

ITERATED ICONS

EUGENIA CHENG AND NICK GURSKI

ABSTRACT.

We study the totality of categories weakly enriched in a monoidal bicategory using a notion of enriched icon as 2-cells. We show that when the monoidal bicategory in question is symmetric then this process can be iterated. We show that starting from the symmetric monoidal bicategory **Cat** and performing the construction twice yields a convenient symmetric monoidal bicategory of partially strict tricategories. We show that restricting to the doubly degenerate ones immediately gives the correct bicategory of “2-tuply monoidal categories” missing from our earlier studies of the Periodic Table. We propose a generalisation to all k -tuply monoidal n -categories.

Introduction

In this paper we use an iterated icon construction to study the degenerate n -categories that appear in the Periodic Table. Unless otherwise specified, the term n -category should be interpreted as weak n -category. We began our analysis of the Periodic Table of n -categories in [5] (degenerate categories and bicategories) and [7] (degenerate tricategories). A k -degenerate n -category is one in which the lowest k dimensions are degenerate, that is, contain only one cell. In this case the lowest non-trivial dimension is k , and we can perform a “dimension shift” to forget the degenerate dimensions and regard the k -cells as the 0-cells of an $(n - k)$ -category. These cells come equipped with k monoidal structures coming from the k types of composition they had in the old structure; thus a degenerate category is regarded as a monoid, and a degenerate bicategory as a monoidal category. In general we have the following shift:

$$\begin{array}{ll} k\text{-degenerate } m\text{-category} & \longrightarrow k\text{-monoidal } (m - k)\text{-category} \\ \text{or } k\text{-degenerate } (n + k)\text{-category} & \longrightarrow k\text{-monoidal } n\text{-category.} \end{array}$$

The first type of indexing is more natural if one is starting with a fixed m and varying the amount of degeneracy as we did in [5, 7]. The second form is more appropriate if one is fixing the *codimension* and varying the number of monoidal structures, which is what we will end up doing in the present work.

The periodic table of n -categories as proposed by Baez and Dolan [1] gives predictions of what sort of structures should arise as k -monoidal n -categories. The idea is that as the

Received by the editors 2013-10-18 and, in revised form, 2014-11-27.

Transmitted by G. Rosolini. Published on 2014-12-22.

2010 Mathematics Subject Classification: Primary 18A05, 18D05, 18D10, 18D20.

Key words and phrases: symmetric monoidal bicategory, icon, enriched category.

© Eugenia Cheng and Nick Gurski, 2014. Permission to copy for private use granted.

k monoidal structures came from different types of composition in the original $(n + k)$ -category, the interchange laws from that structure give interactions between the different monoidal structures which, as a result, amount to weak “commutations” such as braidings and so on. The first few entries are given in the following table.

set	category	2-category	3-category	...
monoid ≡ category with only one object	monoidal category ≡ 2-category with only one object	monoidal 2-category ≡ 3-category with only one object	monoidal 3-category ≡ 4-category with only one object	...
commutative monoid ≡ 2-category with only one object only one 1-cell	braided monoidal category ≡ 3-category with only one object only one 1-cell	braided monoidal 2-category ≡ 4-category with only one object only one 1-cell	braided monoidal 3-category ≡ 5-category with only one object only one 1-cell	...
//	symmetric monoidal category ≡ 4-category with only one object only one 1-cell only one 2-cell	sytleptic monoidal 2-category ≡ 5-category with only one object only one 1-cell only one 2-cell	sytleptic monoidal 3-category ≡ 6-category with only one object only one 1-cell only one 2-cell	...
//	//	symmetric monoidal 2-category ≡ 6-category with only one object only one 1-cell only one 2-cell only one 3-cell	?	...
//	//	//	symmetric monoidal 3-category	...
⋮	⋮	⋮	⋮	

The process of fixing an overall dimension m and increasing the amount of degeneracy amounts to moving diagonally down-and-left in the table. The process of fixing codimension n and adding monoidal structures consists of moving down a column. The hypothesis is then that there is some kind of equivalence between the following three types of structure:

- k -degenerate n -categories

- k -tuply monoidal $(n - k)$ -categories
- the (n, k) th entry of the Periodic Table.

One of the lessons learnt from [5, 7] is that we must take care about the precise formulation of these equivalences. For example, we wish to compare the totality of commutative monoids with the totality of doubly degenerate bicategories, but the former naturally form a category whereas the latter form a tricategory. Now, bicategories do form a category but this is a sort of “accident” and does not generalise into higher dimensions; n -categories and weak maps in the usual sense do not form a category for any higher n , for algebraic definitions. (Garner [8] constructs a category of n -categories using a different type of weak map.) Moreover, the category of doubly degenerate bicategories is in any case not equivalent to the category of commutative monoids, as shown in [5].

The general problem is that we are trying to compare a totality of k -degenerate $(n+k)$ -categories, which is naturally an $(n + k + 1)$ -dimensional structure, with a totality of k -monoidal n -categories, which is naturally an $(n + 1)$ -dimensional structure.

In [5] we investigated ways of

- decreasing the dimensions of the former, by forgetting higher-dimensional maps, or
- increasing the dimensions of the latter, by adding identities as higher cells.

We did this in a somewhat ad hoc way on a case-by-case basis, with mixed results.

We are now able to propose a much more systematic approach based on Lack’s theory of icons [15, 17]. The aim of Lack’s work was to construct a “convenient 2-category of bicategories”. It is well-known that bicategories and homomorphisms form a category, and that bicategories, homomorphisms, transformations and modifications form a tricategory, but simply forgetting the modifications does not give us a 2-category as the transformations do not compose coherently enough.

Lack’s solution is to restrict to “Identity Component Oplax Natural transformations” (ICONS), and re-arrange their composition a little. This gives a 2-dimensional totality of bicategories that is not only coherent (i.e. it is a 2-category) but also convenient—it contains enough 2-dimensional information to express a strong form of the coherence theorem for bicategories.

The key, for our purposes, is that the resulting totality is correct for comparing degenerate bicategories with monoidal categories which, after all, naturally form a bicategory. That is, we consider **Icon**, the 2-category of bicategories, weak functors and icons. The full sub-2-category on the degenerate bicategories is then equivalent to the 2-category of monoidal categories. This is done by Lack [15].

We now generalise this construction in order to iterate it. In [9] various convenient lower-dimensional totalities of tricategories are constructed (that is, lower than the natural four dimensions) but none of these gives the right framework for comparing doubly degenerate tricategories with braided monoidal categories, for example.

Our approach is to consider bicategories as categories “weakly enriched” in **Cat**, and then generalise this to replace **Cat** with any symmetric monoidal bicategory \mathcal{B} . We then make a bicategory with

- 0-cells: weak \mathcal{B} -categories,
- 1-cells: weak \mathcal{B} -functors, and
- 2-cells: \mathcal{B} -icons.

We name this 2-category **\mathcal{B} -Icon** after its 2-cells. Our key theorem 3.8 shows that if \mathcal{B} is symmetric monoidal then so is **\mathcal{B} -Icon**, so that we can iterate the construction.

We then have the following inductive process: starting with $\mathcal{B}_0 = \mathbf{Cat}$ and putting $\mathcal{B}_{k+1} = \mathcal{B}_k\text{-Icon}$, we finally take the full sub-2-category on suitably degenerate objects. This gives correct totalities of monoidal categories as shown in the following schematic diagram.

$$\begin{array}{lcl}
 \mathcal{B}_0 & = & \mathbf{Cat} \\
 \mathcal{B}_1 & = & \mathcal{B}_0\text{-Icon} \xrightarrow{\text{degen}} \mathbf{MonCat} \\
 \mathcal{B}_2 & = & \mathcal{B}_1\text{-Icon} \xrightarrow{2\text{-degen}} \mathbf{BrMonCat} \\
 \mathcal{B}_3 & = & \mathcal{B}_2\text{-Icon} \xrightarrow{3\text{-degen}} \mathbf{SymMonCat} \\
 \mathcal{B}_4 & = & \mathcal{B}_3\text{-Icon} \xrightarrow{4\text{-degen}} \mathbf{SymMonCat} \\
 & & \vdots
 \end{array}$$

Note that the basic definition of **\mathcal{B} -Icon** only requires that \mathcal{B} be monoidal. However, in order to iterate the icon construction

$$\mathcal{B} \mapsto \mathcal{B}\text{-Icon},$$

we need **\mathcal{B} -Icon** to inherit the relevant structure of \mathcal{B} . We have the following structures.

$$\begin{array}{ccc}
 \mathcal{B} & \mapsto & \mathcal{B}\text{-Icon} \\
 \text{monoidal} & & \text{---} \\
 \text{braided monoidal} & & \text{monoidal} \\
 \text{symmetric monoidal} & & \text{symmetric monoidal}
 \end{array}$$

Thus we see that for the iteration to proceed, we need symmetric monoidal bicategories. This is the main technical content of this work. The work was first presented [6] but suffered some delay due to the technical complications in the construction of the symmetric monoidal bicategory **\mathcal{B} -Icon**. These were resolved in [13]. In the meantime, Shulman [20] gave an elegant abstract account of \mathcal{B} -icons via 2-monads, but did not focus on the totality.

