so $(X, \mathcal{P}, \mathcal{Q})$ is complete. ∎

9.6. Remark. Bitopological spaces fulfilling property 3 were called intertwining bispaces by Sergio Salbany [18] and that property was introduced to give an internal characterization of the injective T_0 bispaces introduced and studied in [13].

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