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# THE SPAN CONSTRUCTION

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Abstract. We present two generalizations of the Span construction. The first generalization gives Span of a category with all pullbacks as a (weak) double category. This double category $\mathbb{S}\text{pan}\,\mathbf{A}$ can be viewed as the free double category on the vertical category $\mathbf{A}$ where every vertical arrow has both a companion and a conjoint (and these companions and conjoints are adjoint to each other). Thus defined, $\mathbb{S}\text{pan}\colon \mathbf{Cat} \to \mathbf{Doub}$ becomes a 2-functor, which is a partial left bi-adjoint to the forgetful functor $\text{Vrt} : \mathbf{Doub} \to \mathbf{Cat}$, which sends a double category to its category of vertical arrows.

The second generalization gives Span of an arbitrary category as an oplax normal double category. The universal property can again be given in terms of companions and conjoints and the presence of their composites. Moreover, $\mathbb{S}\text{pan}\,\mathbf{A}$ is universal with this property in the sense that $\mathbb{S}\text{pan}\colon \mathbf{Cat} \to \mathbf{OplaxNDoub}$ is left bi-adjoint to the forgetful functor which sends an oplax double category to its vertical arrow category.

## 1. Introduction

As remarked in [12], the 2-category $\Pi_2\mathbf{A}$ obtained by freely adjoining a right adjoint to each arrow of a category $\mathbf{A}$, studied in [14] and [15], has zig-zag paths of arrows of $\mathbf{A}$ as 1-cells and equivalence classes of fences as 2-cells. Thus, it would appear that this construction can be performed in two steps: first, take spans in $\mathbf{A}$ (that is, one step zig-zags), and then take paths of these. Moreover, each of these constructions is interesting in itself and deserving of further study. And indeed this is so. Paths and spans are also essential building blocks for other localizations such as the hammock localization [10] and categories of fractions [17].

In [12] we made a systematic study of the path construction. There we saw that it was useful, even necessary, to work in the context of double categories and even Leinster's **fc**-multicategories (which we called, more suggestively, lax double categories). The more important construction presented there is $\mathbb{P}\text{ath}_*$, which arises as the solution of the problem of describing the universal oplax normal morphism of double categories. It is in this construction that the equivalence relation on cells in $\Pi_2\mathbf{A}$ first makes its appearance, and this fact certainly clarifies part of the role of the equivalence relation.

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To complete our construction of $\Pi_2\mathbf{A}$ we must study spans in $\mathbf{A}$. If $\mathbf{A}$ is a category with pullbacks, the bicategory $\mathcal{S}pan\,\mathbf{A}$ is well-known. It was introduced by Bénabou in 1967; from there, it is an easy step to pass to the (weak) double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, first introduced in [19], and briefly considered in [13]. This double category of spans is better behaved than the bicategory when it comes to universal properties.

However, localization constructions such as $\Pi_2$ typically do not require categories to have pullbacks. Without pullbacks we cannot compose spans, but we do know what a cell from a span into a path of spans ought to be, because the composite of a path of spans is a limit construction. Generalizations of the Span construction are among the main motivating examples of Hermida's multicategories with several objects [21], and Leinster's **fc**-multicategories [26]. It is our purpose in this paper to elevate these versions of Span from mere examples to the central object of study.

It is clear that all these Span constructions are natural, but it would be reassuring if we had a universal property to validate them, and provide a framework for further extensions and a better understanding of the relationship between the Span construction and other localization constructions. In [13] we established two universal properties of $\mathcal{S}pan\,\mathbf{A}$, one of which did not refer to pullbacks. We will extend these properties in the present context.

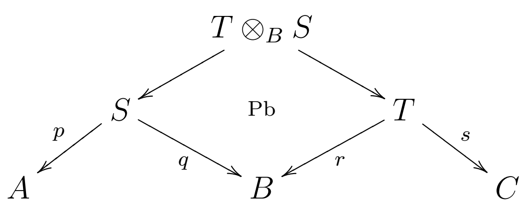
In Section 2 we review the properties of $\mathcal{S}pan$ as a bicategory construction, and give a slightly improved version of our characterization of gregarious morphisms from [13] (which were called 'jointed morphisms' in that paper). In Section 3 we describe the universal properties of the 2-functor $\mathbb{S}\mathrm{pan}\colon \mathbf{Cat} \to \mathbf{Doub}$, defined for categories with pullbacks. We introduce the notions of gregarious double categories and Beck-Chevalley double categories. In Section 4 we extend the $\mathbb{S}\mathrm{pan}$ construction to arbitrary categories and obtain the aforementioned oplax double categories.

## 2. The bicategory of spans

We begin by recalling the basic facts about spans. If $\mathbf{A}$ is a category with pullbacks, the bicategory $\mathcal{S}pan\,\mathbf{A}$ has the same objects as $\mathbf{A}$, but its morphisms are *spans*,

$$A \xleftarrow{\;p\;} S \xrightarrow{\;q\;} B\,. \tag{1}$$

We will denote such a span by a pair $(p, q)$, or by its middle object $S$ when there is no confusion possible. The composite $T \otimes_B S$, also written as $TS$, of $A \xleftarrow{p} S \xrightarrow{q} B$ with $B \xleftarrow{r} T \xrightarrow{s} C$ is given by the pullback



and the identity $I_A$ on an object $A$ is given by $A \xleftarrow{1_A} A \xrightarrow{1_A} A$. Composition, being defined as it is by a universal property, can hardly be expected to be associative or unitary on the nose, but it is up to coherent isomorphism. In order to express this we must have morphisms between spans.

A 2-cell $t\colon S \to S'$ is a commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{p} & S & \xrightarrow{q} & B \\ \| & & \downarrow{\scriptstyle t} & & \| \\ A & \xleftarrow[p']{} & S' & \xrightarrow[q']{} & B\,. \end{array}$$

Vertical composition of 2-cells is defined in the obvious way, whereas horizontal composition can be defined using the universal property of the pullbacks used in the composition of the spans. So we obtain a bicategory $\mathcal{S}pan\,\mathbf{A}$. This construction was the main object of study in our previous paper [13]. In the current paper we will revisit this construction to discuss its functoriality properties and then we will introduce and study several extensions.

The category $\mathbf{A}$ embeds into $\mathcal{S}pan\,\mathbf{A}$ via a morphism of bicategories

$$(\ )_*\colon \mathbf{A} \to \mathcal{S}pan\,\mathbf{A},$$

which assigns to the morphism $f\colon A \to B$ the span $f_* = (A \xleftarrow{1_A} A \xrightarrow{f} B)$. In saying that $(\ )_*$ is a morphism of bicategories we are considering $\mathbf{A}$ as a bicategory with only identity 2-cells. Depending on our choice of pullbacks, a point on which we don't want to commit ourselves, composition may only be preserved up to coherent isomorphism, $(gf)_* \cong g_*f_*$. The morphism $(\ )_*$ is locally full and faithful which means that there is no 2-cell $f_* \Rightarrow f'_*$ unless $f = f'$ and then there is only the identity.

There is another embedding, this time contravariant, of $\mathbf{A}$ into $\mathcal{S}pan\,\mathbf{A}$,

$$(\ )^*\colon \mathbf{A}^{\mathrm{op}} \to \mathcal{S}pan\,\mathbf{A},$$

which takes a morphism $f\colon A \to B$ to the span $f^* = (B \xleftarrow{f} A \xrightarrow{1_A} A)$.

The relationship between the images $f_*$ and $f^*$ is one of adjointness. $\mathcal{S}pan\,\mathbf{A}$, being a bicategory, supports the notion of adjointness, expressed in terms of unit and counit 2-cells satisfying the triangle identities as described in [20], page 137; in $\mathcal{S}pan\,\mathbf{A}$, $f_*$ is left adjoint to $f^*$. All of these spans are further related by the *Beck-Chevalley condition*: If

$$\begin{array}{ccc} P & \xrightarrow{k} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback diagram in $\mathbf{A}$, then the canonical 2-cell

$$\begin{array}{ccc} P & \xrightarrow{k_*} & C \\ {\scriptstyle h^*}\uparrow & \Rightarrow & \uparrow{\scriptstyle g^*} \\ A & \xrightarrow[f_*]{} & B \end{array}$$

is an isomorphism.

Every span $A \xleftarrow{p} S \xrightarrow{q} B$ is isomorphic to $q_* \otimes_S p^*$ so that the arrows of $\mathcal{S}pan\,\mathbf{A}$ are generated by the arrows of $\mathbf{A}$ (*i.e.*, the arrows of the form $f_*$ for $f \in \mathbf{A}$) and their right adjoints. It is also true, although less obvious, that each 2-cell of $\mathcal{S}pan\,\mathbf{A}$ can be factored as a composition of identities, units and counits of adjunctions $f_* \dashv f^*$ and $\mathcal{S}pan\,\mathbf{A}$ is in some sense the free bicategory generated by the arrows of $\mathbf{A}$ and their adjoints. However, it is not so in the obvious way, because it is not equivalent to $\Pi_2\mathbf{A}$ of [14]. Indeed, it was the object of [13] to make this precise, but before we can state the main theorem from that paper we must recall some definitions.

2.1. Gregarious morphisms of bicategories. From the beginning, [2], it was realized that it was too much to require that morphisms of bicategories preserve composition, even up to isomorphism. There are many examples to back this up. We give just one which will be important to us later.

Let $\mathbf{A}$ and $\mathbf{B}$ be categories with pullbacks and $F\colon \mathbf{A} \to \mathbf{B}$ an arbitrary functor. This gives rise to $\mathcal{S}pan(F)\colon \mathcal{S}pan\,\mathbf{A} \to \mathcal{S}pan\,\mathbf{B}$ defined on objects, spans, and 2-cells by the application of $F$. So $F$ assigns to the 2-cell

$$\begin{array}{ccccc} A & \xleftarrow{p} & S & \xrightarrow{q} & B \\ \| & & \downarrow{\scriptstyle t} & & \| \\ A & \xleftarrow[p']{} & S' & \xrightarrow[q']{} & B \end{array},$$

the 2-cell

$$\begin{array}{ccccc} FA & \xleftarrow{Fp} & FS & \xrightarrow{Fq} & FB \\ \| & & \downarrow{\scriptstyle Ft} & & \| \\ FA & \xleftarrow[Fp']{} & FS' & \xrightarrow[Fq']{} & FB \end{array}.$$

Vertical composition of 2-cells is preserved by this functor, but composition of spans will not be, even up to isomorphism, unless $F$ sends pullbacks to pullbacks. In general, the universal property of pullbacks only gives us a comparison cell $\varphi_{T,S}\colon F(T \otimes_B S) \to FT \otimes_{FB} FS$. Such an $F$ will be an oplax morphism of bicategories (the vertical dual of Bénabou's notion of morphism of bicategories [2]).

2.2. Remark. One might object that we should not consider functors $F$ that don't preserve pullbacks in our study of the Span construction, and there is some truth to that. But as noted in [7], one might encounter a pullback-preserving functor $U : \mathbf{B} \to \mathbf{A}$ to which we might want to apply $\mathcal{S}pan$, and this $U$ could have a left adjoint $F$ which does not preserve pullbacks, a common and important situation. Moreover, if this $F$ induces some kind of left adjoint $\mathcal{S}pan\,\mathbf{A} \to \mathcal{S}pan\,\mathbf{B}$ to $\mathcal{S}pan(U)$, we should take it into consideration.

To gain further insight into these questions we will address the 2-functoriality of the Span construction at the end of this section, and further at the end of Sections 3 and 4.

The oplax morphisms $\mathcal{S}pan(F) : \mathcal{S}pan\,\mathbf{A} \to \mathcal{S}pan\,\mathbf{B}$ are all normal. In fact, any morphism of the form $\mathcal{S}pan(F)$ preserves identities on the nose, although this in itself is not enough to ensure normality. The identities must satisfy the unit laws for structure cells of oplax morphisms of bicategories, which they do. Normality plays a fundamental role throughout this paper. One of the features of normal morphisms is that one can easily characterize when they preserve adjunctions.

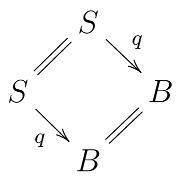
We say that an oplax morphism of bicategories $\Phi : \mathcal{A} \to \mathcal{B}$ preserves adjunctions if for each adjunction $A \underset{u}{\overset{f}{\rightleftarrows}} A'$ in $\mathcal{A}$ with unit $\eta : 1_A \to uf$ and counit $\varepsilon : fu \to 1_{A'}$, there are 2-cells $\bar{\eta} : 1_{\Phi A} \to \Phi(u)\Phi(f)$ and $\bar{\varepsilon} : \Phi(f)\Phi(u) \to 1_{\Phi A'}$ which satisfy the triangle identities and give rise to commutative squares

$$\begin{array}{ccc} \Phi(1_A) & \xrightarrow{\Phi\eta} & \Phi(uf) \\ {\scriptstyle \phi_A}\downarrow & & \downarrow{\scriptstyle \phi_{u,f}} \\ 1_{\Phi A} & \xrightarrow[\bar{\eta}]{} & \Phi(u)\Phi(f) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Phi(fu) & \xrightarrow{\Phi\varepsilon} & \Phi(1'_A) \\ {\scriptstyle \phi_{f,u}}\downarrow & & \downarrow{\scriptstyle \phi_{A'}} \\ \Phi(f)\Phi(u) & \xrightarrow[\bar{\varepsilon}]{} & 1_{\Phi A'} \end{array} .$$

So if $\Phi$ is normal we can express $\bar{\eta}$ directly in terms of $\Phi\eta$ and the structure cells. However, to define $\bar{\varepsilon}$ in terms of $\Phi\varepsilon$ we need that $\Phi$ preserves the composite $fu$ up to isomorphism. Proposition 2.3 below shows that in the presence of normality this condition is also sufficient to ensure that $\Phi$ preserves adjunctions.

We see from this discussion that for bicategories normality is a useful property when considering the preservation of adjunctions, but it is not necessary or sufficient to ensure this preservation. We will see in Proposition 3.8 that in the context of double categories normality is equivalent to the preservation of conjoints, a kind of adjointness between horizontal and vertical morphisms.

The oplax morphisms of the form $\mathcal{S}pan(F)$ are more than normal, they preserve not only identities, but also certain composites, namely those of the form $f_* \otimes S$ and those of the form $S \otimes g^*$, for any span $S$. Indeed, the pullback involved in the composite $f_* \otimes S$ can be chosen to be



and such pullbacks are preserved by any functor. The case of $S \otimes g^*$ is dual. We saw in [13] that the condition that an oplax normal morphism preserve composites of the form $fx$ for every left adjoint $f$ is sufficient to ensure the preservation of adjunctions. However, there is the following slightly stronger result.

2.3. Proposition. *Let $\Phi\colon \mathcal{X} \to \mathcal{Y}$ be an oplax normal morphism of bicategories and let $f \dashv u$ be an adjoint pair of arrows in $\mathcal{X}$. If $\Phi$ preserves the composite $fu$, then $\Phi(f)$ is left adjoint to $\Phi(u)$. Furthermore, in this case $\Phi$ preserves all composites of the form $fx$ and $zu$.*

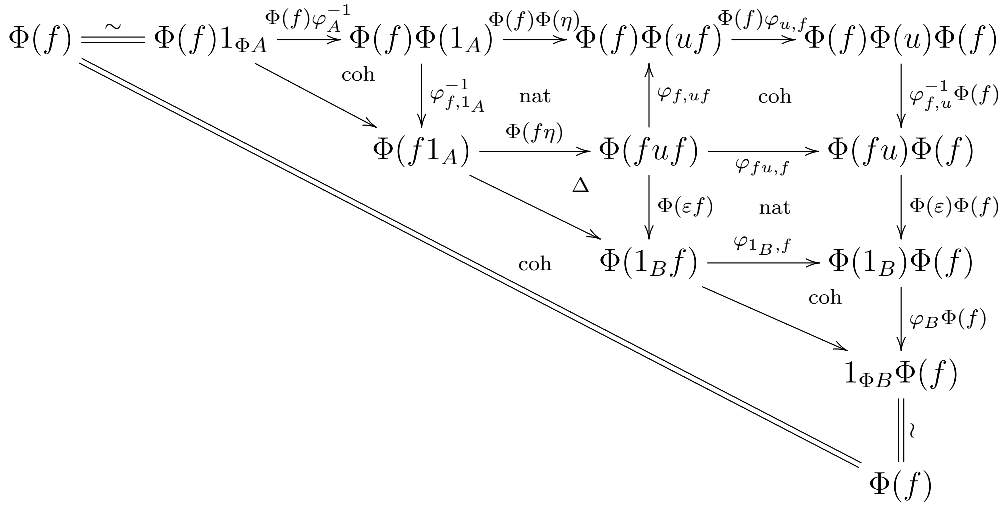
Proof. Let $\varepsilon\colon fu \to 1_B$ and $\eta\colon 1_A \to uf$ be the adjunctions for $f \dashv u$. To say that $\Phi$ preserves the composite $fu$ means that the oplaxity 2-cell $\varphi_{f,u}\colon \Phi(fu) \to \Phi(f)\Phi(u)$ is invertible. Define

$$\overline{\varepsilon} = (\,\Phi(f)\Phi(u) \xrightarrow{\varphi_{f,u}^{-1}} \Phi(fu) \xrightarrow{\Phi(\varepsilon)} \Phi(1_B) \xrightarrow{\varphi_B} 1_{\Phi B}\,)$$

and

$$\overline{\eta} = (\,1_{\Phi A} \xrightarrow{\varphi_A^{-1}} \Phi(1_A) \xrightarrow{\Phi(\eta)} \Phi(uf) \xrightarrow{\varphi_{u,f}} \Phi(u)\Phi(f)\,).$$

Then we obtain commutative diagrams



and

$$
\begin{array}{ccccccc}
\Phi(u) \overset{\sim}{=\!=} 1_{\Phi A}\Phi(u) & \xrightarrow{\varphi_A^{-1}\Phi(u)} & \Phi(1_A)\Phi(u) & \xrightarrow{\Phi(\eta)\Phi(u)} & \Phi(uf)\Phi(u) & \xrightarrow{\varphi_{u,f}\Phi(u)} & \Phi(u)\Phi(f)\Phi(u) \\
 & \text{coh} & \downarrow{\scriptstyle \varphi^{-1}_{1_A,u}} \quad \text{nat} & & \uparrow{\scriptstyle \varphi_{uf,u}} & \text{coh} & \downarrow{\scriptstyle \Phi(u)\varphi^{-1}_{f,u}} \\
 & & \Phi(1_A u) & \xrightarrow{\Phi(\eta u)} & \Phi(ufu) & \xrightarrow[\varphi_{u,fu}]{} & \Phi(u)\Phi(fu) \\
 & & & \Delta & \downarrow{\scriptstyle \Phi(u\varepsilon)} & \text{nat} & \downarrow{\scriptstyle \Phi(u)\Phi(\varepsilon)} \\
 & & \text{coh} & & \Phi(u1_B) & \xrightarrow{\varphi_{u,1_B}} & \Phi(u)\Phi(1_B) \\
 & & & & & \text{coh} & \downarrow{\scriptstyle \Phi(u)\varphi_B} \\
 & & & & & & \Phi(u)1_{\Phi B} \\
 & & & & & & \wr\| \\
 & & & & & & \Phi(u)
\end{array}
$$

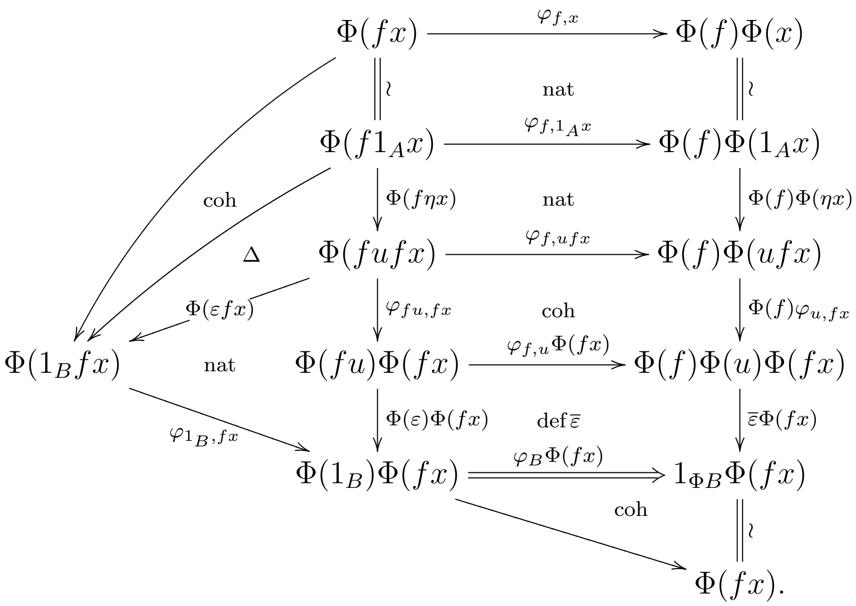
where the diagonal arrows come from unit isomorphisms. Note the arrows $\varphi_{f,uf}$ and $\varphi_{uf,u}$ in the middle which go up. This is necessary, because we do not know whether they are invertible. However, the $\varphi_{f,1}$ and $\varphi_{1,u}$ are invertible, since they are one-sided inverses to $\Phi(f)\varphi_A$ and $\varphi_A\Phi(u)$ respectively, which are invertible by normality. Thus, $\Phi(f) \dashv \Phi(u)$.

Next we will show that composites of the form $fx$ are preserved (up to isomorphism). The inverse for $\varphi_{f,x}\colon \Phi(fx) \to \Phi(f)\Phi(x)$ is the composite on the left of the diagram

$$
\begin{array}{ccc}
\Phi(f)\Phi(x) & & \\
\wr\| & & \\
\Phi(f)\Phi(1_A x) & \xrightarrow{\Phi(f)\varphi_{1_A,x}} & \Phi(f)\Phi(1_A)\Phi(x) \\
{\scriptstyle \Phi(f)\Phi(\eta x)}\downarrow & \text{nat} & \downarrow{\scriptstyle \Phi(f)\Phi(\eta)\Phi(x)} \\
\Phi(f)\Phi(ufx) & \xrightarrow{\Phi(f)\varphi_{uf,x}} & \Phi(f)\Phi(uf)\Phi(x) \\
{\scriptstyle \Phi(f)\varphi_{u,fx}}\downarrow & \text{coh} & \downarrow{\scriptstyle \Phi(f)\varphi_{u,f}\Phi(x)} \\
\Phi(f)\Phi(u)\Phi(fx) & \xrightarrow{\Phi(f)\Phi(u)\varphi_{f,x}} & \Phi(f)\Phi(u)\Phi(f)\Phi(x) \\
{\scriptstyle \bar{\varepsilon}\Phi(fx)}\downarrow & \text{nat} & \downarrow{\scriptstyle \bar{\varepsilon}\Phi(f)\Phi(x)} \\
1_{\Phi B}\Phi(fx) & \xRightarrow{1_{\Phi B}\varphi_{f,x}} & 1_{\Phi B}\Phi(f)\Phi(x) \\
\wr\| & \text{nat} & \|\wr \\
\Phi(fx) & \xrightarrow[\varphi_{f,x}]{} & \Phi(f)\Phi(x)
\end{array}
$$

and $\Phi(f)\varphi_{1_A,x} = \Phi(f)\varphi_A^{-1}\Phi(x)$ by normality and coherence of $\varphi$. Using the first triangle identity we conclude that the right side of the previous diagram, preceded by the top is $\mathrm{id}_{\Phi(f)\Phi(x)}$.

In the other direction we have the diagram




Preservation of $zu$ is dual. ■

The result of Proposition 2.3 leads to the following definition which is fundamental to all that will follow.

2.4. Definition. An oplax morphism $\Phi\colon \mathcal{X} \to \mathcal{Y}$ of bicategories is called *gregarious* if it is normal and preserves all composites of the form $fu$ where $f$ is left adjoint to $u$. (In [13] these morphisms were called *jointed*.)

The following theorem, proved in Proposition 2.8 and 2.9 of [13], establishes gregariousness as a fundamental concept.

2.5. Theorem. *An oplax normal morphism of bicategories is gregarious if and only if it preserves adjoints.*

The importance of this notion has been observed by various authors (cf. [6], [7], [8] and [9]). For a full discussion, see [13].

2.6. The universal property of $\mathcal{S}pan$. As we noted above, the embedding

$$(\ )_*\colon \mathbf{A} \to \mathcal{S}pan(\mathbf{A})$$

sends all arrows in $\mathbf{A}$ to left adjoints in $\mathcal{S}pan(\mathbf{A})$. So in order to state the universal property of $\mathcal{S}pan$ we will use the following property of morphisms of bicategories, which was introduced in [13].

2.7. Definition. A (strong) morphism $\Phi\colon \mathcal{X} \to \mathcal{Y}$ of bicategories is *sinister* if $\Phi(f)$ is a left adjoint for every arrow $f$ of $\mathcal{X}$. A strong transformation $t\colon \Phi \to \Psi$ of sinister morphisms is called *sinister* if for every arrow $f$ of $\mathcal{X}$, the mate of the naturality isomorphism $\Psi(f)t(X) \xrightarrow{\sim} t(X')\Phi(f)$ is itself an isomorphism $t(X)\Phi(f)^* \xrightarrow{\sim} \Psi(f)^* t(X')$. (See [13], Section 1.2.)

The main theorem of [13] now states that $(\ )_*\colon \mathbf{A} \to \mathcal{S}pan\,\mathbf{A}$ is the universal sinister morphism in the following precise sense.

2.8. Theorem. *Let* $\mathbf{A}$ *be a category with pullbacks and* $\mathcal{B}$ *an arbitrary bicategory. Then*

1. *Composing with* $(\ )_*\colon \mathbf{A} \to \mathcal{S}pan\,\mathbf{A}$ *gives an equivalence of categories between the category of gregarious morphisms* $\mathcal{S}pan\,\mathbf{A} \to \mathcal{B}$ *with oplax transformations and the category of sinister morphisms* $\mathbf{A} \to \mathcal{B}$ *with strong transformations.*

2. *Under this correspondence, strong transformations* $F \to G\colon \mathcal{S}pan\,\mathbf{A} \rightrightarrows \mathcal{B}$ *correspond to sinister transformations* $F \circ (\ )_* \to G \circ (\ )_*\colon \mathbf{A} \rightrightarrows \mathcal{B}$.

3. *Strong morphisms* $\mathcal{S}pan\,\mathbf{A} \to \mathcal{B}$ *correspond to sinister morphisms* $\mathbf{A} \to \mathcal{B}$ *satisfying the Beck-Chevalley condition.*

2.9. Functoriality. We write **PBCat** for the category of categories with pullbacks and pullback-preserving functors and **pbCat** for the category of categories with pullbacks and arbitrary functors. Then $\mathcal{S}pan$ can be considered either as a functor $\mathcal{S}pan\colon \mathbf{PBCat} \to \mathbf{Bicat}$ (where the morphisms of the codomain category are homomorphisms, also called pseudo functors or strong morphisms, of bicategories) or as a functor $\mathcal{S}pan\colon \mathbf{pbCat} \to \mathbf{Bicat}_{\mathrm{OplaxN}}$ (where the morphisms of the codomain category are oplax normal morphisms of bicategories). Since we are interested in the preservation of adjunctions, we may want to ask the question whether either of these functors is a 2-functor.

However, this question does not make sense because neither $\mathbf{Bicat}_{\mathrm{OplaxN}}$ nor **Bicat** can be considered as a 2-category (or bicategory) in a way that would include natural transformations with non-identity components on the objects and non-identity 2-cells for the arrows. For $\mathbf{Bicat}_{\mathrm{OplaxN}}$ this is obvious as there is no way of whiskering any type of natural transformations with oplax morphisms. For **Bicat** with homomorphisms there is a notion of whiskering. However, there are several problems with the horizontal and vertical composition of the natural transformations: vertical composition is not necessarily associative on the nose if the codomain object of the homomorphisms is a bicategory, and there are problems with middle-four-interchange and horizontal composition as well. These issues have been discussed in detail in [24].

We will revisit the question of functoriality of the Span construction again at the end of the next section and show that with a slight modification of this construction the problem can be resolved and we will gain further insight into the existence of certain limits and colimits and their properties in the Span categories.

## 3. The double category of spans

There is another version of the Span construction which will be central to what follows. Although a seemingly minor modification of $\mathcal{S}pan\,\mathbf{A}$, the difference will be crucial. We are referring to the (weak) double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$.

