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# WEAKLY MAL'CEV CATEGORIES

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Abstract. We introduce a notion of *weakly Mal'cev category*, and show that: (a) every internal reflexive graph in a weakly Mal'tsev category admits at most one multiplicative graph structure in the sense of [10] (see also [11]), and such a structure always makes it an internal category; (b) (unlike the special case of Mal'tsev categories) there are weakly Mal'tsev categories in which not every internal category is an internal groupoid. We also give a simplified characterization of internal groupoids among internal categories in this context.

## 1. Introduction

A weakly Mal'cev category (WMC) is defined by the following two axioms:

1. Existence of pullbacks of split epis along split epis.
2. Every induced canonical pair of morphisms into a pullback (see Definition 2.3), is jointly epimorphic.

These two simple axioms are exactly what one needs in order to have a unique multiplicative structure, provided it exists, over a reflexive graph, and it then follows that this unique multiplicative structure is in fact an internal category (see [10] and [11]).

The name weakly Mal'cev category is motivated as follows.

A Mal'cev category has finite limits (see [1] Definition 2.2.3, p.142) and the induced canonical pair of morphisms into the pullback (see [1] Lemma 2.3.1, p.151) is strongly epimorphic.

Hence every Mal'cev category is an example of a weakly Mal'cev category.

Examples of weakly Mal'cev categories that are not Mal'cev, are due to G. Janelidze, and are the following:

Commutative monoids with cancelation.

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A category with objects $(A, p, e)$ where $A$ is a set, $p$ a ternary operation, $e$ a unary operation and where the following axioms are satisfied

$$\begin{aligned} p(x,y,y) &= e(x) \quad , p(x,x,y) = e(y) \\ e(x) &= e(y) \implies x = y. \end{aligned}$$

Note that $p$ becomes a Mal'cev operation when $e$ is the identity.

The setting of a weakly Mal'cev category seems to be the most appropriate to study internal categories (see final note for further discussion).

The main purpose of this paper is to introduce the concept of weakly Mal'cev category and describe some of its properties, establishing a convenient notation for *ad hoc* calculations.

In order to stress the significance of the proposed notion, we compare some of its properties with analogous and well known properties in the context of Mal'cev categories. They are the following (see the references, in particular [1],[8],[2],[4],[5],[6],[7] and [9]).

In the context of a Mal'cev category:

1. every reflexive graph admits at most one multiplication;
2. every multiplicative graph is an internal category;
3. every internal category is a groupoid.

In the context of a weakly Mal'cev category:

1. every reflexive graph admits at most one multiplication (here denoted by admissible, in that case);
2. every multiplicative graph (or admissible reflexive graph) is already an internal category;
3. not every admissible reflexive graph (or multiplicative graph, or internal category) is an internal groupoid, nevertheless there is an intrinsic description of the admissible reflexive graphs with the property of being a groupoid.

In commutative monoids with cancelation, an example of a internal category that is not a internal groupoid is the less or equal relation in the natural numbers considered as a preorder.

In a weakly Mal'cev category, given a diagram of the form

$$\begin{tikzcd} & B \arrow[d, shift right, "g"'] \\ A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

with $fr = 1_C = gs$, we may form the pullback (of a split epi along a split epi)

$$\begin{tikzcd} A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

with projections $\pi_1$ and $\pi_2$, and where $e_1, e_2$ are the canonical induced morphisms, that is, they are such that

$$\begin{aligned} \pi_1 e_1 &= 1_A \quad , \quad \pi_2 e_1 = sf \\ \pi_1 e_2 &= rg \quad , \quad \pi_2 e_2 = 1_B. \end{aligned}$$

The pair $(e_1, e_2)$ is jointly epimorphic by definition. Then, for every triple of morphisms $(h, l, k)$

$$\begin{tikzcd} A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

such that $hr = l = ks$, there is at most one morphism

$$\alpha : A \times_C B \longrightarrow D$$

such that

$$\begin{aligned} \alpha e_1 &= h \\ \alpha e_2 &= k \end{aligned}$$

which is denoted by

$$\alpha = \begin{bmatrix} h & l & k \end{bmatrix}$$

when it exists. It is also convenient to specify the morphisms $f$ and $g$; so that in general we will say: the triple of morphisms $(h, l, k)$, as above, has the property (or not) that the morphism

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

exists, with respect to

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd} .$$

In the case $\begin{bmatrix} h & l & k \end{bmatrix}$ exists we will say that the triple $(h, l, k)$ is admissible.

With this notation, the notion of admissible reflexive graph (that is an internal category) is the following:

In a weakly Mal'cev category, a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"' description] \end{tikzcd} \quad , \quad de = 1 = ce$$

is said to be admissible when the triple

$$(1_{C_1}, e, 1_{C_1})$$

is admissible with respect to

$$C_1 \underset{e}{\overset{d}{\rightleftarrows}} C_0 \underset{e}{\overset{c}{\leftrightarrows}} C_1 \; .$$

It is then a multiplicative graph with multiplication

$$C_1 \times_{C_0} C_1 \xrightarrow{[1\, e\, 1]} C_1$$

which automatically satisfies the axioms of an internal category, and furthermore, it has the property of being a groupoid if and only if the triple

$$(\pi_2, 1_{C_1}, \pi_1)$$

is admissible with respect to

$$C_1 \times_{C_0} C_1 \underset{e_2}{\overset{[1\, e\, 1]}{\rightleftarrows}} C_1 \underset{e_1}{\overset{[1\, e\, 1]}{\leftrightarrows}} C_1 \times_{C_0} C_1 \; .$$

In the presence of a Mal'cev operation, $p(x, y, z)$ written formally as in the case of Groups, that is $p(x, y, z) = x - y + z$, a general morphism

$$\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D,$$

in case of existence, is given by

$$\begin{bmatrix} h & l & k \end{bmatrix} (a, c, b) = h\,(a) - l\,(c) + k\,(b)\,,$$

so that in particular the multiplication $\begin{bmatrix} 1 & e & 1 \end{bmatrix}$, in case of existence, is given by

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix} \left( \cdot \xleftarrow{f} x \xleftarrow{g} \cdot \right) = f - 1_x + g,$$

while inverses (assuming $\begin{bmatrix} \pi_2 & 1_{C_1} & \pi_1 \end{bmatrix}$ exists) are given by

$$\begin{aligned} f^{-1} &= \begin{bmatrix} \pi_2 & 1_{C_1} & \pi_1 \end{bmatrix} \begin{bmatrix} e_1 \\ 1_{C_1} \\ e_2 \end{bmatrix} (f) \\ &= (\pi_2 e_1 - 1 + \pi_1 e_2)\,(f) \\ &= (ed - 1 + ec)\,(f) \\ &= ed\,(f) - f + ec\,(f) \\ &= 1_x - f + 1_y \end{aligned}$$

for arrows $x \xrightarrow{f} y$ in $C_1$.

This paper is organized as follows: First we introduce the notion and deduce some properties of weakly Mal'cev categories; next we prove the equivalence between internal categories and admissible reflexive graphs; later we show the connection with Mal'cev categories; at the end we describe internal groupoids in weakly Mal'cev categories.