His focus was on the 0-cells, and he did not study iteration, so he did not need a *symmetric* monoidal structure on \mathcal{B} , or on $\mathcal{B}\text{-Icon}$. Garner and Shulman [10] also study categories weakly enriched in a monoidal bicategory, but construct the tricategory of such, so study the enriched version of pseudo-natural transformations instead of icons.

Weak enrichment is at the centre of higher category theory and so has been studied by many different authors in different ways. Trimble studies weak enrichment via operad actions [21, 16] and this approach is further developed in [4] and [3]. There is a large body of work on weak enrichment via homotopy theory; we refer the reader to Simpson's comprehensive book [19].

We note that there is another point of view one could take when studying degenerate objects, namely that the unique lower-dimensional cells should be treated as basepoints as in topology, with all cells indexed on them required to be identities, but we do not pursue this idea here; for further discussion see [7].

This paper is structured as follows. In Section 1 we recall the basic theory of icons as in [15]. In Section 2 we recall the definitions of various kinds of monoidal bicategory (braided, sylleptic, symmetric), and briefly remind the reader of the relevant coherence theorems that we will use repeatedly in later proofs. Then in Section 3 we generalise the definitions of Section 1 to the enriched setting and prove that if \mathcal{B} is a symmetric monoidal bicategory then so is $\mathcal{B}\text{-Icon}$ so we can iterate the icon construction. We study this iteration in Section 4. We show that this gives correct totalities of degenerate structures to correspond with the second column of the Periodic Table, and make some conjectures about how to generalise this for the higher-dimensional columns.

We note that many of the structures defined in this paper use diagrams which, if written out fully, would each span pages and would likely fail to convey any meaning at all except to the most tricategorically-inclined readers. Thus we have taken a minimalist approach to our diagrams, only including the key cells or those cells which lend some useful geometry to the algebra. That we can do this is a consequence of the coherence theorems in Section 2, but the diligent reader should always be able to reconstruct the full diagrams, complete with all the necessary cells. Two such reconstructions might not exactly coincide, but coherence tells us that they will differ only by unique 2-cell isomorphisms.

1. Icons

In this section we recall Lack's definition of icon [15, 17]. The idea is to form a convenient 2-category of bicategories. Bicategories together with functors, transformations and modifications form a tricategory, and while truncating this to the level of functors does yield a category, truncating to the level of transformations does not yield a coherent 2-dimensional structure. This is because transformations between bicategories do not in general compose in a strictly associative or unital way, nor is interchange strict.

Lack's idea is to introduce a new type of transformation called *icon* which can be thought of as 'Identity Component Oplax Natural transformation'. However, composition

is not merely that of transformations—composing identity 1-cell components I_a would give a component $I_a \circ I_a$, not itself necessarily an identity.

Instead, Lack’s insight is to drop all reference to the 1-cell components of these “transformations” and only retain the 2-cell components as data. This results in the following definition.

1.1. DEFINITION. *Let X and Y be bicategories and let $F, G: X \rightarrow Y$ be weak functors such that $Fa = Ga$ for all objects $a \in X$. An **icon***

$$\begin{array}{ccc}
 & F & \\
 X & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & Y \\
 & G &
 \end{array}$$

is given by, for all pairs of 0-cells $a, b \in X$ a 2-cell

$$\begin{array}{ccc}
 & F & \\
 X(a, b) & \begin{array}{c} \curvearrowright \\ \alpha_{ab} \Downarrow \\ \curvearrowleft \end{array} & \begin{array}{c} Y(Fa, Fb) \\ Y(Ga, Gb) \end{array} \\
 & G &
 \end{array}$$

satisfying the following axioms. Note that to save space we will omit the name of the bi-category when writing hom-categories, thus $X(a, b)$ becomes (a, b) , and $Y(Fa, Fb)$ becomes (Fa, Fb) , and so on.

- *Composition:*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & FF & \\
 (b, c)(a, b) & \begin{array}{c} \curvearrowright \\ \alpha \Downarrow \\ \curvearrowleft \end{array} & (Fb, Fc)(Fa, Fb) \\
 \downarrow m & \begin{array}{c} GG \\ \phi^G \Downarrow \end{array} & \downarrow m' \\
 (a, c) & & (Ga, Gc) \\
 & G &
 \end{array}
 & = &
 \begin{array}{ccc}
 & FF & \\
 (b, c)(a, b) & & (Fb, Fc)(Fa, Fb) \\
 \downarrow m & \begin{array}{c} \phi^F \Downarrow \\ F \\ \alpha \Downarrow \end{array} & \downarrow m' \\
 (a, c) & & \begin{array}{c} (Fa, Fc) \\ (Ga, Gc) \end{array} \\
 & G &
 \end{array}
 \end{array}$$

- *Unit:*

$$\begin{array}{ccc}
 \mathbf{1} & \begin{array}{c} \nearrow I \\ \searrow I' \\ \text{\scriptsize } \phi^F \nearrow \\ \text{\scriptsize } \alpha \end{array} & \begin{array}{c} (a, a) \\ \text{\scriptsize } F \\ \text{\scriptsize } G \\ \text{\scriptsize } (Fa, Fa) = (Ga, Ga) \end{array} \\
 & & \\
 \mathbf{1} & \begin{array}{c} \nearrow I \\ \searrow I' \\ \text{\scriptsize } \phi^G \nearrow \\ \text{\scriptsize } \alpha \end{array} & \begin{array}{c} (a, a) \\ \text{\scriptsize } G \\ \text{\scriptsize } (Fa, Fa) = (Ga, Ga) \end{array}
 \end{array} =$$

Lack notes the following result.

1.2. THEOREM. *Bicategories, weak functors and icons form a 2-category.*

2. Background on monoidal bicategories

In this section we provide the full definition of symmetric monoidal bicategory, which is necessary for the constructions in Section 3. We build this up gradually. We first recall the definition of monoidal bicategory, which can also be found in [7] and [12]. We then define the notion of a braiding, as in [11], followed by syllepsis and symmetry as in [13] and [18]. While these definitions are not new, we feel it is useful to gather them all together in one place.

2.1. DEFINITION. A **monoidal bicategory** M consists of the following data subject to the following axioms:

DATA:

- an underlying bicategory, also denoted M ;
- a functor $\otimes : M \times M \rightarrow M$;
- a functor $I : \mathbf{1} \rightarrow M$, where $\mathbf{1}$ denotes the unit bicategory;
- an adjoint equivalence a

$$\begin{array}{ccc}
 M \times M \times M & \xrightarrow{\otimes \times 1} & M \times M \\
 1 \times \otimes \downarrow & \Downarrow a & \downarrow \otimes \\
 M \times M & \xrightarrow{\otimes} & M
 \end{array}$$

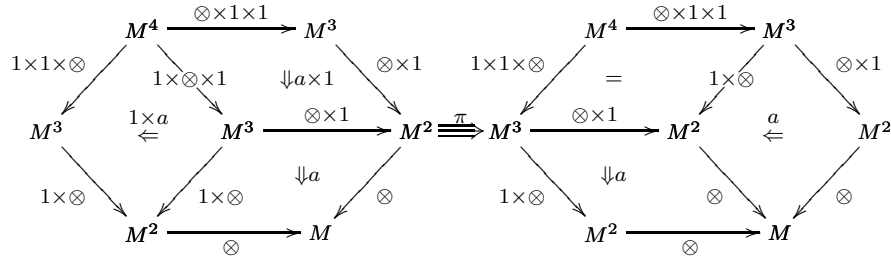
in $\mathbf{Bicat}(M^3, M)$;