As before, let $\mathbf{A}$ be a category with pullbacks. $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is the double category with the same objects as $\mathbf{A}$. The vertical arrows are the morphisms of $\mathbf{A}$, the horizontal arrows are the spans in $\mathbf{A}$ and the double cells are commutative diagrams

$$\begin{array}{ccccc} A & \xleftarrow{p} & S & \xrightarrow{q} & B \\ {\scriptstyle f}\downarrow & & {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \xleftarrow[p']{} & S' & \xrightarrow[q']{} & B' . \end{array}$$

Vertical composition is as in $\mathbf{A}$ and horizontal composition as in $\mathcal{S}pan\,\mathbf{A}$. Of course, horizontal composition is not strictly unitary or associative, so we only get a weak double category. This is most easily expressed by saying that the substructure consisting of objects, horizontal arrows, and (horizontally) special cells, form a bicategory. Here, *special* cells are those which have identity arrows as horizontal domain and codomain, so they are of the form

$$\begin{array}{ccccc} A & \xleftarrow{p} & S & \xrightarrow{q} & B \\ \| & & {\scriptstyle x}\downarrow & & \| \\ A & \xleftarrow[p']{} & S' & \xrightarrow[q']{} & B . \end{array}$$

So there are special associativity and unit cells that satisfy the usual coherence conditions. Where this doesn't lead to confusion we will suppress the occurrence of these cells in our pasting diagrams. Note that, while horizontal composition of spans requires a choice of pullbacks and is only associative up to isomorphism once this choice has been made, the horizontal composition of cells is uniquely determined and as associative as possible.

3.1. CONVENTION. For the rest of this paper, except for Section 4.28, we will refer to weak double categories simply as double categories as they seem to be the more central concept. We will talk of a *strict double category* when the bicategory substructure mentioned above is in fact a 2-category.

An important aspect of this new Span construction is that it retains the morphisms of $\mathbf{A}$ in the structure. And these can be used to formulate concepts and universal properties which $\mathcal{S}pan\,\mathbf{A}$ doesn't support, as we shall see later. Shulman has also noted this important feature in [30].

Let us digress a bit to elaborate on this. It is useful to view internal categories in $\mathbf{A}$ as monads in $\mathcal{S}pan\,\mathbf{A}$. However, this monad approach does not give us internal functors as a natural notion of morphisms between internal categories. Looking for guidance, we might consider a monad in $\mathcal{S}pan\,\mathbf{A}$ as a lax morphism defined on the bicategory $\mathbf{1}$ and so we might consider lax, oplax, or strong transformations between them. However, none of these give functors (nor do any of them give profunctors). In order to get internal functors the usual trick is to restrict to lax transformations whose components are maps. While this works, it is certainly *ad hoc* and does not generalize properly. For example, in the bicategory $V$-**Mat** of $V$-matrices, monads are $V$-categories, but maps between $V$-monads do not give $V$-functors.

Taking the double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ instead, lax morphisms $\mathbb{1} \to \mathbb{S}\mathrm{pan}\,\mathbf{A}$ are still category objects in $\mathbf{A}$, but now we have a good notion of vertical transformation between lax morphisms, and these are exactly the internal functors. Note in passing that lax morphisms $\mathbb{2} \to \mathbb{S}\mathrm{pan}\,\mathbf{A}$ *are* profunctors, and that vertical transformations between them are cells in the double category $\mathbb{C}\mathrm{at}$ of categories, functors, and profunctors.

3.2. Oplax morphisms of double categories. We mentioned lax morphisms of double categories and we should say a word about the definitions of lax and oplax morphisms. It is spelled out in detail in [12], Section 1.9, for the oplax version (which is most useful for our work). An oplax morphism of double categories $F\colon \mathbb{A} \to \mathbb{B}$ assigns objects, horizontal arrows, vertical arrows, and double cells of $\mathbb{B}$ to similar ones in $\mathbb{A}$ respecting all domains and codomains and preserving vertical composition (of arrows and cells). Horizontal identities and composition are not preserved, but *comparison special cells*

$$
\begin{array}{ccc}
FA & \xrightarrow{F1_A} & FA \\
\| & \varphi_A & \| \\
FA & \xrightarrow[1_{FA}]{} & FA
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
FA & & \xrightarrow{F(f'f)} & & FA'' \\
\| & & \varphi_{f',f} & & \| \\
FA & \xrightarrow[Ff]{} & FA' & \xrightarrow[Ff']{} & FA''
\end{array}
$$

are given, satisfying the coherence conditions given in Section 1.9 of [12]. There are three conditions which are straightforward generalizations of the coherence conditions for oplax morphisms of bicategories, and one condition proper to double categories, namely naturality of $\phi_A$ in $A$, with respect to vertical arrows: For every vertical arrow $v\colon A \xrightarrow{\bullet} B$ we require

$$
\begin{array}{ccc}
FA & \xrightarrow{F1_A} & FA \\
{\scriptstyle Fv}\downarrow\bullet & F1_v & \bullet\downarrow{\scriptstyle Fv} \\
FB & \xrightarrow[F1_B]{} & FB \\
\| & \varphi_B & \| \\
FB & \xrightarrow[1_{FB}]{} & FB
\end{array}
\qquad = \qquad
\begin{array}{ccc}
FA & \xrightarrow{F1_A} & FA \\
\| & \varphi_A & \| \\
FA & \xrightarrow[1_{FA}]{} & FA \\
{\scriptstyle Fv}\downarrow\bullet & 1_{Fv} & \bullet\downarrow{\scriptstyle Fv} \\
FB & \xrightarrow[1_{FB}]{} & FB.
\end{array}
$$

Note that this condition is vacuous for bicategories (considered as double categories with only identity vertical arrows).

Vertical transformations $t\colon F \to G$ between oplax morphisms were also defined in [12], Definition 1.13. They associate to each object $A$ of $\mathbb{A}$ a vertical arrow $t_A\colon FA \xrightarrow{\bullet} GA$

and to each horizontal arrow $f\colon A \to A'$ of $\mathbb{A}$ a cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FA' \\ {\scriptstyle t_A}\big\downarrow\!\bullet & t_f & \bullet\!\big\downarrow{\scriptstyle t_{A'}} \\ GA & \xrightarrow[Gf]{} & GA' \end{array}$$

satisfying four conditions, two saying that $t$ is vertically natural and two saying that it is horizontally compatible with the identities and composition.

In this way we get a 2-category $\mathbf{Doub}_{\mathrm{Opl}}$ of double categories, oplax morphisms, and vertical transformations. Because oplax morphisms are vertically functorial and vertical transformations are vertically natural we get a canonical forgetful 2-functor

$$\mathrm{Vrt} : \mathbf{Doub}_{\mathrm{Opl}} \to \mathbf{Cat},$$

which forgets everything but the vertical structure. It can be thought of as the 'category of objects' functor, generalizing $\mathbf{Ob}\colon \mathbf{Cat} \to \mathbf{Set}$.

3.3. Gregarious Double Categories. Returning to our general discussion of $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, apart from $(\ )_*$ and $(\ )^*$ there is a third, more direct, way of including $\mathbf{A}$ in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, *viz.* the inclusion of $\mathbf{A}$ as vertical arrows. We will always consider $\mathbf{A}$ as contained in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ in this way and not use any special notation for the inclusion.

Now the question is, what is the relationship between $f$, $f_*$, and $f^*$ in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$? Recall from [19] the following definitions.

3.4. Definition. Let $\mathbb{D}$ be a double category and consider horizontal morphisms $f\colon A \to B$ and $u\colon B \to A$ and a vertical morphism $v\colon A \to\!\!\!\!\!\bullet\ B$. We say that $f$ and $v$ are *companions* if there exist *binding cells*

$$\begin{array}{ccc} A & = & A \\ \| & \psi & \bullet\!\big\downarrow{\scriptstyle v} \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle v}\big\downarrow\!\bullet & \chi & \| \\ B & = & B, \end{array}$$

such that

$$\begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ \| & \psi & \bullet\!\big\downarrow{\scriptstyle v} & \chi & \| \\ A & \xrightarrow[f]{} & B & = & B \end{array} \;=\; \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & = & A \\ \| & \psi & \bullet\!\big\downarrow{\scriptstyle v} \\ A & \xrightarrow[f]{} & B \\ \bullet\!\big\downarrow{\scriptstyle v} & \chi & \| \\ B & = & B \end{array} \;=\; \begin{array}{ccc} A & = & A \\ \bullet\!\big\downarrow{\scriptstyle v} & 1_v & \bullet\!\big\downarrow{\scriptstyle v} \\ B & = & B \end{array}.$$

Dually, $u$ and $v$ are *conjoints* if there exist *binding cells*

$$\begin{array}{ccc} A & = & A \\ \bullet\downarrow{\scriptstyle v} & \alpha & \| \\ B & \xrightarrow[u]{} & A \end{array} \quad\text{and}\quad \begin{array}{ccc} B & \xrightarrow{u} & A \\ \| & \beta & \bullet\downarrow{\scriptstyle v} \\ B & = & B, \end{array}$$

such that

$$\begin{array}{ccccc} B & \xrightarrow{u} & A & = & A \\ \| & \beta & \bullet\downarrow{\scriptstyle v} & \alpha & \| \\ B & = & B & \xrightarrow[u]{} & A \end{array} = \begin{array}{ccc} B & \xrightarrow{u} & A \\ \| & \mathrm{id}_u & \| \\ B & \xrightarrow[u]{} & A, \end{array} \quad\text{and}\quad \begin{array}{ccc} A & = & A \\ \bullet\downarrow{\scriptstyle v} & \alpha & \| \\ B & \xrightarrow[u]{} & A \\ \| & \beta & \bullet\downarrow{\scriptstyle v} \\ B & = & B \end{array} = \begin{array}{ccc} A & = & A \\ \bullet\downarrow{\scriptstyle v} & 1_v & \bullet\downarrow{\scriptstyle v} \\ B & = & B \end{array}.$$

Finally, $f$ is *left adjoint* to $u$ in $\mathbb{D}$ if there exist special cells

$$\begin{array}{ccccc} A & = & = & = & A \\ \| & & \eta & & \| \\ A & \xrightarrow[f]{} & B & \xrightarrow[u]{} & A \end{array} \quad\text{and}\quad \begin{array}{ccccc} B & \xrightarrow{u} & A & \xrightarrow{f} & B \\ \| & & \varepsilon & & \| \\ B & = & = & = & B, \end{array}$$

such that

$$\begin{array}{ccccccc} A & = & = & = & A & \xrightarrow{f} & B \\ \| & & \eta & & \| & \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B & \xrightarrow[u]{} & A & \xrightarrow[f]{} & B \\ \| & \mathrm{id}_f & \| & & \varepsilon & & \| \\ A & \xrightarrow[f]{} & B & = & = & = & B \end{array} = \begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

and

$$\begin{array}{ccccccc} B & \xrightarrow{u} & A & = & = & = & A \\ \| & \mathrm{id}_u & \| & & \eta & & \| \\ B & \xrightarrow[u]{} & A & \xrightarrow[f]{} & B & \xrightarrow[u]{} & A \\ \| & & \varepsilon & & \| & \mathrm{id}_u & \| \\ B & = & = & = & B & \xrightarrow[u]{} & A \end{array} = \begin{array}{ccc} B & \xrightarrow{u} & A \\ \| & \mathrm{id}_u & \| \\ B & \xrightarrow[u]{} & A \end{array},$$

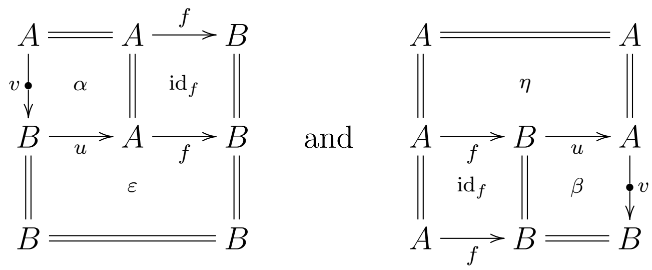
*i.e.*, $f$ is left adjoint to $u$ in the bicategory of horizontal arrows and special cells.

Note that companions in symmetric double categories (where the horizontal and vertical arrow categories are the same) were studied by Brown and Mosa [4] under the name 'connections'.

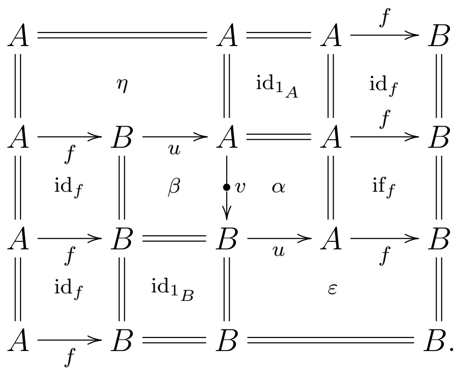
The following result extends that of Proposition 1.4 in [18].

3.5. Proposition. *Given $f$, $u$, and $v$ as above, if any two of the relations in the definition hold, so does the third.*

Proof. The proof is an amusing exercise in pasting diagrams which is left to the reader. We will do one case as an example. Suppose for example that $f$ is left adjoint to $u$ via $\varepsilon$ and $\eta$ and that $u$ and $v$ are conjoints via $\alpha$ and $\beta$. Then define $\chi$ and $\psi$ to be



respectively. Then $\chi\psi$ is the composite

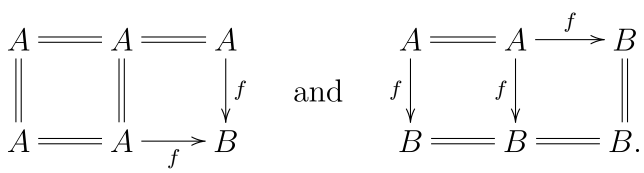


The composite $\alpha\beta$ in the middle is equal to $\mathrm{id}_u$ so the whole middle row is an identity and the big square reduces to a vertical identity cell by the first triangle identity for $\eta$ and $\varepsilon$. ■

Companions are unique up to isomorphism and compose in the sense that if $f$ and $v$ are companions and $f'$ and $v'$ are companions and $f'$ and $f$ are composable, then $f'f$ and $v' \cdot v$ are again companions. A similar statement holds for conjoints by duality and the corresponding statement for adjoints is well known.

3.6. Examples.

1. In $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, $f$ and $f_*$ are companions, the binding cells being



Similarly, $f$ and $f^*$ are conjoints. The fact that $f_*$ is left adjoint to $f^*$ follows by Proposition 3.5.

2. A 2-category $\mathcal{A}$ gives a double category of quintets, $\mathbb{Q}(\mathcal{A})$, where a typical cell looks like

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & \alpha\Downarrow & \big\downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & P\,. \end{array}$$

In $\mathbb{Q}(\mathcal{A})$, the horizontal $f$ and vertical $f$ are companions, whereas a horizontal $f$ and a vertical $v$ are conjoints if and only if $v$ is left adjoint to $f$ in $\mathcal{A}$.

Normal morphisms of double categories preserve companions and conjoints, but more is true.

3.7. Definition. Let $F\colon \mathbb{A} \to \mathbb{B}$ be an oplax (not necessarily normal) morphism of double categories.

1. We say that $F$ *preserves companions* if for each companion pair $f\colon A \to B$ and $v\colon A \longrightarrow\!\!\!\!\!\bullet\,\, B$ with binding cells $\chi$ and $\psi$ in $\mathbb{A}$ there exists a double cell

$$\begin{array}{ccc} FA & = & FA \\ \| & \overline{\psi} & \big\downarrow\!\bullet{\scriptstyle Fv} \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

in $\mathbb{B}$, such that $F\psi = \overline{\psi} \cdot \varphi_A$ and $Ff$ and $Fv$ are again companions with binding cells $\overline{\psi}$ and $\varphi_B \cdot F\chi$.

2. We say that $F$ *preserves conjoints* if for each conjoint pair $f\colon A \to B$ and $v\colon B \longrightarrow\!\!\!\!\!\bullet\,\, A$ with binding cells $\alpha$ and $\beta$ in $\mathbb{A}$ there exists a double cell

$$\begin{array}{ccc} FB & = & FB \\ {\scriptstyle Fv}\bullet\!\big\downarrow & \overline{\alpha} & \| \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

in $\mathbb{B}$, such that $\overline{\alpha} \cdot \varphi_B = F\alpha$ and $Ff$ and $Fv$ are again conjoints with binding cells $\varphi_A \cdot F\beta$ and $\overline{\alpha}$.

3.8. Proposition. *For an oplax morphism $F\colon \mathbb{A} \to \mathbb{B}$ of double categories the following are equivalent:*

1. *$F$ is normal;*
2. *$F$ preserves companions;*
3. *$F$ preserves conjoints.*

PROOF. To see that (1) implies (2), let $f\colon A \longrightarrow B$ and $v\colon A \longrightarrow\!\!\!\!\!\bullet\; B$ be companions with binding cells $\psi$ and $\chi$ as in Definition 3.4. Let $\overline{\psi} = F\psi \cdot \varphi_A^{-1}$. Then obviously, $F\psi = \overline{\psi} \cdot \varphi_A$. So we need to show that

$$
\begin{array}{ccc}
FA & \xrightarrow{1_{FA}} & FA \\
\Vert & \varphi_A^{-1} & \Vert \\
FA & \xrightarrow{F1_A} & FA \\
\Vert & F\psi & \bullet\downarrow Fv \\
FA & \xrightarrow[Ff]{} & FB
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FB \\
Fv \bullet\downarrow & F\chi & \Vert \\
FB & \xrightarrow{F1_B} & FB \\
\Vert & \varphi_B & \Vert \\
FB & \xrightarrow[1_{FB}]{} & FB.
\end{array}
$$

satisfy the conditions to be binding cells for $Ff$ and $Fv$. Their vertical composition is equal to $\varphi_B \cdot F\chi \cdot F\psi \cdot \varphi_A^{-1} = \varphi_B \cdot F(\chi \cdot \psi) \cdot \varphi_A^{-1} = \varphi_B \cdot F(1_v) \cdot \varphi_A^{-1} = 1_{F(v)} \cdot \varphi_A \cdot \varphi_A^{-1} = 1_{F(v)}$. Their horizontal composition is equal to

$$
\begin{array}{ccccc}
FA & \xrightarrow{1_{FA}} & FA & \xrightarrow{Ff} & FB \\
\Vert & \varphi_A^{-1} & \Vert & \mathrm{id}_{Ff} & \Vert \\
FA & \xrightarrow{F1_A} & FA & \xrightarrow{Ff} & FB \\
\Vert & F\psi & \bullet\downarrow Fv & F\chi & \Vert \\
FA & \xrightarrow{Ff} & FB & \xrightarrow{F1_B} & FB \\
\Vert & \mathrm{id}_{Ff} & \Vert & \varphi_B & \Vert \\
FA & \xrightarrow[Ff]{} & FB & \xrightarrow[1_{FB}]{} & FB,
\end{array}
$$

which, by coherence for $F$, is equal to

$$
\begin{array}{ccccc}
FA & & \xrightarrow{\quad Ff \quad} & & FB \\
\Vert & & \varphi_{f,1_A} & & \Vert \\
FA & \xrightarrow{F1_A} & FA & \xrightarrow{Ff} & FB \\
\Vert & F\psi & \bullet\downarrow Fv & F\chi & \Vert \\
FA & \xrightarrow{Ff} & FB & \xrightarrow{F1_B} & FB \\
\Vert & \mathrm{id}_{Ff} & \Vert & \varphi_B & \Vert \\
FA & \xrightarrow[Ff]{} & FB & \xrightarrow[1_{FB}]{} & FB
\end{array}
\quad = \quad
\begin{array}{ccccc}
FA & & \xrightarrow{\quad Ff \quad} & & FB \\
\Vert & & F(\chi\psi) & & \Vert \\
FA & & \xrightarrow{\quad Ff \quad} & & FB \\
\Vert & & \varphi_{1_B,f} & & \Vert \\
FA & \xrightarrow[Ff]{} & FB & \xrightarrow[F1_B]{} & FB \\
\Vert & \mathrm{id}_{Ff} & \Vert & \varphi_B & \Vert \\
FA & \xrightarrow[Ff]{} & FB & = & FB
\end{array}
$$

and $F(\chi\psi) = F(\mathrm{id}_f) = \mathrm{id}_{Ff}$ and $(\varphi_B \mathrm{id}_{Ff}) \cdot \varphi_{1_{Ff}} = \mathrm{id}_{Ff}$ so the whole composite is $\mathrm{id}_{Ff}$, *i.e.*, $Ff$ and $Fv$ are companions with binding cells $\tilde{\psi} = F\psi \cdot \varphi_A^{-1}$ and $\tilde{\chi} = \varphi_B \cdot F\chi$. The corresponding result for conjoints ((1) implies (3)) follows by duality.

To show that (2) implies (1), suppose that $F\colon \mathbb{A} \to \mathbb{B}$ preserves companions. Note that for every object $A$, the horizontal identity arrow $1_A$ and the vertical identity arrow

$\mathrm{Id}_A$ are conjoints with binding cells $\chi = \psi = \iota_A$. This gives rise to the existence of a double cell

$$\begin{array}{ccc} FA & \xrightarrow{1_{FA}} & FA \\ \| & \bar{\iota}_A & \| \\ FA & \xrightarrow[F1_A]{} & FA \end{array}$$

such that

$$\varphi_A \cdot \bar{\iota}_A = \iota_{FA}, \tag{2}$$

and $F1_A$ and the vertical identity on $A$ are companions in $\mathbb{B}$ with binding cells

$$\begin{array}{ccc} FA & \xrightarrow{1_{FA}} & FA \\ \| & \bar{\iota}_A & \| \\ FA & \xrightarrow[F1_A]{} & FA \end{array} \quad \text{and} \quad \begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow[1_{FA}]{} & FA. \end{array}$$

The conditions on the binding cells give us that $\varphi_A \bar{\iota}_A = \mathrm{id}_{F1_A}$ and $\varphi_A \cdot \bar{\iota}_A = \iota_{FA}$. Since

$$\begin{array}{ccccc} FA & \xrightarrow{1_{FA}} & FA & \xrightarrow{F1_A} & FA \\ \| & \bar{\iota}_A & \| & \varphi_A & \| \\ FA & \xrightarrow[F1_A]{} & FA & \xrightarrow[1_{FA}]{} & FA \end{array} = \begin{array}{ccccc} FA & \xrightarrow{1_{FA}} & FA & \xrightarrow{1_{FA}} & FA \\ \| & \iota_{FA} & \| & \varphi_A & \| \\ FA & \xrightarrow{1_{FA}} & FA & \xrightarrow{1_{FA}} & FA \\ \| & \bar{\iota}_A & \| & \iota_{FA} & \| \\ FA & \xrightarrow[F1_A]{} & FA & \xrightarrow[1_{FA}]{} & FA \end{array} = \begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & \varphi_A & \| \\ FA & \xrightarrow{1_{FA}} & FA \\ \| & \bar{\iota}_A & \| \\ FA & \xrightarrow[F1_A]{} & FA \end{array},$$

*i.e.*, $\varphi_A \bar{\iota}_A = \bar{\iota}_A \cdot \varphi_A$, we derive that $\bar{\iota}_A \cdot \varphi_A = \mathrm{id}_{F1_A}$ and $\varphi_A \cdot \bar{\iota}_A = \iota_{FA}$. So $\bar{\iota}_A = \varphi_A^{-1}$, and we conclude that $F$ is normal. The corresponding result for conjoints, *i.e.*, that (3) implies (1), follows again by duality. ■

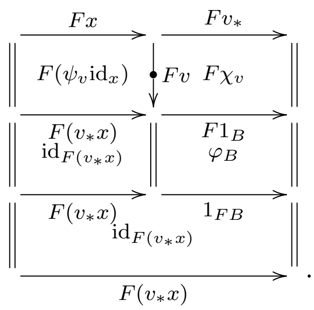
3.9. PROPOSITION. *An oplax normal morphism of double categories preserves composites of the form $v_*x$, where $v_*$ is the companion of a vertical arrow $v$.*

PROOF. We need to show that the double cell

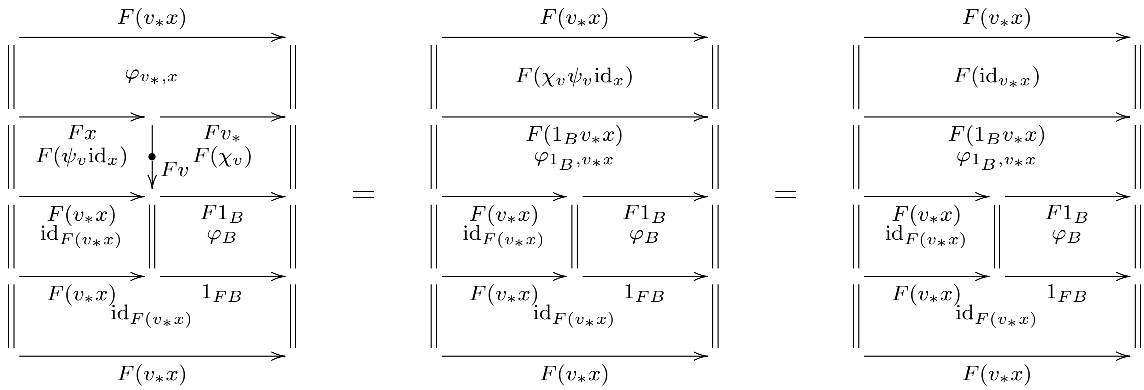
$$\begin{array}{ccccc} \cdot & & \xrightarrow{F(v_*x)} & & \cdot \\ \| & & \varphi_{v_*,x} & & \| \\ \cdot & \xrightarrow[Fx]{} & \cdot & \xrightarrow[Fv_*]{} & \cdot \end{array}$$

is vertically invertible. We claim that the inverse is the pasting of the following diagram:

$\theta_{v_*,x} :=$

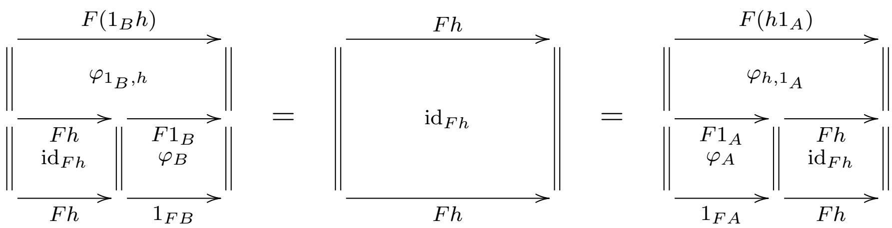



The following calculation shows that $\theta_{v_*,x} \cdot \varphi_{v_*,x} = \mathrm{id}_{F(v_*x)}$. By naturality of $\varphi$ and the horizontal binding cell equality for companion pairs,

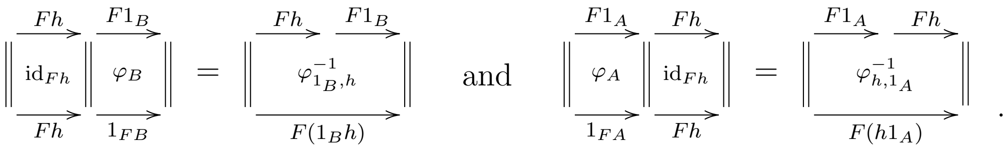



which is equal to $\mathrm{id}_{F(v_*x)}$ by a unit law for $\varphi$.