## 2. The notion of a weakly Mal'cev category

Let $\mathbf{C}$ be a given category.

2.1. Definition. [split span] *A* split span *is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

*such that*

$$fr = 1_C = gs.$$

2.2. Definition. [split square] *A* split square *is a diagram in* $\mathbf{C}$ *of the form*

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

*such that*

$$\begin{aligned} fr &= 1_C = gs \\ gp_2 &= fp_1 \\ e_2 s &= e_1 r \\ p_2 e_2 &= 1_B \\ p_2 e_1 &= sf \\ p_1 e_1 &= 1_A \\ p_1 e_2 &= rg, \end{aligned}$$

*in other words, it is a double split epi, in the sense that it is a split epi in the category of split epis in* $\mathbf{C}$.

The term *split pullback* will be used to refer to a split square as above, such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is a pullback diagram.

2.3. Definition. [weakly Mal'cev category] *A category* $\mathbf{C}$ *is weakly Mal'cev when:*

*1. It has pullbacks of split epis along split epis;*

*2. For every split square*

$$\begin{tikzcd} P \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

*if* $(P, p_1, p_2)$ *is a pullback, then the pair* $(e_1, e_2)$ *is jointly epimorphic, that is, given* $u, v : P \longrightarrow D$,

$$\begin{cases} ue_2 = ve_2 \\ ue_1 = ve_1 \end{cases} \Longrightarrow u = v.$$

2.4. Proposition. *In a weakly Mal'cev category, given a split span*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd} \tag{1}$$

*for every object* $D$*, and a triple of morphisms* $(h, l, k)$

$$\begin{tikzcd} A \arrow[dr, "h"'] & C \arrow[l, "r"'] \arrow[r, "s"] \arrow[d, "l"] & B \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

*such that* $hr = l = ks$*, there exists at most one morphism, denoted by* $\begin{bmatrix} h & l & k \end{bmatrix}$ *when it exists, from the pullback*

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

*to the object* $D$*,*

$$\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D,$$

*with the property that*

$$\begin{aligned} \begin{bmatrix} h & l & k \end{bmatrix} e_1 &= h \\ \begin{bmatrix} h & l & k \end{bmatrix} e_2 &= k, \end{aligned}$$

*where* $e_1 = \langle 1, sf \rangle = \begin{bmatrix} 1 \\ f \\ sf \end{bmatrix} : A \longrightarrow A \times_C B$*, and* $e_2 = \langle rg, 1 \rangle = \begin{bmatrix} rg \\ g \\ 1 \end{bmatrix} : B \longrightarrow A \times_C B$
*are the induced morphisms into the pullback.*

Proof. The pullback $A \times_C B$, being a pullback of a split epi $(g, s)$ along a split epi $(f, r)$, exists in a weakly Mal'cev category, and $e_1,\ e_2$, the induced morphisms into the pullback, make the diagram

$$\begin{tikzcd} A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

a split square; to prove that $\begin{bmatrix} h & l & k \end{bmatrix}$ is unique if it exists, suppose the existence of

$$p, q : A \times_C B \longrightarrow D$$

satisfying

$$\begin{aligned} pe_1 &= h,\ pe_2 = k \\ qe_1 &= h,\ qe_2 = k, \end{aligned}$$

by definition of weakly Mal'cev, the pair $(e_1, e_2)$ is jointly epimorphic and hence $p = q$. ∎

Note that the morphism $l$, being determined by either $h$ or $k$, is explicitly used to avoid always having to choose between $hr$ and $ks$. Also, if $h$ and $k$ do not satisfy $hr = ks$ then there is no morphism $p : A \times_C B \longrightarrow D$ satisfying $pe_1 = h$ and $pe_2 = k$ because if it existed it would imply that $hr = ks$ since $e_1 r = e_2 s$.

Relative to a split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}, \qquad (2)$$

the notation

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_0 \\ \beta_2 \end{bmatrix}$$

for a morphism *into* the pullback

$$\begin{tikzcd} D \arrow[dr, "\beta"] \arrow[drr, bend left, "\beta_2"] \arrow[ddr, bend right, "\beta_1"'] & & \\ & A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"] & C \end{tikzcd}$$

induced by $\beta_1, \beta_0, \beta_2$ with

$$f\beta_1 = \beta_0 = g\beta_2;$$

and

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix}$$

for a morphism *from* the pullback

$$\begin{tikzcd} & C \arrow[dl, "r"'] \arrow[dr, "s"] & \\ A \arrow[r, "e_1"] \arrow[dr, "\alpha_1"'] & A \times_C B \arrow[d, dashed, "?\,\alpha"] & B \arrow[l, "e_2"'] \arrow[dl, "\alpha_2"] \\ & D' & \end{tikzcd}$$

induced (when it exists) by $\alpha_1, \alpha_0, \alpha_2$ with

$$\alpha_1 r = \alpha_0 = \alpha_2 s,$$

seems to be appropriate due to the following facts:

- every split span

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}$$

determines a split pullback

$$\begin{tikzcd} A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

where

$$\begin{aligned}
\pi_1 &= \begin{bmatrix} 1 & r & rg \end{bmatrix} \\
\pi_0 &= \begin{bmatrix} f & 1 & g \end{bmatrix} \\
\pi_2 &= \begin{bmatrix} sf & s & 1 \end{bmatrix} \\
e_1 &= \begin{bmatrix} 1 \\ f \\ sf \end{bmatrix},\ e_0 = \begin{bmatrix} r \\ 1 \\ s \end{bmatrix},\ e_2 = \begin{bmatrix} rg \\ g \\ 1 \end{bmatrix} \\
1_{A\times_C B} &= \begin{bmatrix} \pi_1 \\ \pi_0 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} e_1 & e_0 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & r & rg \\ f & 1 & g \\ sf & s & 1 \end{bmatrix};
\end{aligned}$$

- for every $u : D' \longrightarrow D''$, the composite $u\alpha : A \times_C B \longrightarrow D''$ is given by the formula

$$u \begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} u\alpha_1 & u\alpha_0 & u\alpha_2 \end{bmatrix},$$

whenever both sides are defined, in the sense that from $\begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix}$ we can deduce the existence of $\begin{bmatrix} u\alpha_1 & u\alpha_0 & u\alpha_2 \end{bmatrix}$, but given $\begin{bmatrix} u\alpha_1 & u\alpha_0 & u\alpha_2 \end{bmatrix}$ we can only write $u \begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix}$ provided that the existence of $\begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix}$ is already ensured.