- adjoint equivalences l and r

$$\begin{array}{ccc}
 & M \times M & \\
 I \times 1 \nearrow & & \searrow \otimes \\
 M & \xrightarrow{1} & M \\
 & \Downarrow l & \\
 & M \times M & \\
 1 \times I \nearrow & & \searrow \otimes \\
 M & \xrightarrow{1} & M \\
 & \Downarrow r &
 \end{array}$$

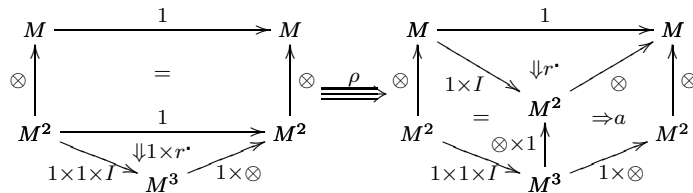
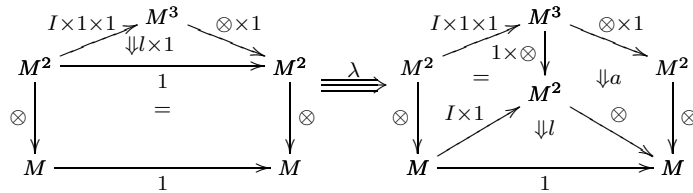
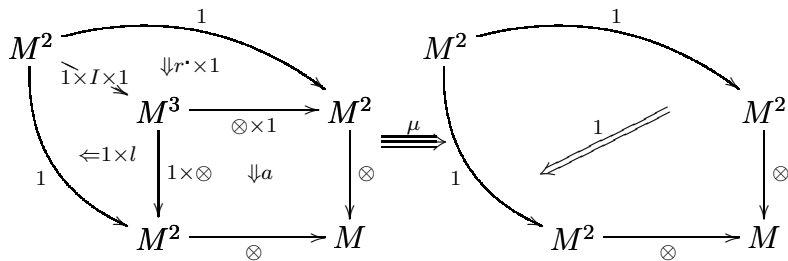
in $\mathbf{Bicat}(M, M)$;

- an invertible 2-cell π (i.e., an invertible modification)



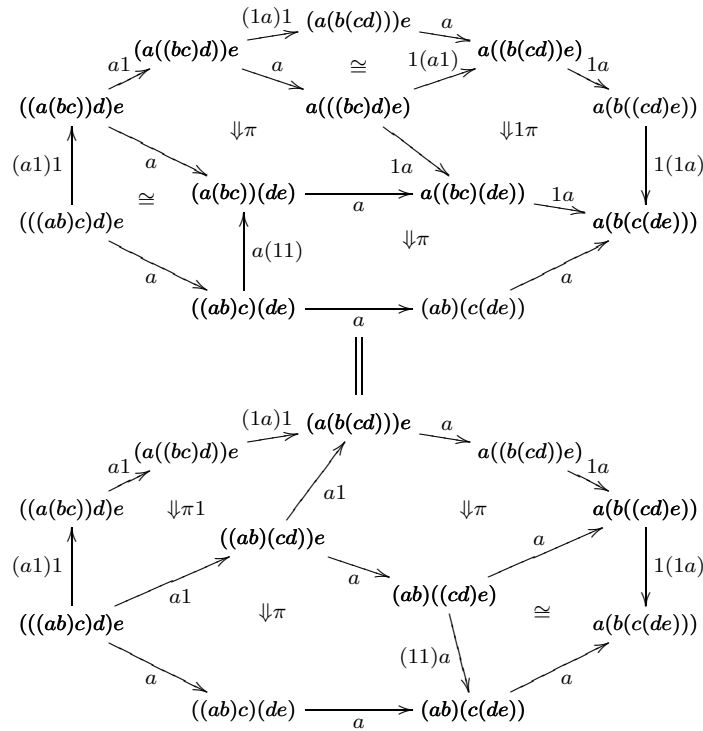
in the bicategory $\mathbf{Bicat}(M^4, M)$;

- invertible modifications

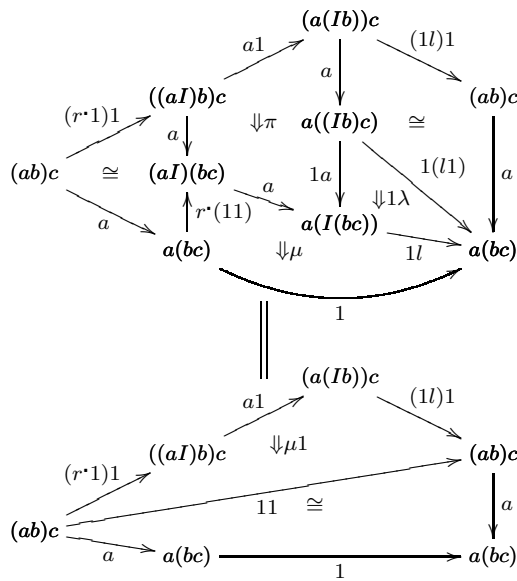


AXIOMS:

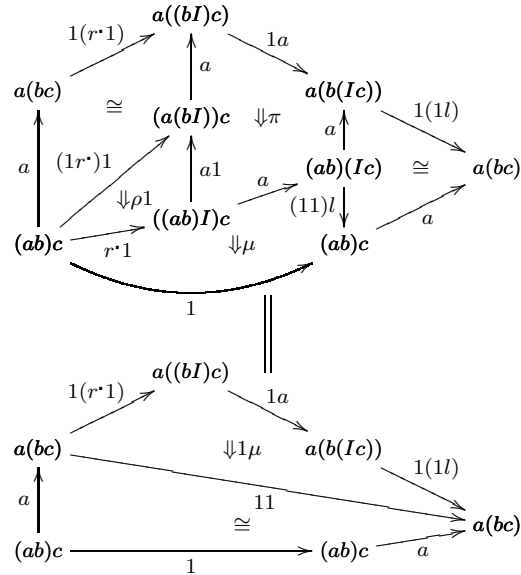
- The following equation of 2-cells holds for all objects a, b, c, d, e in M , where we have used parentheses instead of \otimes for compactness and the unmarked isomorphisms are naturality isomorphisms for a .



- The following equation of 2-cells holds for all objects a, b, c in M , where the unmarked isomorphisms are either naturality isomorphisms for a or unique coherence isomorphisms from the hom-bicategory.

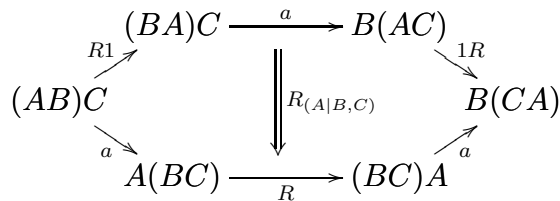


- The following equation of 2-cells holds for all objects a, b, c in M .