In order to show that $\varphi_{v_*,x} \cdot \theta_{v_*,x} = \mathrm{id}_{Fv_*}\mathrm{id}_{Fx}$, we first make a couple of observations about normal oplax morphisms of double categories. Let $h\colon A \to B$ be any horizontal arrow in $\mathbb{A}$. Since $\varphi_A$ and $\varphi_B$ are vertically invertible in $\mathbb{B}$ and

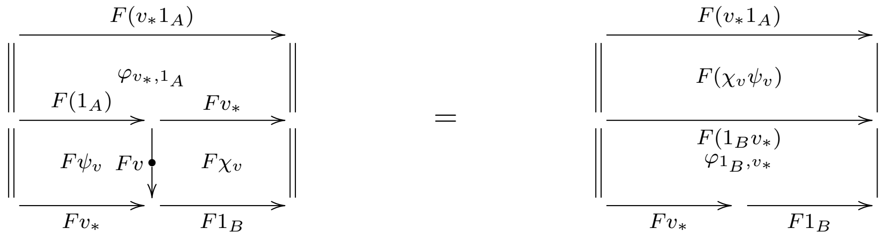



we derive that

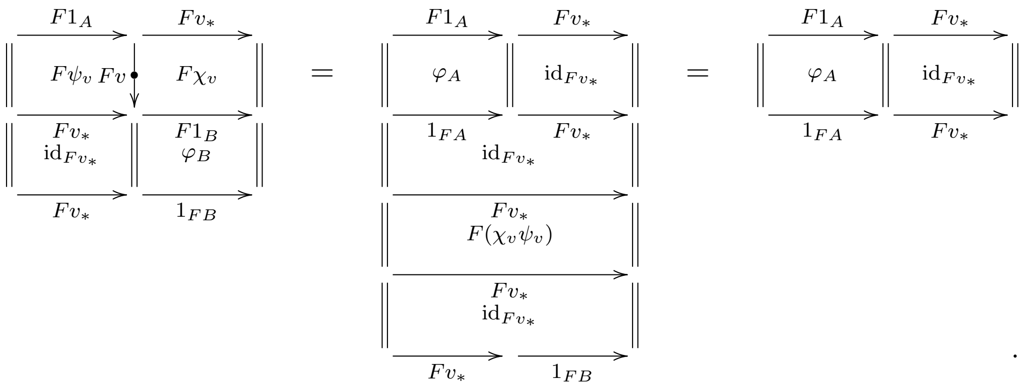


(3)

We can apply this to the binding cells of a companion pair to obtain the following. By naturality of $\varphi$ we have



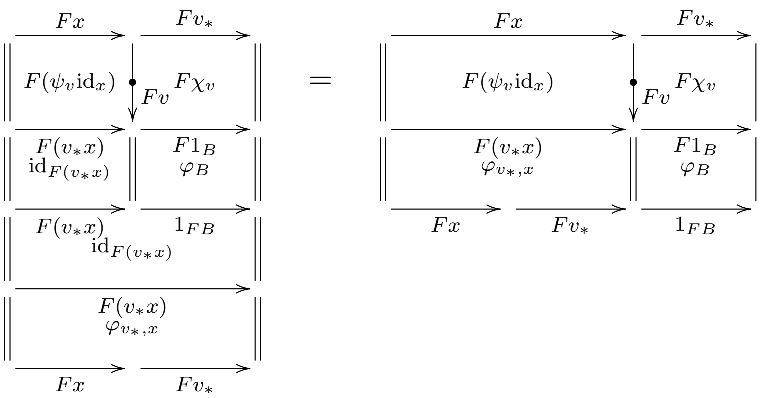

and by vertical composition with inverses we obtain



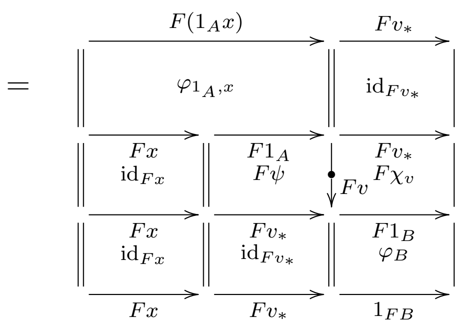
(4)

.

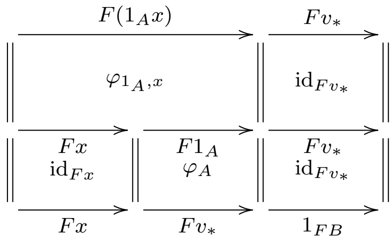
We will now use this to calculate the composition $\varphi_{v_*,x} \cdot \theta_{v_*,x}$:



by associativity



by naturality of $\varphi$

$=$  by (4)

and this is equal to $\mathrm{id}_{Fv_*}\mathrm{id}_{Fx}$ by (3). ∎

3.10. Remark. Note that we can apply Proposition 2.3 to the horizontal bicategory of a double category to show that an oplax normal morphism preserves composites of the form $v_*x$ if and only if it preserves composites of the form $zv^*$ if and only if it preserves composites of the form $v_*v^*$. So we conclude that oplax normal morphisms preserve all these composites.

3.11. Definition. A double category is *gregarious* if every vertical arrow has a companion and a conjoint. We write $\mathbf{GregDoub}_{\mathrm{OplaxN}}$ for the 2-category of gregarious double categories, oplax normal morphisms and vertical transformations.

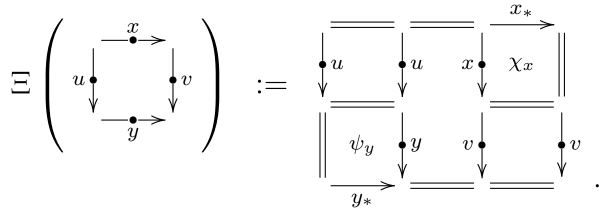
Gregarious double categories were introduced and studied as *framed bicategories* by Shulman in [30].

3.12. Examples.

1. The double category of commuting squares in a groupoid is gregarious.
2. If $\mathcal{A}$ is a 2-category in which every arrow has a right adjoint, then the double category $\mathbb{Q}(\mathcal{A})$ of quintets is gregarious.
3. For any category $\mathbf{A}$ with pullbacks, the double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is gregarious.

3.13. The universal property of $\mathbb{S}$pan. Let $\mathbb{A}$ be a gregarious double category with $\mathbf{A}_0$ as category of objects and vertical arrows and $\mathbf{A}_1$ as category of horizontal arrows and cells (with vertical composition of cells). Any choice of companions for the vertical arrows in $\mathbf{A}$ gives rise to a functor $\Xi\colon \mathbf{A}_0^{\mathbf{2}} \to \mathbf{A}_1$, defined by:

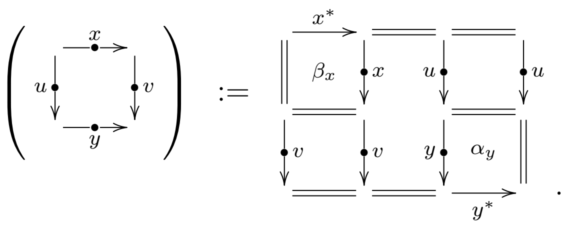
$$\Xi(\xrightarrow{x}) \;:=\; \xrightarrow{x_*} \tag{5}$$



(6)

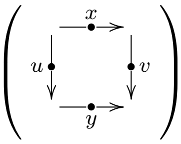
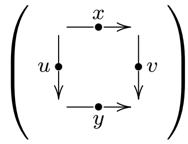
Dually, a choice of conjoints gives rise to a functor $Z\colon \mathbf{A}_0^{\mathbf{2}} \to \mathbf{A}_1$, defined by:

$$Z(\xrightarrow{x}) := \xrightarrow{x^*} \tag{7}$$

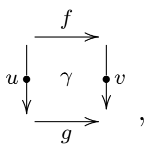
$Z$  (8)



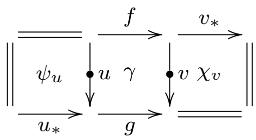
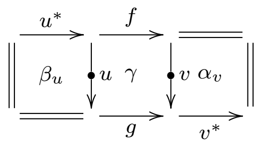
As we will use the cells from these definitions frequently, we will use the following abbreviations:

$\xi(x,u,v,y) = \Xi$  and $\zeta(x,u,v,y) = Z$  (9)

There are corresponding folding operations on the cells of the double category that send cells to special horizontal cells. By abuse of notation we will write for a cell

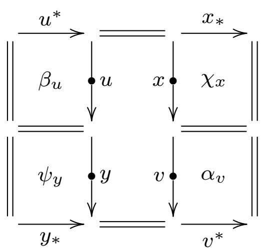




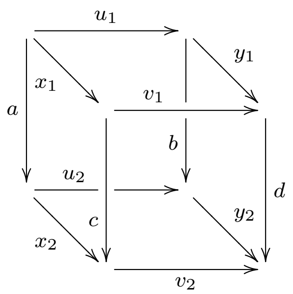
$\xi(\gamma) =$  and $\zeta(\gamma) =$  .



Note that $\zeta(\xi(x,u,v,y)) = \xi(\zeta(u,x,y,v))$. This cell will play an important role in our study of a Beck-Chevalley type condition for double categories, so we will give it a special name:

$\Upsilon(x,u,v,y) :=$  (10)

3.14. Lemma. *For any commutative cube of vertical arrows*




*the cells defined in (9) and (10) satisfy*

$$(\zeta(y_1,b,d,y_2)\xi(v_1,c,d,v_2))\cdot\Upsilon(u_1,x_1,y_1,v_1) = \\ \Upsilon(u_2,x_2,y_2,v_2)\cdot(\xi(u_1,a,b,u_2)\zeta(x_1,a,c,x_2)).$$

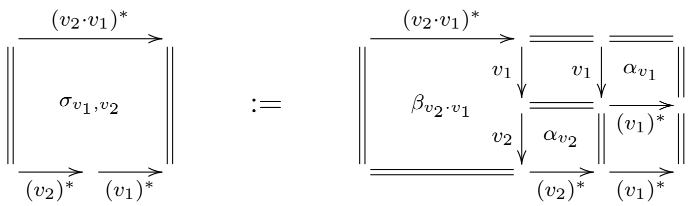
Proof. Straightforward calculation using the cancelation properties of the companion and conjoint binding cells. ■

We can now state a variant of part 1 of Theorem 2.8 for the double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$.

3.15. Theorem. *Let* $\mathbf{A}$ *be a category with pullbacks and* $\mathbb{B}$ *be a gregarious double category. Then composing with the inclusion* $\mathbf{A}\hookrightarrow\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *gives an equivalence of categories between the category of oplax normal morphisms* $\mathbb{S}\mathrm{pan}\,\mathbf{A}\to\mathbb{B}$ *with vertical transformations, and the category of functors* $\mathbf{A}\to\mathrm{Vrt}\,(\mathbb{B})$ *with natural transformations,*

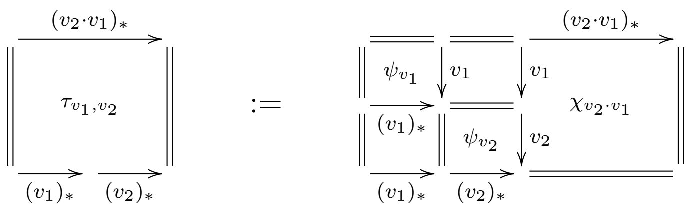
$$\mathbf{GregDoub}_{\mathrm{OplaxN}}(\mathbb{S}\mathrm{pan}\,\mathbf{A},\mathbb{B})\simeq\mathbf{Cat}\,(\mathbf{A},\mathrm{Vrt}\,(\mathbb{B})).$$

Proof. For every vertical arrow $v$ in $\mathbb{B}$, choose a companion $v_*$ with cells $\chi_v$ and $\psi_v$ and a conjoint $v^*$ with cells $\alpha_v$ and $\beta_v$. For the vertical identity arrows, choose the companions and conjoints to be the corresponding horizontal identity arrows with the $\iota$ cells as binding cells. Note that for composable vertical morphisms $v_1$ and $v_2$, we don't get necessarily that $(v_2\cdot v_1)_* = (v_2)_*(v_1)_*$ or $(v_2\cdot v_1)^* = (v_1)^*(v_2)^*$, but there are canonical vertical isomorphisms $\tau_{v_1,v_2}$ and $\sigma_{v_1,v_2}$ defined by



(11)

and



(12)

With these choices of companions and conjoints, a functor $F\colon \mathbf{A} \to \mathrm{Vrt}\,(\mathbb{B})$ can be lifted to an oplax normal morphism $\tilde{F}\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ defined as follows. On objects and vertical arrows, $\tilde{F} = F$, and on horizontal arrows, $\tilde{F}(A \xleftarrow{q} S \xrightarrow{p} B) = (Fp)_*(Fq)^*$. Further, $\tilde{F}$ sends a cell

$$\begin{array}{ccccc} A_1 & \xleftarrow{q_1} & S_1 & \xrightarrow{p_1} & B_1 \\ {\scriptstyle v_1}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v_2} \\ A_2 & \xleftarrow[q_2]{} & S_2 & \xrightarrow[p_2]{} & B_2 \end{array}$$

in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ to the following horizontal composition of cells in $\mathbb{B}$ (*cf.* (9)),

$$\begin{array}{ccccc} \cdot & \xrightarrow{(Fq_1)^*} & \cdot & \xrightarrow{(Fp_1)_*} & \cdot \\ {\scriptstyle Fv_1}\downarrow\!\bullet & \zeta(Fq_1,Fw,Fv_1,Fq_2) & \bullet\!\downarrow{\scriptstyle Fw} & \xi(Fp_1,Fw,Fv_2,Fp_2) & \bullet\!\downarrow{\scriptstyle Fv_2} \\ \cdot & \xrightarrow[(Fq_2)^*]{} & \cdot & \xrightarrow[(Fp_2)_*]{} & \cdot \end{array}\;.$$

Note that by our choice of companions and conjoints for the vertical identity arrows, $\tilde{F}$ preserves horizontal identities on the nose. To obtain the comparison cells for horizontal composites, consider a composable pair of horizontal arrows

$$A \xleftarrow{q} S \xrightarrow{p} B \xleftarrow{q'} S' \xrightarrow{p'} C$$

in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, with pullback

$$\begin{array}{ccc} S \times_B S' & \xrightarrow{\overline{p}} & S' \\ {\scriptstyle \overline{q'}}\downarrow & & \downarrow{\scriptstyle q'} \\ S & \xrightarrow[p]{} & B \end{array}$$

in $\mathbf{A}$. The comparison cell $\varphi_{S,S'}$ is defined as the composite

$$\begin{array}{ccccccccc} \cdot & & \xrightarrow{F(q\overline{q'})^*} & & \cdot & & \xrightarrow{F(p'\overline{p})_*} & & \cdot \\ \| & & \sigma_{F\overline{q'},Fq} & & \| & & \tau_{F\overline{p},Fp'} & & \| \\ \cdot & \xrightarrow{(Fq)^*} & \cdot & \xrightarrow{(F\overline{q'})^*} & \cdot & \xrightarrow{(F\overline{p})_*} & \cdot & \xrightarrow{(Fp')_*} & \cdot \\ \| & & \| & & \Upsilon(F\overline{p},F\overline{q'},Fq',Fp) & & \| & & \| \\ \cdot & \xrightarrow[(Fq)^*]{} & \cdot & \xrightarrow[(Fp)_*]{} & \cdot & \xrightarrow[(Fq')^*]{} & \cdot & \xrightarrow[(Fp')_*]{} & \cdot \end{array}\;, \qquad (13)$$

with $\Upsilon$ as defined in (10). Straightforward calculations show that these comparison cells satisfy the coherence conditions.

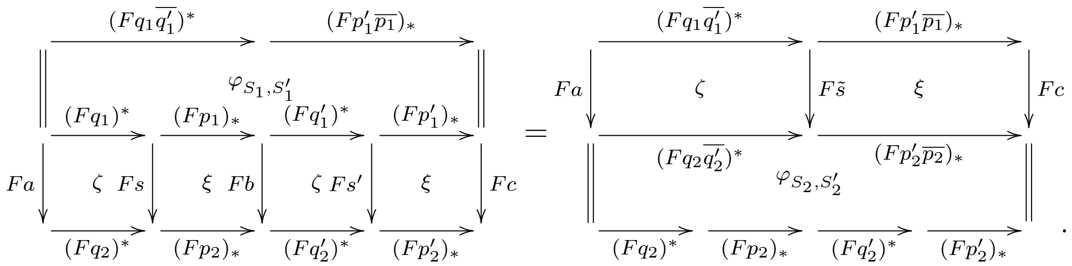
To check that these comparison cells satisfy the naturality conditions with respect to cells, consider the horizontally composable cells

$$\begin{array}{ccccccccc} A_1 & \xleftarrow{q_1} & S_1 & \xrightarrow{p_1} & B_1 & \xleftarrow{q_1'} & S_1' & \xrightarrow{p_1'} & C_1 \\ {\scriptstyle a}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle c} \\ A_2 & \xleftarrow[q_2]{} & S_2 & \xrightarrow[p_2]{} & B_2 & \xleftarrow[q_2']{} & S_2' & \xrightarrow[p_2']{} & C_2 \end{array}$$

in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ with composition

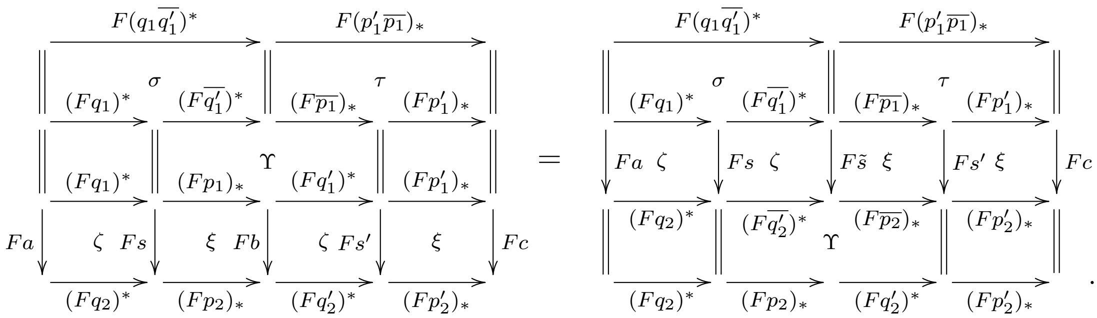
$$\begin{array}{ccccc} A_1 & \xleftarrow{q_1\overline{q_1'}} & S_1 \times_{B_1} S_1' & \xrightarrow{p_1'\overline{p_1}} & C_1 \\ {\scriptstyle a}\downarrow & & \tilde{b}\downarrow & & \downarrow{\scriptstyle c} \\ A_2 & \xleftarrow[q_2\overline{q_2'}]{} & S_2 \times_{B_2} S_2' & \xrightarrow[p_2'\overline{p_2}]{} & C_2 \end{array} \quad .$$

We need to show that

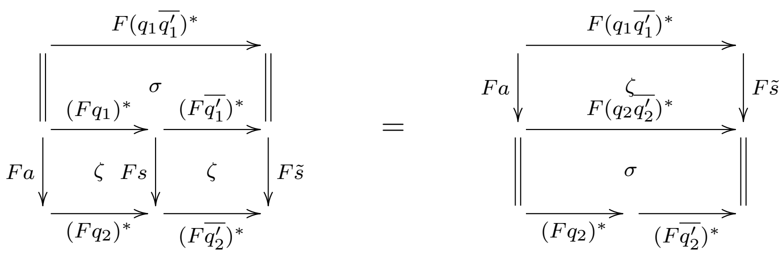



(14)

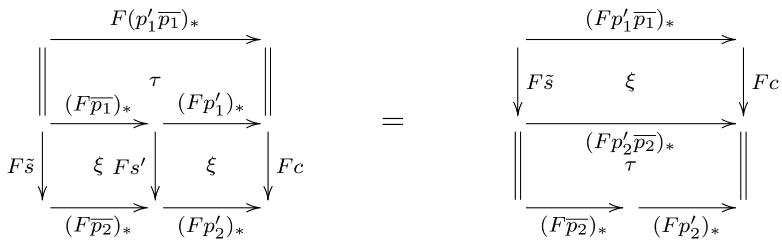
By Lemma 3.14 we get that the left-hand side of this equation can be rewritten as




It is now straightforward to check that




and



and this gives us (14).

We conclude that $\tilde{F}$ is oplax normal. Also note that $\tilde{F} \circ I_A = F$, so composition with $I_A$ is essentially surjective.

A natural transformation $\gamma\colon F \Rightarrow F'\colon \mathbf{A} \rightrightarrows \mathrm{Vrt}\,(\mathbb{B})$ lifts to a vertical transformation $\tilde{\gamma}\colon \tilde{F} \Rightarrow \tilde{F}'\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \rightrightarrows \mathbb{B}$ defined as follows. For an object $A$ in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, $\tilde{\gamma}_A$ is the vertical arrow $\gamma_A$ in $\mathbb{B}$. For a horizontal arrow $A \xleftarrow{q} S \xrightarrow{p} B$, the cell

$$\begin{array}{ccc} \tilde{F}A & \xrightarrow{\tilde{F}(S)} & \tilde{F}B \\ {\scriptstyle \tilde{\gamma}_A}\downarrow\bullet & \tilde{\gamma}_S & \bullet\downarrow{\scriptstyle \tilde{\gamma}_B} \\ \tilde{F}'A & \xrightarrow[\tilde{F}'(S)]{} & \tilde{F}'B \end{array}$$

is defined as

$$\begin{array}{ccccc} \cdot & \xrightarrow{(Fq)^*} & \cdot & \xrightarrow{(Fp)_*} & \cdot \\ {\scriptstyle \gamma_A}\downarrow\bullet & \zeta(Fq,\gamma_S,\gamma_A,F'q) & \bullet\downarrow{\scriptstyle \gamma_S} & \xi(Fp,\gamma_S,\gamma_B,F'p) & \bullet\downarrow{\scriptstyle \gamma_B} \\ \cdot & \xrightarrow[(F'q)^*]{} & \cdot & \xrightarrow[(F'p)_*]{} & \cdot \end{array}.$$

Note that if $S$ is the identity span on $A$, then this diagram becomes the identity on $\gamma_A$.

To show that $\tilde{\gamma}$ satisfies the (oplax) functoriality property with respect to horizontal composition, let $A \xleftarrow{q} S \xrightarrow{p} B \xleftarrow{q'} S' \xrightarrow{p'} C$ be a composable pair of horizontal arrows in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ with composition $A \xleftarrow{q\overline{q'}} T \xrightarrow{p'\overline{p}} C$. We need to show that

$$\begin{array}{ccccc} \cdot & & \xrightarrow{F(T)} & & \cdot \\ \| & & \varphi_{S,S'} & & \| \\ \cdot & \xrightarrow{F(S)} & \cdot & \xrightarrow{F(S')} & \cdot \\ {\scriptstyle \tilde{\gamma}_A}\downarrow\bullet & \tilde{\gamma}_S & \bullet\downarrow{\scriptstyle \tilde{\gamma}_B} & \tilde{\gamma}_{S'} & \bullet\downarrow{\scriptstyle \tilde{\gamma}_C} \\ \cdot & \xrightarrow[F'(S)]{} & \cdot & \xrightarrow[F'(S')]{} & \cdot \end{array} \equiv \begin{array}{ccccc} \cdot & & \xrightarrow{F(T)} & & \cdot \\ {\scriptstyle \tilde{\gamma}_A}\downarrow\bullet & & \tilde{\gamma}_T & & \bullet\downarrow{\scriptstyle \tilde{\gamma}_C} \\ \cdot & & \xrightarrow{F'(T)} & & \cdot \\ \| & & \varphi'_{S,S'} & & \| \\ \cdot & \xrightarrow[F'(S)]{} & \cdot & \xrightarrow[F'(S')]{} & \cdot \end{array} \qquad (15)$$

and the proof of this equation is completely analogous to that of (14).

The transformation $\tilde{\gamma}$ commutes trivially with the identity structure cells of $\tilde{F}$, since this functor preserves identities on the nose. Finally, note that $\tilde{\gamma} \circ I_A = \gamma$, so composition with $I_A$ is full. To conclude that it is also faithful, note that $\tilde{\gamma}$ is completely determined by $\gamma$. ∎

3.16. Remarks. Note that Theorem 3.15 states that $\mathbb{S}\mathrm{pan}$ is a partial left biadjoint to the forgetful 2-functor

$$\mathrm{Vrt} : \mathbf{Doub}_{\mathrm{OplaxN}} \to \mathbf{Cat}.$$

This is better than the bicategory version of the universal property of Span, Theorem 2.8, where we did not get a partial adjoint. That is because using the double category structure we were able to replace the condition on the morphism $\mathbf{A} \to \mathcal{B}$, *i.e.*, to be sinister, with a condition on the receiving double category $\mathbb{B}$, *i.e.*, to be gregarious. Or, put differently, when we pass from $\mathbf{A}$ to $\mathcal{S}pan\,\mathbf{A}$ we add right adjoints for the arrows of $\mathbf{A}$, but this is not a completion. There are new arrows in $\mathcal{S}pan\,\mathbf{A}$, which do not have right adjoints. However, passing from $\mathbf{A}$ to $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is a completion process. Every arrow in $\mathbf{A}$ gets a companion and conjoint and there are no new vertical arrows.

However, there are two points on which Theorem 3.15 is unsatisfactory. The first is the fact that as a left adjoint $\mathbb{S}\mathrm{pan}$ is only defined for categories with pullbacks, not for all categories. The other issue is more subtle. Gregarious oplax morphisms of double categories do not preserve all horizontal composites. Generally, one would like morphisms to be structure preserving maps, so we would like to change the structure. Both of these issues will be resolved in Section 4.

3.17. Beck-Chevalley double categories. The most general equivalence of categories in the description of the universal property of the bicategory $\mathcal{S}pan\,\mathbf{A}$ in Theorem 2.8,

$$\mathbf{Bicat}_{\mathrm{Greg}}(\mathcal{S}pan\,\mathbf{A}, \mathcal{B}) \simeq \mathbf{Bicat}_{\mathrm{Sinister}}(\mathbf{A}, \mathcal{B}), \tag{16}$$

restricts to

$$\mathbf{Bicat}_{\mathrm{Strong}}(\mathcal{S}pan\,\mathbf{A}, \mathcal{B}) \simeq \mathbf{Bicat}_{\mathrm{Beck}}(\mathbf{A}, \mathcal{B})$$

because strong morphisms from $\mathcal{S}pan\,\mathbf{A}$ to $\mathcal{B}$ correspond under (16) to sinister morphisms that satisfy the Beck-Chevalley condition.

So we may ask the question whether there is a similar restriction of the equivalence of categories in Theorem 3.15. In other words, what do strong morphisms $\mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ correspond to? It is clear that we will need to consider functors $\mathbf{A} \to \mathrm{Vrt}\,(\mathbb{B})$ that preserve pullbacks, so we will need $\mathbb{B}$ to have at least pullbacks of vertical arrows. This leads us to consider the following concept.

3.18. Definition. A double category

$$\mathbf{A}_2 \overset{\pi_1}{\underset{\pi_2}{\overset{m}{\rightrightarrows}}} \mathbf{A}_1 \overset{s}{\underset{t}{\overset{u}{\leftrightarrows}}} \mathbf{A}_0$$

is said to *have vertical pullbacks* if this diagram lies within the category $\mathbf{PBCat}$ of categories with pullbacks and pullback-preserving functors between them.