- for every $v : \overline{D} \longrightarrow D$ the composite $\beta v : \overline{D} \longrightarrow A \times_C B$ is given by the formula

$$\begin{bmatrix} \beta_1 \\ \beta_0 \\ \beta_2 \end{bmatrix} v = \begin{bmatrix} \beta_1 v \\ \beta_0 v \\ \beta_2 v \end{bmatrix};$$

- it is sometimes useful to write the composite $\alpha\beta : D \longrightarrow D'$ as a formal formula

$$\begin{bmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_0 \\ \beta_2 \end{bmatrix} = \alpha_1\beta_1 - \alpha_0\beta_0 + \alpha_2\beta_2,$$

it is not the case that it defines a Mal'cev operation, but for the following special cases one has

$$\begin{aligned}
\alpha_1 &= \alpha e_1 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}1\\ f\\ sf\end{bmatrix} = \alpha_1 - \alpha_0 f + \alpha_2 sf = \alpha_1 - \alpha_0 f + \alpha_0 f\\
\alpha_0 &= \alpha e_0 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}r\\ 1\\ s\end{bmatrix} = \alpha_1 r - \alpha_0 + \alpha_2 s = \alpha_0 - \alpha_0 + \alpha_0\\
\alpha_2 &= \alpha e_2 = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}rg\\ g\\ 1\end{bmatrix} = \alpha_1 rg - \alpha_0 g + \alpha_2 = \alpha_0 g - \alpha_0 g + \alpha_2
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \pi_1\beta = \begin{bmatrix}1 & r & rg\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = \beta_1 - r\beta_0 + rg\beta_2 = \beta_1 - r\beta_0 + r\beta_0\\
\beta_0 &= \pi_0\beta = \begin{bmatrix}f & 1 & g\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = f\beta_1 - \beta_0 + g\beta_2 = \beta_0 - \beta_0 + \beta_0\\
\beta_2 &= \pi_2\beta = \begin{bmatrix}sf & s & 1\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = sf\beta_1 - s\beta_0 + \beta_2 = s\beta_0 - s\beta_0 + \beta_2,
\end{aligned}$$

and in general, for a triple of morphisms

$$x, y, z : D \longrightarrow D'$$

such that there exists

$$\beta : D \longrightarrow A \times_C B \quad\text{and}\quad \alpha : A \times_C B \longrightarrow D'$$

with

$$\alpha_1\beta_1 = x\ ,\ \alpha_0\beta_0 = y\ ,\ \alpha_2\beta_2 = z,$$

and writing

$$\alpha\beta = \begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_0\\ \beta_2\end{bmatrix} = \alpha_1\beta_1 - \alpha_0\beta_0 + \alpha_2\beta_2 = x - y + z$$

it is clear that there is a partially defined (relative to the split span (2)) ternary operation in $\hom(D, D')$,

$$(x, y, z) \longmapsto x - y + z,$$

but in general there is no reason for this to satisfy the Mal'cev axioms $x - z + z = x$ and $x - x + z = z$; however, it does so if the category $\mathbf{C}$ is a Mal'cev variety of universal algebras, since in that case

$$\begin{bmatrix}\alpha_1 & \alpha_0 & \alpha_2\end{bmatrix}(a, c, b) = p\left(\alpha_1(a), \alpha_0(c), \alpha_2(b)\right),$$

with $p$ the Mal'cev operation on $D'$;

- also, given another split span

$$\begin{tikzcd} A' \arrow[r, shift left, "f'"] & C' \arrow[l, shift left, "r'"] \arrow[r, shift right, "s'"'] & B' \arrow[l, shift right, "g'"'] \end{tikzcd}$$

and a morphism $\gamma = \begin{bmatrix}\gamma_1 & \gamma_0 & \gamma_2\end{bmatrix} : A' \times_{C'} B' \longrightarrow D$, the composite $\beta\gamma$,

$$\begin{tikzcd} A' \arrow[dr, "\gamma_1"'] & C' \arrow[l, "r'"'] \arrow[r, "s'"] \arrow[d, "\gamma_0"] & B' \arrow[dl, "\gamma_2"] \\ & D \arrow[dl, "\beta_1"'] \arrow[d, "\beta_0"] \arrow[dr, "\beta_2"] & \\ A \arrow[r, "f"] & C & B \arrow[l, "g"'] \end{tikzcd} \tag{3}$$

is given by

$$\begin{bmatrix}\beta_1 \\ \beta_0 \\ \beta_2\end{bmatrix}\begin{bmatrix}\gamma_1 & \gamma_0 & \gamma_2\end{bmatrix} = \begin{bmatrix}\beta_1\gamma_1 & \beta_1\gamma_0 & \beta_1\gamma_2 \\ \beta_0\gamma_1 & \beta_0\gamma_0 & \beta_0\gamma_2 \\ \beta_2\gamma_1 & \beta_2\gamma_0 & \beta_2\gamma_2\end{bmatrix} : A' \times_{C'} B' \longrightarrow A \times_C B$$

with components relative to the specified split spans, as displayed in (3).

Moreover, the following proposition describes the form of the morphisms between pullbacks, relative to the specified split spans.

2.5. Proposition. *In a WMC (Weakly Mal'cev Category), given two split spans*

$$\begin{tikzcd} A' \arrow[r, shift left, "f'"] & C' \arrow[l, shift left, "r'"] \arrow[r, shift right, "s'"'] & B' \arrow[l, shift right, "g'"'] \end{tikzcd}$$

*and*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & B \arrow[l, shift right, "g"'] \end{tikzcd},$$

*a morphism $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ is of the form*

$$\varphi = \begin{bmatrix}\pi_1\varphi e'_1 & \pi_1\varphi e'_0 & \pi_{1\varphi} e'_2 \\ \pi_0\varphi e'_1 & \pi_0\varphi e'_0 & \pi_0\varphi e'_2 \\ \pi_2\varphi e'_1 & \pi_2\varphi e'_0 & \pi_2\varphi e'_2\end{bmatrix} = \begin{bmatrix}\varphi_{11} & \varphi_{10} & \varphi_{12} \\ \varphi_{01} & \varphi_{00} & \varphi_{02} \\ \varphi_{21} & \varphi_{20} & \varphi_{22}\end{bmatrix}$$

*and it determines the following commutative diagram*

$$\begin{tikzcd}
A' \arrow[d, "\varphi_{11}"'] \arrow[dr, "\varphi_{01}"] \arrow[drr, "\varphi_{21}"] & C' \arrow[l, "r'"'] \arrow[r, "s'"] \arrow[d, "\varphi_{00}"] \arrow[dl, "\varphi_{10}"'] \arrow[dr, "\varphi_{20}"] & B' \arrow[d, "\varphi_{22}"] \arrow[dl, "\varphi_{02}"'] \arrow[dll, "\varphi_{12}"'] \\
A \arrow[r, "f"'] & C & B \arrow[l, "g"]
\end{tikzcd} \quad . \tag{4}$$

*Conversely, given a commutative diagram as above, it determines a morphism* $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ *of the form*

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \\ \varphi_{01} & \varphi_{00} & \varphi_{02} \\ \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$$

*if and only if the morphisms* $\begin{bmatrix} \varphi_{11} & \varphi_{10} & \varphi_{12} \end{bmatrix}$, $\begin{bmatrix} \varphi_{01} & \varphi_{00} & \varphi_{02} \end{bmatrix}$ *and* $\begin{bmatrix} \varphi_{21} & \varphi_{20} & \varphi_{22} \end{bmatrix}$ *exist. In particular, given a commutative diagram*

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "h"'] & C' \arrow[d, "l"] & B' \arrow[l, "g'"'] \arrow[d, "k"] \\
A \arrow[r, "f"'] & C & B \arrow[l, "g"]
\end{tikzcd}$$