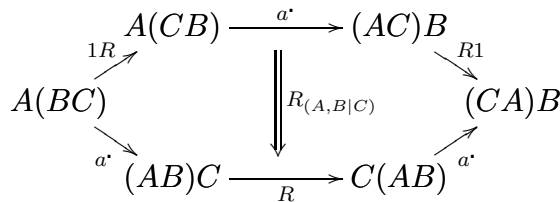


2.2. DEFINITION. Let $B = (B, \otimes, I, a, l, r, \pi, \mu, \rho, \lambda)$ be a monoidal bicategory. Then a **braiding** for B consists of

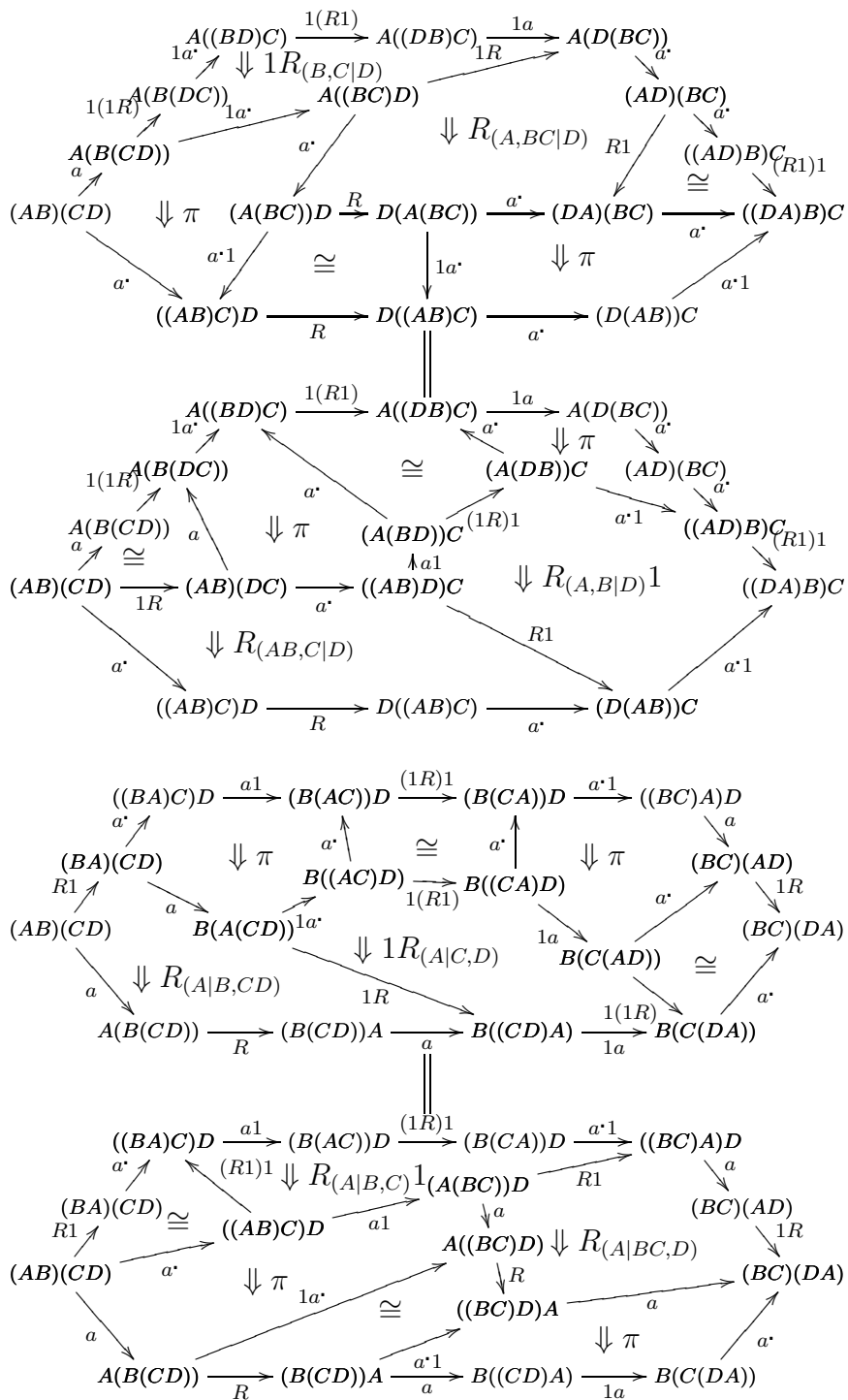
- an adjoint equivalence $\mathbf{R} : \otimes \Rightarrow \otimes \circ \tau$ in $\mathbf{Bicat}(B \times B, B)$, where we define $\tau : B \times B \rightarrow B \times B$ to interchange the coordinates;
- an invertible modification $R_{(-|-,-)}$ as displayed below;

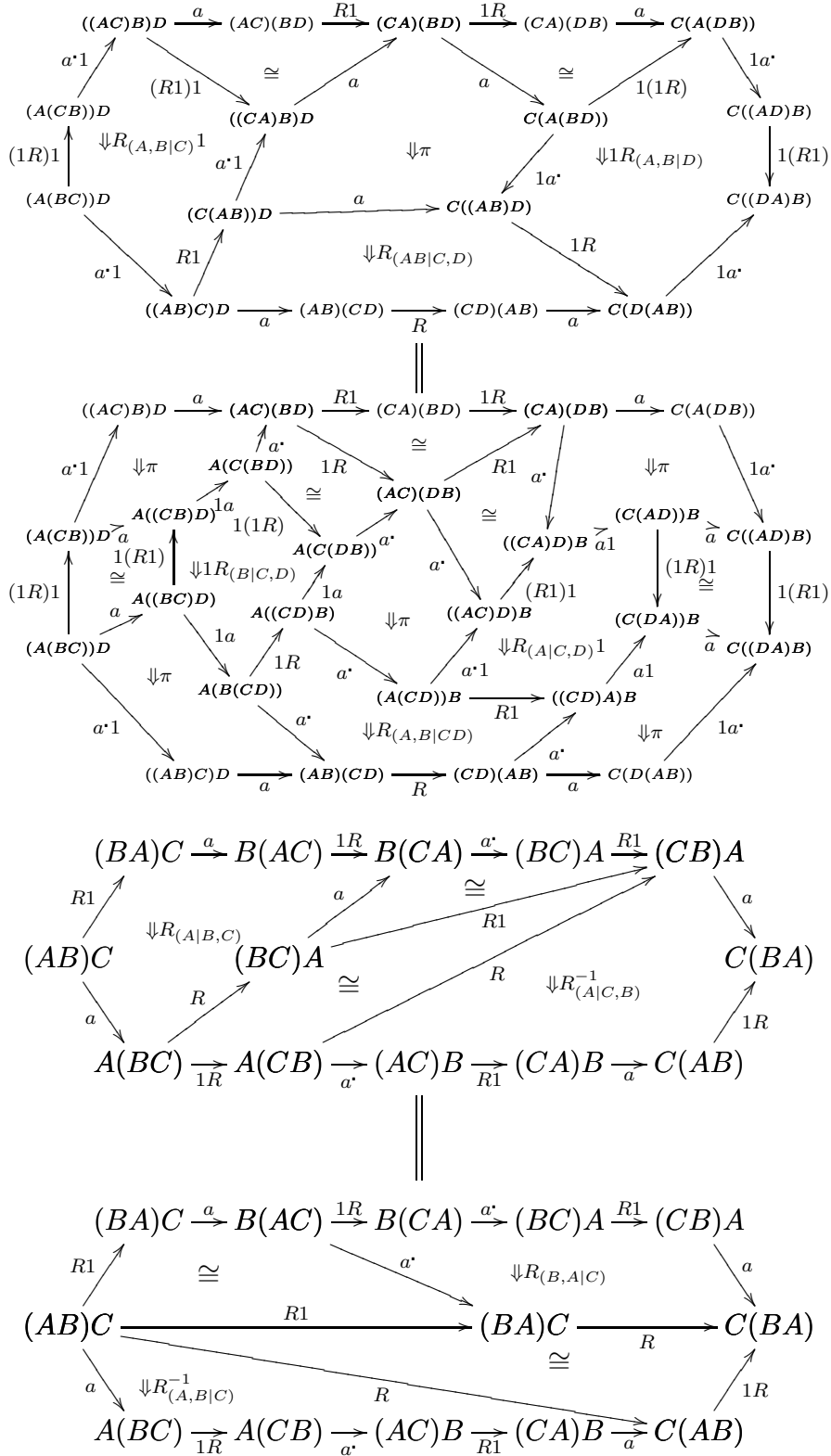


- and an invertible modification $R_{(-,-|-)}$ as displayed below;



all subject to the following four axioms.





A **braided monoidal bicategory** is a monoidal bicategory equipped with a braiding.

While these axioms might look quite daunting, they are just algebraic expressions of the notion that a pair of homotopies that each start at the braid γ and end at the braid γ' are in fact themselves homotopic. For instance, the first axiom concerns the case of a braid with four strands in which the first three strands are braided past the final one. This can be done either all at once (this is the 1-cell target of the pasting diagram) or it can be done step-by-step in which strand 3 is braided past strand 4, then strand 2 is braided past strand 4, and finally strand 1 is braided past strand 4 (this is the 1-cell source). The two different composite 2-cells which are claimed to be equal in this axiom are just two different ways to transform the step-by-step method into the all-at-once method using the algebra available in a braided monoidal bicategory. The other three axioms also have similar interpretations. In fact, one could take the presentation of these axioms seriously, and view them as certain three-dimensional polytopes in which the two-dimensional faces are precisely the 2-cells in each equation above. Doing so produces polytopes discovered by Bar-Natan in [2].