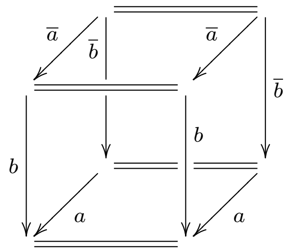
A morphism $F\colon \mathbb{A} \to \mathbb{B}$ between categories with vertical pullbacks is said to *preserve* these pullbacks if its components $F_i\colon \mathbf{A}_i \to \mathbf{B}_i$ are pullback-preserving functors. When there is no confusion possible, we will also call such an $F$ a *pullback-preserving morphism.*

Let us spell out what it means for a double category $\mathbb{B}$ to have vertical pullbacks. If we represent $\mathbb{B}$ by a diagram of categories and functors as in the definition above, the fact that $u$ preserves pullbacks means that for every cospan of vertical arrows $A \xrightarrow{a} C \xleftarrow{b} B$

with pullback

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\bar{a}} & B \\ {\scriptstyle \bar{b}}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[a]{} & C \end{array}$$

the corresponding square of unit cells



is a pullback in $\mathbf{B}_1$. As a consequence we obtain the following 2-dimensional universal property for pullback squares in $\mathbf{B}_0$. For any horizontal arrow $X \xrightarrow{h} Y$ with cells

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle x_A}\downarrow & \gamma_A & \downarrow{\scriptstyle y_A} \\ A & = & A \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle x_B}\downarrow & \gamma_B & \downarrow{\scriptstyle y_B} \\ B & = & B \end{array}$$

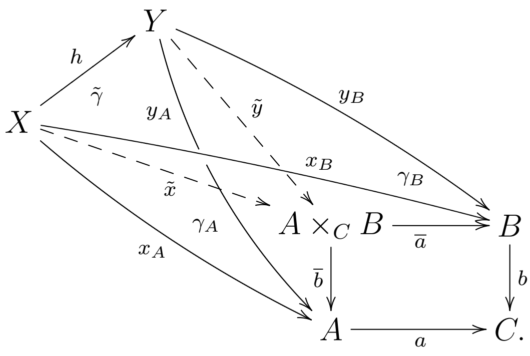
such that

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle x_A}\downarrow & \gamma_A & \downarrow{\scriptstyle y_A} \\ A & = & A \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ C & = & C \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle x_B}\downarrow & \gamma_B & \downarrow{\scriptstyle y_B} \\ B & = & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ C & = & C \end{array}$$

there is a unique cell

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle \tilde{x}}\downarrow & \tilde{\gamma} & \downarrow{\scriptstyle \tilde{y}} \\ A \times_C B & = & A \times_C B \end{array}$$

such that $1_{\bar{b}} \cdot \tilde{\gamma} = \gamma_A$ and $1_{\bar{a}} \cdot \tilde{\gamma} = \gamma_B$, as depicted in



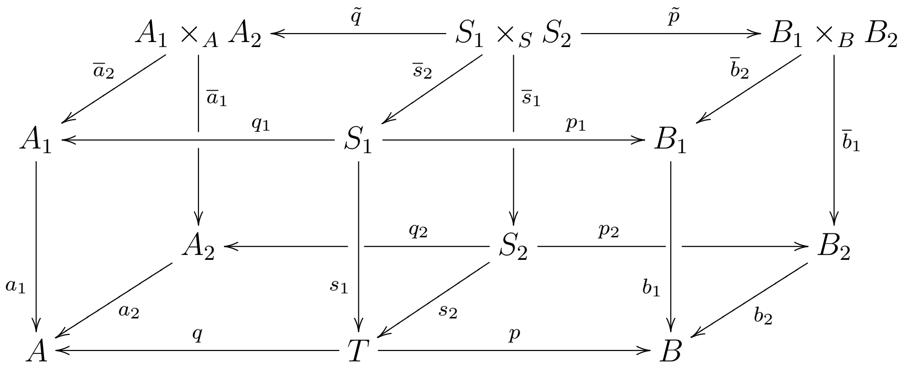

Note that this is a generalization of the 2-dimensional universal property of pullbacks in 2-categories.

The fact that $m$ preserves pullbacks means that one can horizontally compose two vertical pullback squares of cells and obtain again a vertical pullback square of cells. This is what was called "having a functorial choice of pullbacks" in [19].

3.19. Examples.

1. If $\mathbf{A}$ is a category with pullbacks, then the double category of pullback squares in $\mathbf{A}$ has vertical pullbacks.

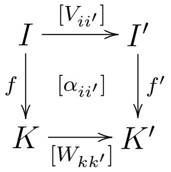
2. If $\mathbf{A}$ is a category with pullbacks, then the double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has vertical pullbacks. The pullback of cells is depicted in the following diagram.




(17)

With a straightforward diagram chase using the universal properties of all pullbacks involved the reader may verify that all the structure maps are pullback-preserving.

3. Let $V$ be a monoidal category with sums (that distribute over tensor) and pullbacks. Then there is a double category $V$-$\mathbb{M}\mathrm{at}$ with sets as objects, functions as vertical arrows, and $V$-matrices $[V_{ij}]\colon I \to J$ as horizontal arrows. The cells are of the form



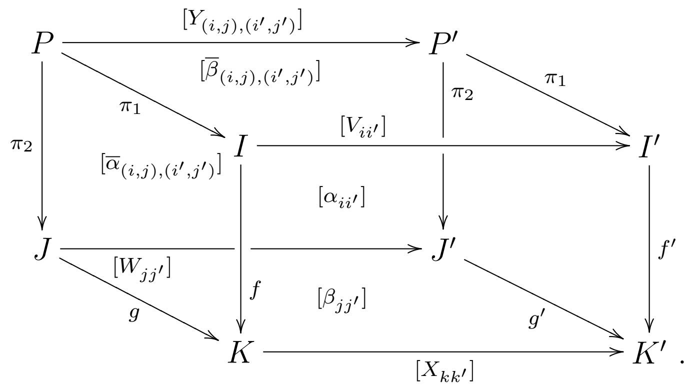
where the $\alpha_{ii'}\colon V_{ii'} \to W_{f(i)f'(i')}$ are arrows in $V$. To define the vertical pullback of squares

$$\begin{array}{ccc} I & \xrightarrow{[V_{ii'}]} & I' \\ {\scriptstyle f}\downarrow & [\alpha_{ii'}] & \downarrow{\scriptstyle f'} \\ K & \xrightarrow[{[X_{kk'}]}]{} & K' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} J & \xrightarrow{[W_{jj'}]} & J' \\ {\scriptstyle g}\downarrow & [\beta_{jj'}] & \downarrow{\scriptstyle g'} \\ K & \xrightarrow[{[X_{kk'}]}]{} & K' \end{array}$$

let $P = I \times_K J$ and $P' = I' \times_{K'} J'$ in Sets. Moreover, for $(i, j) \in P$ and $(i', j') \in P'$, let

$$\begin{array}{ccc} Y_{(i,j),(i',j')} & \xrightarrow{\overline{\alpha}_{(i,j),(i',j')}} & W_{jj'} \\ {\scriptstyle \overline{\beta}_{(i,j),(i',j')}}\downarrow & & \downarrow{\scriptstyle \beta_{jj'}} \\ V_{ii'} & \xrightarrow[\alpha_{ii'}]{} & X_{kk'} \end{array}$$

(where $k = f(i)$ and $k = g(j)$, and $k' = f'(i')$ and $k' = g'(j')$) be a pullback in $V$. Then the pullback of the cells $[\alpha_{ii'}]$ and $[\beta_{jj'}]$ is




We see now that the horizontal composition of vertical pullback squares of cells will only give a pullback square of cells if the sums and tensors distribute over the pullbacks.

Moreover, this double category is gregarious. For a vertical arrow $f\colon I \to K$, we define the companion horizontal arrow as $f_*\colon I \to K$ with $(f_*)_{ik} = I$ if $f(i) = k$ and $(f_*)_{ik} = 0$ otherwise. The binding cells are defined in the obvious way. For example, $\chi_{ik} = 1_I$ if $f(i) = k$ and $\chi_{ik} = 1_0$ otherwise. The conjoint $f^*\colon K \to I$ is defined by $(f^*)_{ki} = I$ if $f(i) = k$ and $(f^*)_{ki} = 0$ otherwise.

The fact that the vertical pullback of cells in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is formed as in (17) generalizes to the following result for gregarious double categories.

3.20. Proposition. *Let $\mathbb{A}$ be a gregarious double category. Then the functor $\Xi\colon \mathbf{A}_0^{\mathbf{2}} \to \mathbf{A}_1$, defined in (5) and (6), and the functor $Z\colon \mathbf{A}_0^{\mathbf{2}} \to \mathbf{A}_1$, defined in (7) and (8), preserve all limits that the functor $U\colon \mathbf{A}_0 \to \mathbf{A}_1$ (which sends objects and vertical arrows to their horizontal identities) preserves.*

Proof. We will show this for $\Xi$. The proof for $Z$ is its dual. Let $\mathbf{P}(\mathbb{A})$ be the category whose objects are diagrams of the form

$$A \xrightarrow{x} \bullet B \xrightarrow{f} C$$

in $\mathbb{A}$, and arrows are commutative diagrams of the form

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{f} & C \\ {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & \alpha & {\scriptstyle v_3}\downarrow \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[f']{} & C' \end{array}$$

in $\mathbb{A}$.

The functor $\Xi$ factors through $\mathbf{P}(\mathbb{A})$ in the following way:

$$\begin{array}{ccc} \mathbf{A}_0^{\mathbf{2}} & \xrightarrow{\Xi} & \mathbf{A}_1 \\ & {\scriptstyle L}\searrow \quad \nearrow{\scriptstyle \Gamma} & \\ & \mathbf{P}(\mathbb{A}) & \end{array}$$

where $L$ is defined by

$$\left( A \xrightarrow{x} B \right) \mapsto \left( A \xrightarrow{x} B = B \right)$$

$$\left( \begin{array}{ccc} A & \xrightarrow{x} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{x'} & B' \end{array} \right) \mapsto \left( \begin{array}{ccccc} A & \xrightarrow{x} & B & = & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & 1_v & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{x'} & B' & = & B' \end{array} \right)$$

and $\Gamma$ is defined by

$$\left( A \xrightarrow{x} B \xrightarrow{f} C \right) \mapsto \left( A \xrightarrow{fx_*} C \right)$$

$$\left( \begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{f} & C \\ {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & \alpha & {\scriptstyle v_3}\downarrow \\ A' & \xrightarrow[x']{} & B' & \xrightarrow[f']{} & C' \end{array} \right) \mapsto \left( \begin{array}{ccc} A & \xrightarrow{fx_*} & C \\ {\scriptstyle v_1}\downarrow & \alpha\xi(x,v_1,v_2,x') & {\scriptstyle v_3}\downarrow \\ A' & \xrightarrow[f'(x')_*]{} & C' \end{array} \right).$$

A straightforward calculation shows that $L$ preserves all limits that $U$ preserves.

We will now show that $\Gamma$ has a left adjoint and therefore preserves all limits. The left adjoint is the functor $\Delta\colon \mathbf{A}_1 \to \mathbf{P}(\mathbb{A})$ defined by

$$\left( A \xrightarrow{f} B \right) \;\mapsto\; \left( A = A \xrightarrow{f} B \right)$$

$$\left( \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ A' & \xrightarrow[f']{} & B' \end{array} \right) \;\mapsto\; \left( \begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow\!\bullet & & {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ A' & = & A' & \xrightarrow[f']{} & B' \end{array} \right).$$

Note that the composite $\Delta\Gamma\colon \mathbf{P}(\mathbb{A}) \to \mathbf{P}(\mathbb{A})$ is given by

$$\left( A \xrightarrow{x}\!\!\!\!\!\bullet\;\; B \xrightarrow{f} C \right) \;\mapsto\; \left( A = A \xrightarrow{fx_*} B \right)$$

$$\left( \begin{array}{ccccc} A & \xrightarrow{x}\!\!\bullet & B & \xrightarrow{f} & C \\ {\scriptstyle v_1}\bullet\!\downarrow & & {\scriptstyle v_2}\bullet\!\downarrow & \alpha & \bullet\!\downarrow{\scriptstyle v_3} \\ A' & \xrightarrow{x'}\!\!\bullet & B' & \xrightarrow[f']{} & C' \end{array} \right) \;\mapsto\; \left( \begin{array}{ccccc} A & = & A & \xrightarrow{fv_*} & C \\ {\scriptstyle v_1}\bullet\!\downarrow & & {\scriptstyle v_1}\bullet\!\downarrow & \alpha\xi(x,v_1,v_2,x') & \bullet\!\downarrow{\scriptstyle v_3} \\ A' & = & A' & \xrightarrow[f'(x')_*]{} & B' \end{array} \right).$$

We define the counit to be the natural transformation $\varepsilon\colon \Delta\Gamma \Rightarrow 1_{\mathbf{P}(\mathbb{A})}$ by

$$\varepsilon_{(x,f)} = \begin{array}{ccccc} \cdot & = & \cdot & \xrightarrow{fx_*} & \cdot \\ \| & & {\scriptstyle x}\bullet\!\downarrow & \mathrm{id}_f\chi_x & \| \\ \cdot & \xrightarrow[x]{}\!\!\bullet & \cdot & \xrightarrow[f]{} & \cdot \end{array}$$

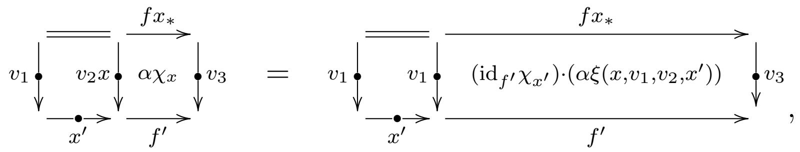
To check that this is indeed natural, let

$$\begin{array}{ccccc} A & \xrightarrow{x}\!\!\bullet & B & \xrightarrow{f} & C \\ {\scriptstyle v_1}\bullet\!\downarrow & & {\scriptstyle v_2}\bullet\!\downarrow & \alpha \;\; {\scriptstyle v_3} & \bullet\!\downarrow \\ A' & \xrightarrow[x']{}\!\!\bullet & B' & \xrightarrow[f']{} & C' \end{array}$$

be any arrow in $\mathbf{P}(\mathbb{A})$. Then we need to show that

$$\begin{array}{ccccc} \cdot & = & \cdot & \xrightarrow{fx_*} & \cdot \\ \| & & {\scriptstyle x}\bullet\!\downarrow & \mathrm{id}_f\chi_x & \| \\ \cdot & \xrightarrow[x]{}\!\!\bullet & \cdot & \xrightarrow[f]{} & \cdot \\ {\scriptstyle v_1}\bullet\!\downarrow & & {\scriptstyle v_2}\bullet\!\downarrow & \alpha & \bullet\!\downarrow{\scriptstyle v_3} \\ \cdot & \xrightarrow[x']{}\!\!\bullet & \cdot & \xrightarrow[f']{} & \cdot \end{array} \quad = \quad \begin{array}{ccccc} \cdot & = & \cdot & \xrightarrow{fx_*} & \cdot \\ {\scriptstyle v_1}\bullet\!\downarrow & & {\scriptstyle v_1}\bullet\!\downarrow & \alpha\xi(x,v_1,v_2,x') & \bullet\!\downarrow{\scriptstyle v_3} \\ \cdot & = & \cdot & \xrightarrow{f'(x')_*} & \cdot \\ \| & & {\scriptstyle x'}\bullet\!\downarrow & \mathrm{id}_{f'}\chi_{x'} & \| \\ \cdot & \xrightarrow[x']{}\!\!\bullet & \cdot & \xrightarrow[f']{} & \cdot \end{array},$$

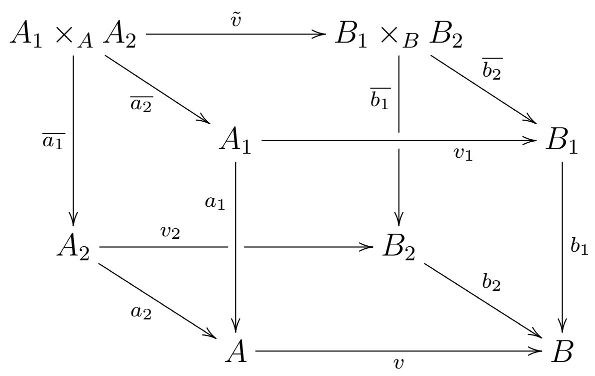
*i.e.*, that



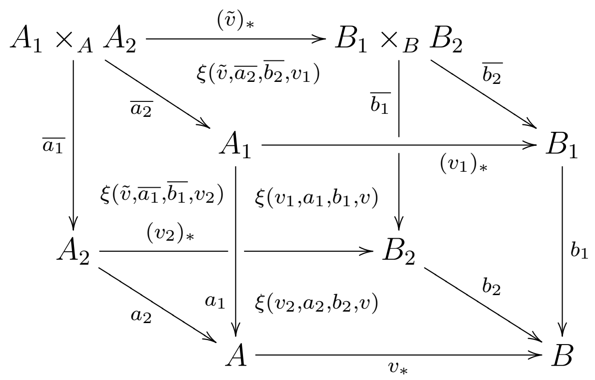

and this follows immediately from the definition of $\xi(x, v_1, v_2, x')$ and one of the binding cell identities for $x'$.

Note that we can take $(\mathrm{id}_A)_* = 1_A$ and $\xi(\mathrm{id}_A, u, u, \mathrm{id}_B) = 1_u$, so it follows that $\Gamma\Delta = 1_{\mathbf{A}_1}$, and we may take the unit of the adjunction to be the identity transformation. This completes the proof of this proposition. ∎

3.21. COROLLARY. *Let $\mathbb{B}$ be a gregarious double category with vertical pullbacks. Any commutative cube of vertical arrows*

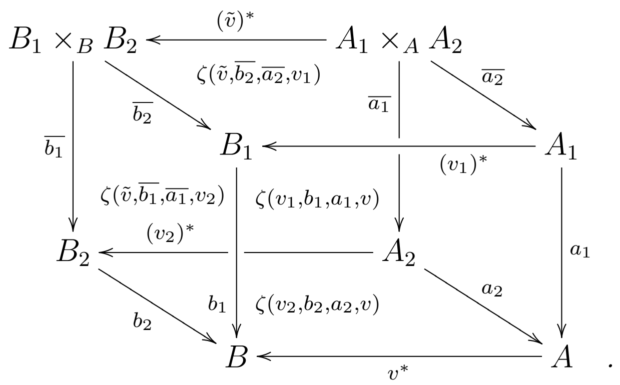



*where the left and right faces are pullback squares, gives rise to the following vertical pullbacks of cells:*



(18)

*and*

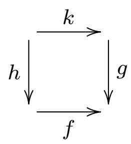



(19)

We would like to restrict the equivalence in Theorem 3.15 in such a way that we obtain pullback-preserving strong morphisms $\mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ as objects on the left-hand side of the equivalence.

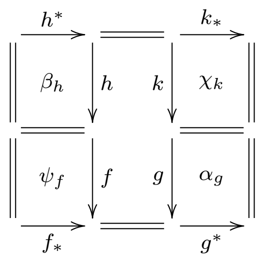
Note that we cannot put the Beck-Chevalley condition on the functors from $\mathbf{A}$ into $\mathrm{Vrt}\,(\mathbb{B})$, since $\mathrm{Vrt}\,(\mathbb{B})$ doesn't contain the information about the companions and conjoints in $\mathbb{B}$. However, just as with the notion of gregariousness, we can make the Beck-Chevalley condition a property of gregarious double categories rather than a property of the morphisms between them.

3.22. Definition. A *Beck-Chevalley double category* is a gregarious double category which has vertical pullbacks and for which the following version of the Beck-Chevalley condition holds: For each pullback square of vertical arrows




the induced cell

$\Upsilon(k, h, g, f) \;=$




(as defined in (10)) is vertically invertible. We will call the cells of the form $\Upsilon(k, h, g, f)$ *Beck-Chevalley cells.*

We write **BCDoub** for the 2-category of Beck-Chevalley double categories with strong pullback-preserving morphisms and vertical transformations.

3.23. Examples.

1. For any category $\mathbf{A}$ with pullbacks, the double category $\mathbb{S}\mathrm{pan}(\mathbf{A})$ is Beck-Chevalley.

2. The double category $\mathbb{R}\mathrm{el}$ with sets as objects, functions as vertical arrows and relations as horizontal arrows is a Beck-Chevalley double category.

3. In $V$-$\mathbb{M}\mathrm{at}$ (with $V$ as in Example 3.19 (3)) let

$$\begin{array}{ccc} P & \overset{q}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} & J \\ {\scriptstyle p}\bullet\!\!\downarrow & & \downarrow\!\!\bullet{\scriptstyle g} \\ I & \underset{f}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} & K \end{array}$$

be a vertical pullback square. Then

$$(g^* f_*)_{ij} = \sum_{k \in K} (f_*)_{ik} \otimes (g^*)_{kj} = \begin{cases} I & \text{if } f(i) = g(j) \\ 0 & \text{otherwise} \end{cases}$$

and

$$(q_* p^*)_{ij} = \sum_{\substack{r,s \\ f(r)=g(s)}} (q_*)_{i,(r,s)} \otimes (p^*)_{(r,s),j} = \begin{cases} I & \text{if } f(i) = g(j) \\ 0 & \text{otherwise} \end{cases}$$

and $\Upsilon(q, p, g, f)$ is the vertical identity cell. So $V$-$\mathbb{M}\mathrm{at}$ is a Beck-Chevalley double category (if $V$ satisfies the conditions in Example 3.19 (3)).

We are now ready to state the restriction of the equivalence in Theorem 3.15 to pullback-preserving strong morphisms of double categories.

3.24. Theorem. *Let* $\mathbf{A}$ *be a category with pullbacks and* $\mathbb{B}$ *be a Beck-Chevalley double category. Then composing with the inclusion* $\mathbf{A} \hookrightarrow \mathbb{S}\mathrm{pan}\,\mathbf{A}$ *gives an equivalence of categories between the category of strong pullback-preserving morphisms* $\mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ *with vertical transformations, and the category of pullback-preserving functors* $\mathbf{A} \to \mathrm{Vrt}\,(\mathbb{B})$ *with natural transformations,*

$$\mathbf{BCDoub}\,(\mathbb{S}\mathrm{pan}\,\mathbf{A}, \mathbb{B}) \simeq \mathbf{PBCat}\,(\mathbf{A}, \mathrm{Vrt}\,(\mathbb{B})).$$

Proof. It is clear that for any pullback-preserving strong morphism $F\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$, the composition with the inclusion $\mathbf{A} \hookrightarrow \mathbb{S}\mathrm{pan}\,\mathbf{A}$ induces a pullback-preserving functor of categories $\mathbf{A} \to \mathrm{Vrt}\,\mathbb{B}$.

To show that this functor $\mathbf{BCDoub}(\mathbb{S}\mathrm{pan}\,\mathbf{A}, \mathbb{B}) \to \mathbf{PBCat}\,(\mathbf{A}, \mathrm{Vrt}\,(\mathbb{B}))$ is essentially surjective, let $F\colon \mathbf{A} \to \mathrm{Vrt}\,\mathbb{B}$ be a pullback-preserving functor and let $\tilde{F}\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ be the lifting constructed in the proof of Theorem 3.15.

Since $\mathbb{B}$ is a Beck-Chevalley double category and $F$ preserves pullbacks, the cell $\Upsilon(F\overline{p}, F\overline{q'}, Fq', Fp)$ is vertically invertible for every pullback diagram

$$\begin{array}{ccc} S \times_B S' & \xrightarrow{\overline{p}} & S' \\ {\scriptstyle \overline{q'}}\downarrow & & \downarrow{\scriptstyle q'} \\ S & \xrightarrow[p]{} & B \end{array}$$

in **A**. Diagram (13) shows that this implies that the comparison cells for $\tilde{F}$ are all isomorphisms (since $\sigma$ and $\tau$ are vertical isomorphisms), so $\tilde{F}$ is a strong morphism.

We also need to show that $\tilde{F}$ preserves all vertical pullbacks. It preserves pullbacks of vertical arrows by the assumption that $F$ preserves pullbacks. We will consider the image of the pullback diagram (17) under $\tilde{F}$. To make our notation more manageable, let $\zeta_i = \zeta(Fq_i, Fs_i, Fa_i, Fq)$, and $\xi_i = \xi(Fp_i, Fs_i, Fb_i, Fp)$ for $i = 1, 2$. Also, let $\overline{\zeta}_1 = \zeta(F\tilde{q}, F\overline{s}_1, F\overline{a}_1, Fq_2)$, $\overline{\zeta}_2 = \zeta(F\tilde{q}, F\overline{s}_2, F\overline{a}_2, Fq_1)$, $\overline{\xi}_1 = \xi(F\tilde{p}, F\overline{s}_1, F\overline{b}_1, Fp_2)$, and $\overline{\xi}_2 = \xi(F\tilde{p}, F\overline{s}_2, F\overline{b}_2, Fp_1)$. Now we need to show that the vertical square of cells

$$\begin{array}{ccc} \cdot & \xrightarrow{\overline{\xi}_2\overline{\zeta}_2} & \cdot \\ {\scriptstyle \overline{\xi}_1\overline{\zeta}_1}\downarrow & & \downarrow{\scriptstyle \xi_1\zeta_1} \\ \cdot & \xrightarrow[\xi_2\zeta_2]{} & \cdot \end{array} \tag{20}$$

is a pullback square. Corollary 3.21 gives us that the squares

$$\begin{array}{ccc} \cdot & \xrightarrow{\overline{\zeta}_2} & \cdot \\ {\scriptstyle \overline{\zeta}_1}\downarrow & & \downarrow{\scriptstyle \zeta_1} \\ \cdot & \xrightarrow[\zeta_2]{} & \cdot \end{array} \quad \text{and} \quad \begin{array}{ccc} \cdot & \xrightarrow{\overline{\xi}_2} & \cdot \\ {\scriptstyle \overline{\xi}_1}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ \cdot & \xrightarrow[\xi_2]{} & \cdot \end{array}$$

are vertical pullbacks. The fact that (20) is a pullback follows now from the fact that the horizontal composition of two vertical pullback squares of cells is again a vertical pullback square. ■

3.25. FUNCTORIALITY. We saw in Section 2.9 that the bicategory version of Span is not a 2-functor. The double category version, however, is. As implied by Theorem 3.15 and Theorem 3.24, the $\mathbb{S}$pan construction should give bifunctors from **pbCat** to $\mathbf{Doub}_{\mathrm{OplaxN}}$ and from **PBCat** to **BCDoub**. In fact, more is true.

3.26. PROPOSITION. *The Span construction gives a 2-functor* $\mathbb{S}\mathrm{pan}\colon \mathbf{pbCat} \to \mathbf{Doub}_{OplaxN}$ *which restricts to a 2-functor* $\mathbb{S}\mathrm{pan}\colon \mathbf{PBCat} \to \mathbf{BCDoub}$.

One consequence of 2-functoriality is that $\mathbb{S}$pan preserves adjunctions. This will give us completeness results for double categories of the form $\mathbb{S}\mathrm{pan}\,\mathbf{A}$. There are various equivalent ways of discussing limits and colimits in double categories (see for example [19]). Definition 3.28 below is best suited to our purpose.

3.27. PROPOSITION. *The 2-categories* **pbCat**, **PBCat**, $\mathbf{Doub}_{\mathrm{OplaxN}}$, *and* **BCDoub** *are cotensored and the 2-functors*

$$\begin{array}{ccc} \mathbf{PBCat} & \xrightarrow{\mathbb{S}pan} & \mathbf{BCDoub} \\ \downarrow & & \downarrow \\ \mathbf{pbCat} & \xrightarrow{\mathbb{S}pan} & \mathbf{Doub}_{OplaxN} \end{array}$$

*preserve cotensors.*

PROOF. If $\mathbf{A}$ is a category with pullbacks and $\mathbf{I}$ an arbitrary category, the functor category $\mathbf{A}^{\mathbf{I}}$ also has pullbacks, and cartesian closedness of **Cat** gives the cotensor property for **pbCat**:

$$\frac{\mathbf{I} \longrightarrow \mathbf{pbCat}\,(\mathbf{B},\mathbf{A})}{\mathbf{B} \longrightarrow \mathbf{A}^{\mathbf{I}} \text{ in } \mathbf{pbCat}}$$

The bijection restricts to a bijection

$$\frac{\mathbf{I} \longrightarrow \mathbf{PBCat}\,(\mathbf{B},\mathbf{A})}{\mathbf{B} \longrightarrow \mathbf{A}^{\mathbf{I}} \text{ in } \mathbf{PBCat}}$$

This is quite general and would work for any class of limits or colimits in place of pullbacks.

Given a weak double category $\mathbb{A}$, it can be considered a weak category object in **Cat**,

$$\mathbf{A}_2 \equiv\!\!\!\Rrightarrow \mathbf{A}_1 \rightleftarrows \mathbf{A}_0.$$

Since $(\ )^{\mathbf{I}}$ is a 2-functor, it follows that

$$\mathbf{A}_2^{\mathbf{I}} \equiv\!\!\!\Rrightarrow \mathbf{A}_1^{\mathbf{I}} \rightleftarrows \mathbf{A}_0^{\mathbf{I}}.$$

is again a weak double category object, which we shall denote by $\mathbb{A}^{\mathbf{I}}$. A basic calculus of adjoints shows that $\mathbb{A}^{\mathbf{I}}$ is indeed the cotensor of $\mathbb{A}$ by $\mathbf{I}$ in $\mathbf{Doub}_{\mathrm{OplaxN}}$, *i.e.*, we have an isomorphism of categories

$$\mathbf{Doub}_{\mathrm{OplaxN}}(\mathbb{B},\mathbb{A}^{\mathbf{I}}) \cong \mathbf{Cat}(\mathbf{I},\mathbf{Doub}_{\mathrm{OplaxN}}(\mathbb{B},\mathbb{A})).$$

This isomorphism is easily seen to restrict to an isomorphism

$$\mathbf{BCDoub}(\mathbb{B},\mathbb{A}^{\mathbf{I}}) \cong \mathbf{Cat}(\mathbf{I},\mathbf{BCDoub}(\mathbb{B},\mathbb{A})).$$

It is clear from the constructions that cotensors are preserved by the 2-functors $\mathbb{S}\mathrm{pan}$ and 'inclusion'. ■

3.28. DEFINITION. A double category $\mathbb{A}$ has *oplax* $\mathbf{I}$*-colimits* if the diagonal $\Delta\colon \mathbb{A} \to \mathbb{A}^{\mathbf{I}}$ has a left adjoint in the 2-category $\mathbf{Doub}_{\mathrm{OplaxN}}$. The $\mathbf{I}$-colimits are *strong* if $\Delta$ has a left adjoint in **Doub**. A dual definition gives *lax* and *strong* $\mathbf{I}$*-limits*.

