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ON ENDOMORPHISM ALGEBRAS OF SEPARABLE MONOIDAL FUNCTORS

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ABSTRACT. We show that the (co)endomorphism algebra of a sufficiently separable "fibre" functor into $\mathbf{Vect}_k$, for k a field of characteristic 0, has the structure of what we call a "unital" von Neumann core in $\mathbf{Vect}_k$. For $\mathbf{Vect}_k$, this particular notion of algebra is weaker than that of a Hopf algebra, although the corresponding concept in **Set** is again that of a group.

1. Introduction

Let $\mathscr{C} = (\mathscr{C}, \otimes, I, c)$ be a braided (or even symmetric) monoidal category. Recall that an *algebra* in $\mathscr{C}$ is an object $A \in \mathscr{C}$ equipped with a multiplication $\mu : A \otimes A \to A$ and a unit $\eta : I \to A$ satisfying $\mu_3 = \mu(1 \otimes \mu) = \mu(\mu \otimes 1) : A^{\otimes 3} \to A$ (associativity) and $\mu(\eta \otimes 1) = 1 = \mu(1 \otimes \eta) : A \to A$ (unit conditions). Dually, a *coalgebra* in $\mathscr{C}$ is an object $C \in \mathscr{C}$ equipped with a comultiplication $\delta : C \to C \otimes C$ and a counit $\epsilon : C \to I$ satisfying $\delta_3 = (1 \otimes \delta)\delta = (\delta \otimes 1)\delta : C \to C^{\otimes 3}$ (coassociativity) and $(\epsilon \otimes 1)\delta = 1 = (1 \otimes \epsilon)\delta : C \to C$ (counit conditions).

A *very weak bialgebra* in $\mathscr{C}$ is an object $A \in \mathscr{C}$ with both the structure of an algebra and a coalgebra in $\mathscr{C}$ related by the axiom

$$\delta\mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) : A \otimes A \to A \otimes A.$$

For example, when $\mathscr{C} = \mathbf{Vect}_k$, any k-bialgebra or weak k-bialgebra is a very weak bialgebra in this sense.

We note briefly that, if A is such a structure, but has no unit or counit, we simply call A a *semibialgebra*, or *core* for short. This minimal structure on A is then called a *von Neumann core* in $\mathscr{C}$ if it also is equipped with an endomorphism $S : A \to A$ in $\mathscr{C}$ satisfying the axiom

$$\mu_3(1 \otimes S \otimes 1)\delta_3 = 1 : A \to A.$$

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A von Neumann regular semigroup is precisely a von Neumann core in **Set**, while the free k-vector space on it is a special type of von Neumann core in $\mathbf{Vect}_k$. However, within this article we shall always suppose that A has both a unit and a counit. For example, when $\mathscr{C} = \mathbf{Vect}_k$, a Hopf k-algebra or a weak Hopf k-algebra is a von Neumann core in this somewhat stronger sense.

Since groups A in **Set** are characterized by the (stronger) axiom

$$1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_3 : A \to A \otimes A, \qquad (\dagger)$$

a very weak bialgebra A satisfying (†), in the general $\mathscr{C}$, will be called a *unital von Neumann core* in $\mathscr{C}$. Such a unital von Neumann core A always has a left inverse to the "fusion" operator [9]

$$(1 \otimes \mu)(\delta \otimes 1) : A \otimes A \to A \otimes A,$$

namely

$$(1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1) : A \otimes A \to A \otimes A.$$

Any Hopf algebra in $\mathscr{C}$ satisfies the stronger axiom (†), but a weak Hopf algebra does not necessarily do so. In this article we are mainly interested in producing a unital von Neumann core, namely $\mathrm{End}^{\vee}U$, associated to a certain type of split monoidal functor U into $\mathbf{Vect}_k$. It seems unlikely that all unital von Neumann cores in $\mathbf{Vect}_k$ may be reproduced as such.

We will tacitly assume throughout the article that the ground category [8] is $\mathbf{Vect} = \mathbf{Vect}_k$, for k a field of characteristic 0, so that the categories and functors considered here are all k-linear (although any reasonable category $[\mathscr{D}, \mathbf{Vect}]$ of parameterized vector spaces would suffice). We denote by $\mathbf{Vect}_f$ the full subcategory of **Vect** consisting of the finite dimensional vector spaces, and we further suppose that $\mathscr{C} = (\mathscr{C}, \otimes, I, c)$ is a braided monoidal category with a "fibre" functor

$$U : \mathscr{C} \to \mathbf{Vect},$$

with both a monoidal structure (U, r, r_0) and a comonoidal structure (U, i, i_0), which need not be inverse to one another. We call U *separable*[1] if $ri = 1$ and $i_0 r_0 = \dim(UI) \cdot 1$; i.e., for all $A, B \in \mathscr{C}$, the diagrams

$$\begin{array}{ccc} U(A \otimes B) & \xrightarrow{\ i\ } & UA \otimes UB \\ & \searrow^{1} & \downarrow r \\ & & U(A \otimes B) \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\ r_0\ } & UI \\ & \searrow^{\dim UI \cdot 1} & \downarrow i_0 \\ & & k \end{array}$$

[1]Strictly, we should also require the conditions (cf. [1, 10])

$$(r \otimes 1)(1 \otimes i) = ir : UA \otimes U(B \otimes C) \to U(A \otimes B) \otimes UC, \text{ and}$$
$$(1 \otimes r)(i \otimes 1) = ir : U(A \otimes B) \otimes UC \to UA \otimes U(B \otimes C)$$

in order for U to be called "separable", but we do not need these here.

commute.

First we produce an algebra structure (μ, η) on

$$\mathrm{End}^{\vee}U = \int^{C} U(C)^{*} \otimes UC$$

using the monoidal and comonoidal structures on U. Secondly, we suppose that $\mathscr{C}$ has a suitable small generating set $\mathscr{A}$ of objects, and produce a coalgebra structure (δ, ϵ) on $\mathrm{End}^{\vee}U$ when each value UA, $A \in \mathscr{A}$, is finite dimensional. Finally, we assume that each $A \in \mathscr{A}$ has a $\otimes$-dual A^{*} which also lies in $\mathscr{A}$, and that U is equipped with an isomorphism

$$U(A^{*}) \cong U(A)^{*}$$

for all $A \in \mathscr{A}$. This isomorphism should be suitably related to the evaluation and coevaluation maps of $\mathscr{C}$ and $\mathbf{Vect}_f$ which then allows us to define a natural non-degenerate form

$$U(A^{*}) \otimes UA \to k.$$

This last assumption is sufficient to provide $\mathrm{End}^{\vee}U$ with an automorphism S so that it becomes a unital von Neumann core in the above sense whenever (U, r, r_0) is a braided monoidal functor.