2.3. DEFINITION. Let B be a braided monoidal bicategory. A **syllepsis** for B consists of an invertible modification with components $v_{xy} : R_{yx} \circ R_{xy} \Rightarrow 1$ such that the following two axioms hold; here we use the convention that if α is a 2-cell, then $\hat{\alpha}$ denotes its mate.

The first equation shows a 2-cell equality between two 3-cell diagrams. The left side is a 3-cell with 1-cells R^*1 and $1R^*$, and 2-cells $\Downarrow \hat{v}_{ab,1}$ and $\Downarrow 1\hat{v}_{ac}$. The right side is a 3-cell with 1-cells R^* and R , and 2-cells $\Downarrow \hat{R}_{(a,b|c)}$ and $\Downarrow \hat{v}_{a,bc}$.

The second equation is similar but uses primed objects and 2-cells. The left side has 1-cells $1R^*$ and R^*1 , and 2-cells $\Downarrow 1\hat{v}_{bc}$ and $\Downarrow \hat{v}_{ac,1}$. The right side has 1-cells $1R$ and $R1$, and 2-cells $\Downarrow \hat{R}_{(a|b,c)}$ and $\Downarrow \hat{v}_{ab,c}$.

A **sylleptic monoidal bicategory** is a braided monoidal bicategory equipped with a syllepsis.

2.4. DEFINITION. A **symmetric monoidal bicategory** is a sylleptic monoidal bicategory in which the following axiom holds for every pair of objects a, b .

The diagram shows two 2-cells between $R: ab \rightarrow ab$ and $R: ba \rightarrow ba$. The left 2-cell has 1-cells R and 1 , and a 2-cell $\Downarrow v_{ab}$. The right 2-cell has 1-cells R and 1 , and a 2-cell $\Downarrow v_{ba}$. The two 2-cells are shown to be equal.

The proofs in the next section rely heavily on various coherence theorems for monoidal and symmetric monoidal bicategories; see [12, 11, 13]. Essentially, these say that “every diagram of 2-cell constraints commutes”. We collect these theorems together as the following.

2.5. THEOREM. *In the free monoidal bicategory, braided monoidal bicategory, or symmetric monoidal bicategory generated by a bicategory B , every diagram of constraint 2-cells commutes.*

In practice, this means we can simplify calculations involving 2-cells in the following ways.

- We can disregard lower-dimensional coherence cells in the sources and targets of 2-cell diagrams, as there is a unique coherence isomorphism between any two possible interpretations.
- We can manipulate coherence 2-cells in the diagram, more-or-less at will.
- When using only the braided structure, there is a unique 2-cell between any pair of parallel 1-cells built using the braid constraint if and only if those 1-cells have the same associated geometric braid, and if not there is no 2-cell at all. Using the full symmetric structure, there is a unique 2-cell between such a pair if and only if those 1-cells have the same underlying permutation, and if not there is no 2-cell between them at all.

We will at times record some of these coherence 2-cells in our diagrams, but it is largely done for either clarity (if omitting them would make the diagram difficult to parse) or aesthetic purposes.

3. Generalised icons

In this section we are going to generalise the definition of icon in the following sense. Given a monoidal bicategory \mathcal{B} we will define a bicategory $\mathcal{B}\text{-Icon}$ with

- 0-cells: categories weakly enriched in \mathcal{B}
- 1-cells: weak functors
- 2-cells: icons enriched in \mathcal{B} .

The case $\mathcal{B} = \mathbf{Cat}$ gives the original definition of Lack in [15]; Shulman gives an abstract account of the case where \mathcal{B} is a tensor distributive monoidal 2-category in [20]. Furthermore we will show that if \mathcal{B} is symmetric monoidal then $\mathcal{B}\text{-Icon}$ is also symmetric monoidal, so the construction can be iterated.

We begin by defining the appropriate bicategory of graphs. For the rest of this section we fix a monoidal bicategory \mathcal{B} , with tensor product \otimes (or omitted) and unit object $\mathbf{1}$. Motivating examples include \mathbf{Cat} and \mathbf{Icon} .

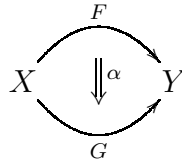
3.1. DEFINITION. A \mathcal{B} -graph X is given by

- a set X_0 of objects, and
- for all objects $x, y \in X_0$ a hom-object $X(x, y) \in \mathcal{B}$.

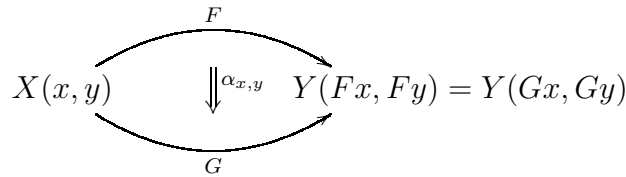
A **morphism of \mathcal{B} -graphs** $F : X \rightarrow Y$ is given by

- a function $F : X_0 \rightarrow Y_0$, and
- for all objects $x, y \in X_0$ a morphism $F = F_{x,y} : X(x, y) \rightarrow Y(Fx, Fy) \in \mathcal{B}$.

A **2-morphism of \mathcal{B} -graphs** $\alpha : F \Rightarrow G$



exists only when $Fx = Gx$ for all $x \in X_0$. When this holds, α is given by, for every pair $x, y \in X_0$ a 2-cell



It is straightforward to check that \mathcal{B} -graphs, their morphisms and their 2-morphisms form a bicategory, which we write as $\mathcal{B}\text{-Gph}$. Note that horizontal and vertical composition of 2-morphisms come from the horizontal and vertical composition of 2-cells in \mathcal{B} .

3.2. DEFINITION. Let \mathcal{B} be a braided monoidal bicategory. A **category weakly enriched in \mathcal{B}** or **weak \mathcal{B} -category** X is given by

- an underlying \mathcal{B} -graph X ,
- composition: for all $a, b, c \in X_0$ a 1-cell in \mathcal{B}

$$m_{abc} : X(b, c) \otimes X(a, b) \rightarrow X(a, c),$$

- identities: for all $a \in X_0$ a 1-cell in \mathcal{B}

$$I_a : \mathbf{1} \rightarrow X(a, a),$$

- *associativity constraints: for all $a, b, c, d \in X_0$ an invertible 2-cell in \mathcal{B}*

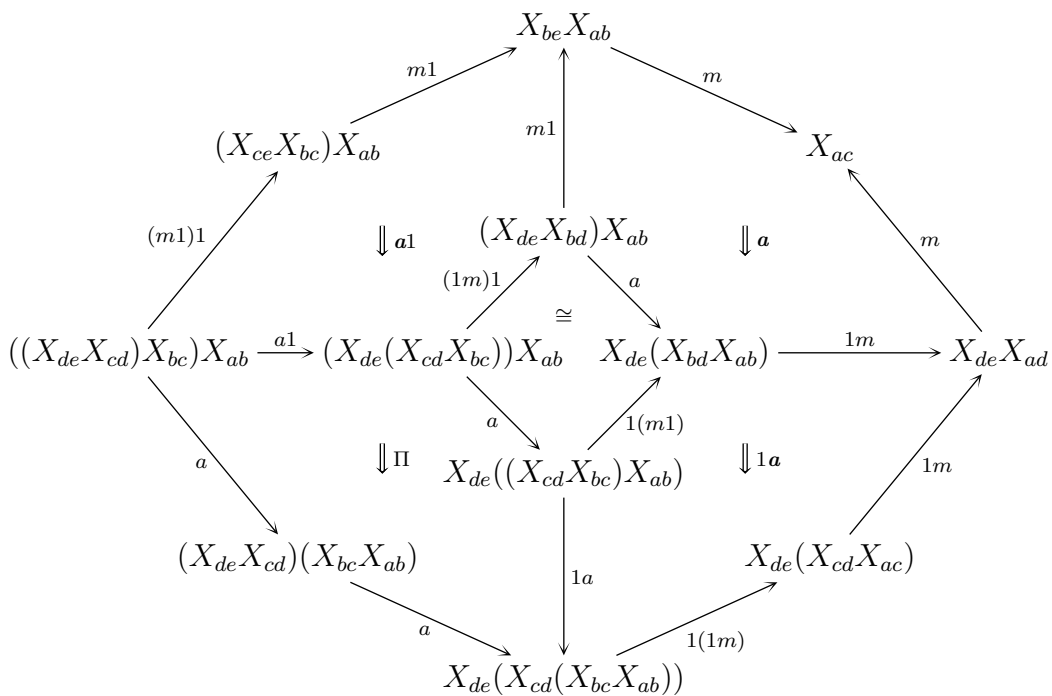
$$\begin{array}{ccc}
 (X(c, d) \otimes X(b, c)) \otimes X(a, b) & \xrightarrow{m_{b,c,d} \otimes 1} & X(b, d) \otimes X(a, b) \\
 \simeq \downarrow & & \downarrow m_{a,b,d} \\
 X(c, d) \otimes (X(b, c) \otimes X(a, b)) & & \\
 \downarrow 1 \otimes m_{a,b,c} & \swarrow \mathbf{a}_{a,b,c,d} & \\
 X(c, d) \otimes X(a, c) & \xrightarrow{m_{a,c,d}} & X(a, d)
 \end{array}$$