*the induced morphism* $h \times_l k : A' \times_{C'} B' \longrightarrow A \times_C B$ *is given by*

$$h \times_l k = \begin{bmatrix} h\pi'_1 \\ l\pi'_0 \\ k\pi'_2 \end{bmatrix} = \begin{bmatrix} h & hr' & hr'g' \\ fh & l & gk \\ ks'f' & ks' & k \end{bmatrix}.$$

Proof. A given morphism $\varphi : A' \times_{C'} B' \longrightarrow A \times_C B$ is always determined as a morphism into the pullback

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_0"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

by the components

$$\varphi = \begin{bmatrix} \pi_1\varphi \\ \pi_0\varphi \\ \pi_2\varphi \end{bmatrix}.$$

Since each one of the components is a morphism from the pullback $A' \times_{C'} B'$, they are determined by the canonical morphisms $e'_1, e'_0, e'_2$

$$\begin{tikzcd}
A' \times'_C B' & B' \arrow[l, "e'_2"'] \\
A' \arrow[u, "e'_1"] & C' \arrow[l, "r'"] \arrow[u, "s'"'] \arrow[ul, "e'_0"']
\end{tikzcd}$$

and

$$\varphi=\begin{bmatrix}\pi_1\varphi\\ \pi_0\varphi\\ \pi_2\varphi\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}\pi_1\varphi e'_1 & \pi_1\varphi e'_0 & \pi_1\varphi e'_2\end{bmatrix}\\ \begin{bmatrix}\pi_0\varphi e'_1 & \pi_0\varphi e'_0 & \pi_0\varphi e'_2\end{bmatrix}\\ \begin{bmatrix}\pi_2\varphi e'_1 & \pi_2\varphi e'_0 & \pi_2\varphi e'_2\end{bmatrix}\end{bmatrix}.$$

In the same way one obtains

$$\varphi=\begin{bmatrix}\varphi e'_1 & \varphi e'_0 & \varphi e'_2\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}\pi_1\varphi e'_1\\ \pi_0\varphi e'_1\\ \pi_2\varphi e'_1\end{bmatrix} & \begin{bmatrix}\pi_1\varphi e'_0\\ \pi_0\varphi e'_0\\ \pi_2\varphi e'_0\end{bmatrix} & \begin{bmatrix}\pi_1\varphi e'_2\\ \pi_0\varphi e'_2\\ \pi_2\varphi e'_2\end{bmatrix}\end{bmatrix}$$

so that one can simply write

$$\varphi=\begin{bmatrix}\pi_1\varphi e'_1 & \pi_1\varphi e'_0 & \pi_{1\varphi}e'_2\\ \pi_0\varphi e'_1 & \pi_0\varphi e'_0 & \pi_0\varphi e'_2\\ \pi_2\varphi e'_1 & \pi_2\varphi e'_0 & \pi_2\varphi e'_2\end{bmatrix}.$$

To prove that

$$\varphi=\begin{bmatrix}\varphi_{11} & \varphi_{10} & \varphi_{12}\\ \varphi_{01} & \varphi_{00} & \varphi_{02}\\ \varphi_{21} & \varphi_{20} & \varphi_{22}\end{bmatrix}$$

with the components $\varphi_{ij}$ as in the commutative diagram (4), exists if and only if

$$\begin{bmatrix}\varphi_{11} & \varphi_{10} & \varphi_{12}\end{bmatrix},\begin{bmatrix}\varphi_{01} & \varphi_{00} & \varphi_{02}\end{bmatrix},\begin{bmatrix}\varphi_{21} & \varphi_{20} & \varphi_{22}\end{bmatrix}$$

exists, observe that given $\varphi$, they are respectively $\pi_1\varphi,\pi_0\varphi,\pi_2\varphi$, conversely, given such morphisms, there is $\varphi$ and it is given by

$$\varphi=\begin{bmatrix}\begin{bmatrix}\varphi_{11} & \varphi_{10} & \varphi_{12}\end{bmatrix}\\ \begin{bmatrix}\varphi_{01} & \varphi_{00} & \varphi_{02}\end{bmatrix}\\ \begin{bmatrix}\varphi_{21} & \varphi_{20} & \varphi_{22}\end{bmatrix}\end{bmatrix}.$$

Finally, given $h\times_l k : A'\times_{C'}B'\longrightarrow A\times_C B$, by the previous argument, it is of the form

$$h\times_l k=\begin{bmatrix}\pi_1(h\times_l k)\\ \pi_0(h\times_l k)\\ \pi_2(h\times_l k)\end{bmatrix}$$

and by the properties of $h\times_l k$ one has

$$\begin{bmatrix}\pi_1(h\times_l k)\\ \pi_0(h\times_l k)\\ \pi_2(h\times_l k)\end{bmatrix}=\begin{bmatrix}h\pi'_1\\ l\pi'_0\\ k\pi'_2\end{bmatrix}=\begin{bmatrix}h\begin{bmatrix}1 & r' & r'g'\end{bmatrix}\\ l\begin{bmatrix}f' & 1 & g'\end{bmatrix}\\ k\begin{bmatrix}s'f' & s' & 1\end{bmatrix}\end{bmatrix}$$

and since $fh=lf'$ and $lg\prime=gk$ one has

$$h\times_l k=\begin{bmatrix}h & hr' & hr'g'\\ fh & l & gk\\ ks'f' & ks' & k\end{bmatrix}.$$

■

## 3. Internal categories in weakly Mal'cev categories

The abbreviation WMC stands for Weakly Mal'cev Category. A triple of morphisms $(h, l, k)$ as in Proposition 2.4 is said to be *admissible* with respect to the split span (1) if the morphism $\begin{bmatrix} h & l & k \end{bmatrix}$ exists. By abuse of notation we will also say that a reflexive graph is admissible when the triple $(1, e, 1)$ is admissible.

3.1. Definition. [Admissible Reflexive Graph] *In a WMC, a reflexive graph*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

$$de = 1_{C_0} = ce$$

*is said to be* admissible *if the triple* $(1, e, 1)$ *is admissible, that is, the morphism*

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix}$$

*exists relative to the split span*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[r, shift right, "e"'] & C_1 \arrow[l, shift right, "c"'] \end{tikzcd} .$$

3.2. Theorem. *In a WMC, every admissible reflexive graph is an internal category. More specifically, given the admissible reflexive graph*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

*it is possible to construct the internal category*

$$\begin{tikzcd} C_2 \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=4, "\pi_1"'] & C_1 \arrow[l, shift right=2, "e_2"'] \arrow[l, shift left=2, "e_1"] \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

*where*

$$\begin{aligned}
1_{C_2} &= \begin{bmatrix} 1_{C_1} & e & ec \\ d & 1_{C_0} & c \\ ed & e & 1_{C_1} \end{bmatrix} \\
\pi_1 &= \begin{bmatrix} 1 & e & ec \end{bmatrix} \\
\pi_2 &= \begin{bmatrix} ed & e & 1 \end{bmatrix} \\
e_1 &= \begin{bmatrix} 1 \\ d \\ ed \end{bmatrix} \quad , \quad e_2 = \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} \\
m &= \begin{bmatrix} 1 & e & 1 \end{bmatrix} .
\end{aligned}$$