By way of examples, we note that many separable monoidal functors are constructable from separable monoidal categories, i.e., from monoidal categories $\mathscr{C}$ for which the tensor product map

$$\otimes : \mathscr{C}(A, B) \otimes \mathscr{C}(C, D) \to \mathscr{C}(A \otimes C, B \otimes D)$$

is a naturally split epimorphism (as is the case for some finite cartesian products such as $\mathbf{Vect}_f^n$). A closely related source of examples is the notion of a weak dimension functor on $\mathscr{C}$ (cf. [6]); this is a comonoidal functor

$$(d, i, i_0) : \mathscr{C} \to \mathbf{Set}_f$$

for which the comonoidal transformation components

$$i = i_{C,D} : d(C \otimes D) \to dC \times dD$$

are injective functions, while the unique map $i_0 : dI \to 1$ is surjective. Various examples are described at the conclusion of the paper.

We suppose the reader is familiar to some extent with the standard Tannaka reconstruction problem when restricted to the case of U strong monoidal (see [7] for example).

2. The very weak bialgebra $\mathrm{End}^{\vee}U$

If $\mathscr{C}$ is a (k-linear) monoidal category and

$$U : \mathscr{C} \to \mathbf{Vect}$$

has a monoidal structure (U, r, r_0) and a comonoidal structure (U, i, i_0), then $\mathrm{End}^{\vee}U$, when it exists, has an associative and unital k-algebra structure whose multiplication μ is the composite map

$$\begin{array}{ccc}
\int^{C} U(C)^* \otimes UC \otimes \int^{D} U(D)^* \otimes UD & \xrightarrow{\mu} & \int^{B} U(B)^* \otimes UB \\
\downarrow{\cong} & & \\
\int^{C,D} U(C)^* \otimes U(D)^* \otimes UC \otimes UD & & \uparrow{\int^{\otimes}} \\
\downarrow{\mathrm{can}} & & \\
\int^{C,D} (UC \otimes UD)^* \otimes UC \otimes UD & \xrightarrow[\int i^* \otimes r]{} & \int^{C,D} U(C \otimes D)^* \otimes U(C \otimes D)
\end{array}$$

while the unit η is given by

$$\begin{array}{ccc}
k & \xrightarrow{\eta} & \int^{C} U(C)^* \otimes UC \\
\downarrow{\cong} & & \uparrow{\mathrm{copr}_{C=I}} \\
k^* \otimes k & \xrightarrow[i_0^* \otimes r_0]{} & UI^* \otimes UI.
\end{array}$$

The associativity and unit axioms for $(\mathrm{End}^{\vee}U, \mu, \eta)$ now follow directly from the corresponding associativity and unit axioms for (U, r, r_0) and (U, i, i_0). An augmentation ϵ is given by

$$\begin{array}{ccc}
\int^{C} U(C)^* \otimes UC & \xrightarrow{\epsilon} & k \\
\nwarrow{\mathrm{copr}_{C=D}} & & \nearrow{e} \\
& U(D)^* \otimes UD &
\end{array}$$

in **Vect**, where e denotes evaluation in **Vect**.

We also observe that the coend

$$\mathrm{End}^{\vee}U = \int^{C} U(C)^* \otimes UC$$

actually exists in **Vect** if $\mathscr{C}$ contains a small full subcategory $\mathscr{A}$ with the property that the family

$$\{Uf : UA \to UC \mid f \in \mathscr{C}(A, C), A \in \mathscr{A}\}$$

is epimorphic in **Vect** for each object $C \in \mathscr{C}$. In fact, we shall use the stronger condition that the maps

$$\alpha_C : \int^{A \in \mathscr{A}} \mathscr{C}(A, C) \otimes UA \to UC$$

should be isomorphisms, not just epimorphisms. This stronger condition implies that we can effectively replace $\int^{C\in\mathscr{C}}$ by $\int^{A\in\mathscr{A}}$ since by the Yoneda lemma

$$\begin{aligned}\int^{C} U(C)^{*}\otimes UC &\cong \int^{C} U(C)^{*}\otimes (\int^{A}\mathscr{C}(A,C)\otimes UA)\\ &\cong \int^{A} U(A)^{*}\otimes UA.\end{aligned}$$

If we furthermore ask that each value UA be finite dimensional for A in $\mathscr{A}$, then

$$\mathrm{End}^{\vee}U \cong \int^{A\in\mathscr{A}} U(A)^{*}\otimes UA$$

is canonically a k-coalgebra with counit the augmentation ϵ, and comultiplication δ given by

$$\begin{array}{ccc}\int^{A} U(A)^{*}\otimes UA & \xrightarrow{\ \delta\ } & \int^{A} U(A)^{*}\otimes UA\otimes \int^{A} U(A)^{*}\otimes UA\\ \uparrow{\scriptstyle \mathrm{copr}} & & \uparrow{\scriptstyle \mathrm{copr}\otimes\mathrm{copr}}\\ U(A)^{*}\otimes UA & \xrightarrow[1\otimes n\otimes 1]{} & U(A)^{*}\otimes UA\otimes U(A)^{*}\otimes UA,\end{array}$$

where n denotes the coevaluation morphism in $\mathbf{Vect}_f$.

2.1. Proposition. *If U is separable then* $\mathrm{End}^{\vee}U$ *satisfies the k-bialgebra axiom expressed by the commutativity of*

$$\begin{array}{ccc}\mathrm{End}^{\vee}U\otimes\mathrm{End}^{\vee}U & \xrightarrow{\ \delta\otimes\delta\ } & (\mathrm{End}^{\vee}U)^{\otimes 4}\\ & & \downarrow{\scriptstyle 1\otimes c\otimes 1}\\ \downarrow{\scriptstyle \mu} & & (\mathrm{End}^{\vee}U)^{\otimes 4}\\ & & \downarrow{\scriptstyle \mu\otimes\mu}\\ \mathrm{End}^{\vee}U & \xrightarrow{\ \delta\ } & \mathrm{End}^{\vee}U\otimes\mathrm{End}^{\vee}U.\end{array}$$

Proof. Let $\mathscr{B}$ denote the monoidal full subcategory of $\mathscr{C}$ generated by $\mathscr{A}$ (we will essentially replace $\mathscr{C}$ by this small category $\mathscr{B}$). Then, for all C, D in $\mathscr{B}$, we have, by induction on the tensor lengths of C and D, that $U(C\otimes D)$ is finite dimensional since it is a retract of $UC\otimes UD$. Moreover, we have

$$\int^{A\in\mathscr{A}} U(A)^{*}\otimes UA \;\cong\; \int^{B\in\mathscr{B}} U(B)^{*}\otimes UB$$

by the Yoneda lemma, since the natural transformation

$$\alpha=\alpha_B : \int^{A\in\mathscr{A}} \mathscr{C}(A,B)\otimes UA \to UB$$

is an isomorphism for all $B \in \mathscr{B}$. Since $ri = 1$, the triangle

$$\begin{array}{ccc}
 & \overset{n}{\nearrow} & (UC \otimes UD) \otimes (UC \otimes UD)^* \\
k & & \Big\downarrow {\scriptstyle r \otimes i^*} \\
 & \underset{n}{\searrow} & U(C \otimes D) \otimes U(C \otimes D)^*,
\end{array}$$

commutes in $\mathbf{Vect}_f$, where n denotes the coevaluation maps. The asserted bialgebra axiom then holds on $\mathrm{End}^\vee U$ since it reduces to the following diagram on filling in the definitions of μ and δ (where, for the moment, we have dropped the symbol "$\otimes$"):