3.29. THEOREM. *If* $\mathbf{A}$ *has* $\mathbf{I}$*-colimits then* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *has oplax* $\mathbf{I}$*-colimits. If the* $\mathbf{I}$*-colimits commute with pullbacks in* $\mathbf{A}$ *then* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *has strong* $\mathbf{I}$*-colimits. If* $\mathbf{A}$ *has* $\mathbf{I}$*-limits, then* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *has strong* $\mathbf{I}$*-limits.*

PROOF. If $\mathbf{A}$ has $\mathbf{I}$-colimits we obtain a functor $\varinjlim\colon \mathbf{A}^{\mathbf{I}} \to \mathbf{A}$ which is left adjoint to $\Delta\colon \mathbf{A} \to \mathbf{A}^{\mathbf{I}}$ in **pbCat** and $\mathbb{S}\mathrm{pan}\colon \mathbf{pbCat} \to \mathbf{Doub}_{\mathrm{OplaxN}}$ preserves this adjunction. As $\mathbb{S}\mathrm{pan}(\mathbf{A}^{\mathbf{I}}) \cong \mathbb{S}\mathrm{pan}(\mathbf{A})^{\mathbf{I}}$ we see that $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has oplax $\mathbf{I}$-colimits.

If $\mathbf{I}$-colimits commute with pullbacks in $\mathbf{A}$, the adjunction $\varinjlim \dashv \Delta$ lives in **PBCat** and $\mathbb{S}\mathrm{pan}$ maps **PBCat** to **BCDoub**. The arrows of **BCDoub** are strong morphisms, so $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has strong $\mathbf{I}$-colimits.

The result for $\mathbf{I}$-limits follows from the fact that the adjunction $\Delta \dashv \varprojlim$ is always in **PBCat**. ■

3.30. Remarks.

1. This proposition extends our result about pullbacks in the previous subsection to arbitrary small limits. Another important special case occurs when **A** has products. It is well-known that products in **A** give rise to a tensor structure on $\mathcal{S}pan(\mathbf{A})$. However, in $\mathbb{S}\mathrm{pan}(\mathbf{A})$ they give rise to vertical products with a universal property.

2. The fact that $\mathbb{S}\mathrm{pan}$ of a left adjoint in **Cat** only gives rise to an oplax morphism of double categories fits in the general theory of adjunctions between double categories as developed by Grandis and Paré in [18]; generally, left adjoints are oplax whereas right adjoints are lax.

3. There are important cases when the colimit functor preserves pullbacks and thus these vertical limits in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ will be strong. A category **A** is *lextensive* if it has finite limits and finite coproducts with the property that the coproduct functor

$$+\colon \mathbf{A}/I \times \mathbf{A}/J \to \mathbf{A}/(I+J)$$

is an equivalence (and also $\mathbf{A}/0 \simeq \mathbb{1}$). Thus, $+$ preserves pullbacks so that $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has strong finite coproducts.

In a similar vein, if **A** is a Grothendieck topos, the double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ will have arbitrary strong coproducts. In fact, disjoint and universal coproducts suffice.

If **A** is locally finitely presentable, then filtered colimits commute with finite limits, so $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has strong filtered colimits.

3.31. Equipments. A different way to make the Span construction 2-functorial was presented in [7]. This paper makes a very careful study of the types of morphisms one may want to consider between categories of spans. The **spn** construction of [7] produces what is called a pointed, starred equipment. An equipment has objects, scalar arrows, vector arrows, and vector transformations. The objects and vector arrows form a category, and for any pair of objects $A$ and $B$, there is a category with the vector arrows from $A$ to $B$ as objects and the vector transformations between them as morphisms. Finally, the scalar arrows act on the vector arrows and vector transformations from both the left and the right in such a way that if $k\colon K \to K'$ and $l\colon L \to L'$ are scalar arrows, $\mu, \nu\colon K \longrightarrow\!\!\!\!\!\!\!+\;\; L$ are vector arrows, and and $\Phi\colon \mu \to \nu$ is a vector transformation, then there are vector transformations $\Phi k\colon \mu k \to \nu k$ and $l\Phi\colon l\mu \to l\nu$, and these actions are functorial in $\Phi$, strictly unitary, and coherently associative in all three possible senses.

An equipment is called *starred* if for each scalar $k$ the action $(-)k$ has a left adjoint, denoted by $(-)k^*$ and for each scalar $l$, the action $l(-)$ has a right adjoint, denoted by $l^*(-)$, and these actions need to satisfy a Beck-Chevalley type of condition.

An equipment is called *pointed* if it comes a with a family of distinguished vector arrows $\iota_K\colon K \longrightarrow\!\!\!\!\!\!\!+\;\; K$ for each object $K$ and an invertible vector transformation $f\iota_K \xrightarrow{\simeq} \iota_L f$ for each scalar $f\colon K \to L$, such that $(\,1\iota_K \xrightarrow{\simeq} \iota_K 1\,) = 1_{\iota_K}$.

In order to compare gregarious double categories to equipments, we note that analogously to the case of adjunctions, companions can also be characterized by the following universal properties:

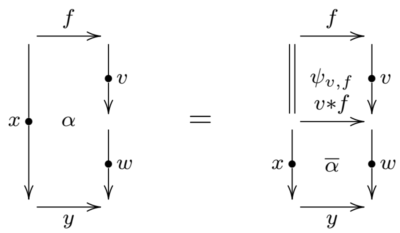
3.32. LEMMA. *For a vertical arrow $A \overset{v}{\longrightarrow\!\!\!\!\bullet\!\!\!\!\to} B$ in a double category the following are equivalent:*

1. *$v$ is a companion;*

2. *For each horizontal arrow $X \xrightarrow{f} A$, there is a horizontal arrow $X \xrightarrow{v*f} B$ and a cell*

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ \| & \psi_{v,f} & \downarrow{\scriptstyle\bullet}\, v \\ X & \xrightarrow[v*f]{} & B \end{array}$$

*which is initial in the sense that for each cell $\alpha$ as in (21) there is a unique cell $\overline{\alpha}$:*

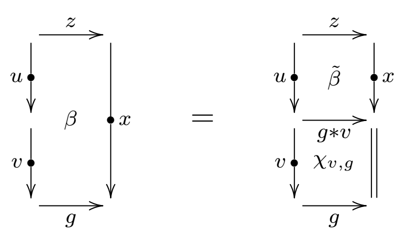


(21)

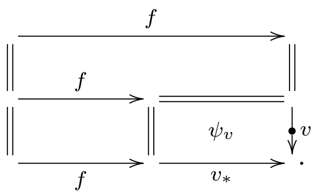
3. *For each horizontal arrow $B \xrightarrow{g} Y$ there is a horizontal arrow $A \xrightarrow{g*v} Y$ and a cell*

$$\begin{array}{ccc} A & \xrightarrow{g*v} & Y \\ v\,{\scriptstyle\bullet}\!\downarrow & \chi_{v,g} & \| \\ B & \xrightarrow[g]{} & Y \end{array}$$

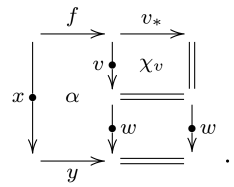
*which is terminal in the the sense that for each cell $\beta$ as in the following diagram there is a unique cell $\tilde{\beta}$:*



PROOF. To show that 1 implies 2, let $v$ have a horizontal companion $v_*$ with binding cells $\psi_v$ and $\chi_v$, and let $f$ be as in 2. Then take $v * f := v_* f$ and the cell

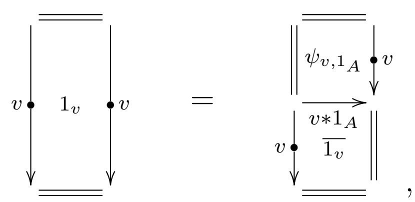
$\psi_{v,f} :=$ 


For any cell $\alpha$ as in (21) the induced cell $\overline{\alpha}$ is

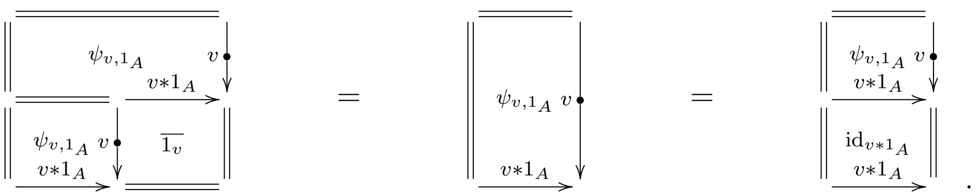



It is clear that this is the unique cell that fits in the diagram (21).

To show that 2 implies 1, let $v$ be a vertical arrow with the property described in 2. Define the horizontal companion of $v$ to be $v_* := v * 1_A$, with binding cells $\psi_v = v$ and $\chi_v = \overline{1_v}$, where $\overline{1_v}$ is the unique cell such that




as in (21). This establishes right away that these binding cells satisfy the second binding cell identity in Definition 3.4. To prove that they also satisfy the first one, note that




By the uniqueness of $\overline{\alpha}$ in (21) we conclude that $\overline{1_v}\psi_{v,1_A} = \mathrm{id}_{v*1_A}$.

The proof that 1 is equivalent to 3 is dual to the one that 1 is equivalent to 2 and left for the reader. ∎

By duality, conjoints can be characterized in the following way:

3.33. LEMMA. *For a vertical arrow $A \overset{v}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} B$ in a double category the following are equivalent:*

1. *$v$ is a conjoint;*

2. *For each horizontal arrow $A \xrightarrow{f} X$, there is a horizontal arrow $B \xrightarrow{f \star v} X$ and a cell*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle v}\bullet\big\downarrow & \alpha_{v,f} & \big\| \\ B & \xrightarrow[f\star v]{} & X \end{array}$$

*which is initial in the sense that for each cell $\zeta$ as in Figure 1 there is a unique cell $\overline{\zeta}$ as in Figure 1.*

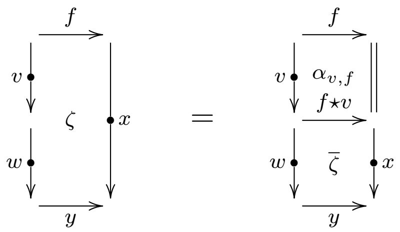



Figure 1: For any cell $\zeta$ there is an induced cell $\overline{\zeta}$.

3. *For each horizontal arrow $Y \xrightarrow{g} B$ there is a horizontal arrow $Y \xrightarrow{v \star g} A$ and a cell*

$$\begin{array}{ccc} Y & \xrightarrow{v\star g} & A \\ \big\| & \beta_{v,g} & \big\downarrow\bullet{\scriptstyle v} \\ Y & \xrightarrow[g]{} & B \end{array}$$

*which is terminal in the the sense that for each cell $\theta$ as in Figure 2 there is a unique cell $\tilde{\theta}$ as in Figure 2.*

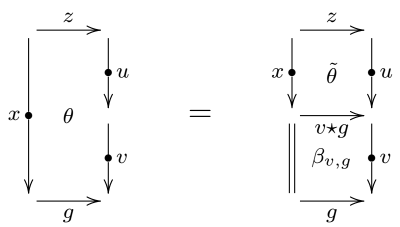



Figure 2: For any cell $\theta$ there is an induced cell $\tilde{\theta}$.

Given a double category with companions, one can construct a pointed equipment by using the same objects, defining the scalar arrows to be the vertical arrows of the double category, the vector arrows to be the horizontal arrows of the double category, and the vector transformations correspond to special cells (where the horizontal domain and codomain are identity arrows). Composition is defined as in the double category, and the action of the scalar arrows is defined using companions and horizontal composition. With the notation as above, $l\mu := l_*\mu$ (where the concatenation on the left-hand side stands for horizontal composition), $\mu k := \mu k_*$, and $l\Phi := \mathrm{id}_{l_*}\Phi$ and $\Phi k := \Phi\mathrm{id}_{k_*}$. The pointing is defined by the horizontal identity arrows (the universal properties in Lemma 3.32 imply the existence of a canonical vertically invertible special cell from $v * 1_A$ to $1_B * v$ for any vertical arrow $v\colon A \longrightarrow\!\!\!\!\bullet\!\!\to B$ ).

For a Beck-Chevalley double category this construction can be extended to obtain a starred pointed equipment. Oplax normal morphisms between double categories give rise to right homomorphisms of equipments under this construction.

However, since equipments do not have a notion of horizontal composition, we cannot reverse this operation to create a double category out of an equipment. The best we could hope for is an oplax double category and we will return to this question in the next section.

## 4. Spans without pullbacks

Notice that Theorem 2.8 gives a characterization of $\mathcal{S}pan\,\mathbf{A}$ that does not refer to pullbacks, and Theorem 3.15 gives a similar characterization for $\mathbb{S}\mathrm{pan}\,\mathbf{A}$. So we might use these universal properties to define $\mathcal{S}pan\,\mathbf{A}$ and $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ for general categories. This would be useful as we could then apply the results of [12] to $\mathcal{S}pan\,\mathbf{A}$ and $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ for further study of the $\Pi_2$-construction.

Of course, pullbacks are necessary in the composition of spans and so we take the drastic step of discarding composition. There are several justifications for such an action. The most immediate is that without pullbacks the composites just are not there, but there is sufficient extra structure at the 2-dimensional level to salvage a good part of what composition does for us. Another consideration is that the universal property expressed in Theorem 2.8 refers to oplax morphisms out of $\mathcal{S}pan\,\mathbf{A}$, and if the relevant morphisms do not preserve composition, we should not require it to be there. Of course, oplax morphisms do involve composition and the 2-dimensional structure referred to before is exactly what is needed to express this.

Along the same lines, for categories $\mathbf{A}$ and $\mathbf{B}$ with pullbacks and a functor $F\colon \mathbf{A} \to \mathbf{B}$ that does not preserve them, $\mathcal{S}pan(F)\colon \mathcal{S}pan\,\mathbf{A} \to \mathcal{S}pan\,\mathbf{B}$ is only oplax. Clearly, we are in need of oplax bicategories or some such concept. There exist two notions which might serve in this capacity, the dual of Hermida's bicategory with several objects [21] and the dual of Leinster's **fc**-multicategory [26]. Both Hermida and Leinster use spans in a category without pullbacks as a basic example. It is our contention that the second notion is the most useful one, at least for the work that follows. Leinster's **fc**-multicategories can also be viewed as Burroni's $\mathbb{T}$-multicategories in $\mathbf{Set}^{\bullet\rightrightarrows\bullet}$, where $\mathbb{T}$ is the free category monad [5], or as the 2-dimensional truncation of Hermida's multicategories in $\omega$-graphs, introduced in Section 10.2 of [22]. Hermida defines a notion of representability for composites which differs a bit from ours, as we will discuss in Section 4.6.

In [12], we called **fc**-multicategories lax double categories. We showed that lax morphisms of double categories are the Kleisli morphisms for the comonad $\mathbb{P}\mathrm{ath}$ on the 2-category **Doub** of double categories with double functors and vertical transformations, and that the Eilenberg-Moore algebras are precisely the lax double categories. (The anonymous referee has made the interesting suggestion that this result follows from Theorem 6.1 in [22].) We took this fact as justification that lax double categories are the right structure for discussing lax morphisms.

4.1. Oplax double categories. Recall from [12] that an *oplax double category* has objects, horizontal and vertical arrows, and cells. There are two differences with double categories. The first is that there is no composition given for horizontal arrows. This is somewhat compensated for by the fact that cells are more general. They have vertical arrows as horizontal domain and codomain and although the vertical domain is a horizontal arrow, the vertical codomain is a finite string of compatible horizontal arrows (possibly of length 0). So a cell is of the form

$$\begin{array}{ccccccccc} & & A & & \xrightarrow{\quad f \quad} & & B & & \\ & \overset{v}{\swarrow} & & & \alpha & & & \overset{w}{\searrow} & \\ C_0 & \xrightarrow[g_1]{} & C_1 & \xrightarrow[g_2]{} & C_2 & \xrightarrow[g_3]{} \cdots & & \xrightarrow[g_n]{} & C_n . \end{array} \qquad (22)$$

Vertical arrows compose and form a category. There are identity cells

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \mathrm{id}_f & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

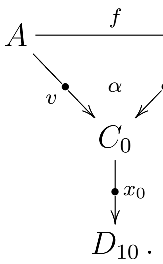
and cells compose vertically in the way that functions of several variables do. To be precise, given cells

$$\begin{array}{ccccccc} & C_{i-1} & & \xrightarrow{\quad g_i \quad} & & C_i & \\ \overset{x_{i-1}}{\swarrow} & & & \beta_i & & & \overset{x_i}{\searrow} \\ D_{i0} & \longrightarrow & D_{i1} & \longrightarrow \cdots & & \longrightarrow & D_{im_i} \end{array}$$

for $i = 1, \ldots, n$, we can compose them with $\alpha$ in (22) to obtain a cell

$$\begin{array}{ccccccc} & A & & \xrightarrow{\quad f \quad} & & B & \\ \overset{x_0 \cdot v}{\swarrow} & & & (\beta_1, \ldots, \beta_n) \cdot \alpha & & & \overset{x_n \cdot w}{\searrow} \\ D_{10} & \longrightarrow & D_{11} & \longrightarrow \cdots & & \longrightarrow & D_{nm_n} . \end{array}$$

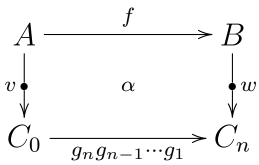
Note that there are $n + 1$ vertical arrows $x_i$ and $n$ cells $\beta_i$ involved in such a composite, so when $n = 0$ our composite looks like



This composition is unitary and associative in the obvious sense.

A morphism of oplax double categories is a map that preserves all the structure (on the nose).

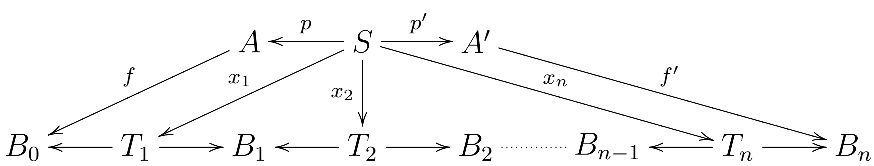
Every double category $\mathbb{A}$ gives rise to a canonical oplax double category $\mathbb{O}\mathrm{plax}\,\mathbb{A}$ with the same objects and horizontal and vertical arrows, but where a cell $\alpha$ as in (22) corresponds to a cell



in $\mathbb{A}$. Then morphisms of oplax double categories $\mathbb{O}\mathrm{plax}\,\mathbb{A} \to \mathbb{O}\mathrm{plax}\,\mathbb{B}$ correspond precisely to oplax morphisms $\mathbb{A} \to \mathbb{B}$. All of this is explained in full detail in [12].

An *oplax bicategory* is an oplax double category where all the vertical arrows are identities. Lax double categories and lax bicategories are defined dually.

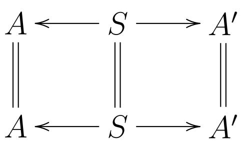
4.2. Definition. Let $\mathbf{A}$ be an arbitrary category. Then $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is the oplax double category whose objects and vertical arrows are the objects and arrows of $\mathbf{A}$, respectively. The horizontal arrows of $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ are spans in $\mathbf{A}$ and its cells are commutative diagrams of the form



(23)

4.3. Proposition. *With the above arrows and cells,* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *can be made into an oplax double category.*

Proof. Identities are given by



and composition is defined as follows. Suppose we are given $n$ cells

$$\begin{array}{ccccccccccc}
 & & B_{i-1} & \longleftarrow & T_i & \longrightarrow & B_i & & & & \\
 & \swarrow^{g_{i-1}} & & \swarrow^{y_{i1}} & \downarrow{\scriptstyle y_{i2}} & \searrow^{y_{im_i}} & & \searrow^{g_i} & & & \\
C_{i,0} & \longleftarrow & U_{i1} & \longrightarrow & C_{i,1} \longleftarrow U_{i2} & \longrightarrow & \cdots & U_{im_i} & \longrightarrow & C_{i,m_i}, &
\end{array}$$

with $C_{i,0} = C_{i-1,m_{i-1}}$, the composite with the cell (23) above is

$$\begin{array}{ccccccc}
 & & A & \longleftarrow S \longrightarrow & A' & & \\
 & \swarrow^{g_0 f} & & \downarrow{\scriptstyle y_{ij}x_i} & & \searrow^{g_n f'} & \\
C_{10} & \longleftarrow & \cdots \longleftarrow & U_{ij} & \longrightarrow \cdots & \longrightarrow & C_{n,m_n}.
\end{array}$$

Verification of associativity is straightforward. ∎

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor. We get a morphism of oplax double categories,

$$\mathbb{S}\mathrm{pan}(F)\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{S}\mathrm{pan}\,\mathbf{B},$$

simply by applying $F$ to each of the components of arrows or cells. So, for example, $\mathbb{S}\mathrm{pan}(F)$ applied to the span $A \xleftarrow{p} S \xrightarrow{p'} A'$ gives the span $FA \xleftarrow{Fp} FS \xrightarrow{Fp'} FA'$, and so on.

For a natural transformation $t\colon F \to G$ we get a *vertical* transformation

$$\mathbb{S}\mathrm{pan}\, t\colon \mathbb{S}\mathrm{pan}\, F \to \mathbb{S}\mathrm{pan}\, G,$$

defined by

$$(\mathbb{S}\mathrm{pan}\, t)_A := \begin{array}{c} FA \\ \downarrow{\bullet\, t_A} \\ GA \end{array}$$

$$(\mathbb{S}\mathrm{pan}\, t)_S := \begin{array}{ccccc}
FA & \xleftarrow{Fp} & FS & \xrightarrow{Fp'} & FA' \\
{\scriptstyle t_A}\bullet\downarrow & & {\scriptstyle t_S}\bullet\downarrow & & \downarrow\bullet{\scriptstyle t_{A'}} \\
GA & \xleftarrow[Gp]{} & GS & \xrightarrow[Gp']{} & GA.
\end{array}$$

We have the following result.

4.4. Proposition. *Let* **Oplax** *be the 2-category of oplax double categories, oplax morphisms and vertical transformations. Then* $\mathbb{S}\mathrm{pan}$: **Cat** $\to$ **Oplax** *is a locally fully faithful 2-functor.*

PROOF. The fact that $\mathbb{S}\mathrm{pan}$ is a 2-functor can be verified by a straightforward calculation. (See [12], following Theorem 2.1 for the definition of vertical transformation.) Local faithfulness is also obvious. To check that $\mathbb{S}\mathrm{pan}$ is locally full, let $\tau\colon \mathbb{S}\mathrm{pan}\,F \to \mathbb{S}\mathrm{pan}\,G$ be any vertical transformation, where $F, G\colon \mathbf{A} \rightrightarrows \mathbf{B}$ are functors. Let $\tau_{(p,q)}$ be of the form

$$\begin{array}{ccccc} FA & \xleftarrow{Fp} & FS & \xrightarrow{Fq} & FB \\ {\scriptstyle \tau_A}\downarrow & & {\scriptstyle \tau_{(p,q)}}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ GA & \xleftarrow[Gp]{} & GS & \xrightarrow[Gq]{} & GB \end{array}$$

We want to show that $\tau_{(p,q)} = \tau_S$. First note that $\tau_{(1_S,1_S)} = \tau_S$. Now consider the 2-cell

$$\begin{array}{ccccc} S & \xleftarrow{1} & S & \xrightarrow{1} & S \\ {\scriptstyle p}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xleftarrow[p]{} & S & \xrightarrow[q]{} & B \end{array}$$

in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$. By vertical naturality of $\tau$, we have that

$$\begin{array}{ccccc} FS & \xleftarrow{F1} & FS & \xrightarrow{F1} & FS \\ {\scriptstyle Fp}\downarrow & & {\scriptstyle F1}\downarrow & & \downarrow{\scriptstyle Fq} \\ F & & FS & \xrightarrow{Fq} & FB \\ {\scriptstyle \tau_A}\downarrow & & {\scriptstyle \tau_{(p,q)}}\downarrow & & \downarrow{\scriptstyle \tau_B} \\ GA & \xleftarrow[Gp]{} & GS & \xrightarrow[Gq]{} & GB \end{array} \quad = \quad \begin{array}{ccccc} FS & \xleftarrow{F1} & FS & \xrightarrow{F1} & FS \\ {\scriptstyle \tau_S}\downarrow & & {\scriptstyle \tau_{(1,1)}=\tau_S}\downarrow & & \downarrow{\scriptstyle \tau_S} \\ GS & \xleftarrow[G1]{} & GS & \xrightarrow[G1]{} & GS \\ {\scriptstyle Gp}\downarrow & & {\scriptstyle G1}\downarrow & & \downarrow{\scriptstyle Gq} \\ GA & \xleftarrow[Gp]{} & GS & \xrightarrow[Gq]{} & GB \end{array} .$$

We conclude that $\tau_{(p,q)} = \tau_S$, as required. ∎

4.5. PROPOSITION. *The inclusion* $\mathbf{A} \hookrightarrow \mathbb{S}\mathrm{pan}\,\mathbf{A}$ *as vertical arrows is locally full and faithful.*

PROOF. Given $f, g\colon A \rightrightarrows B$ in $\mathbf{A}$, there is at most one cell

$$\begin{array}{ccccc} A & = & A & = & A \\ {\scriptstyle f}\downarrow & & \vdots & & \downarrow{\scriptstyle g} \\ B & = & B & = & B \end{array}$$

and there is one if and only if $f = g$. ∎

4.6. OPLAX NORMAL DOUBLE CATEGORIES. Not much can be done with oplax morphisms that are not normal, and the more interesting construction of [12] was $\mathbb{P}\mathrm{ath}_*$ for oplax normal double categories. This leads us to consider the notions of (strong) representability of composites and identities in oplax double categories. The definition of strongly representable composites in lax double categories was given in [12], Definition 2.7. We will give the oplax version here for convenience:

4.7. Definition. Let $\mathbb{A}$ be an oplax double category and

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$$

a path of horizontal arrows of $\mathbb{A}$. We say that *the composite of this path is strongly representable* if there is an arrow $f \colon A_0 \to A_n$ and a cell

$$\begin{array}{ccc} A_0 & \xrightarrow{\quad f \quad} & A_n \\ \| & \eta_{\langle f_n, \ldots, f_1 \rangle} & \| \\ A_0 & \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} & A_n \end{array}$$

such that for any paths $\langle x_1, \ldots, x_m \rangle$, and $\langle y_p, \ldots, y_1 \rangle$, and cell $\alpha$ as below

$$\begin{array}{ccc} B & \xrightarrow{\qquad\qquad g \qquad\qquad} & C \\ v \big\downarrow & \alpha & \big\downarrow w \\ X_m & \xrightarrow{x_m} \cdots \longrightarrow X_1 \xrightarrow{x_1} A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} \cdots \xrightarrow{y_p} & Y_p \end{array}$$

there is a unique cell $\overline{\alpha}$

$$\begin{array}{ccc} B & \xrightarrow{\qquad\qquad g \qquad\qquad} & C \\ v \big\downarrow & \overline{\alpha} & \big\downarrow w \\ X_m & \xrightarrow{x_m} \cdots \longrightarrow X_1 \xrightarrow{x_1} A_0 \xrightarrow{f} A_n \xrightarrow{y_1} Y_1 \xrightarrow{y_2} \cdots \longrightarrow & Y_p \end{array}$$

such that the composition of

$$\begin{array}{ccccccccc} B & & & & \xrightarrow{\qquad\qquad g \qquad\qquad} & & & & C \\ v \big\downarrow & & & & \overline{\alpha} & & & & \big\downarrow w \\ X_m & \xrightarrow{x_m} X_{m-1} \xrightarrow{x_{m-1}} \cdots & X_1 & \xrightarrow{x_1} & A_0 \xrightarrow{\qquad f \qquad} A_n & \xrightarrow{y_1} & Y_1 & \xrightarrow{y_2} \cdots \xrightarrow{y_p} & Y_p \\ \| & \text{id} \quad \| & \| & \text{id} & \| \quad \eta_{\langle f_m, \ldots, f_1 \rangle} \quad \| & \text{id} & \| & & \| \\ X_m & \xrightarrow{x_m} X_{m-1} \xrightarrow{x_{m-1}} \cdots & X_1 & \xrightarrow{x_1} & A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n & \xrightarrow{y_1} & Y_1 & \xrightarrow{y_2} \cdots \xrightarrow{y_p} & Y_p \end{array}$$

is equal to $\alpha$.