and

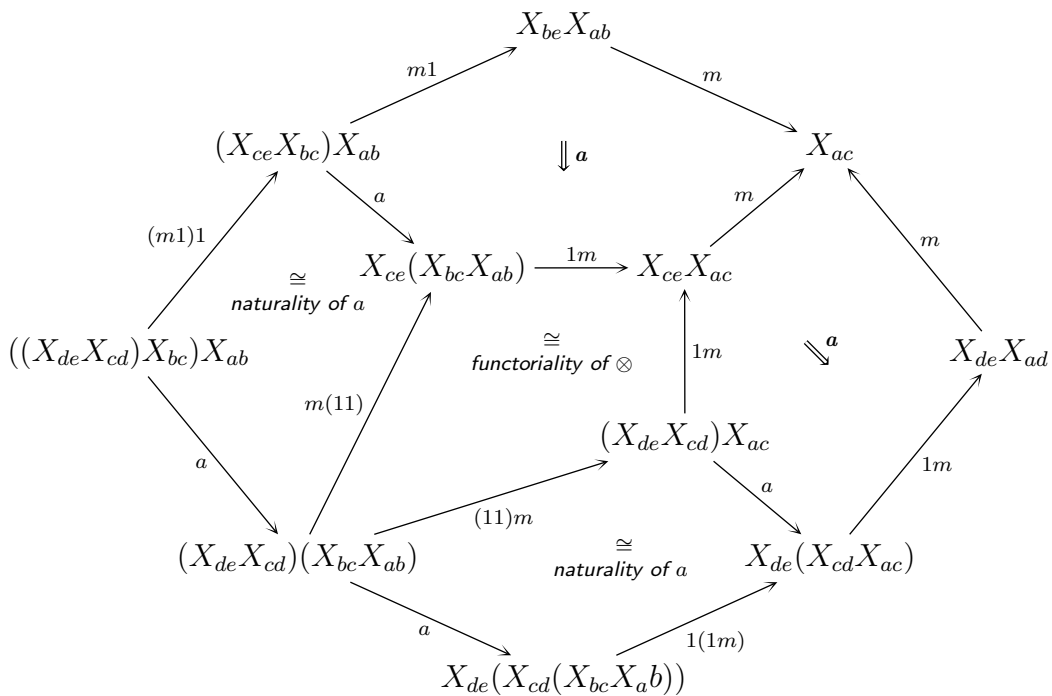
- *unit constraints: for all a, b invertible 2-cells in \mathcal{B}*

$$\begin{array}{ccc}
 X(a, b) \otimes \mathbf{1} & \xrightarrow{1 \otimes I_a} & X(a, b) \otimes X(a, a) \\
 \searrow \simeq & \swarrow r_{a,b} & \downarrow m_{a,a,b} \\
 & & X(a, b)
 \end{array}
 \qquad
 \begin{array}{ccc}
 X(b, b) \otimes X(a, b) & \xleftarrow{I_b \otimes 1} & \mathbf{1} \otimes X(a, b) \\
 \downarrow m_{a,b,b} & \swarrow l_{a,b} & \searrow \simeq \\
 X(a, b) & &
 \end{array}$$

This data satisfies the following axioms. We write (a, b) for $X(a, b)$ except in the first diagram where we write X_{ab} to space. We also omit all \otimes symbols. Note also that the 2-cell \mathbf{a} is the associativity constraint for the weak \mathcal{B} -category X , whereas the 1-cell a is the associativity constraint for the monoidal structure of \mathcal{B} .



=



Note that ignoring \mathcal{B} coherence this axiom becomes (more tractably):

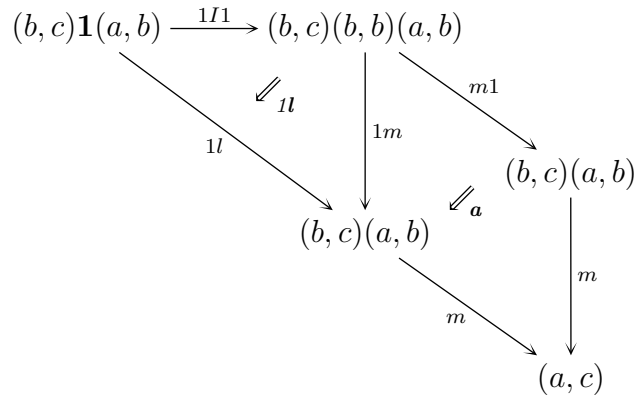
$$\begin{array}{ccccc}
 (d, e)(c, d)(b, c)(a, b) & \xrightarrow{m11} & (c, e)(b, c)(a, b) & & \\
 \downarrow 11m & \searrow 1m1 & \Downarrow a1 & \searrow m1 & \\
 & (d, e)(b, d)(a, b) & \xrightarrow{m1} & (b, e)(a, b) & \\
 & \downarrow 1m & & \downarrow m & \\
 (d, e)(c, d)(a, c) & \Downarrow 1a & & & \\
 \searrow 1m & & \Downarrow a & & \\
 & (d, e)(a, d) & \xrightarrow{m} & (a, e) &
 \end{array}$$

=

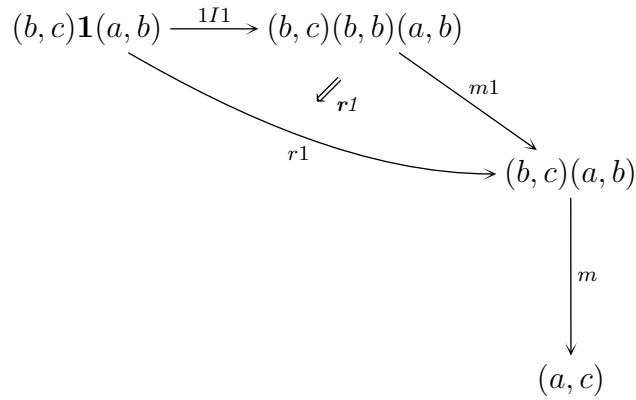
$$\begin{array}{ccccc}
 (d, e)(c, d)(b, c)(a, b) & \xrightarrow{m11} & (c, e)(b, c)(a, b) & & \\
 \downarrow 11m & \xrightarrow{\cong} & \downarrow 1m & \searrow m1 & \\
 & \text{functoriality of } \otimes & & (b, e)(a, b) & \\
 & & & \Downarrow a & \\
 (d, e)(c, d)(a, c) & \xrightarrow{m1} & (c, e)(a, c) & & \\
 \searrow 1m & & \searrow m & \downarrow m & \\
 & \Downarrow a & & (a, e) & \\
 & (d, e)(a, d) & \xrightarrow{m} & &
 \end{array}$$

where we now write (a, b) for $X(a, b)$. This is the notation that we will use for the rest of this work.

The unit axiom is as follows.