$$\begin{array}{ccc}
UC\ U(C)^*\ UD\ U(D)^* & \xrightarrow{1\ n\ 1\ 1\ n\ 1} & UC\ (UC\ U(C)^*)\ U(C)^*\ UD\ (UD\ U(D)^*)\ U(D)^* \\
\cong \Big\downarrow & & \Big\downarrow \cong \\
UC\ UD\ U(C)^*\ U(D)^* & & UC\ UD\ UC\ UD\ U(C)^*\ U(D)^*\ U(C)^*\ U(D)^* \\
\cong \Big\downarrow & & \Big\downarrow \cong \\
UC\ UD\ (UC\ UD)^* & \xrightarrow{1\ n\ 1} & UC\ UD\ UC\ UD\ (UC\ UD)^*\ (UC\ UD)^* \\
{\scriptstyle r\ i^*} \Big\downarrow & & \Big\downarrow {\scriptstyle r\ r\ i^*\ i^*} \\
U(C\ D)\ U(C\ D)^* & \xrightarrow{1\ n\ 1} & U(C\ D)\ U(C\ D)\ U(C\ D)^*\ U(C\ D)^*
\end{array}$$

for all $C, D \in \mathscr{B}$. ∎

Notably the bialgebra axiom expressed by the commutativity of

$$\begin{array}{ccc}
\mathrm{End}^\vee U \otimes \mathrm{End}^\vee U & \xrightarrow{\mu} & \mathrm{End}^\vee U \\
 & {\scriptstyle \epsilon \otimes \epsilon} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\
 & k &
\end{array}$$

does not hold in general, while the form of the axiom expressed by

$$\begin{array}{ccc}
\mathrm{End}^\vee U & \xrightarrow{\delta} & \mathrm{End}^\vee U \otimes \mathrm{End}^\vee U \\
 & {\scriptstyle \eta} \nwarrow \quad \nearrow {\scriptstyle \eta \otimes \eta} & \\
 & k &
\end{array}$$

holds only if $\eta = r_0 \otimes i_0^*$. Also $\epsilon\eta = \dim UI \cdot 1$ for U separable.

The single k-bialgebra axiom established in the above proposition implies that the "fusion" operator $(1 \otimes \mu)(\delta \otimes 1) : A \otimes A \to A \otimes A$ satisfies the fusion equation (see [9] for details).

The k-linear dual of $\mathrm{End}^\vee U$ is of course

$$[\int^C U(C)^* \otimes U(C), k] \cong \int_C [U(C)^*, U(C)^*]$$

which is the endomorphism k-algebra of the functor

$$U(-)^* : \mathscr{C}^{\mathrm{op}} \to \mathbf{Vect}.$$

If $\mathrm{ob}\,\mathscr{A}$ is finite, so that

$$\int^A U(A)^* \otimes UA$$

is finite dimensional, then

$$\int_C [U(C)^*, U(C)^*] \cong \int_A [U(A)^*, U(A)^*]$$

is also a k-coalgebra.

3. The unital von Neumann core $\mathrm{End}^\vee U$

We now take $\mathscr{C} = (\mathscr{C}, \otimes, I, c)$ to be a braided monoidal category and $\mathscr{A} \subset \mathscr{C}$ to be a small full subcategory of $\mathscr{C}$ for which the monoidal and comonoidal functor $U : \mathscr{C} \to \mathbf{Vect}$ induces

$$U : \mathscr{A} \to \mathbf{Vect}_f$$

on restriction to $\mathscr{A}$. We suppose that $\mathscr{A}$ is such that

- the identity I of $\otimes$ lies in $\mathscr{A}$, and each object of $A \in \mathscr{A}$ has a $\otimes$-dual A^* lying in $\mathscr{A}$.

With respect to U, we suppose $\mathscr{A}$ has the properties

- "U-irreducibility": $\mathscr{A}(A, B) \neq 0$ implies $\dim UA = \dim UB$ for all $A, B \in \mathscr{A}$,
- "U-density": the canonical map

$$\alpha_C : \int^{A \in \mathscr{A}} \mathscr{C}(A, C) \otimes UA \to UC$$

  is an isomorphism for all $C \in \mathscr{C}$,
- "U-trace": each object of $\mathscr{A}$ has a U-trace in $\mathscr{C}(I, I)$, where by U-trace of $A \in \mathscr{A}$ we mean an isomorphism $d(A)$ in $\mathscr{C}(I, I)$ such that the following two diagrams commute.

$$\begin{array}{ccc} I & \xrightarrow{d(A)} & I \\ {\scriptstyle n}\downarrow & & \uparrow{\scriptstyle e} \\ A \otimes A^* & \xrightarrow{c} & A^* \otimes A \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\dim UA} & k \\ {\scriptstyle r_0}\downarrow & & \downarrow{\scriptstyle r_0} \\ UI & \xrightarrow{\dim UI \cdot U(d(A))} & UI \end{array}$$

  We shall assume $\dim UI \neq 0$ so that the latter assumption implies $\dim UA \neq 0$, for all $A \in \mathscr{A}$.

We require also a natural isomorphism

$$u = u_A : U(A^*) \xrightarrow{\cong} U(A)^*$$

such that

$$\begin{array}{ccccc} k & & \xrightarrow{r_0} & & UI \\ \downarrow{\scriptstyle n} & & & & \downarrow{\scriptstyle Un} \\ UA \otimes U(A)^* & & & & U(A \otimes A^*) \\ & \searrow{\scriptstyle 1\otimes u^{-1}} & & \nearrow{\scriptstyle r} & \\ & & UA \otimes U(A^*) & & \end{array} \qquad (n, r, r_0)$$

commutes, and

$$\begin{array}{ccccc} UI & & \xrightarrow{i_0} & & k \\ \uparrow{\scriptstyle Ue} & & & & \uparrow{\scriptstyle e} \\ U(A^* \otimes A) & & & & U(A)^* \otimes UA \\ & \searrow{\scriptstyle i} & & \nearrow{\scriptstyle u\otimes 1} & \\ & & U(A^*) \otimes UA & & \end{array} \qquad (e, i, i_0)$$

commutes. This means that U "preserves duals" when restricted to $\mathscr{A}$.

An endomorphism

$$\sigma : \mathrm{End}^{\vee}U \to \mathrm{End}^{\vee}U$$

may be defined by components

$$\begin{array}{ccc} \int^A U(A)^* \otimes UA & \xrightarrow{\sigma} & \int^A U(A)^* \otimes UA \\ \uparrow{\scriptstyle \mathrm{copr}} & & \uparrow{\scriptstyle \mathrm{copr}} \\ U(A)^* \otimes UA & \xrightarrow{\sigma_A} & U(A^*)^* \otimes U(A^*), \end{array}$$

each σ_A being given by commutativity of

$$\begin{array}{ccc} U(A)^* \otimes UA & \xrightarrow{\sigma_A} & U(A^*)^* \otimes U(A^*) \\ \downarrow{\scriptstyle 1\otimes\rho} & & \uparrow{\scriptstyle c} \\ U(A)^* \otimes U(A)^{**} & \xrightarrow{u^{-1}\otimes u^*} & U(A^*) \otimes U(A^*)^*, \end{array}$$

where ρ denotes the canonical isomorphism from a finite dimensional vector space to its double dual. Clearly each component σ_A is invertible.