We say that the *composite of the path* $\langle f_n, \ldots, f_1 \rangle$ *is representable* if we require the above condition only in the case where $m = p = 0$, *i.e.*, when there are no $x$'s or $y$'s.

This definition was originally given by Lambek [25] for multicategories and by Hermida [21] for lax bicategories (there called multicategories with several objects) and by Leinster [26] for lax double categories (there called **fc**-multicategories). A different approach is given in Penon's article [29] describing and extending Burroni's work [5] on $T$-categories. Hermida's work on representable multicategories [22] gives a more general notion of representability where the universal cell is not required to be special. However, since we want to think of the domain of the universal cell as the composite of its codomain sequence of composable arrows, we require the cells to be special. If vertical isomorphisms have both companions and conjoints, the two definitions are equivalent. It can also be verified that if all $n$-tuple composites are representable and the structure cells are closed under composition as required in [22], then all these composites are strongly representable.

When the composite of a string $A_0 \xrightarrow{f_1} A_1 \longrightarrow \cdots \xrightarrow{f_n} A_n$ is strongly representable we will choose a representing morphism and call it $f_n f_{n-1} \cdots f_1$. We will also say that the *composite exists* in this case. The representing cell is denoted

$$\begin{array}{ccc} A_0 & \xrightarrow{\quad f_n \cdots f_1 \quad} & A_n \\ \| & \iota_{f_n, f_{n-1}, \ldots, f_1} & \| \\ A_0 & \xrightarrow[f_1]{} A_1 \xrightarrow[f_2]{} \cdots \xrightarrow[f_n]{} & A_n. \end{array}$$

For $n = 0$, the composite, when it exists, will be denoted by $1_A$ and we say that the identity on $A$ exists or is strongly representable. The universal cell is denoted

$$\begin{array}{ccc} A & \xrightarrow{\quad 1_A \quad} & A \\ & \iota_A & \\ & A & \end{array} .$$

As is historically the case, we are mostly interested in identities and composites of two arrows.

The following proposition is due to Hermida [21], Theorem 11.6, although it is stated for oplax bicategories in that paper.

4.8. Proposition. $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ *has all composites if and only if* $\mathbf{A}$ *has pullbacks.*

We shall need refinements of this result for categories which may not have all pullbacks.

4.9. Proposition. *For any* $\mathbf{A}$*, horizontal identities exist in* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$*. Moreover, for any functor* $F\colon \mathbf{A} \to \mathbf{B}$*, the functor* $\mathbb{S}\mathrm{pan}\,(F)$ *preserves these identities.*

Proof. It is easily verified that the cell

$$\begin{array}{ccccc} A & \xleftarrow{1_A} & A & \xrightarrow{1_A} & A \\ & & A & & \end{array}$$

has the required universal property. ∎

Following Definition 3.15 of [12] we say that an oplax double category is *normal* if all identities exist and an oplax morphism is *normal* if it preserves these identities in the sense that

$$\begin{array}{ccc} FA & \xrightarrow{F1_A} & FA \\ \| & F\iota_A & \| \\ & FA & \end{array}$$

has the universal property making $F1_A$ an identity on $FA$. Thus, $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ and $\mathbb{S}\mathrm{pan}\,(F)$ are normal.

4.10. Proposition. *Let $A \xleftarrow{s_1} S \xrightarrow{s_2} B$ and $B \xleftarrow{t_1} T \xrightarrow{t_2} C$ be spans in $\mathbf{A}$. Then the following are equivalent.*

1. *$TS$ is representable;*
2. *$TS$ is strongly representable;*
3. *The pullback*
$$\begin{array}{ccc} & P & \\ {}^{s}\swarrow & & \searrow^{t} \\ S & & T \\ {}_{s_2}\searrow & & \swarrow_{t_1} \\ & B & \end{array}$$
*exists.*

*Furthermore, $\mathbb{S}\mathrm{pan}\,(F)\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{S}\mathrm{pan}\,\mathbf{B}$ preserves the composite $TS$ if $F$ preserves the pullback $S \times_B T$.*

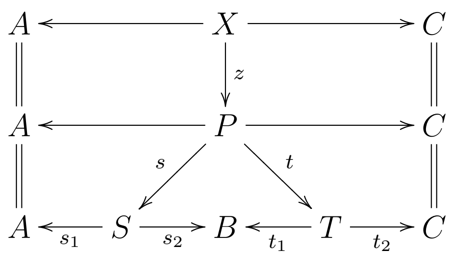
Proof. To prove the implication (1)⇒(3), suppose that $TS$ is representable with universal cell $\iota$ as in the diagram

$$\begin{array}{ccccccccc} A & \longleftarrow & & & P & & & \longrightarrow & C \\ \| & & & {}^{s}\swarrow & & \searrow^{t} & & & \| \\ A & \xleftarrow[s_1]{} & S & \xrightarrow[s_2]{} & B & \xleftarrow[t_1]{} & T & \xrightarrow[t_2]{} & C \end{array} .$$

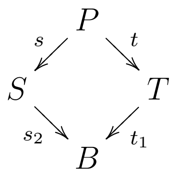
For a commutative square
$$\begin{array}{ccc} & X & \\ {}^{x}\swarrow & & \searrow^{y} \\ S & & T \\ {}_{s_2}\searrow & & \swarrow_{t_1} \\ & B & \end{array}$$
we get a cell

$$\begin{array}{ccccccccc} A & \xleftarrow{s_1 x} & & & X & & & \xrightarrow{t_2 y} & C \\ \| & & & {}^{x}\swarrow & & \searrow^{y} & & & \| \\ A & \xleftarrow[s_1]{} & S & \xrightarrow[s_2]{} & B & \xleftarrow[t_1]{} & T & \xrightarrow[t_2]{} & C \end{array} \tag{24}$$

which must factor uniquely through $\iota$, *i.e.*, there is a unique $z$ such that




is equal to (24). Thus,




is a pullback.

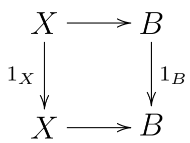
The implication (3)⇒(2) is immediate from the universal property of pullbacks, and the implication (2)⇒(1) is obvious.

Finally, that $\mathbb{S}\mathrm{pan}\, F$ preserves the composite $TS$ when $F$ preserves the pullback in Condition (3) is also obvious. ■

4.11. Corollary.

1. *If* $g\colon B \to C$ *is an arrow in* $\mathbf{A}$*, then* $g_* S$ *is strongly representable in* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$*.*
2. *If* $f\colon B \to A$ *is an arrow in* $\mathbf{A}$*, then* $Tf^*$ *is strongly representable in* $\mathbb{S}\mathrm{pan}\,\mathbf{A}$*.*
3. *For any functor* $F\colon \mathbf{A} \to \mathbf{B}$*, the induced morphism* $\mathbb{S}\mathrm{pan}\,(F)$ *preserves composites of the form* $g_* S$ *and* $Tf^*$*.*

Proof. The arrow $g_*$ is the span $B \xleftarrow{1_B} B \xrightarrow{g} C$ and $f^*$ is the span $A \xleftarrow{f} B \xrightarrow{1_B} B$. In either case, the pullback of Condition (3) in Proposition 4.10 is of the form

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which always exists for trivial reasons and is preserved by any functor. ■

4.12. Corollary. *The composites* $1_B S$ *and* $S 1_A$ *exist and are canonically isomorphic to* $S$*.*

We conclude that for any category $\mathbf{A}$, the oplax double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is normal and for any functor $F$, $\mathbb{S}\mathrm{pan}\,(F)$ is normal. Moreover, $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ has composites of the form $f_* S$ and $Tf^*$ and $\mathbb{S}\mathrm{pan}\,(F)$ preserves them. If we want to extend the universal properties of Theorems 2.8 and 3.15 to oplax bicategories and double categories, representability of such composites has to be considered. For this we need a notion of gregarious oplax double category. There is a very natural extension of the notion of being gregarious to oplax normal double categories in terms of companions and conjoints. However, in order to be able to talk about companions and conjoints in oplax normal double categories we first need to consider composition of cells in more detail.

4.13. Composition of cells. We begin with a couple of results on composition of horizontal arrows.

4.14. Proposition. *Let $\mathbb{A}$ be an oplax double category and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ horizontal arrows in $\mathbb{A}$.*

1. *If $gf$ and $hg$ are strongly representable, the following are equivalent:*

   *(a) $h(gf)$ is strongly representable;*

   *(b) $(hg)f$ is strongly representable;*

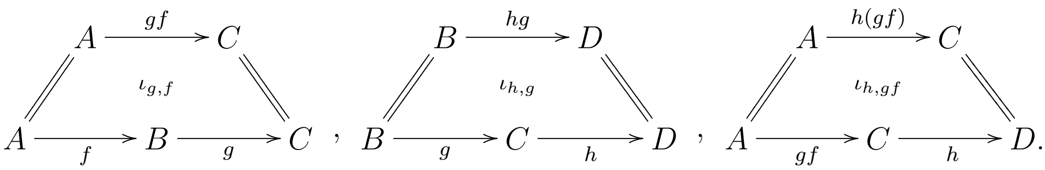
   *(c) $hgf$ is strongly representable.*

   *Furthermore, if any one of these is strongly representable, then we have $h(gf) \cong (hg)f \cong hgf$ by vertically invertible special cells.*

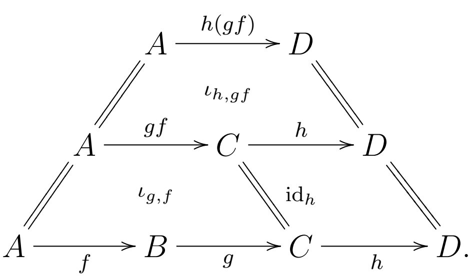
2. *If $1_B$ is strongly representable, then so are the composites $1_B f$ and $g 1_B$. Moreover, $1_B f \cong f$ and $g 1_B \cong g$.*

3. *If any oplax morphism preserves $1_B$ then it preserves the composites $1_B f$ and $g 1_B$.*

Proof. These are all more or less standard representability results. Let us first show (1a) ⇔ (1c). Assume that $gf$, $gh$, and $h(gf)$ are strongly representable with universal cells



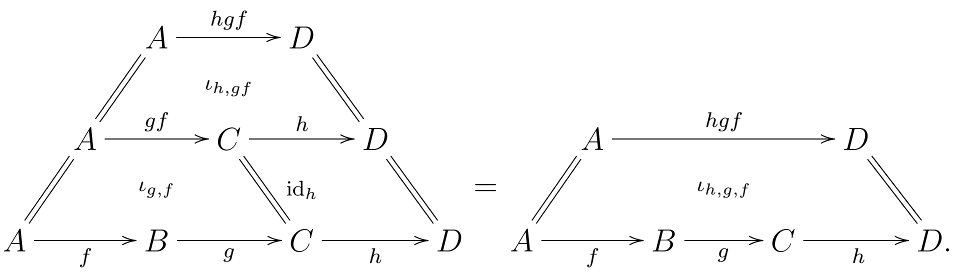

Define $\iota_{h,g,f}$ to be the composite




The universal property of $\iota_{h,g,f}$ is easily checked, but note that strong representability of $gf$ is needed, even if we only want mere representability of $hgf$. So we see that $h(gf)$

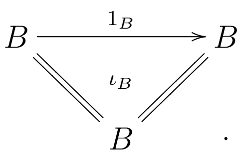
strongly represents $hgf$. By the universal property for representability, if we had chosen any other arrow $hgf$ to represent this composite, we conclude that $h(gf) \cong hgf$ by a vertically invertible special cell.

Assume now that $gf$, $hg$, and $hgf$ are strongly representable. Strong representability of $gf$ tells us that there is a unique cell $\iota_{h,gf}$ such that

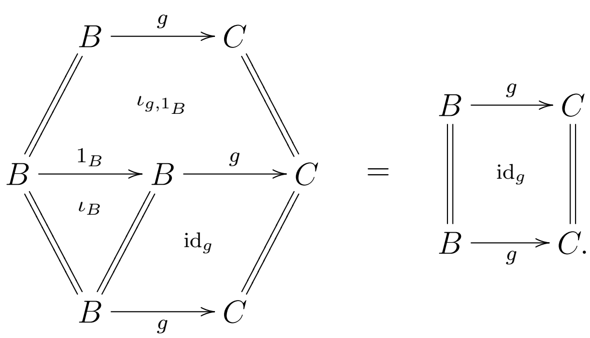



It is again straightforward to check that $\iota_{h,gf}$ has the universal property making $h(gf)$ strongly represented by $hgf$, and $h(gf) \cong hgf$. The rest of (1) follows by duality.

For (2), assume that $1_B$ is strongly representable with universal cell

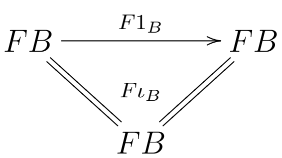



Then there is a unique cell $\iota_{g,1_B}$ such that

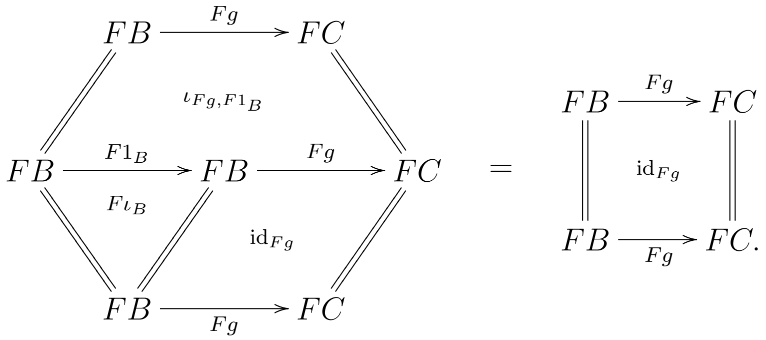



It is easily seen that $\iota_{g,1_B}$ has the universal property which makes $g1_B$ strongly represented by $g$. The proof of the fact that $1_Bf$ is strongly representable is dual to this.

For (3), assume that $F$ preserves $1_B$. Then

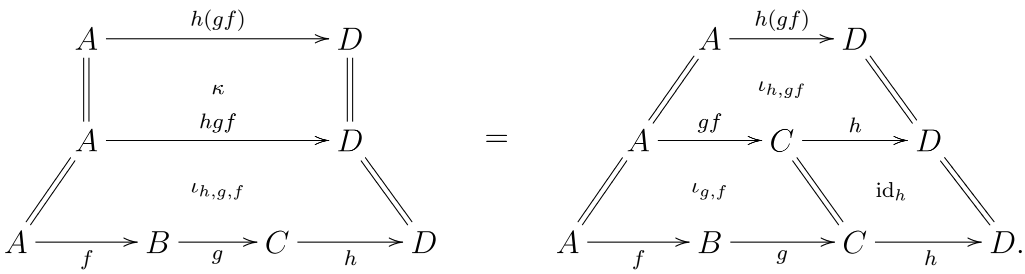


strongly represents the horizontal identity on $FB$. So there is a unique cell $\iota_{Fg,F1_B}$ such that




Just as in the proof of the previous part, it is easily seen that $\iota_{Fg,F1_B}$ has the universal property which makes $FgF1_B$ strongly represented by $Fg$. The proof for the preservation of $1_Bf$ is dual to this one. ■

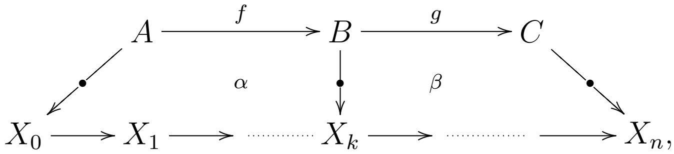
Note that the above proof shows that $h(gf)$ satisfies the universal property of $hgf$, so we could choose $hgf$ to be $h(gf)$. However, we cannot choose it to be $(hg)f$ at the same time. It is best to make an independent choice of $hgf$. Then we get a unique isomorphism $\kappa\colon h(gf) \to hgf$ satisfying



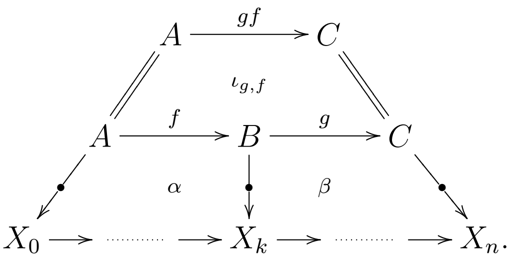

The horizontal composition of arrows in an oplax double category requires the solution of a universal mapping problem, and when it does exist associativity and the unit laws only hold up to comparison cells which are vertical isomorphisms if we have strong representability. However, we will see that horizontal composition of cells works perfectly well provided that the horizontal composites of the relevant domains and codomains exist. All reasonable laws hold on the nose once the canonical isomorphisms are factored in.

There are two ways we might define horizontal composition of cells in an oplax double category. The first is more general and a graded kind of composition.

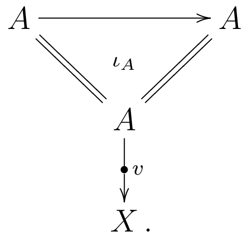
Given cells $\alpha$ and $\beta$ as below



where $gf$ is strongly representable, the horizontal composite $\beta * \alpha$ is defined to be the vertical composite $(\beta, \alpha) \cdot \iota_{g,f}$,

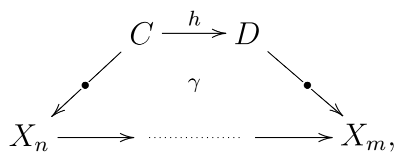


Similarly, given a vertical arrow $v\colon A \to X$, where $1_A$ is strongly representable, the horizontal identity cell $1_v^*$ is defined to be the composite

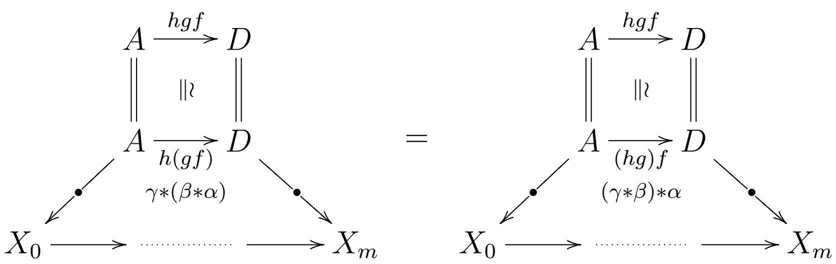


4.15. Proposition.

1. *Given cells $\alpha$, and $\beta$ as above and a further one*



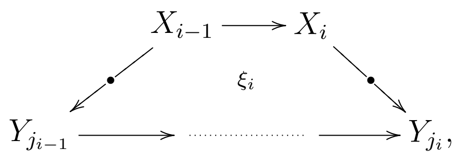
*such that $gf$, $hg$, and $hgf$ are strongly representable, then $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$, in the sense that*



*where the top cells are the canonical isomorphisms mentioned in Proposition 4.14.*

2. *If $1_B$ is strongly representable, then $1_v^* * \alpha = \alpha$ and $\beta * 1_v^* = \beta$, modulo the canonical isomorphisms $1_B f \cong f$ and $g 1_B \cong g$ as in Part* (1).

3. *With cells $\alpha$ and $\beta$ as above, and $gf$ representable, there is also an interchange law. Given cells*



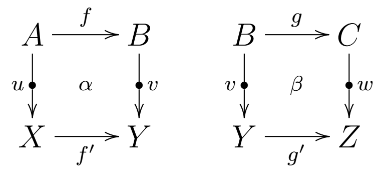
*we have*

$$[(\xi_n, \xi_{n-1}, \dots, \xi_k) \cdot \beta] * [(\xi_k, \dots, \xi_1) \cdot \alpha] = (\xi_n, \dots, \xi_1) \cdot (\beta * \alpha).$$

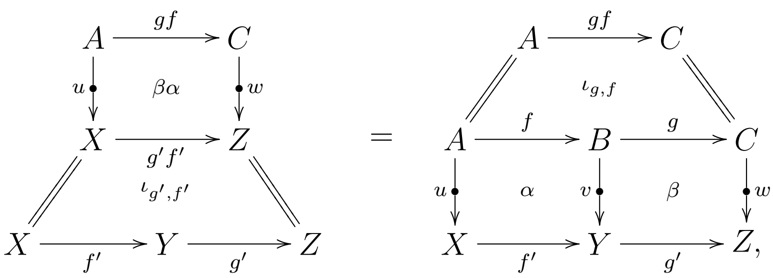
PROOF. Straightforward. (The proof of Part (3) is easiest, being nothing but associativity of vertical composition in $\mathbb{A}$.) ■

There is another more special kind of horizontal composition on cells with a square boundary, *i.e.*, for which the vertical codomain has length 1. This notion of composition is the one which is more useful for us.

Given cells $\alpha$ and $\beta$ as in



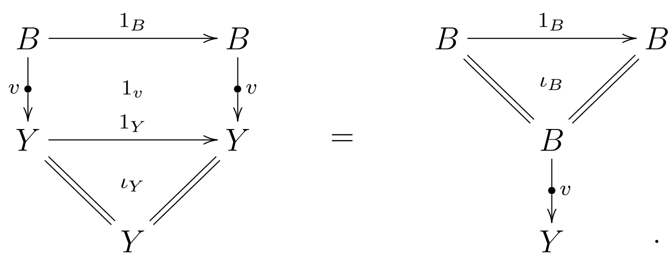
such that $gf$ and $g'f'$ are strongly representable, the horizontal composite $\beta\alpha$ is the unique cell such that



(25)

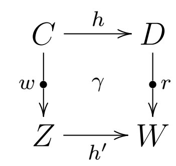
*i.e.*, $\iota \cdot (\beta\alpha) = \beta * \alpha$.

Also, given a vertical arrow $v\colon B \multimap Y$, for which $1_B$ and $1_Y$ are both strongly representable, we define the horizontal identity cell $1_v$ to be the unique cell such that



4.16. Proposition.

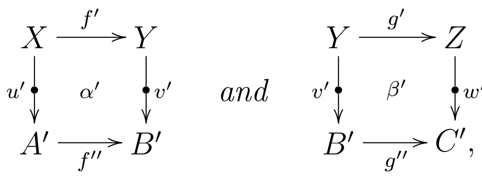
1. *Given cells* $\alpha$ *and* $\beta$ *as above, and a further one*




*for which* $gf$, $hg$, $hgf$, $g'f'$, $h'g'$, *and* $h'g'f'$ *are all strongly representable, then* $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ *(modulo the canonical isomorphisms).*

2. *If* $1_B$ *and* $1_Y$ *are strongly representable, then* $1_v\alpha = \alpha$ *and* $\beta 1_v = \beta$.

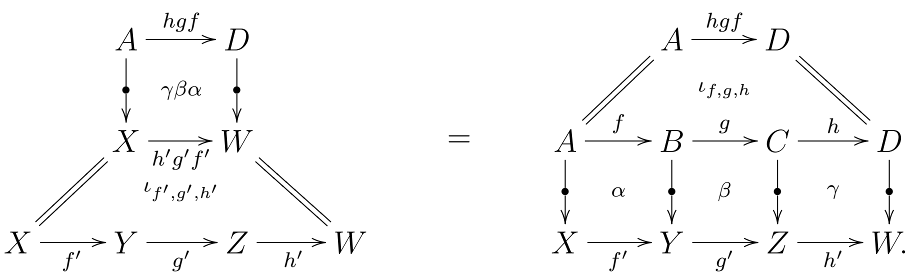
3. *(The Interchange Law) If we also have cells*



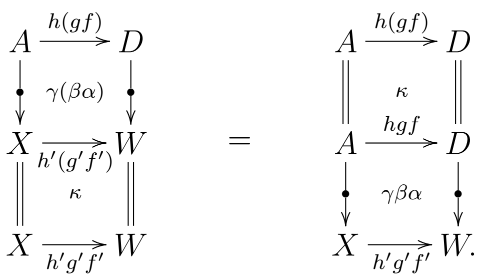

*and* $gf$, $g'f'$, *and* $g''f''$ *are all strongly representable, then*

$$(\beta'\alpha') \cdot (\beta\alpha) = (\beta' \cdot \beta)(\alpha' \cdot \alpha).$$

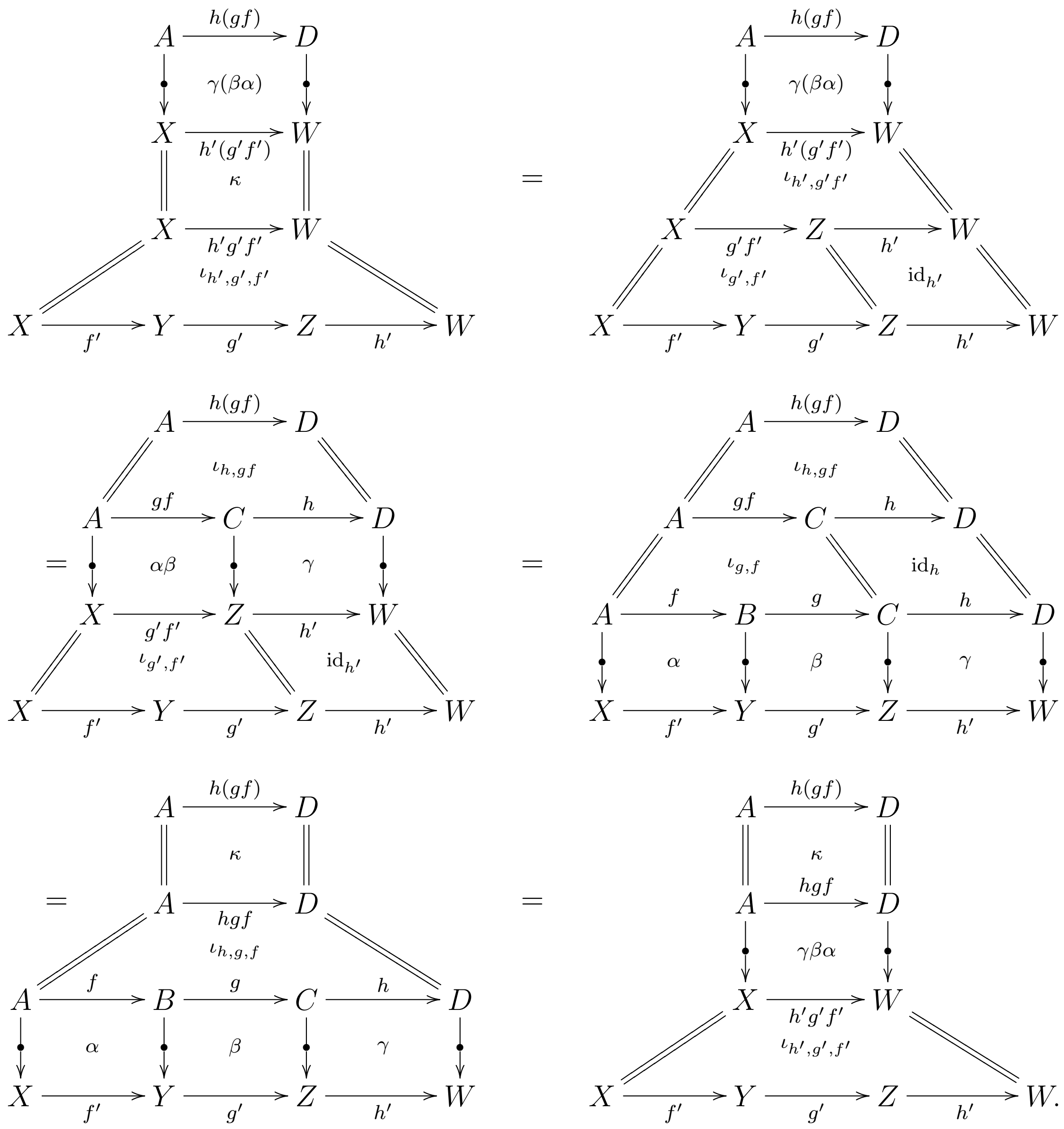
Proof. There is a triple horizontal composite of cells $\gamma\beta\alpha$ defined to be the unique cell such that




We will show that $\gamma(\beta\alpha) = \gamma\beta\alpha$ in the sense that



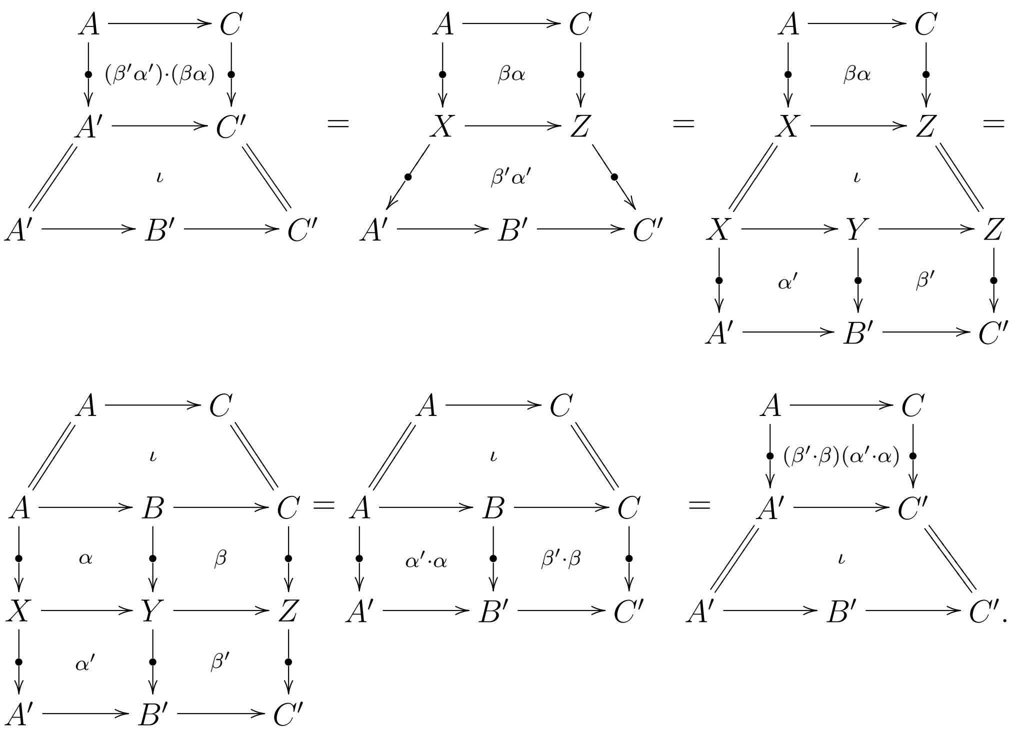
Then the result will follow by duality. These cells will be equal if they are equal when composed (vertically) with $\iota_{h',g',f'}$. Consider the following diagrams whose composites are all equal.