*Furthermore, every internal category is of this form.*

PROOF. The pullback $C_2$ always exists in a WMC, since $d$ and $c$ are split epis.
In a WMC, every split pullback is of the form presented above, and $m = \begin{bmatrix} 1 & e & 1 \end{bmatrix}$ is well defined because the reflexive graph is admissible by hypothesis.
In order to be an internal category we have to prove that the following conditions hold

$$\begin{aligned} dm &= d\pi_2 \\ cm &= c\pi_1 \\ me_1 &= 1 \\ me_2 &= 1 \\ m\left(1 \times_{C_0} m\right) &= m\left(m \times_{C_0} 1\right) \end{aligned}$$

and we observe:

$$\begin{aligned} dm &= d\begin{bmatrix} 1 & e & 1 \end{bmatrix} \\ &= \begin{bmatrix} d & 1 & d \end{bmatrix} \\ &= \begin{bmatrix} ded & de & d \end{bmatrix} \\ &= d\begin{bmatrix} ed & e & 1 \end{bmatrix} \\ &= d\pi_2; \end{aligned}$$

$$\begin{aligned} cm &= c\begin{bmatrix} 1 & e & 1 \end{bmatrix} \\ &= \begin{bmatrix} c & 1 & c \end{bmatrix} \\ &= \begin{bmatrix} c & ce & cec \end{bmatrix} \\ &= c\begin{bmatrix} 1 & e & ec \end{bmatrix} \\ &= c\pi_1; \end{aligned}$$

$$\begin{aligned} me_1 &= \begin{bmatrix} 1 & e & 1 \end{bmatrix} e_1 = 1 \\ me_2 &= \begin{bmatrix} 1 & e & 1 \end{bmatrix} e_2 = 1. \end{aligned}$$

In order to compare $m\left(1 \times m\right)$ and $m\left(m \times 1\right)$, from the split span

$$\begin{tikzcd} C_2 \arrow[r, shift left, "\pi_2"] & C_1 \arrow[l, shift left, "e_2"] \arrow[r, shift right, "e_1"'] & C_2 \arrow[l, shift right, "\pi_1"'] \end{tikzcd}$$

construct the split pullback

$$\begin{tikzcd} C_3 \arrow[r, shift left, "p_2'"] \arrow[d, shift right, "p_1'"'] & C_2 \arrow[l, shift left, "e_2'"] \arrow[d, shift right, "\pi_1"'] \\ C_2 \arrow[u, shift right, "e_1'"'] \arrow[r, shift left, "\pi_2"] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"'] \end{tikzcd}$$

and observe that from Proposition 2.5 and the commutativity of the following diagram

$$\begin{tikzcd} C_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "c"] & C_2 \arrow[l, "\pi_1"'] \arrow[d, "m"] \\ C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"'] \end{tikzcd}$$

we have

$$\begin{aligned}
(1 \times_{C_0} m) &= \pi_1 \times_c m = \begin{bmatrix} \pi_1 p_1' \\ c\pi_2 p_1' \\ m p_2' \end{bmatrix} \\
&= \begin{bmatrix} \pi_1 & ec & \pi_1 e_2 \pi_1 \\ d\pi_1 & c & cm \\ me_1\pi_2 & me_1 & m \end{bmatrix} \\
&= \begin{bmatrix} \pi_1 & ec & ecm \\ \pi_0 & c & cm \\ \pi_2 & 1 & m \end{bmatrix} \\
&= \begin{bmatrix} 1_{C_2} & e_2 & e_2 m \end{bmatrix}
\end{aligned}$$

and similarly form the commutativity of the diagram

$$\begin{tikzcd} C_2 \arrow[r, "\pi_2"] \arrow[d, "m"'] & C_1 \arrow[d, "d"] & C_2 \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] \\ C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"] \end{tikzcd}$$

we have

$$(m \times_{C_0} 1) = m \times_d \pi_2 = \begin{bmatrix} e_1 m & e_1 & 1_{C_2} \end{bmatrix}$$

so that

$$m \begin{bmatrix} 1_{C_2} & e_2 & e_2 m \end{bmatrix} = \begin{bmatrix} m & me_2 & me_2 m \end{bmatrix} = \begin{bmatrix} m & 1 & m \end{bmatrix}$$

and

$$m \begin{bmatrix} e_1 m & e_1 & 1_{C_2} \end{bmatrix} = \begin{bmatrix} me_1 m & me_1 & m \end{bmatrix} = \begin{bmatrix} m & 1 & m \end{bmatrix}.$$

To see that every internal category is obtained in this way simply observe that the morphism $m$ is determined by $me_1$ and $me_2$. ■

Next we show that the forgetful functor from the category of admissible reflexive graphs to the category of reflexive graphs is full, as it is the case in a Mal'cev category (see [11]).

3.3. PROPOSITION. *In a WMC, a morphism of admissible reflexive graphs is also a morphism of internal categories. More specifically, given a morphism of admissible reflexive graphs*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"'] & C_0 \arrow[l, "e" description] \arrow[d, "f_0"] \\ C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description] \end{tikzcd}$$

*then*

$$\begin{tikzcd} C_2 \arrow[r, shift left=3, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=3, "\pi_1"'] \arrow[d, "f_2"'] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "f_1"] & C_0 \arrow[l, "e" description] \arrow[d, "f_0"] \\ C_2' \arrow[r, shift left=3, "\pi_2"] \arrow[r, "m" description] \arrow[r, shift right=3, "\pi_1"'] & C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e" description] \end{tikzcd}$$

*is a morphism of internal categories, where*

$$f_2 = \begin{bmatrix} f_1 & f_1e & f_1ec \\ f_0d & f_0 & f_0c \\ f_1ed & f_1e & f_1 \end{bmatrix}.$$

PROOF. The morphism $f_2 : C_2 \longrightarrow C_2'$, being a morphism between pullbacks, by Proposition 2.5, it is of the form

$$\begin{aligned} f_2 &= \begin{bmatrix} f_1 & f_1e & f_1ec \\ f_0d & f_0 & f_0c \\ f_1ed & f_1e & f_1 \end{bmatrix} \\ &= \begin{bmatrix} e_1f_1 & e_1ef_0 & e_2f_1 \end{bmatrix} \end{aligned}$$

and hence we have $f_1m = mf_2$ :

$$\begin{aligned} f_1m &= f_1 \begin{bmatrix} 1 & e & 1 \end{bmatrix} = \begin{bmatrix} f_1 & f_1e & f_1 \end{bmatrix} \\ mf_2 &= m \begin{bmatrix} e_1f_1 & e_1ef_0 & e_2f_1 \end{bmatrix} \\ &= \begin{bmatrix} me_1f_1 & me_1ef_0 & me_2f_1 \end{bmatrix} \\ &= \begin{bmatrix} f_1 & ef_0 & f_1 \end{bmatrix}. \end{aligned}$$

■

3.4. THEOREM. *In a WMC,* **C**, *the following is an equivalence of categories:*

$$Cat(\mathbf{C}) \sim AdmRGrph(\mathbf{C}).$$

PROOF. The equivalence is established by the previous results. ■

3.5. COROLLARY. *In a WMC, a internal natural transformation $t : f \longrightarrow g$, corresponds to a morphism $t : C_0 \longrightarrow C_1'$, as displayed in the following picture*