3.1. THEOREM. *Let $\mathscr{C}$, $\mathscr{A}$, and U be as above, and suppose that U is braided and separable as a monoidal functor. Then there is an automorphism S on* $\mathrm{End}^\vee U$ *such that* $(\mathrm{End}^\vee U, \mu, \eta, \delta, \epsilon, S)$ *is a unital von Neumann core in* $\mathbf{Vect}_k$.

PROOF. A family of maps $\{S_A \mid A \in \mathscr{A}\}$ is defined by

$$S_A = \dim UI \cdot (\dim UA)^{-1} \cdot \sigma_A.$$

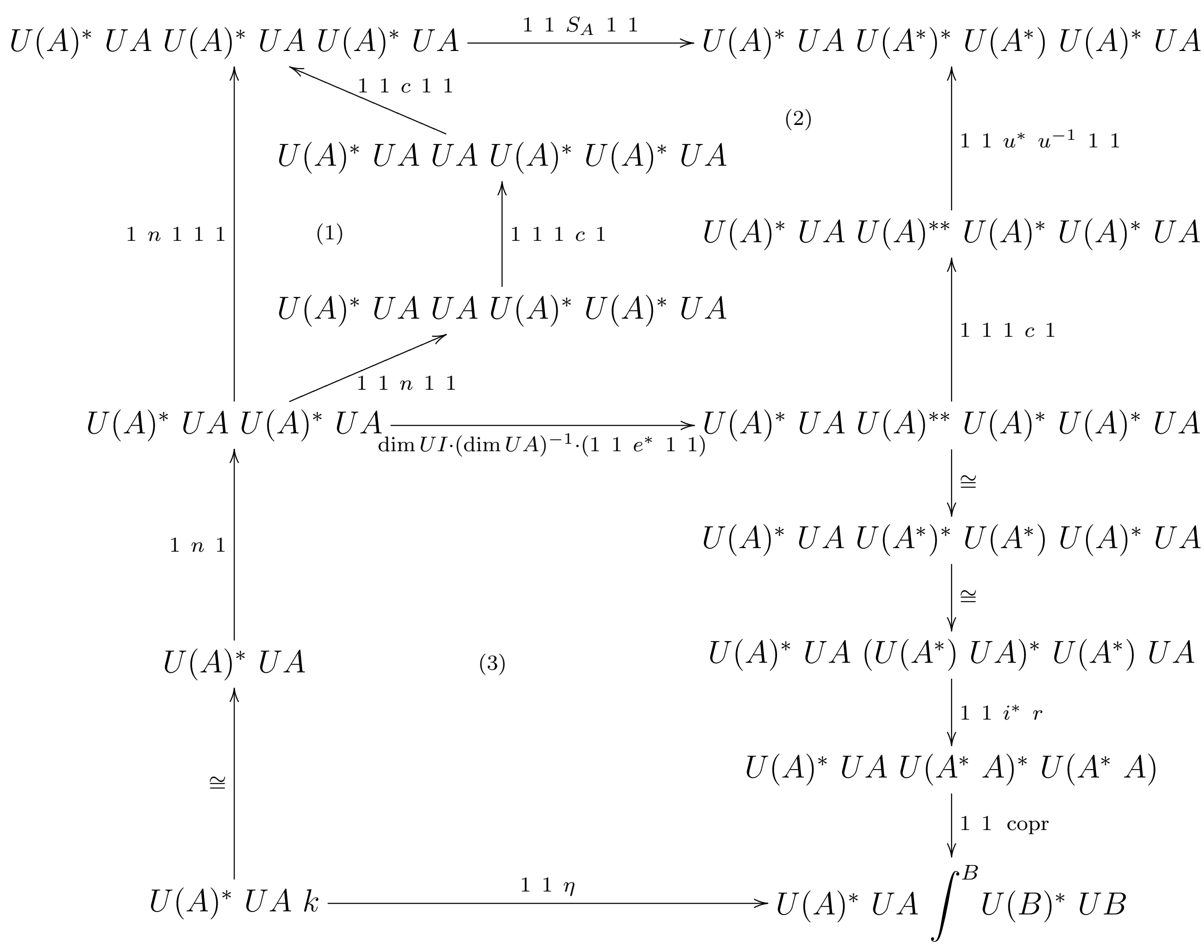
Then, by the U-irreducibility assumption on the category $\mathscr{A}$, this family induces an automorphism S on the coend

$$\mathrm{End}^\vee U \cong \sum_{n=1}^{\infty} \int^{A \in \mathscr{A}_n} U(A)^* \otimes UA,$$

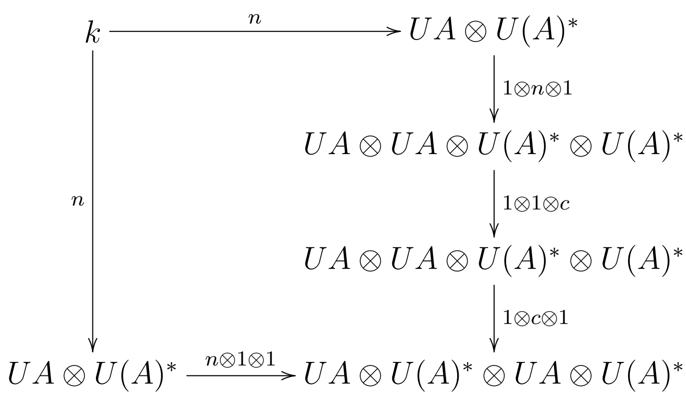
where $\mathscr{A}_n$ is the full subcategory of $\mathscr{A}$ determined by $\{A \mid \dim UA = n\}$. We now take S to be the prospective core endomorphism on $\mathrm{End}^\vee U$ and check that

$$1 \otimes \eta = (1 \otimes \mu)(1 \otimes S \otimes 1)\delta_3.$$

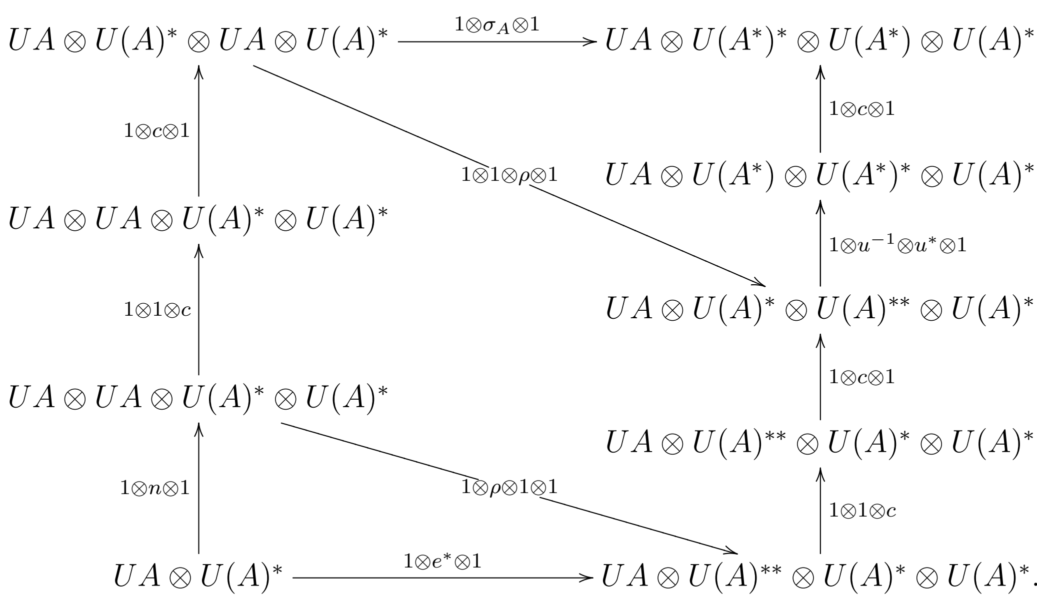
From the definition of μ and δ, we require commutativity of the exterior of the following diagram (where, again, we have dropped the symbol "$\otimes$"):