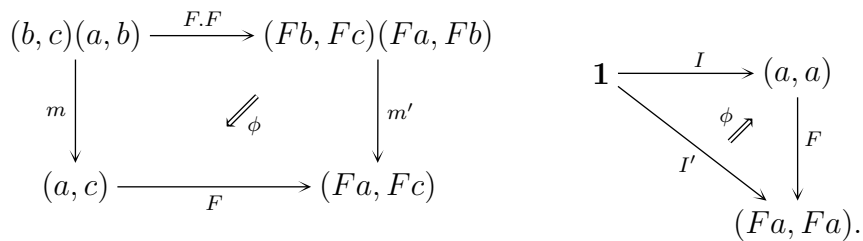


=



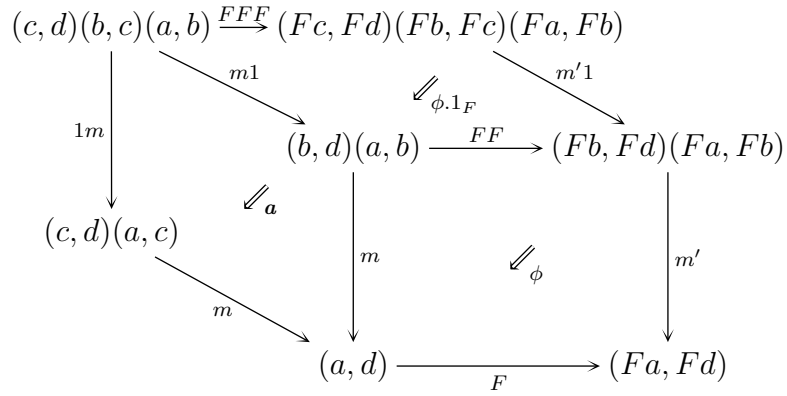
3.3. DEFINITION. A **weak \mathcal{B} -functor** $F : X \rightarrow X'$ is given by

- a function $F : \text{ob } X \rightarrow \text{ob } X'$,
- 1-cells $F_{ab} : X(a, b) \rightarrow X'(Fa, Fb) \in \mathcal{B}$, and
- 2-cells

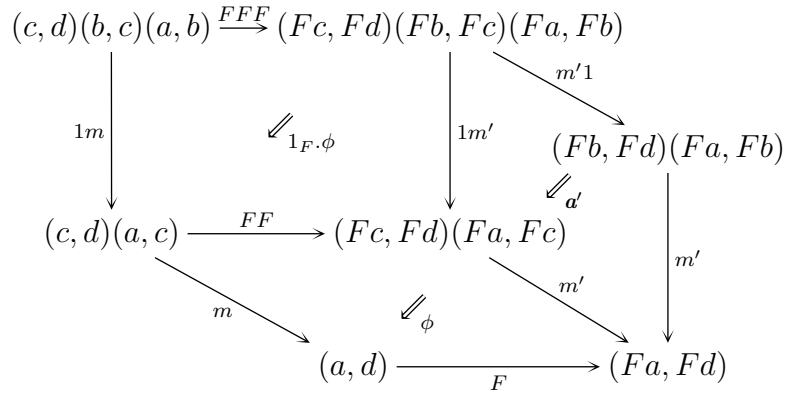


This data satisfies the following axioms:

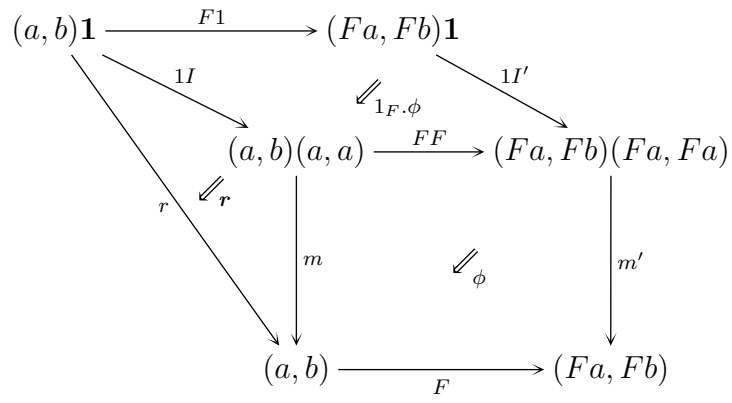
- *associativity*



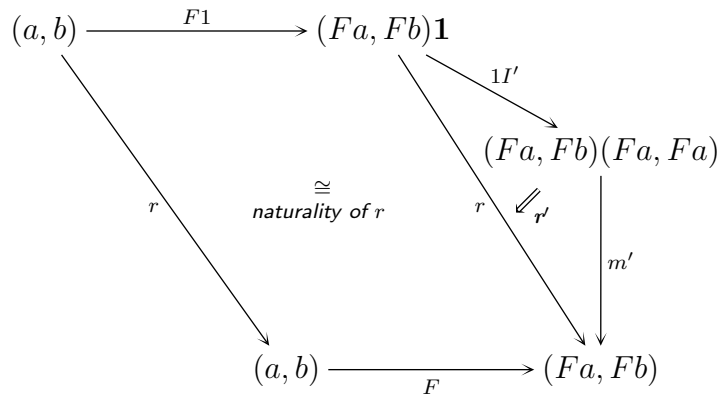
=



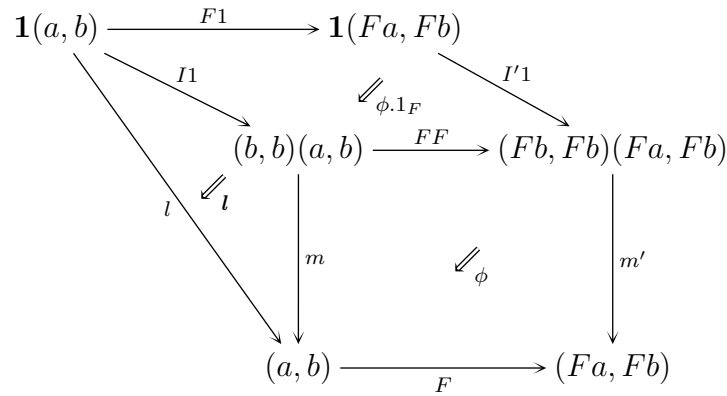
- *right unit*



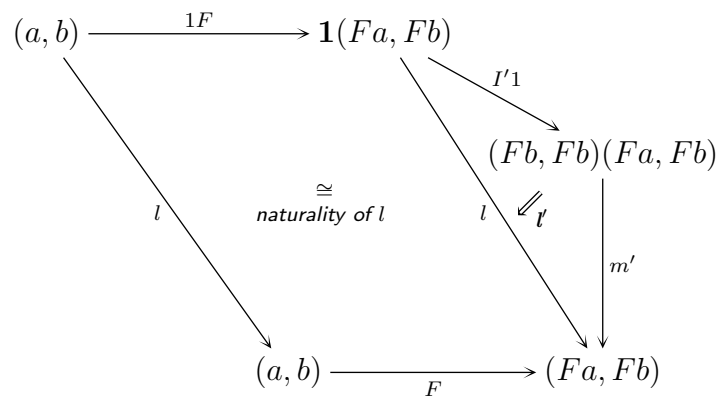
=



- *left unit*



=



3.4. DEFINITION. *Composition of weak \mathcal{B} -functors is defined as follows. Given composable weak \mathcal{B} -functors as below*

$$X \xrightarrow{(F, \phi^F)} Y \xrightarrow{(G, \phi^G)} Z$$

we define the composite GF .