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & C_0 \arrow[l, "e"' description] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \arrow[dl, "t"'] \\ C_1' \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0' \arrow[l, "e"' description] \end{tikzcd}$$

*such that*

$$dt = f_0 \quad , \quad ct = g_0$$

*and in addition*

$$\begin{bmatrix} 1 & e & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ ectd \\ td \end{bmatrix} = \begin{bmatrix} 1 & e & 1 \end{bmatrix} \begin{bmatrix} tc \\ edtc \\ f_1 \end{bmatrix}.$$

PROOF. It is simply the interpretation of the condition

$$m \langle tc, f_1 \rangle = m \langle g_1 td \rangle$$

for a internal natural transformation $t : f \longrightarrow g$ in the weakly Mal'cev context. ■

3.6. COROLLARY. *If $\mathbf{B}$ is a WMC then Cat($\mathbf{B}$) is a WMC .*

We observe that Cat($\mathbf{B}$) is in fact a weakly Mal'cev sesquicategory (as introduced in the author's PhD thesis) with internal transformations as 2-cells.

## 4. The connection with Mal'cev categories

The following is a result from [1] p.151, adapted to correspond to the present notation. This result was first established by D. Bourn in [8].

4.1. LEMMA. *Let $\mathbf{C}$ be a Mal'cev category. Consider the following diagram*

$$\begin{tikzcd} A \times_C B \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & B \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & C \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

*where:*

$$fr = 1_C = gs;$$
*the up and left square is a pullback;*
$$e_1 = \langle 1, ed \rangle \ , \ e_2 = \langle ec, 1 \rangle .$$

*Then the pair* $(e_1, e_2)$ *is jointly strongly epimorphic in* $\mathbf{C}$.

From here one concludes that every Mal'cev category is a weakly Mal'cev category.

An interesting particular case of a weakly Mal'cev category is obtained when the pullback $A \times_C B$ is also a pushout (of $r$ and $s$). The result is very close to the concept of naturally Mal'cev category (see [13]) and it has the property that the morphism

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

always exists relative to a split span

$$A \underset{r}{\overset{f}{\rightleftarrows}} C \underset{s}{\overset{g}{\leftrightarrows}} B$$

and hence, we have the following equivalence of categories

$$Cat\,(\mathbf{B}) \sim RGrph\,(\mathbf{B})\,.$$

4.2. Proposition. *In a Mal'cev variety of universal algebras, with Mal'cev operation*

$$\begin{aligned} p &: X \times X \times X \longrightarrow X \\ p\,(x,x,z) &= z\ ,\ p\,(x,z,z) = x, \end{aligned}$$

*the morphism*

$$\begin{bmatrix} h & l & k \end{bmatrix}$$

*exists relative to the split span*

$$A \underset{r}{\overset{f}{\rightleftarrows}} C \underset{s}{\overset{g}{\leftrightarrows}} B$$

*if and only if*

$$\begin{aligned} &p\,(\theta\,(h\,(a_1)\,,...,h\,(a_n))\,,\theta\,(l\,(c_1)\,,...,l\,(c_n))\,,\theta\,(k\,(b_1)\,,...,k\,(b_n))) = \\ &\qquad = \theta\,(p\,(h\,(a_1)\,,l\,(c_1)\,,k\,(b_1))\,,....,p\,(h\,(a_n)\,,l\,(c_n)\,,k\,(b_n))) \end{aligned}$$

*for all* $n$*-ary operation* $\theta$*, for all* $n \in \mathbb{N}_0$*, and for all*

$$\begin{aligned} a_1,...,a_n &\in A \\ b_1,...,b_n &\in B \\ c_1,...,c_n &\in C \end{aligned}$$

*with*

$$f\,(a_i) = c_i = g\,(b_i)\,.$$

PROOF. First observe that if $\begin{bmatrix} h & l & k \end{bmatrix} : A \times_C B \longrightarrow D$ exists, it is given by

$$\begin{bmatrix} h & l & k \end{bmatrix} (a, c, b) = p\left(h\left(a\right), l\left(c\right), k\left(b\right)\right).$$

In fact we have

$$\begin{aligned} (a, c, b) &= \left(p\left(a, r\left(c\right), rg\left(b\right)\right), p\left(f\left(a\right), c, g\left(b\right)\right), p\left(sf\left(a\right), s\left(c\right), b\right)\right) \\ &= p\left(\left(a, f\left(a\right), sf\left(a\right)\right), \left(r\left(c\right), c, s\left(c\right)\right), \left(rg\left(b\right), g\left(b\right), b\right)\right) \\ &= p\left(e_1\left(a\right), e_1 r\left(c\right), e_2\left(b\right)\right) \end{aligned}$$

so that

$$\begin{aligned} \begin{bmatrix} h & l & k \end{bmatrix} (a, c, b) &= \begin{bmatrix} h & l & k \end{bmatrix} p\left(e_1\left(a\right), e_1 r\left(c\right), e_2\left(b\right)\right) \\ &= p\left(h\left(a\right), l\left(c\right), k\left(b\right)\right). \end{aligned}$$

In order to be an homomorphism of universal algebras one also has

$$\begin{aligned} &\begin{bmatrix} h & l & k \end{bmatrix} \left(\theta\left(a_1, ..., a_n\right), \theta\left(c_1, ..., c_n\right), \theta\left(b_1, ..., b_n\right)\right) = \\ &\qquad = \theta\left(\begin{bmatrix} h & l & k \end{bmatrix} \left(a_1, c_1, b_1\right), ..., \begin{bmatrix} h & l & k \end{bmatrix} \left(a_n, c_n, b_n\right)\right) \end{aligned}$$

and by definition of $\begin{bmatrix} h & l & k \end{bmatrix}$ it becomes

$$\begin{aligned} &p\left(h\theta\left(a_1, ..., a_n\right), l\theta\left(c_1, ..., c_n\right), k\theta\left(b_1, ..., b_n\right)\right) = \\ &\qquad = \theta\left(p\left(ha_1, lc_1, kb_1\right), ..., p\left(ha_n, lc_n, kb_n\right)\right) \end{aligned}$$

and since $h, l, k$ are homomorphisms of universal algebra one obtains the result. ∎

As a simple observation one concludes that an internal category in a Mal'cev variety of universal algebras is a reflexive graph

$$C_1 \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} C_0 \quad , \quad de = 1 = ce,$$

such that

$$\begin{aligned} &p\left(\theta\left(a_1, ..., a_n\right), \theta\left(ed\left(a_1\right), ..., ed\left(a_n\right)\right), \theta\left(b_1, ..., b_n\right)\right) = \\ &\qquad = \theta\left(p\left(a_1, ed\left(a_1\right), b_1\right), ...., p\left(a_n, ed\left(a_n\right), b_n\right)\right) \end{aligned}$$

for all $n$-ary operation $\theta$, for all $n \in \mathbb{N}_0$, and for all

$$\begin{aligned} a_1, ..., a_n &\in C_1 \\ b_1, ..., b_n &\in C_1 \end{aligned}$$

with

$$d\left(a_i\right) = c\left(b_i\right).$$

## 5. Internal groupoids in weakly Mal'cev categories

See [1] p.420 and [2].