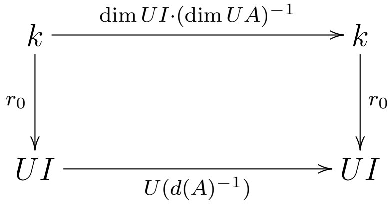
The region labelled by (1) commutes on composition with $1 \otimes n \otimes 1$ since



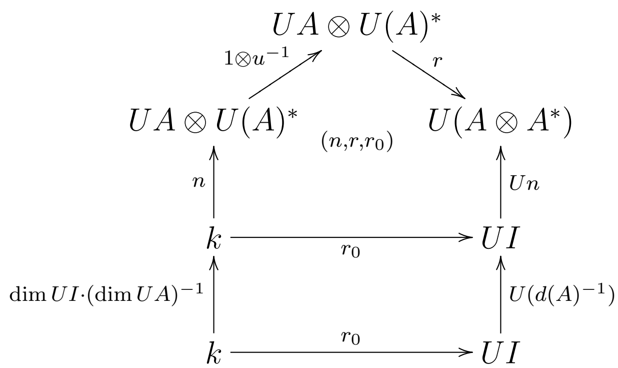
commutes (choose a basis for UA). The region labelled by (2) now commutes by inspection of



From the definition of the U-trace $d(A)$ of $A \in \mathscr{A}$, we have that

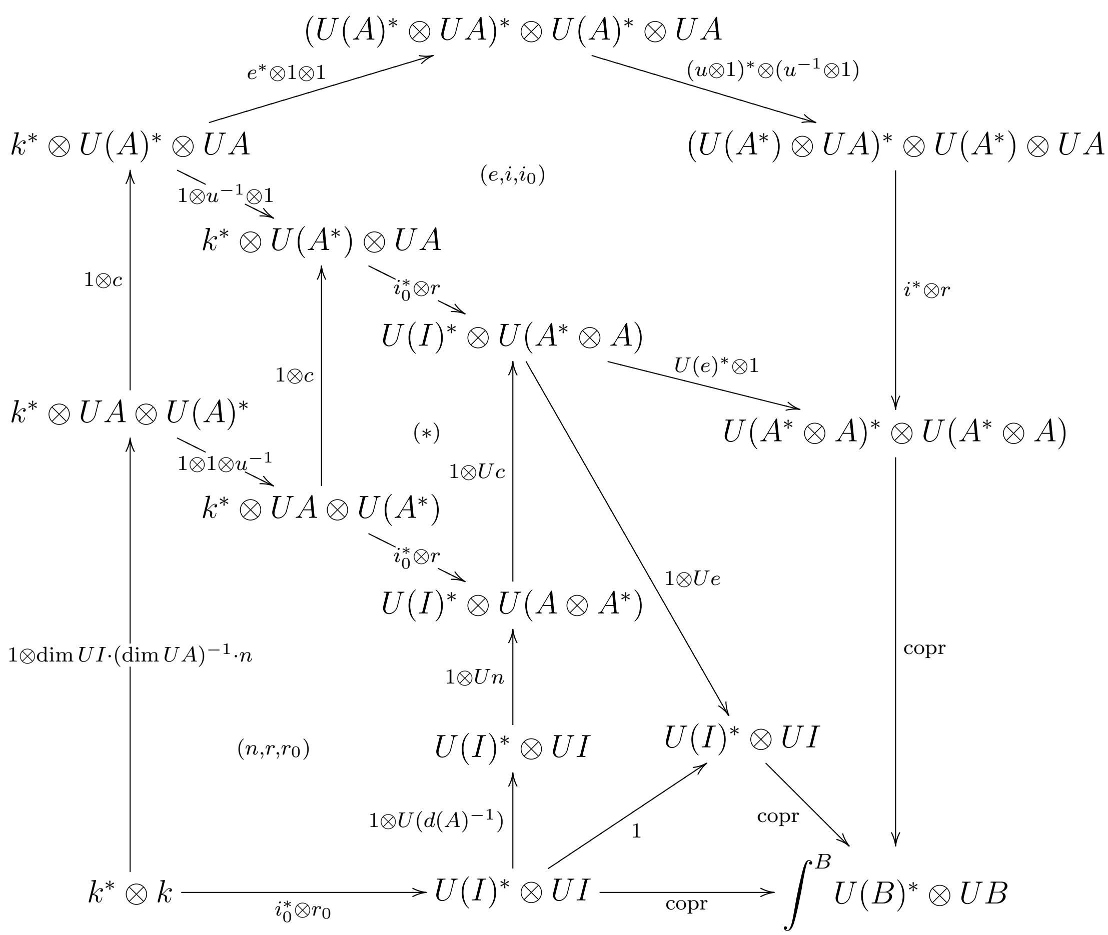


commutes, so that the exterior of



commutes.

Finally, the region labelled by (3) commutes on examination of the following diagram



whose commutativity depends on the hypothesis that (U, r, r_0) is braided monoidal in order for

$$\begin{array}{ccc} UA \otimes U(A^*) & \xrightarrow{\;c\;} & U(A^*) \otimes UA \\ {\scriptstyle r}\downarrow & (*) & \downarrow{\scriptstyle r} \\ U(A \otimes A^*) & \xrightarrow[\;Uc\;]{} & U(A^* \otimes A) \end{array}$$

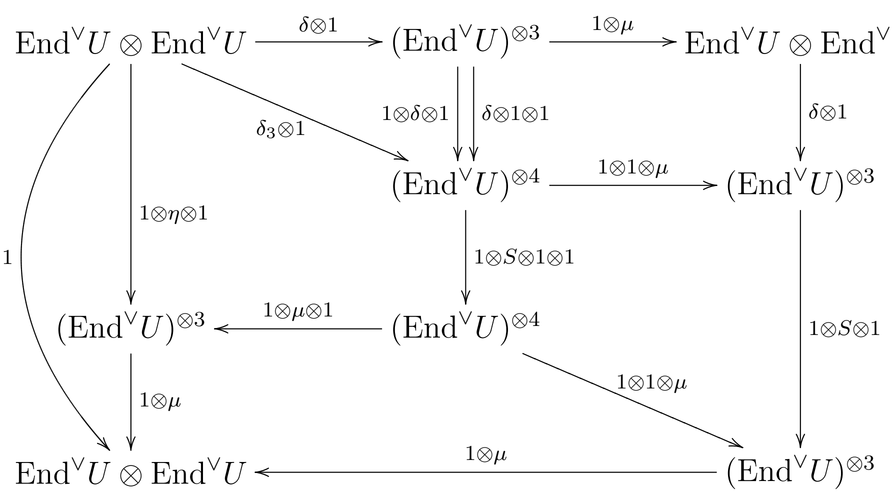
to commute. ■

4. The fusion operator

The unital von Neumann axiom on $\mathrm{End}^\vee U$ implies that the fusion operator

$$f = (1 \otimes \mu)(\delta \otimes 1) : \mathrm{End}^\vee U \otimes \mathrm{End}^\vee U \to \mathrm{End}^\vee U \otimes \mathrm{End}^\vee U$$

has a left inverse, namely $g = (1 \otimes \mu)(1 \otimes S \otimes 1)(\delta \otimes 1)$. For this we consider the following diagram.