This proves Part (1).

The proof of Part (2) is similar and left as an exercise.

We give the proof of Part (3) which is a bit different. The cells $(\beta'\alpha') \cdot (\beta\alpha)$ and $(\beta' \cdot \beta)(\alpha' \cdot \alpha)$ are equal if they are equal when composed with $\iota_{g',f'}$. This is indeed the case, as the following sequence of diagrams with equal composites shows.

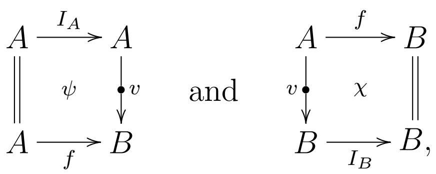


■

4.17. Corollary. *If in any oplax double category all composites and all identities are strongly representable, the square cells (with vertical codomains of length one) form a (weak) double category.*

4.18. Gregarious oplax double categories. For an oplax normal double category the definitions of companions and conjoints given in Definition 3.4 for double categories still make sense:

4.19. Definition. Let $\mathbb{B}$ be an oplax normal double category. We say that a horizontal arrow $f\colon A \longrightarrow B$ and a vertical arrow $v\colon A \multimap B$ in $\mathbb{B}$ are *companions* if there are *binding cells*



such that

$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ \| & \psi & \big\downarrow\bullet v \\ A & \xrightarrow[f]{} & B \\ v\bullet\big\downarrow & \chi & \| \\ B & \xrightarrow[I_B]{} & B \end{array} = 1_v \qquad \text{and} \qquad \begin{array}{ccccc} A & \xrightarrow{I_A} & A & \xrightarrow{f} & B \\ \| & \psi & \big\downarrow\bullet v & \chi & \| \\ A & \xrightarrow[f]{} & B & \xrightarrow[I_B]{} & B \end{array} = \mathrm{id}_f \tag{26}$$

,

*i.e.*, $\chi \cdot \psi = 1_v$ and $\chi\psi = \mathrm{id}_f$.

Dually, we say that a horizontal arrow $u\colon B \longrightarrow A$ and a vertical arrow $v\colon A \relbar\!\!\bullet\!\!\rightarrow B$ are *conjoints* if there are *binding cells*

$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ v\bullet\big\downarrow & \alpha & \| \\ B & \xrightarrow[u]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B & \xrightarrow{u} & A \\ \| & \beta & \big\downarrow\bullet v \\ B & \xrightarrow[I_B]{} & B, \end{array}$$

such that

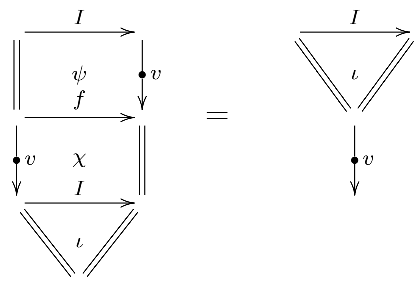
$$\begin{array}{ccc} A & \xrightarrow{I_A} & A \\ v\bullet\big\downarrow & \alpha & \| \\ B & \xrightarrow[u]{} & A \\ \| & \beta & \big\downarrow\bullet v \\ B & \xrightarrow[I_B]{} & B \end{array} = 1_v \qquad \text{and} \qquad \begin{array}{ccccc} B & \xrightarrow{u} & A & \xrightarrow{I_A} & A \\ \| & \beta & \big\downarrow\bullet v & \alpha & \| \\ B & \xrightarrow{I_B} & B & \xrightarrow[u]{} & A \end{array} = \mathrm{id}_f$$

,

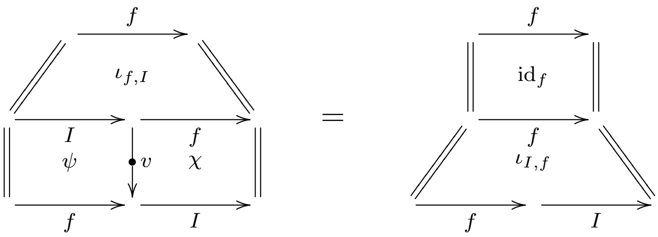
*i.e.*, $\beta \cdot \alpha = 1_v$ and $\alpha\beta = \mathrm{id}_f$.

Note that these definitions only use composites of two horizontal arrows in which one is an identity and these exist by Proposition 4.14 Part (2). They also use horizontal composition of cells and horizontal identities which exist and have the required properties by Proposition 4.16. Note however, that the definition of adjoint given in Definition 3.4 does not carry over to the present context as it uses the triple composites $fuf$ and $ufu$.

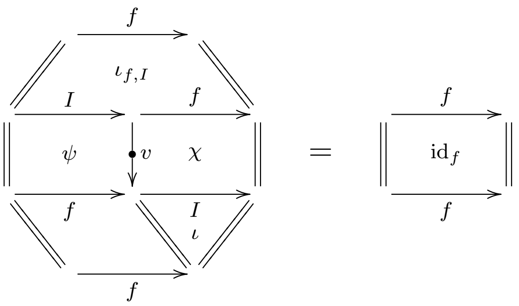
To illustrate how the special horizontal composition of cells works here, we spell out what is meant by the equations in (26). The left-hand equation means that



whereas the right-hand equation means that




We derive from this that




(27)

The following properties of conjoints and companions hold in any oplax normal double category.

1. If $h$ and $h'$ are both companions to $v$ then $h \cong h'$ by a special vertical isomorphism.
2. If $h$ is companion to $v$ and $v'$ then $v \cong v'$ by a special horizontal isomorphism.
3. If $A \xrightarrow{v} B \xrightarrow{w} C$ are vertical morphisms with companions $h$ and $k$ respectively, then $w \cdot v$ has a companion if and only if the composite $kh$ is strongly representable and that composite is the companion of $w \cdot v$.
4. The vertical arrow $\text{id}_A$ has the horizontal arrow $1_A$ as companion.
5. Any morphism of oplax normal double categories preserves companions.
6. Dual statements hold for conjoints.

We are now ready to introduce gregarious oplax double categories and discuss the universal properties of $\mathbb{S}\text{pan}\,\mathbf{A}$ for an arbitrary category $\mathbf{A}$.

4.20. Definition. An oplax double category is *gregarious* if it satisfies the following conditions.

1. It is normal.
2. Every vertical arrow $v$ has a horizontal companion $v_*$ and a horizontal conjoint $v^*$.

3. For each span of vertical arrows $A \xleftarrow{v} B \xrightarrow{w} C$, the composite $w_* v^*$ is strongly representable.

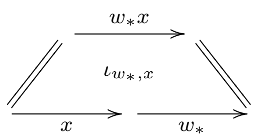
We write **GregOplax** for the 2-category of gregarious oplax double categories, normal morphisms and vertical transformations.

4.21. Example. For any category $\mathbf{A}$, the oplax double category $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is gregarious. Also, for any functor $F\colon \mathbf{A} \to \mathbf{B}$, the induced functor $\mathbb{S}\mathrm{pan}\,(F)$ of oplax double categories is a normal morphism, as was shown above.

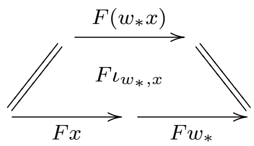
The morphisms between gregarious oplax double categories are not explicitly required to preserve the composites of the form $w_* v^*$, because normal morphisms will do this automatically.

4.22. Proposition. *A normal morphism $F\colon \mathbb{C} \to \mathbb{D}$ between oplax normal double categories preserves all composites of the forms $w_* x$ and $y w^*$ that exist in $\mathbb{C}$.*

Proof. We will show that $F$ preserves the composites of the form $w_* x$ that exist in $\mathbb{C}$. The dual of this proof shows that $F$ also preserves the composites of the form $y w^*$ that exist in $\mathbb{C}$. Let

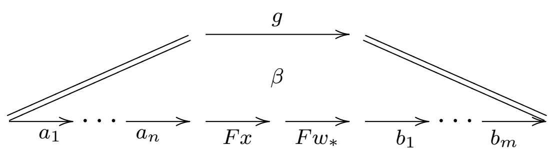



be a universal cell with the strong representability property in $\mathbb{C}$ for the composite $w_* x$. We will now show that

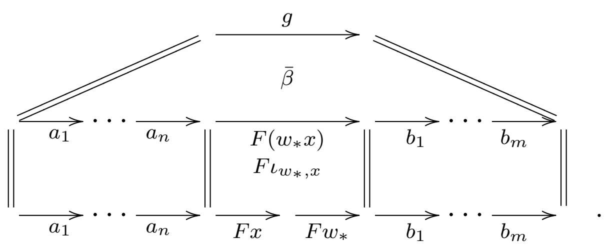



has the strong representability property in $\mathbb{D}$.

Let

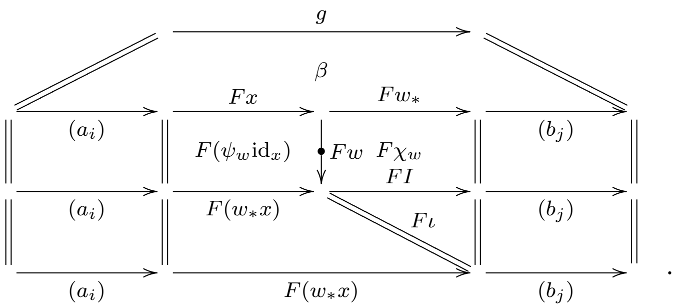



be any cell in $\mathbb{D}$. We need to show that there is a unique cell $\bar{\beta}$ such that $\beta$ is the composite,

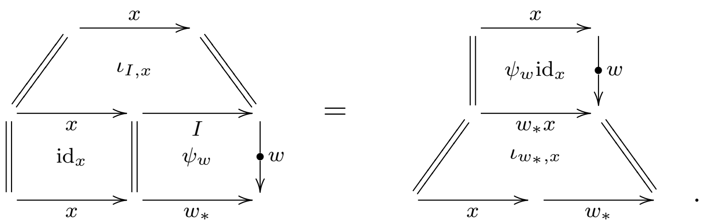



(28)

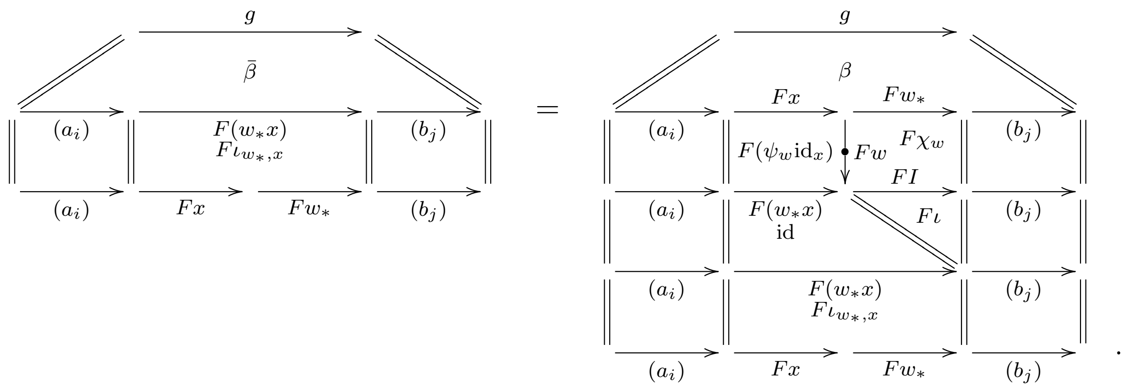
For the sake of space, we will from now on abbreviate the path $\xrightarrow[a_1]{}\cdots\xrightarrow[a_n]{}$ by $\xrightarrow[(a_i)]{}$ (and analogously for the $(b_j)$ path.) We define $\bar{\beta}$ to be the composite of



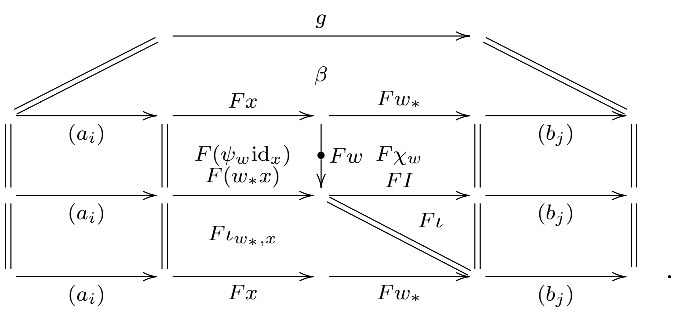

Recall that the cell $\psi_w \mathrm{id}_x$ is defined as the unique cell such that

 (29)

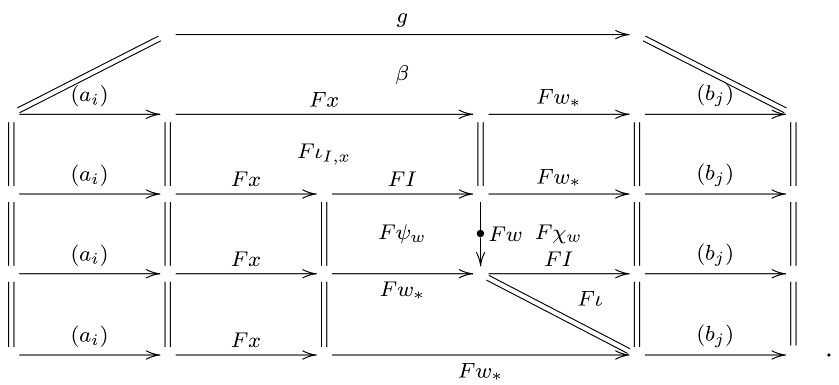

We will now calculate the composite (28) and show that it is indeed equal to $\beta$.



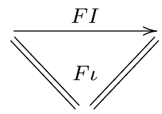

By the associativity and identity properties of the composition of cells in an oplax double category, this is equal to



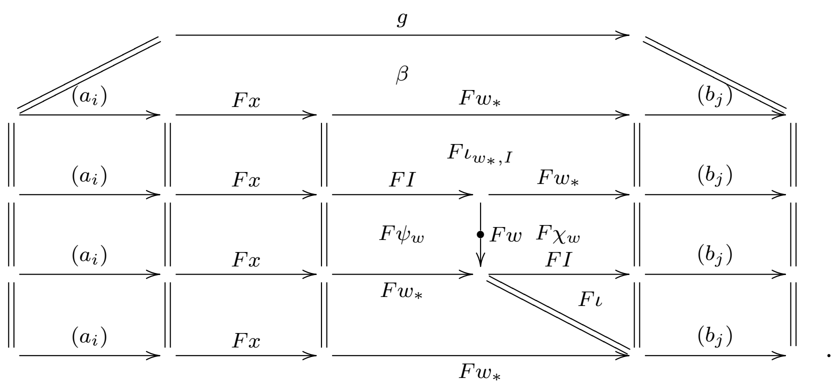
By (29) this is equal to




By the universal property of

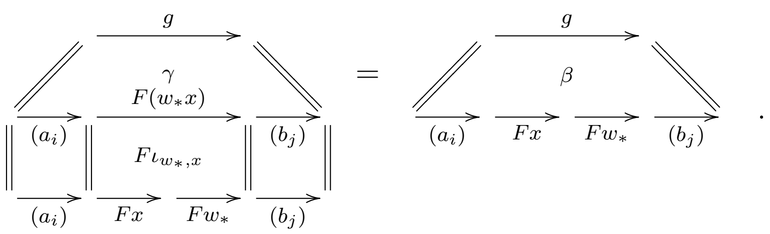



this is equal to



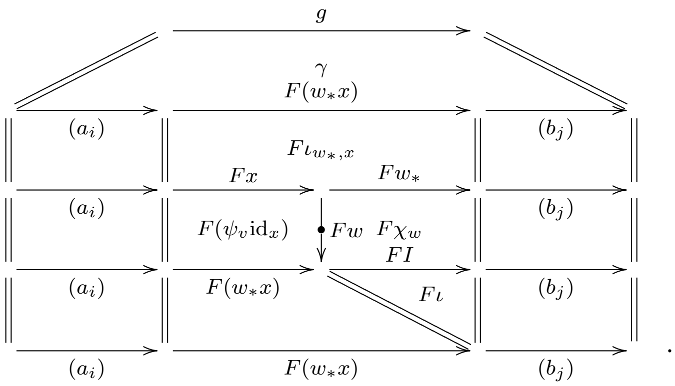

By (27) this is equal to $\beta$.

It remains to be shown that $\bar{\beta}$ is unique with this property. So suppose that $\gamma$ is a cell satisfying



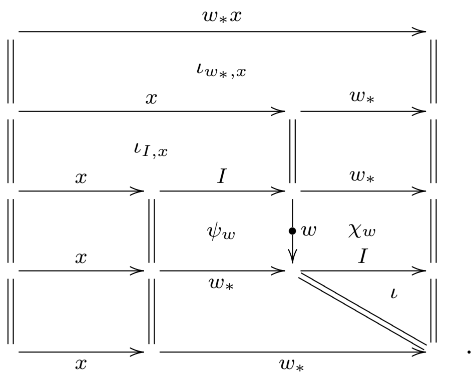

(30)

This implies that $\bar{\beta}$ is equal to the composite



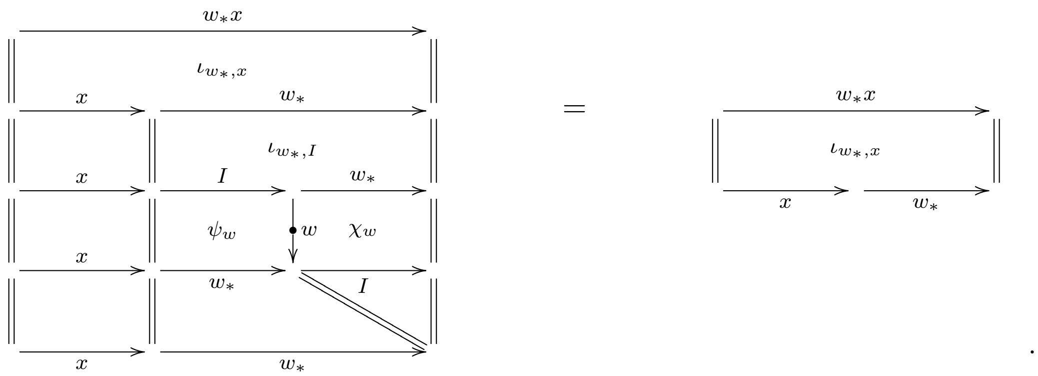

(31)

Now consider the following composite in $\mathbb{C}$:

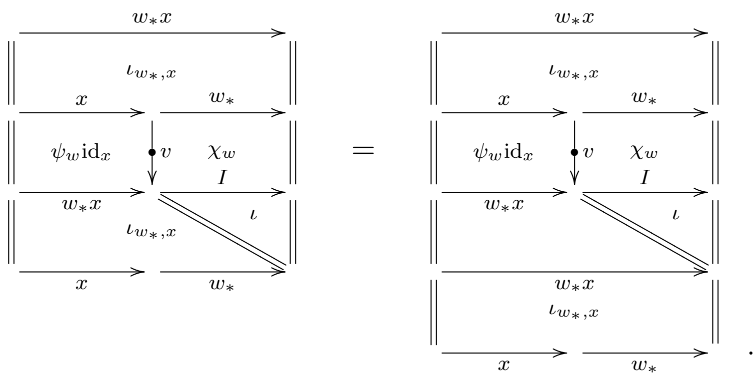



(32)

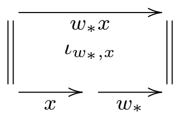
By coherence and (27) this is equal to



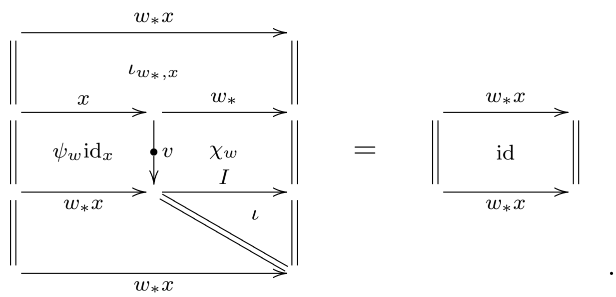
By (29) the composite (32) is also equal to




By the universal property of




this implies that




Substituting the image under $F$ of this into (31), we get that that composite is equal to $\gamma$. So we conclude that $\bar{\beta} = \gamma$ as desired. ■

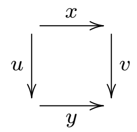
4.23. Corollary. *Any normal morphism between gregarious oplax double categories preserves all composites of the form $w_*v^*$ for vertical arrows $w$ and $v$ with a common domain.*

4.24. The Main Theorem. We will now show that $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ is the universal gregarious oplax double category in the following sense.

4.25. Theorem. *Let $\mathbb{B}$ be a gregarious oplax double category. Composition with the vertical inclusion $\mathbf{A} \hookrightarrow \mathbb{S}\mathrm{pan}\,\mathbf{A}$ induces an equivalence of categories between the category of normal morphisms $\mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$ and (vertical) transformations and the category of functors $\mathbf{A} \to \mathrm{Vrt}\,\mathbb{B}$ and natural transformations,*

$$\mathbf{GregOplax}(\mathbb{S}\mathrm{pan}\,\mathbf{A}, \mathbb{B}) \simeq \mathbf{Cat}\,(\mathbf{A}, \mathrm{Vrt}\,\mathbb{B}).$$

Before we prove this theorem we want to introduce the canonical cells associated with a commutative square of vertical arrows

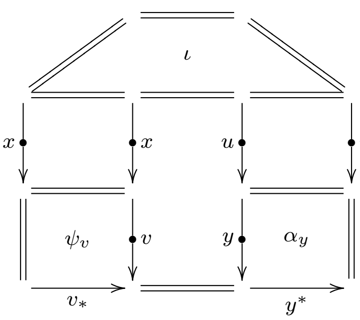



in a gregarious oplax double category. We first reintroduce some of the cells from (9). Note that both $\zeta(x, u, v, y)$ and $\xi(x, u, v, y)$ can be defined using the special horizontal composition of square cells defined in (25):

$$\zeta(x, v, u, y) = (\alpha_y \cdot 1_u)(1_v \cdot \beta_x) \text{ and } \xi(x, v, u, y) = (1_y \cdot \chi_u)(\psi_v \cdot 1_x).$$

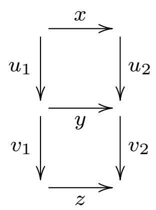
In addition to these cells, we will need the cells

$\Lambda(x, u, v, y) :=$




.

Note that for any commutative diagram of vertical arrows



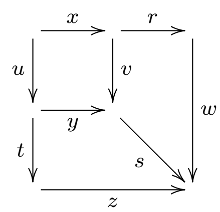

we have that

$$\zeta(y, v_1, v_2, z) \cdot \zeta(x, u_1, u_2, y) = \zeta(x, v_1 \cdot u_1, v_2 \cdot u_2, z), \tag{33}$$

and

$$\xi(y, v_1, v_2, z) \cdot \xi(x, u_1, u_2, y) = \xi(x, v_1 \cdot u_1, v_2 \cdot u_2, z). \tag{34}$$

Also, for a commutative diagram of vertical arrows




it is straightforward to check that

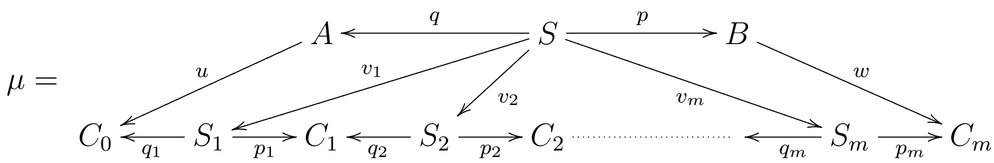
$$(\zeta(y, t, s, z), \xi(v, s, r, w)) \cdot \Lambda(x, u, v, y) = \Lambda(r \cdot x, t \cdot u, w, z). \tag{35}$$

PROOF. (of Theorem 4.25) Since $\mathbb{B}$ is gregarious, start by choosing companions and conjoints with their binding cells for all vertical arrows. For vertical identities choose $(\mathrm{id}_A)^* = 1_A = (\mathrm{id}_A)_*$ with $\iota_A$ as binding cells. Also choose the composites $w_* v^*$ for all spans of vertical arrows.

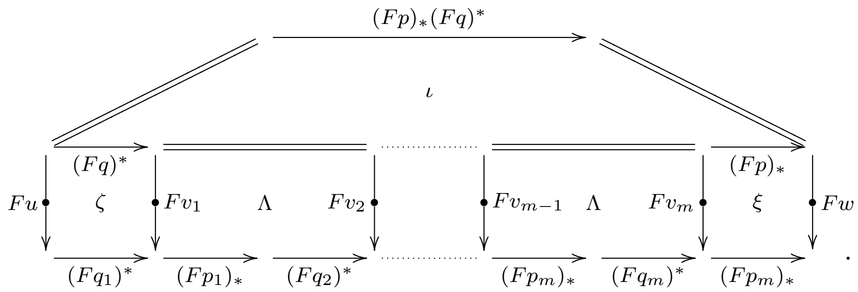
To show that the functor defined by composition is essentially surjective, let $F\colon \mathbf{A} \to \mathrm{Vrt}\,\mathbb{B}$ be any functor. We will now construct its extension to a normal morphism of oplax double categories $\tilde{F}\colon \mathbb{S}\mathrm{pan}\,\mathbf{A} \to \mathbb{B}$. Obviously, on objects and vertical arrows $\tilde{F}$ has the same values as $F$.

On horizontal arrows, $\tilde{F}\left(\xleftarrow{\;q\;}\xrightarrow{\;p\;}\right) = (Fp)_*(Fq)^*$ (the chosen representative for this strongly representable horizontal composition).