5.1. Definition. *Let* $\mathbf{C}$ *be a category with pullbacks of split epis along split epis. An internal groupoid is an internal category*

$$\begin{tikzcd} C_2 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & C_0 \arrow[l, "e"' description] \end{tikzcd}$$

*together with a morphism* $t : C_1 \longrightarrow C_1$*, satisfying the following conditions*

$$\begin{aligned} dt &= c \\ ct &= d \\ m\begin{bmatrix} t \\ c \\ 1 \end{bmatrix} &= ed \quad , \quad m\begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec. \end{aligned}$$

It is well known that, for an internal category, being a groupoid is a property, not an additional structure (see for instance [1], p.149).

In a WMC, given an admissible reflexive graph $(C_1, C_0, d, e, c)$ one may try to find the morphism

$$t : C_1 \longrightarrow C_1$$

provided it exists.

In the case of a Mal'cev variety it would be of the form (see [2])

$$t(x) = ed(x) - x + ec(x)$$

which suggests us, in a WMC, to look for something of the form

$$t = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix} \begin{bmatrix} e_1 \\ 1 \\ e_2 \end{bmatrix} = \pi_2 e_1 - 1 + \pi_1 e_2 = ed - 1 + ec$$

and a suitable configuration where it makes sense.

The challenge is to find an appropriate split span that agrees with the following diagram

$$\begin{tikzcd} & C_1 \arrow[dl, "e_1"'] \arrow[d, "1"] \arrow[dr, "e_2"] & \\ C_2 \arrow[dr, "\pi_2"'] & C_1 \arrow[d, "1"] & C_2 \arrow[dl, "\pi_1"] \\ & C_1 & \end{tikzcd} .$$

The answer is

$$C_2 \underset{e_2}{\overset{m}{\rightleftarrows}} C_1 \underset{e_1}{\overset{m}{\leftrightarrows}} C_2 \tag{5}$$

with the respective pullback denoted by

$$\begin{tikzcd} C_m \arrow[r, shift left, "p_2'"] \arrow[d, shift right, "p_1'"'] & C_2 \arrow[l, shift left, "e_2'"] \arrow[d, shift right, "m"'] \\ C_2 \arrow[u, shift right, "e_1'"'] \arrow[r, shift left, "m"] & C_1 \arrow[l, shift left, "e_2"] \arrow[u, shift right, "e_1"'] \end{tikzcd} .$$

Observe that in Sets the object $C_m$ is exactly the set of commutative squares. The only question is whether or not the morphism

$$\begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}$$

exists with respect to (5). The answer is given by the following proposition.

5.2. PROPOSITION. *In a WMC, for an internal category*

$$\begin{tikzcd} C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, "m"] \arrow[r, shift right=6, "\pi_1"'] & C_1 \arrow[l, shift right=3, "e_2"'] \arrow[l, shift left=3, "e_1"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & C_0 \arrow[l, "e"'] \end{tikzcd}$$

*the following are equivalent:*

*1. It is a groupoid.*

*2. There is a morphism $t : C_1 \longrightarrow C_1$ such that*

$$\begin{aligned} ct &= d \\ m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} &= ec. \end{aligned}$$

*3. The morphism*

$$\begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}$$

*exists with respect to the split span*

$$C_2 \underset{e_2}{\overset{m}{\rightleftarrows}} C_1 \underset{e_1}{\overset{m}{\leftrightarrows}} C_2 \; .$$

PROOF. We will prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2) is trivial by definition of groupoid.

(2) $\Longrightarrow$ (3) First observe that conditions

$$ct = d \; , \; m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec \; , \; me_2 = 1 \text{ and } de = 1 = ce$$

imply $te = e$:

$$\begin{aligned} me_2 &= 1 \Leftrightarrow m \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} = 1 \implies m \begin{bmatrix} ec \\ c \\ 1 \end{bmatrix} te = te \Leftrightarrow m \begin{bmatrix} ecte \\ cte \\ te \end{bmatrix} = te \Leftrightarrow \\ &\Leftrightarrow m \begin{bmatrix} e \\ de \\ te \end{bmatrix} = te \Leftrightarrow m \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} e = te \Leftrightarrow ece = te \Leftrightarrow e = te; \end{aligned}$$

Now, the commutativity of the following diagram

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "d"] & C_2 \arrow[l, "m"'] \arrow[d, "t\pi_2"] \\ C_1 \arrow[r, "d"'] & C_0 & C_1 \arrow[l, "c"'] \end{tikzcd}$$

induces the morphism

$$\pi_2 \times_d t\pi_2 : C_m \longrightarrow C_2$$

and we have

$$m\,(\pi_2 \times_d t\pi_2) = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix},$$

since by Proposition 2.5

$$\begin{aligned} (\pi_2 \times_d t\pi_2) &= \begin{bmatrix} \pi_2 & \pi_2 e_2 & \pi_2 e_2 m \\ d\pi_2 & d & ct\pi_2 \\ t\pi_2 e_1 m & t\pi_2 e_1 & t\pi_2 \end{bmatrix} \\ &= \begin{bmatrix} \pi_2 & 1 & m \\ d\pi_2 & d & d\pi_2 \\ tedm & ted & t\pi_2 \end{bmatrix} = \begin{bmatrix} \pi_2 & 1 & m \\ d\pi_2 & d & d\pi_2 \\ ed\pi_2 & ed & t\pi_2 \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} 1 & e & 1 \\ d & 1 & d \\ ted & te & t \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} 1 & e & 1 \\ d & 1 & d \\ ed & e & t \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} e_1 & e_1 e & \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} \end{bmatrix} \end{bmatrix} \end{aligned}$$

and composing with $m$ gives the result

$$\begin{aligned} m\,(\pi_2 \times_d t\pi_2) &= m \begin{bmatrix} e_1\pi_2 & e_1 & \begin{bmatrix} e_1 & e_1 e & \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \pi_2 & 1 & \begin{bmatrix} 1 & e & ec \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \pi_2 & 1 & \pi_1 \end{bmatrix}. \end{aligned}$$

(3) $\Longrightarrow$ (1) Given $\begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix}$ define

$$t = \begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix}\begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix}$$

and we have to prove

$$dt = c\ ,\ ct = d\ ,\ m\begin{bmatrix}t\\ c\\ 1\end{bmatrix} = ed\ ,\ m\begin{bmatrix}1\\ d\\ t\end{bmatrix} = ec.$$

First observe that $m = \begin{bmatrix}1 & e & 1\end{bmatrix}$ and

$$1_{C_m} = \begin{bmatrix}1_{C_2} & e_2 & e_2 m\\ m & 1_{C_1} & m\\ e_1 m & e_1 & 1_{C_2}\end{bmatrix} = \left[\begin{array}{ccc|c|ccc} 1_{C_1} & e & ec & ec & ec & e & ec\\ d & 1_{C_0} & c & e & c & 1 & c\\ ed & e & 1_{C_1} & 1 & 1 & e & 1\\ \hline 1 & e & 1 & 1_{C_0} & 1 & e & 1\\ \hline 1 & e & 1 & 1 & 1_{C_1} & e & ec\\ d & 1 & d & d & d & 1_{C_0} & c\\ ed & e & ed & ed & ed & e & 1_{C_1}\end{array}\right].$$