In particular $f = (1 \otimes \mu)(\delta \otimes 1)$ is a partial isomorphism, i.e., $fgf = f$ and $gfg = g$.

5. Examples of separable monoidal functors in the present context

Unless otherwise indicated, categories, functors, and natural transformations shall be k-linear, for k a field of characteristic 0.

For these examples we recall that a (small) k-linear promonoidal category $(\mathscr{A}, p, j)$ (previously called "premonoidal" in [2]) consists of a k-linear category $\mathscr{A}$ and two k-linear functors

$$p : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathbf{Vect}$$
$$j : \mathscr{A} \to \mathbf{Vect}$$

equipped with associativity and unit constraints satisfying axioms (as described in [2]) analogous to those used to define a monoidal structure on $\mathscr{A}$. The notion of a symmetric promonoidal category (also introduced in [2]) was extended in [4] to that of a braided promonoidal category.

The main point is that (braided) promonoidal structures on $\mathscr{A}$ correspond to cocontinuous (braided) monoidal structures on the functor category $[\mathscr{A}, \mathbf{Vect}]$. This latter monoidal structure is often called the convolution product of $\mathscr{A}$ and $\mathbf{Vect}$ and is given explicity by the coend formula

$$(f * g)(c) = \int^{a,b} p(a,b,c) \otimes fa \otimes gb$$

in $\mathbf{Vect}$. The unit of this convolution product is given by j.

5.1. EXAMPLE. Let $(\mathscr{A}, p, j)$ be a small braided promonoidal category with

$$\mathscr{A}(I,I) \cong k \qquad \text{where} \qquad j = \mathscr{A}(I,-),$$

and suppose that each hom-space $\mathscr{A}(a,b)$ is finite dimensional. Let $f : \mathscr{A} \to \mathbf{Vect}_f$ be a very weak bialgebra in the convolution $[\mathscr{A}, \mathbf{Vect}]$ so that we have maps

$$\mu : f * f \to f \qquad \text{and} \qquad \eta : j \to f$$

and

$$\delta : f \to f * f \qquad \text{and} \qquad \epsilon : f \to j,$$

satisfying associativity and unital axioms, plus the very weak bialgebra axiom. Suppose also that $\mathscr{A} \subset \mathscr{C}$ where $\mathscr{C}$ is a separable braided monoidal category, with

$$p(a,b,c) \cong \mathscr{C}(a \otimes b, c) \quad \text{and} \quad j(a) \cong \mathscr{C}(I,a)$$

naturally, and suppose the induced maps

$$\int^{c \in \mathscr{A}} p(a,b,c) \otimes \mathscr{C}(c,C) \to \mathscr{C}(a \otimes b, C)$$

are isomorphisms (e.g., $\mathscr{A}$ monoidal). We also suppose that each $a \in \mathscr{A}$ has a dual $a^* \in \mathscr{A}$.

Define a functor $U : \mathscr{C} \to \mathbf{Vect}$ by

$$UC = \int^{a \in \mathscr{A}} fa \otimes \mathscr{C}(a,C);$$

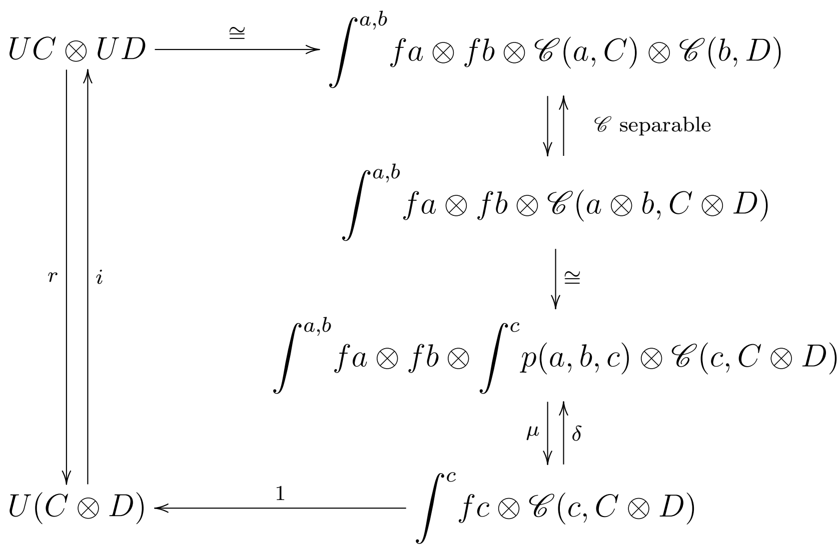
then, by the Yoneda lemma, $U(a^*) \cong U(a)^*$ if $f(a^*) \cong f(a)^*$ for $a \in \mathscr{A}$. Furthermore, by the Yoneda lemma,

$$UI = \int^{a \in \mathscr{A}} fa \otimes \mathscr{C}(a,I) \cong fI$$

so that, by our assumption $\mathscr{A}(I,I) \cong k$ the maps η and ϵ induce respectively maps

$$r_0 : k \to UI \qquad \text{and} \qquad i_0 : UI \to k.$$

Maps r and i are described in the following diagram.



These then produce a braided monoidal and comonoidal structure on U. Moreover, we have $i_0 r_0 = \dim UI \cdot 1$ if and only if $\epsilon_I \eta_I = \dim fI \cdot 1$, and if f is a separable very weak bialgebra, then U is separable since $ri = 1$ if $\mu\delta = 1$.

Therefore, Theorem 3.1 may be applied when $\mathscr{A}$ and U satisfy the "U-irreducibility" and "U-trace" criteria.