The image $\tilde{F}(\mu)$ of a cell


(36)

under $\tilde{F}$ is defined as



It is straightforward to check that $\tilde{F}$ is normal.

It is obvious that composition with the inclusion of $\mathbf{A}$ into $\mathbb{S}\mathrm{pan}\,\mathbf{A}$ defines a faithful functor $\mathbf{GregOplax}(\mathbb{S}\mathrm{pan}\,\mathbf{A}, \mathbb{B}) \to \mathbf{Cat}\,(\mathbf{A}, \mathrm{Vrt}\,\mathbb{B})$. To show that it is also full, let $\gamma\colon F \to G$ be a natural transformation. We will now construct its extension $\tilde{\gamma}\colon \tilde{F} \to \tilde{G}$. For objects $A$ in $\mathbb{S}\mathrm{pan}\,\mathbf{A}$, $\tilde{\gamma}_A = \gamma_A$, taken as a vertical arrow. For a span $A \xleftarrow{\;q\;} S \xrightarrow{\;p\;} B$, define

$$\tilde{\gamma}_{(q,S,p)} = \xi(Fp, \gamma_S, \gamma_B, Gp)\zeta(Fq, \gamma_S, \gamma_A, Gq),$$

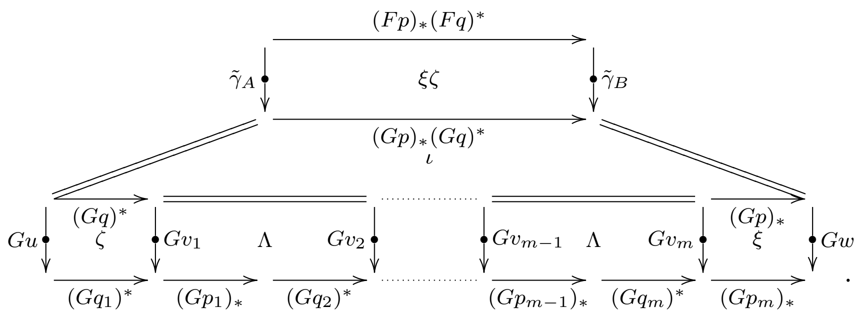
*i.e.*, the special horizontal composition of

$$\begin{array}{ccccc} FA & \xrightarrow{(Fq)^*} & FS & \xrightarrow{(Fp)_*} & FB \\ \gamma_A \big\downarrow & \zeta & \gamma_S \big\downarrow & \xi & \big\downarrow \gamma_B \\ GA & \xrightarrow[(Gq)^*]{} & GS & \xrightarrow[(Gp)_*]{} & GB \end{array} .$$

To verify vertical naturality of $\tilde{\gamma}$, we need to show that

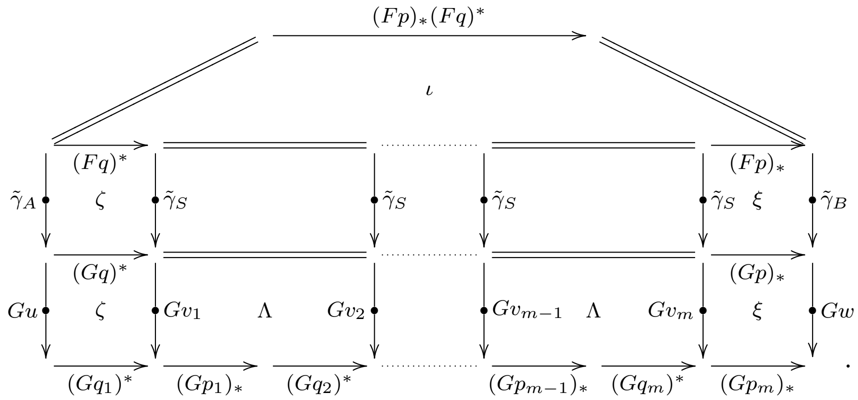
$$\tilde{G}(\mu) \cdot \tilde{\gamma}_{(q,S,p)} = (\tilde{\gamma}_{(q_m,S_m,p_m)}, \cdots, \tilde{\gamma}_{(q_1,S_1.p_1)}) \cdot \tilde{F}(\mu) \tag{37}$$

for any cell $\mu$ as in (36). The left-hand side of this equation is the following cell:




By the definition of special horizontal composition in (25), this can be rewritten as

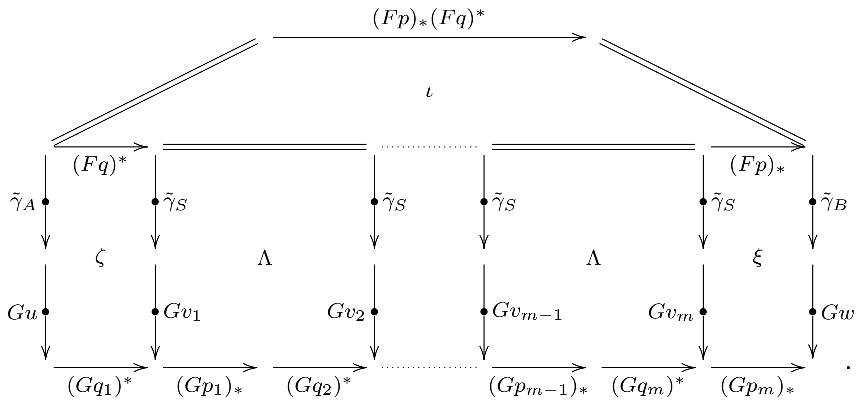
(38)




Note that

$$\Lambda(Gv_i, Gv_{i+1}, Gp_i, Gq_{i+1}) \cdot 1_{\tilde{\gamma}_S} = \Lambda(Gv_i \cdot \tilde{\gamma}_S, Gv_{i+1} \cdot \tilde{\gamma}_S, Gp_i, Gq_{i+1}).$$

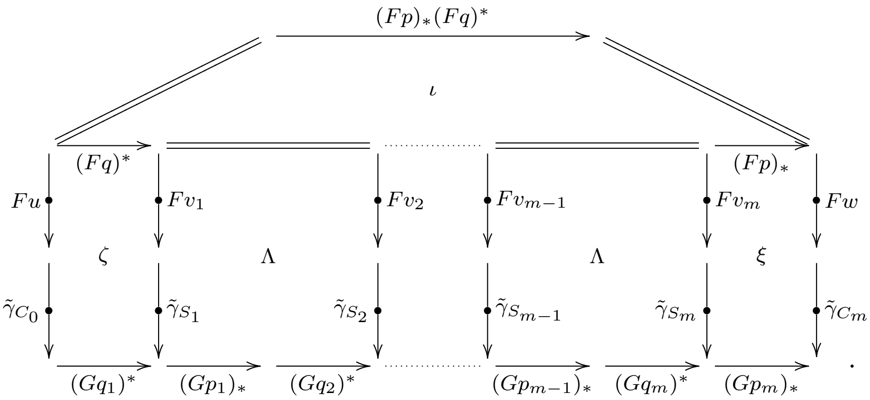
Using this result, together with (33) and (34), we can rewrite (38) as




Since $Gu \cdot \tilde{\gamma}_A = \tilde{\gamma}_{C_0} \cdot FU$, $Gw \cdot \tilde{\gamma}_B = \tilde{\gamma}_{C_m} \cdot Fw$, and

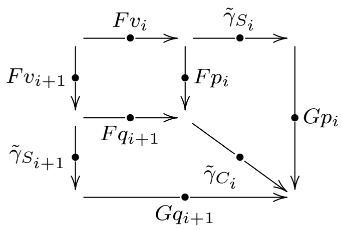
$$\Lambda(Gv_i \cdot \tilde{\gamma}_S, Gv_{i+1} \cdot \tilde{\gamma}_S, Gp_i, Gq_{i+1}) = \Lambda(\tilde{\gamma}_{S_i} \cdot Fv_i, \tilde{\gamma}_S \cdot Fv_{i+1}, Gp_i, Gq_{i+1})$$

we can further rewrite this as

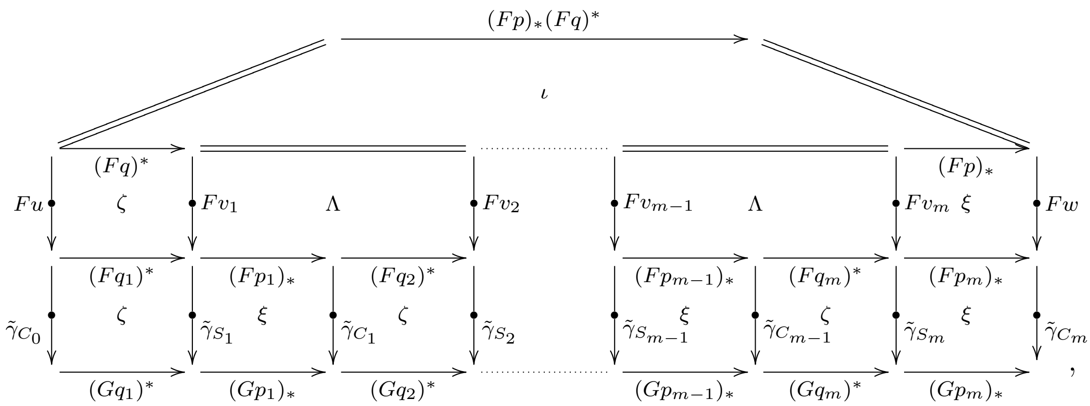



(39)

Now we use (35) for the commutative diagram




together with (33) and (34) to rewrite (39) as




and this is the cell on the right-hand side of (37). So $\tilde{\gamma}$ forms a vertical natural transformation from $\tilde{F}$ to $\tilde{G}$. This concludes the proof of Theorem 4.25. ∎

4.26. Corollary. $\mathbb{S}$pan: **Cat** → **GregOplax** *is locally fully faithful.*

4.27. Observation. Since the embedding $\mathbf{GregDoub}_{\mathrm{OplaxN}} \hookrightarrow$ **GregOplax** is 2-full and faithful, we can also view Theorem 3.15 as a corollary of Theorem 4.25. This gives us an alternative proof for Theorem 3.15, although at the cost of introducing additional concepts which are not needed in the context of that proof.

4.28. Free Adjoints Revisited. In this section we combine our $\mathbb{S}\mathrm{pan}$ construction with the $\mathbb{P}\mathrm{ath}_*$ construction of [12] to get a double category version of the $\Pi_2$ construction of [14]. It will be important to distinguish between strict double categories and weak double categories and between strict and strong morphisms of double categories. The universal properties of $\mathbb{P}\mathrm{ath}$ and $\mathbb{P}\mathrm{ath}_*$ are with respect to strict double categories and strict morphisms between them, whereas the universal property of $\mathbb{S}\mathrm{pan}$ is of necessity the weak version. To combine the two constructions they should be of the same type. However, the strong version does not follow from the weak one and neither does the weak property automatically follow from the strong one.

In order to get the weak property from the strong one, we will use two strictification results. The first is a variation on the coherence theorem for bicategories [27]. Let $\mathbb{B}$ be a weak double category and define $\bar{\mathbb{B}}$, the *strictification* of $\mathbb{B}$, to have the same objects and vertical arrows as $\mathbb{B}$, but with paths of horizontal arrows in $\mathbb{B}$ as its horizontal arrows. A cell

$$\begin{array}{ccccccccc} B_0 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_2} & B_2 & \xrightarrow{b_3} & \cdots & \xrightarrow{b_n} & B_n \\ {\scriptstyle x}\big\downarrow\!\bullet & & & & \alpha & & & & \bullet\!\big\downarrow{\scriptstyle y} \\ C_0 & \xrightarrow[c_1]{} & C_1 & \xrightarrow[c_2]{} & C_2 & \xrightarrow[c_3]{} & \cdots & \xrightarrow[c_m]{} & C_m \end{array}$$

is a cell in $\mathbb{B}$ of the form

$$\begin{array}{ccc} B_0 & \xrightarrow{\prod b_i} & B_n \\ {\scriptstyle x}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle y} \\ C_0 & \xrightarrow[\prod c_j]{} & C_m \end{array} \quad ,$$

where

$$\prod_{i=1}^{n} b_i = \begin{cases} 1_{B_0} & \text{if} \quad n = 0 \\ b_1 & \text{if} \quad n = 1 \\ b_n \left(\prod_{i=1}^{n-1} b_i\right) & \text{if} \quad n > 1. \end{cases}$$

So $\prod_{i=1}^{n} b_i = b_n(\cdots(b_3(b_2b_1)\cdots)$, *i.e.*, all brackets are at the right. Vertical composition of cells in $\bar{\mathbb{B}}$ is the same as in $\mathbb{B}$. Horizontal compostion of paths is by concatenation, whereas horizontal composition of cells uses the associativity isomorphisms of $\mathbb{B}$. The coherence theorem for bicategories says that there are canonical isomorphisms (special cells)

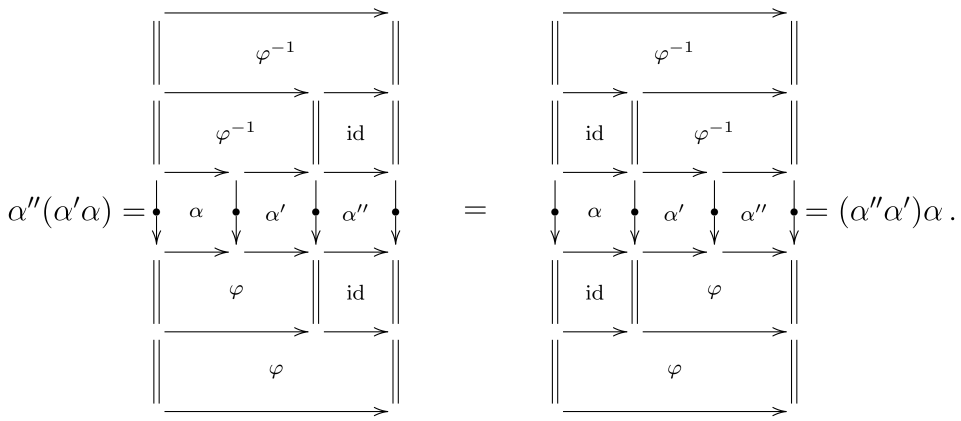
$$\varphi_{m,n} \colon \left(\prod_{i=n+1}^{n+m} b_i\right)\left(\prod_{i=1}^{n} b_i\right) \to \prod_{i=1}^{n+m} b_i$$

which satisfy all reasonable commutation properties, in particular the obvious associativity and unit laws. Then the horizontal composition of $\bar{\mathbb{B}}$ is given by

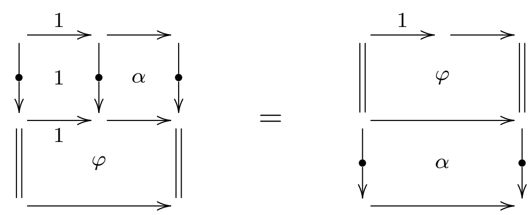
$$
\begin{array}{ccccc}
B_0 & & \xrightarrow{\prod b_i} & & B_{m+n} \\
\| & & \varphi_{m,n}^{-1} & & \| \\
B_0 & \xrightarrow{\prod b_i} & B_n & \xrightarrow{\prod b_i} & B_{m+n} \\
\downarrow\!\bullet & \alpha & \downarrow\!\bullet & \alpha' & \downarrow\!\bullet \\
C_0 & \xrightarrow[\prod c_j]{} & C_p & \xrightarrow[\prod c_j]{} & C_{p+q} \\
\| & & \varphi_{p,q} & & \| \\
C_0 & & \xrightarrow[\prod c_j]{} & & C_{p+q}
\end{array} \quad .
$$

4.29. THEOREM. *With these operations, $\bar{\mathbb{B}}$ is a strict double category which is equivalent to $\mathbb{B}$ in the 2-category* **psDoub** *of pseudo double categories, pseudo functors and vertical transformations.*

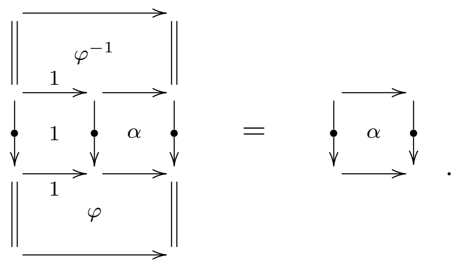
PROOF. Vertical composition, being the same as in $\mathbb{B}$, gives no problem. Interchange is easy - the $\varphi_{p,q}$ and $\varphi_{p,q}^{-1}$ in the middle cancel. Horizontal composition of arrows, being concatenation, is strictly associative. The only thing to check are the associativity and unit laws for horizontal composition of cells. Associativity is represented schematically by




The unit law from $\mathbb{B}$ says that



which implies that



There are pseudo functors $\Phi\colon \mathbb{B} \to \bar{\mathbb{B}}$ and $\Psi\colon \bar{\mathbb{B}} \to \mathbb{B}$ which are the identity on objects and vertical arrows. For a horizontal morphism $f\colon B \to B'$ in $\mathbb{B}$, $\Phi(f) = (\,B \xrightarrow{f} B'\,)$ the path of length 1. $\Phi$ on cells is the identity. $\Psi$ of a path $B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} B_n$ is $\prod f_i\colon B_0 \to B_n$ and $\Psi$ of a cell is the same cell in $\mathbb{B}$ now considered as a cell with vertical domain and codomain $\prod f_i$ and $\prod g_j$ rather than the paths $\langle f_i\rangle$ and $\langle g_j\rangle$.

That $\Phi$ and $\Psi$ are pseudo functors is easy to verify. The isomorphisms $\Phi(f'f) \to \Phi(f')\Phi(f)$ and $\Phi(1_B) \to 1_{\Phi B}$ are special cells in $\bar{\mathbb{B}}$ coming from identities in $\mathbb{B}$, whereas the isomorphisms $\Psi\langle f_i\rangle \to \Psi\langle f'_j\rangle \to \Psi(\langle f_i\rangle\langle f'_j\rangle)$ are the $\varphi_{p,q}$ above and $1_B \to \Psi\langle\ \rangle$ is the identity.

The composite $\Psi\Phi$ is obviously the identity on $\mathbb{B}$ but the isomorphism $\Phi\Psi \cong 1_{\bar{\mathbb{B}}}$ can use some elaboration. This isomorphism is given by the vertical transformations $t\colon \Phi\Psi \longrightarrow 1_{\bar{\mathbb{B}}}$ and $u\colon 1_{\bar{\mathbb{B}}} \longrightarrow \Phi\Psi$, defined by $t(B) = \mathrm{id}_B\colon B \rightarrow\!\!\!\!\!\bullet\;\, B$ and $t(\langle f_i\rangle)$ is

$$\begin{array}{ccccccc}
B_0 & & & \xrightarrow{\prod f_i} & & & B_n \\
\| & & & \mathrm{id}_{\prod f_i} & & & \| \\
B_0 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[f_2]{} & \cdots & \xrightarrow[f_n]{} & B_n
\end{array} \quad .$$

It is straightforward to check that $t$ is a vertical transformation with inverse $u$ defined analogously. ∎

4.30. Corollary. *For any pseudo double category $\mathbb{A}$, $\Phi$ and $\Psi$ induce an equivalence of categories*

$$\mathbf{psDoub}(\mathbb{A}, \mathbb{B}) \simeq \mathbf{psDoub}(\mathbb{A}, \bar{\mathbb{B}}).$$

4.31. Remarks.

1. $\bar{\mathbb{B}}$ is the object part of a 2-functor $(\bar{\ })\colon$ **psDoub** $\to$ **stDoub** which is a 2-left adjoint to the inclusion **stDoub** $\to$ **psDoub**. It is not, however, a 2-equivalence or even a biequivalence as one might be tempted to believe. This is because for strict $\mathbb{B}$, although $\Phi\colon \mathbb{B} \to \bar{\mathbb{B}}$ is an equivalence, it is not an equivalence in **stDoub**. On the other hand, we do get a biequivalence between **psDoub** and its full sub 2-category determined by the strict double categories.

2. If $\mathbb{B}$ is gregarious then so is $\bar{\mathbb{B}}$.

$\mathbb{P}\mathrm{ath}\,\mathbb{A}$, $\mathbb{P}\mathrm{ath}_*\mathbb{A}$, and $\bar{\mathbb{B}}$ all share the property that they are strict double categories whose horizontal categories are free on a graph. This is at the heart of our next strictification result.

4.32. Proposition. *Let $\mathbb{A}$ and $\mathbb{B}$ be strict double categories and assume that the category $Hor(\mathbb{A})$ is free on a graph. Then the inclusion*

$$\mathbf{stDoub}(\mathbb{A},\mathbb{B}) \hookrightarrow \mathbf{psDoub}(\mathbb{A},\mathbb{B})$$

*is an equivalence of categories.*

Proof. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a pseudo double functor. Define $\overline{F}\colon \mathbb{A} \to \mathbb{B}$ to be the same as $F$ on objects and vertical arrows. A horizontal morphism in $\mathbb{A}$ is represented by a unique path $A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} A_n$ of edges in the generating graph. Define $\overline{F}(\langle a_i \rangle)$ to be the composite $F(a_n)\cdots F(a_1)$ in $\mathbb{B}$. This makes $\overline{F}$ strict at the level of horizontal arrows. For a cell

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{a_1} & A_1 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_n} & A_n \\
\big\downarrow & & & \alpha & & & \big\downarrow \\
A'_0 & \xrightarrow[a'_1]{} & A'_1 & \xrightarrow[a'_2]{} & \cdots & \xrightarrow[a'_m]{} & A'_m
\end{array} \quad ,$$

define $\overline{F}(\alpha)$ to be

$$\begin{array}{ccccccc}
FA_0 & \xrightarrow{Fa_1} & FA_1 & \xrightarrow{Fa_2} & \cdots & \xrightarrow{Fa_n} & FA_n \\
\| & & & \varphi & & & \| \\
FA_0 & & & \xrightarrow{F(a_n\cdots a_1)} & & & FA_n \\
\big\downarrow & & & F\alpha & & & \big\downarrow \\
FA'_0 & & & \xrightarrow{F(a'_m\cdots a'_1)} & & & FA'_m \\
\| & & & \varphi^{-1} & & & \| \\
FA'_0 & \xrightarrow[Fa'_1]{} & FA'_1 & \xrightarrow[Fa'_2]{} & \cdots & \xrightarrow[Fa'_m]{} & FA'_m
\end{array}$$

where $\varphi$ is the structural isomorphism.

It is straightforward to check that vertical transformations from $F$ to $G$ are in bijective correspondence with vertical transformations between $\overline{F}$ and $\overline{G}$, since they are completely determined by their components for the edges of the generating graph. ∎

4.33. Remark. In [16] such double categories were shown to be the flexible algebras for a 2-monad on **Cat**(**Graph**), of internal categories in the category **Graph** of non-reflexive directed graphs, whose strict algebras are (strict) double categories. The result above can also be seen as fitting in with known results about flexible algebras: strict morphisms from a flexible algebra $A$ to a strict algebra $B$ are in bijective correspondence with pseudo morphisms from $A$ to $B$ (see also [3]).

Recall from [12] that the universal properties of $\mathbb{P}\mathrm{ath}$ and $\mathbb{P}\mathrm{ath}_*$ are given by 2-adjunctions. Involved in these adjunctions are **Oplax**, the 2-category of oplax double categories, oplax morphisms and vertical transformations, and **OplaxN**, its locally full sub 2-category of normal oplax double categories and normal morphisms. Then $\mathbb{P}\mathrm{ath}\colon \mathbf{Oplax} \to \mathbf{stDoub}$ is left 2-adjoint to the 'inclusion' $\mathbb{O}\mathrm{plax}\colon \mathbf{stDoub} \to \mathbf{Oplax}$ and $\mathbb{P}\mathrm{ath}_*\colon \mathbf{OplaxN} \to \mathbf{stDoub}$ is left 2-adjoint to the 'inclusion' $\mathbb{O}\mathrm{plax}_*\colon \mathbf{stDoub} \to \mathbf{OplaxN}$. This means that if $\mathbb{A}$ is oplax and $\mathbb{B}$ strict we have an isomorphism of categories

$$\mathbf{stDoub}(\mathbb{P}\mathrm{ath}\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{Oplax}(\mathbb{A}, \mathbb{B}),$$

and if $\mathbb{A}$ is oplax normal, we have an isomorphism of categories,

$$\mathbf{stDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\cong} \mathbf{OplaxN}(\mathbb{A}, \mathbb{B}).$$

The 'inclusions' $\mathbb{O}\mathrm{plax}$ and $\mathbb{O}\mathrm{plaxN}$ extend to **psDoub** in the obvious way.

4.34. Theorem. *The functors* $\mathbb{P}\mathrm{ath}\colon \mathbf{Oplax} \to \mathbf{psDoub}$ *and* $\mathbb{P}\mathrm{ath}_*\colon \mathbf{OplaxN} \to \mathbf{psDoub}$ *satisfy the following universal properties. If* $\mathbb{A}$ *is oplax and* $\mathbb{B}$ *is a pseudo double category, then the morphism* $\mathbb{A} \to \mathbb{P}\mathrm{ath}\mathbb{A}$ *induces, by composition, an equivalence of categories*

$$\mathbf{psDoub}(\mathbb{P}\mathrm{ath}\mathbb{A}, \mathbb{B}) \xrightarrow{\simeq} \mathbf{Oplax}(\mathbb{A}, \mathbb{B})$$

*and if* $\mathbb{A}$ *is oplax normal, composition with* $\mathbb{A} \to \mathbb{P}\mathrm{ath}_*\mathbb{A}$ *induces an equivalence of categories*

$$\mathbf{psDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) \xrightarrow{\simeq} \mathbf{OplaxN}(\mathbb{A}, \mathbb{B})$$

Proof. The proofs are similar, so we only do the case of $\mathbb{P}\mathrm{ath}_*$ as it is the one we will use. We get the following equivalences

$$\begin{array}{rcll} \mathbf{psDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \mathbb{B}) & \xrightarrow{\simeq} & \mathbf{psDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \overline{\mathbb{B}}) & \text{by Corollary 4.30} \\ & \xrightarrow{\simeq} & \mathbf{stDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{A}, \overline{\mathbb{B}}) & \text{by the univ. prop. of } \mathbb{P}\mathrm{ath}_* \\ & \xrightarrow{\simeq} & \mathbf{OplaxN}(\mathbb{A}, \overline{\mathbb{B}}) & \text{by composition with } \Psi. \end{array}$$

■

4.35. Theorem. *Let* $\mathbf{A}$ *be a category. The composite* $\mathbf{A} \to \mathbb{S}\mathrm{pan}\mathbf{A} \to \mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}\mathbf{A}$ *induces, by composition, an equivalence of categories,*

$$\mathbf{GregDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}\mathbf{A}, \mathbb{B}) \xrightarrow{\simeq} \mathbf{Cat}(\mathbf{A}, \mathrm{Vrt}\,(\mathbb{B})),$$

*for any gregarious pseudo double category* $\mathbb{B}$*, i.e.,* $\mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}\colon \mathbf{Cat} \to \mathbf{GregDoub}$ *is a left biadjoint of* $\mathrm{Vrt}: \mathbf{GregDoub} \to \mathbf{Cat}$.

Proof. By the previous theorem,

$$\mathbf{psDoub}(\mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}\mathbf{A}, \mathbb{B}) \xrightarrow{\simeq} \mathbf{OplaxN}(\mathbb{S}\mathrm{pan}\mathbf{A}, \mathbb{B}).$$

The oplax double category $\mathbb{S}\mathrm{pan}\mathbf{A}$ is gregarious and normal morphisms between gregarious double categories are automatically gregarious, so

$$\mathbf{OplaxN}(\mathbb{S}\mathrm{pan}\mathbf{A}, \mathbb{B}) = \mathbf{GregOplax}(\mathbb{S}\mathrm{pan}\mathbf{A}, \mathbb{B}) \xrightarrow{\simeq} \mathbf{Cat}(\mathbf{A}, \mathrm{Vrt}\,(\mathbb{B})),$$

by Theorem 4.25. ■

We denote the composition of functors $\mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}$ by $\mathbb{I}\mathbb{I}_2$. It is the double category version of the $\Pi_2$-construction studied in [14]. When we take the bicategory of horizontal arrows and special cells in $\mathbb{P}\mathrm{ath}_*\mathbb{S}\mathrm{pan}\mathbf{A}$ we obtain a bicategory which is biequivalent to $\Pi_2\mathbf{A}$ as introduced before.

ACKNOWLEDGEMENT. We would like to thank the anonymous referee for various helpful comments.

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