Denote by $l_i$ the i$^{th}$ line in the $(7\times 7)$ identity of $C_m$ and observe the following

$$l_i\begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix} = l_i\left[\begin{array}{c}1_{C_1}\\ d\\ ed\\ \hline 1_{C_1}\\ \hline ec\\ c\\ 1_{C_1}\end{array}\right] = \begin{cases}1_{C_1} & \text{if} \quad \text{i=1}\\ d & \text{if} \quad \text{i=2}\\ ed & \text{if} \quad \text{i=3}\\ 1_{C_1} & \text{if} \quad \text{i=4}\\ ec & \text{if} \quad \text{i=5}\\ c & \text{if} \quad \text{i=6}\\ 1_{C_1} & \text{if} \quad \text{i=7}\end{cases}.$$

Also let

$$\begin{aligned} l &= \begin{bmatrix}\pi_2 & 1 & \pi_1\end{bmatrix} = \left[\begin{array}{ccc|c|ccc} ed & e & 1 & 1 & 1 & e & ec\end{array}\right]\\ \varepsilon &= \begin{bmatrix}e_1\\ 1\\ e_2\end{bmatrix}\end{aligned}$$

so that

$$t = l\varepsilon.$$

To show that $dt = c$ we have

$$dt = dl\varepsilon = \left[\begin{array}{ccc|c|ccc} d & 1 & d & d & d & 1_{C_0} & c\end{array}\right]\varepsilon = l_6\varepsilon = c;$$

To show that $ct = d$ we have

$$ct = cl\varepsilon = \left[\begin{array}{ccc|c|ccc} d & 1_{C_0} & c & e & c & 1 & c \end{array}\right]\varepsilon = l_2\varepsilon = d.$$

To prove that $m\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} = ed$ first observe that

$$\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} = \begin{bmatrix} l\varepsilon \\ dl\varepsilon \\ l_7\varepsilon \end{bmatrix}$$

(note that we could also try to choose $1_{C_1} = l_1\varepsilon$ or $1_{C_1} = l_4\varepsilon$ but then

$$\begin{bmatrix} l \\ dl \\ l_1 \end{bmatrix} \text{ and } \begin{bmatrix} l \\ dl \\ l_4 \end{bmatrix}$$

would not be defined), and then

$$\begin{aligned} \begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} &= \begin{bmatrix} l \\ dl \\ l_7 \end{bmatrix}\varepsilon = \left[\begin{array}{ccc|c|ccc} ed & e & 1 & 1 & 1_{C_1} & e & ec \\ d & 1 & d & d & d & 1_{C_0} & c \\ ed & e & ed & ed & ed & e & 1_{C_1} \end{array}\right]\varepsilon \\ &= \begin{bmatrix} e_1\pi_2 & e_1 & 1_{C_2} \end{bmatrix}\varepsilon \end{aligned}$$

so that

$$\begin{aligned} m\begin{bmatrix} t \\ dt \\ 1 \end{bmatrix} &= m\begin{bmatrix} e_1\pi_2 & e_1 & 1_{C_2} \end{bmatrix}\varepsilon \\ &= \begin{bmatrix} \pi_2 & 1 & m \end{bmatrix}\varepsilon \\ &= \left[\begin{array}{ccc|c|ccc} ed & e & 1_{C_1} & 1 & 1 & e & 1 \end{array}\right]\varepsilon \\ &= l_3\varepsilon = ed. \end{aligned}$$

A similar calculation shows $m\begin{bmatrix} 1 \\ d \\ t \end{bmatrix} = ec$, in fact

$$\begin{aligned} \begin{bmatrix} 1 \\ d \\ t \end{bmatrix} &= \begin{bmatrix} 1 \\ ct \\ t \end{bmatrix} = \begin{bmatrix} l_1\varepsilon \\ cl\varepsilon \\ l\varepsilon \end{bmatrix} = \begin{bmatrix} l_1 \\ cl \\ l \end{bmatrix}\varepsilon \\ &= \left[\begin{array}{ccc|c|ccc} 1_{C_1} & e & ec & ec & ec & e & ec \\ d & 1_{C_0} & c & e & c & 1 & c \\ ed & e & 1_{C_1} & 1 & 1 & e & ec \end{array}\right]\varepsilon \\ &= \begin{bmatrix} 1_{C_2} & e_2 & e_2\pi_1 \end{bmatrix}\varepsilon \end{aligned}$$

and hence

$$
\begin{aligned}
m\begin{bmatrix}1\\ d\\ t\end{bmatrix} &= m\begin{bmatrix}1_{C_2} & e_2 & e_2\pi_1\end{bmatrix}\varepsilon \\
&= \begin{bmatrix}m & 1 & \pi_1\end{bmatrix}\varepsilon \\
&= \left[\begin{array}{ccc|c|ccc} 1 & e & 1 & 1 & 1_{C_1} & e & ec\end{array}\right]\varepsilon \\
&= l_5\varepsilon = ec.
\end{aligned}
$$

∎

## 6. Conclusion

We conclude by saying once again that the notion of weakly Mal'cev category is introduced with the unique purpose to have a setting (easy to handle) where a multiplicative graph, that is a diagram of the form

$$
\begin{tikzcd}
C_2 \arrow[r, shift left=6, "\pi_2"] \arrow[r, shift right=6, "\pi_1"'] \arrow[r, "m" description] & C_1 \arrow[l, shift right=3, "e_2"'] \arrow[l, shift left=3, "e_1"] \arrow[r, shift left=3, "d"] \arrow[r, shift right=3, "c"'] & C_0 \arrow[l, "e" description]
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
C_2 \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] & C_1 \arrow[l, shift left, "e_2"] \arrow[d, shift right, "c"'] \\
C_1 \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "e"] \arrow[u, shift right, "e"']
\end{tikzcd}
$$

is a split pullback and $me_1 = 1_{C_1} = me_2$, is already an internal category, that is, the conditions

$$
\begin{aligned}
dm &= d\pi_2 \\
cm &= c\pi_1 \\
m\,(1\times m) &= m\,(m\times 1)
\end{aligned}
$$

are automatically satisfied.

However, the original motivation was to have a setting where a reflexive graph would admit at most one multiplication, so that the two axioms in the definition of a WMC are thus explained:

– the existence of pullbacks of split epis along split epis is used to construct the pullback $C_2$ of the split epi $(c, e)$ along the split epi $(d, e)$;

– the requirement that the pair $(e_1, e_2)$ is jointly epimorphic is used to uniquely determine the morphism $m$, provided it exists, from the two components $me_1$ and $me_2$.

It was a happy surprise to observe that preservation of domain and codomain as well as associativity would automatically follow.

There are still many comparisons to be made in order to decide if this is in fact a good notion for a category and if it does not coincide with something already known. For example, it would be interesting to find out what other conditions are needed in order to have that every internal category is an internal groupoid. We leave this and other questions for a future work.

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