5.2. Example. Suppose that $(\mathscr{A}^{\mathrm{op}}, p, j)$ is a small braided promonoidal category with $I \in \mathscr{A}$ such that $j \cong \mathscr{A}(-,I)$ and with each $x \in \mathscr{A}$ an "atom" in $\mathscr{C}$ (i.e., an object $x \in \mathscr{C}$ for which $\mathscr{C}(x,-)$ preserves all colimits) where $\mathscr{C}$ is a cocomplete and cocontinuous braided monoidal category containing $\mathscr{A}$ and each $x \in \mathscr{A}$ has a dual $x^* \in \mathscr{A}$. Suppose that the inclusion $\mathscr{A} \subset \mathscr{C}$ is dense over **Vect** (that is, the canonical evaluation morphism

$$\int^a \mathscr{C}(a,C) \cdot a \to C$$

is an isomorphism for all $C \in \mathscr{C}$), and

$$x \otimes y \cong \int^z p(x,y,z) \cdot z \qquad \text{(naturally in } x, y \in \mathscr{A})$$

so that

$$\begin{aligned} \mathscr{C}(a, x \otimes y) &= \mathscr{C}(a, \int^z p(x,y,z) \cdot z) \\ &\cong \int^z p(x,y,z) \otimes \mathscr{C}(a,z) \quad \text{since } a \in \mathscr{A} \text{ is an atom in } \mathscr{C}, \\ &\cong p(x,y,a) \qquad \text{by the Yoneda lemma applied to } z \in \mathscr{A}. \end{aligned}$$

Let $W : \mathscr{A} \to \mathbf{Vect}$ be a strong braided promonoidal functor on $\mathscr{A}$. This means that we have structure isomorphisms

$$\begin{aligned} Wx \otimes Wy &\cong \int^z \mathscr{C}(z, x \otimes y) \otimes Wz \quad \text{and} \\ k &\cong WI \end{aligned}$$

satisfying suitable associativity and unital coherence axioms. Define a functor $U : \mathscr{C} \to \mathbf{Vect}$ by

$$UC = \int^a \mathscr{C}(a, C) \otimes Wa.$$

Then, if we suppose that $W(x^*) \cong W(x)^*$ for all $x \in \mathscr{A}$, we have

$$\begin{aligned} U(x^*) &= \int^a \mathscr{C}(a, x^*) \otimes Wa \\ &\cong W(x^*) \\ &\cong W(x)^* \\ &\cong \Big(\int^a \mathscr{C}(a, x) \otimes Wa \Big)^* \\ &= U(x)^*, \end{aligned}$$

so that $U(x^*) \cong U(x)^*$, and

$$\begin{aligned} i_0 : UI &= \int^a \mathscr{C}(a, I) \otimes Wa \\ &\cong WI \\ &\cong k, \end{aligned}$$

so that $i_0 r_0 = 1$ and $r_0 i_0 = 1$. Also there are mutually inverse composite maps r and i given by:

$$\begin{aligned} r : UC \otimes UD &\cong \int^{x,y} \mathscr{C}(x, C) \otimes \mathscr{C}(y, D) \otimes Ux \otimes Uy \\ &\cong \int^{x,y} \mathscr{C}(x, C) \otimes \mathscr{C}(y, D) \otimes Wx \otimes Wy \\ &\cong \int^{x,y} \mathscr{C}(x, C) \otimes \mathscr{C}(y, D) \otimes \int^z \mathscr{C}(z, x \otimes y) \otimes Wz \\ &\cong \int^z \mathscr{C}(z, C \otimes D) \otimes Wz \\ &\cong U(C \otimes D), \end{aligned}$$

which uses the assumptions that $\mathscr{C}$ is cocontinuous monoidal and $\mathscr{A} \subset \mathscr{C}$ is dense. Thus, $ri = 1$ and $ir = 1$ so that U is a braided strong monoidal functor.

5.3. Example. (See [6] Proposition 3.) Let $\mathscr{C}$ be a braided compact monoidal category and let $\mathscr{A} \subset \mathscr{C}$ be a full finite discrete Cauchy generator of $\mathscr{C}$ which contains I and is closed under dualization in $\mathscr{C}$. As in the Häring-Oldenburg case [6], we suppose that each hom-space $\mathscr{C}(C, D)$ is finite dimensional with a chosen natural isomorphism $\mathscr{C}(C^*, D^*) \cong \mathscr{C}(C, D)^*$.

Then we have a separable monoidal functor

$$UC = \bigoplus_{a,b \in \mathscr{A}} \mathscr{C}(a, C \otimes b),$$

whose structure maps are given by the composites

$$\begin{aligned} UC \otimes UD &\cong \bigoplus_{a,b,c,d} \mathscr{C}(c, C \otimes b) \otimes \mathscr{C}(a, D \otimes d) \\ &\underset{\text{adjoint}}{\overset{c=d}{\rightleftarrows}} \bigoplus_{a,b,c} \mathscr{C}(c, C \otimes b) \otimes \mathscr{C}(a, D \otimes c) \\ &\cong \bigoplus_{a,b} \mathscr{C}(a, D \otimes (C \otimes b)) \\ &\cong \bigoplus_{a,b} \mathscr{C}(a, (D \otimes C) \otimes b) \\ &\cong \bigoplus_{a,b} \mathscr{C}(a, (C \otimes D) \otimes b) \\ &= U(C \otimes D), \end{aligned}$$

and $r_0 : k \to UI$ the diagonal, with i_0 its adjoint. Moreover

$$\begin{aligned} U(C^*) &= \bigoplus_{a,b} \mathscr{C}(a, C^* \otimes b) \\ &\cong \bigoplus_{a,b} \mathscr{C}(a^*, C^* \otimes b^*) \\ &\cong \bigoplus_{a,b} \mathscr{C}(a, C \otimes b)^* \\ &\cong UC^* \end{aligned}$$

for all $C \in \mathscr{C}$.

5.4. Example. Let $(\mathscr{A}, p, j)$ be a finite braided promonoidal category over $\mathbf{Set}_f$ with $I \in \mathscr{A}$ such that $j \cong \mathscr{A}(I, -)$ and with a braided promonoidal functor

$$d : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}_f$$

for which each structure map

$$u : \int^z p(x, y, z) \times dz \to dx \times dy$$

is an injection, and $u_0 : dI \to 1$ is a surjection. Then we have corresponding maps

$$\int^z k[p(x,y,z)] \otimes k[dz] \rightleftarrows k[dx] \otimes k[dy]$$

and

$$k[dI] \rightleftarrows k[1],$$

where $k[s]$ denotes the free k-vector space on the (finite) set s, in $\mathbf{Vect}_f$. Define the functor $U : \mathscr{C} \to \mathbf{Vect}_f$ by

$$Uf = \int^x fx \otimes k[dx]$$

for $f \in \mathscr{C} = [k_*\mathscr{A}, \mathbf{Vect}_f]$, with the convolution braided monoidal closed structure, where $k_*\mathscr{A}$ is the free k-linear category on $\mathscr{A}$ so that

$$\begin{aligned} r : Uf \otimes Ug = & \left(\int^x fx \otimes k[dx]\right) \otimes \left(\int^y gx \otimes k[dy]\right) \\ \cong & \int^{x,y} fx \otimes gy \otimes (k[dx] \otimes k[dy]) \\ \rightleftarrows & \int^{x,y} fx \otimes gy \otimes \left(\int^z k[p(x,y,z)] \otimes k[dz]\right) \\ \cong & \int^z \left(\int^{x,y} fx \otimes gy \otimes k[p(x,y,z)]\right) \otimes k[dz] \\ = & \int^z (f \otimes g)(z) \otimes k[dz] \\ = & \int^z U(f \otimes g) \end{aligned}$$

and

$$\begin{aligned} i_0 : UI = & \int^x k[\mathscr{A}(I,x)] \otimes k[dx] \\ & \cong k[dI] \\ & \rightleftarrows k[1] \cong k. \end{aligned}$$

Hence $i_0 r_0 = \dim UI \cdot 1 = |dI| \cdot 1$. Thus, U becomes a braided separable monoidal functor.

5.5. Example. Let $\mathscr{A}$ be a finite (discrete) set and give the cartesian product $\mathscr{A} \times \mathscr{A}$ the $\mathbf{Set}_f$-promonoidal structure corresponding to bimodule composition (i.e., to matrix multiplication). If

$$d : \mathscr{A} \times \mathscr{A} \to \mathbf{Set}_f$$

is a braided promonoidal functor, then its associated structure maps

$$\begin{aligned}\sum_{z,z'} p((x,x'),(y,y'),(z,z')) \times d(z,z') &= \sum_{z,z'} \mathscr{A}(z,x) \times \mathscr{A}(x',y) \times \mathscr{A}(y',z') \times d(z,z') \\ &\cong \mathscr{A}(x',y) \times d(x,y') \\ &\to d(x,x') \times d(y,y'),\end{aligned}$$

and

$$\begin{aligned}\sum_{z,z'} j(z,z') \times d(z,z') &= \sum_{z,z'} \mathscr{A}(z,z') \times d(z,z') \\ &\cong \sum_{z} d(z,z) \\ &\to 1,\end{aligned}$$

are determined by components

$$\begin{aligned} d(x,y') &\rightarrowtail d(x,y) \times d(y,y') \\ d(z,z) &\twoheadrightarrow 1 \end{aligned}$$

which give $\mathscr{A}$ the structure of a discrete cocategory over $\mathbf{Set}_f$.

Define the functor $U : \mathscr{C} = [k_*(\mathscr{A} \times \mathscr{A}), \mathbf{Vect}_f] \to \mathbf{Vect}_f$ by

$$Uf = \bigoplus_{x,y} (f(x,y) \otimes k[d(x,y)]).$$

Then we obtain monoidal and comonoidal structure maps

$$U(f \otimes g) \underset{i}{\overset{r}{\rightleftarrows}} Uf \otimes Ug$$

$$UI \underset{i_0}{\overset{r_0}{\rightleftarrows}} k \cong k[1]$$

from the canonical maps

$$\begin{aligned}&\bigoplus_{x,y,z} f(x,z) \otimes g(z,y) \otimes k[d(x,y)] \\ &\qquad \underset{z=u=v}{\overset{\text{adjoint}}{\rightleftarrows}} \bigoplus_{x,u} \big(f(x,u) \otimes k[d(x,u)]\big) \otimes \bigoplus_{v,y} \big(g(v,y) \otimes k[d(v,y)]\big)\end{aligned}$$

and

$$\bigoplus_{z} k[d(z,z)] \rightleftarrows k \cong k[1].$$

These give U the structure of a separable braided monoidal functor on $\mathscr{C}$.

6. Concluding remarks

If the original "fibre" functor U is faithful and exact then the Tannaka equivalence (duality)

$$\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect}) \simeq \mathbf{Comod}(\mathrm{End}^{\vee}U)$$

is available, where $\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect})$ is the category of k-linear left exact functors from $\mathscr{C}^{\mathrm{op}}$ to $\mathbf{Vect}$. (See [3] for example.) Thus, since $\mathscr{C}$ is braided monoidal, so is $\mathbf{Comod}(\mathrm{End}^{\vee}U)$ with the tensor product and unit induced by the convolution product on $\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect})$; for convenience we recall [3] that, for $\mathscr{C}$ compact, this convolution product is given by the restriction to $\mathrm{Lex}(\mathscr{C}^{\mathrm{op}}, \mathbf{Vect})$ of the coend

$$\begin{aligned} F * G &= \int^{C,D} FC \otimes GD \otimes \mathscr{C}(-, C \otimes D) \\ &\cong \int^{C} FC \otimes G(C^* \otimes -) \end{aligned}$$

computed in the whole functor category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Vect}]$. Moreover, when U is separable monoidal, the category $\mathbf{Co}(\mathrm{End}^{\vee}U)$ of cofree coactions of $\mathrm{End}^{\vee}U$ (as constructed in [7] for example) also has a monoidal structure $(\mathbf{Co}(\mathrm{End}^{\vee}U), \otimes, k)$, this time obtained from the algebra structure of $\mathrm{End}^{\vee}U$. The forgetful inclusion

$$\mathbf{Comod}(\mathrm{End}^{\vee}U) \subset \mathbf{Co}(\mathrm{End}^{\vee}U)$$

preserves colimits while $\mathbf{Comod}(\mathrm{End}^{\vee}U)$ has a small generator, namely $\{UC \mid C \in \mathscr{C}\}$, and thus, from the special adjoint functor theorem, this inclusion has a right adjoint. The value of the adjunction's counit at the functor $F \otimes G$ in $\mathbf{Co}(\mathrm{End}^{\vee}U)$ is then a split monomorphism and, in particular, the monoidal forgetful functor

$$\mathbf{Comod}(\mathrm{End}^{\vee}U) \to \mathbf{Vect},$$

which is the composite $\mathbf{Comod}(\mathrm{End}^{\vee}U) \subset \mathbf{Co}(\mathrm{End}^{\vee}U) \to \mathbf{Vect}$, is a separable braided monoidal functor extension of the given functor $U : \mathscr{C} \to \mathbf{Vect}$.

References

[1] Aurelio Carboni. Matrices, relations, and group representations, J. Algebra 136 no. 2 (1991): 497–529.

[2] Brian Day. On closed categories of functors, in *Reports of the Midwest Category Seminar IV*, Lecture Notes in Mathematics 137 (1970): 1–38.

[3] Brian J. Day. Enriched Tannaka reconstruction, J. Pure Appl. Algebra 108 no. 1 (1996): 17–22.

[4] B. Day, E. Panchadcharam, and R. Street. Lax braidings and the lax centre, in *Hopf Algebras and Generalizations*, Contemporary Mathematics 441 (2007): 1–17.

[5] Brian Day and Ross Street. Quantum categories, star autonomy, and quantum groupoids, in *Galois Theory, Hopf Algebras, and Semiabelian Categories*, Fields Institute Communications 43 (2004): 187–226.

[6] Reinhard Häring-Oldenburg. Reconstruction of weak quasi-Hopf algebras, J. Algebra 194 no. 1 (1997): 14–35.

[7] André Joyal and Ross Street. An Introduction to Tannaka Duality and Quantum Groups, Lecture Notes in Mathematics 1488 (Springer-Verlag, Berlin, 1991): 411–492.

[8] G. M. Kelly. Basic concepts of enriched category theory, London Mathematical Society Lecture Note Series 64. Cambridge University Press, Cambridge, 1982. Also at http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html

[9] Ross Street. Fusion operators and cocycloids in monoidal categories, Appl. Categ. Struct. 6 (1998): 177–191.

[10] Kornél Szlachányi. Finite quantum groupoids and inclusions of finite type, Fields Inst. Comm. 30 (2001): 393–407.

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