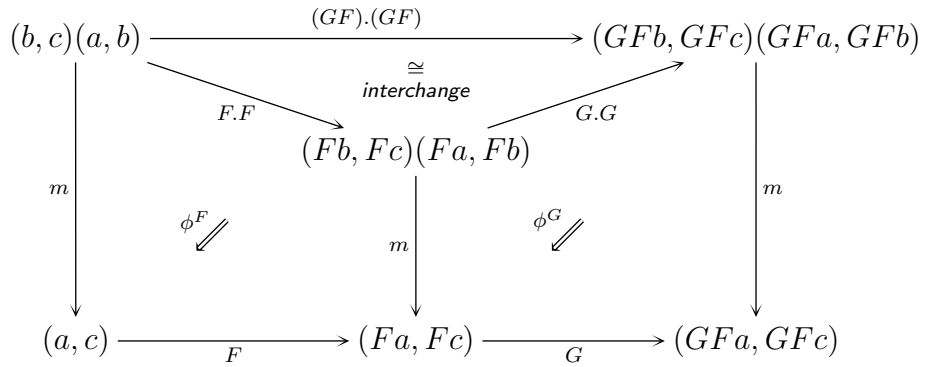
- *The action on objects is given by the composite*

$$\text{ob } X \xrightarrow{F} \text{ob } Y \xrightarrow{G} \text{ob } Z.$$

- *The action on hom-categories is given by the composite*

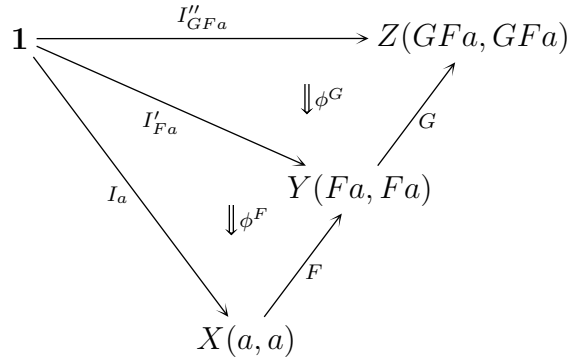
$$X(a, b) \xrightarrow{F_{ab}} Y(Fa, Fb) \xrightarrow{G_{Fa, Fb}} Z(GFa, GFb).$$

- The composition constraint ϕ^{GF} is given by the following composite 2-cell.



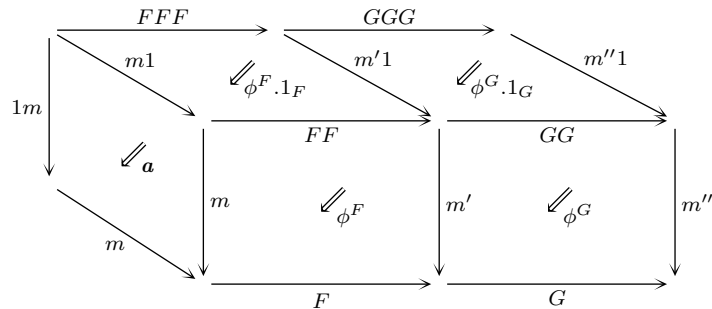
In what follows below, we will omit the interchange cells, noting that these are uniquely determined by coherence.

- The unit constraint is given by the following composite 2-cell.

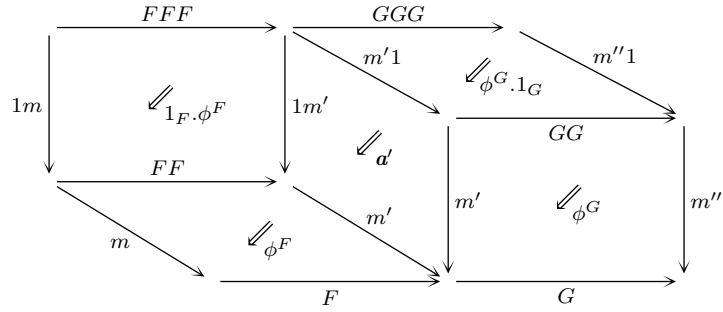


We immediately check that this satisfies the axioms as follows.

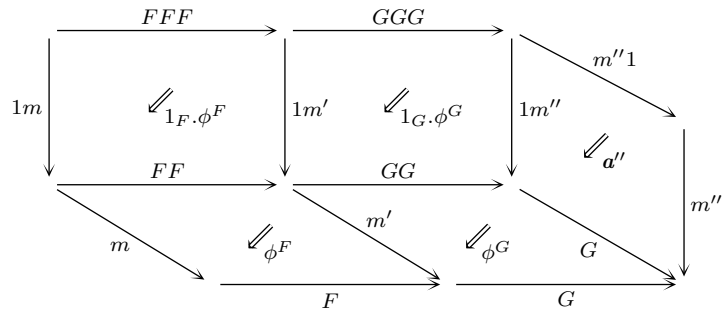
- *Associativity:*



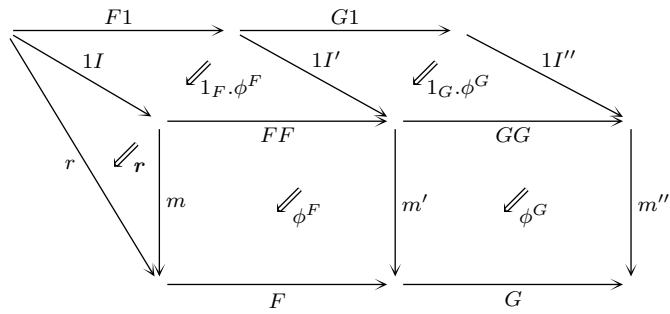
=



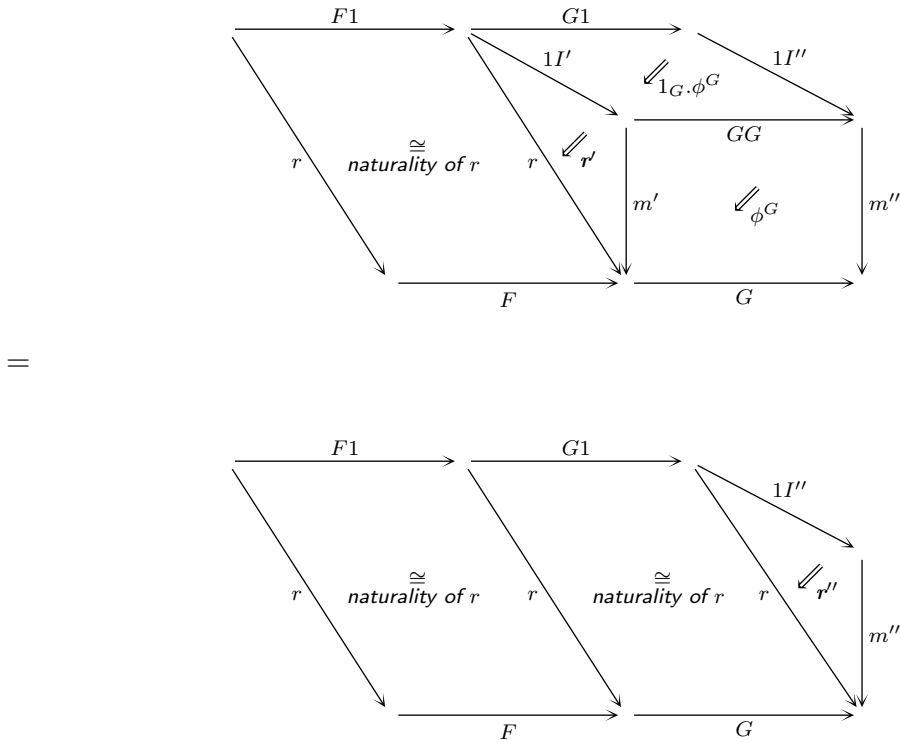
=



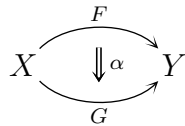
- *Right unit (and similarly left unit):*



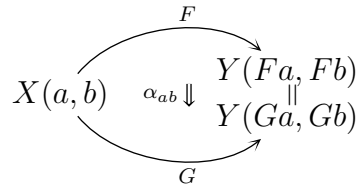
=



3.5. DEFINITION. Let $F, G: X \rightarrow Y$ be weak \mathcal{B} -functors such that $Fa = Ga$ for all objects $a \in X$. A \mathcal{B} -icon



is given by, for all pairs of objects $a, b \in X$ a 2-cell



satisfying the following axioms.

- Composition:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \xrightarrow{FF} & \\
 (b,c)(a,b) & \xrightarrow{\alpha\alpha\Downarrow} & (Fb,Fc)(Fa,Fb) \\
 & \xrightarrow{GG} & (Gb,Gc)(Ga,Gb) \\
 \downarrow m & \searrow \phi^G\Downarrow & \downarrow m' \\
 (a,c) & & (Ga,Gc) \\
 & \xrightarrow{G} &
 \end{array}
 & = &
 \begin{array}{ccc}
 & \xrightarrow{FF} & \\
 (b,c)(a,b) & \xrightarrow{\phi^F\Downarrow} & (Fb,Fc)(Fa,Fb) \\
 & \xrightarrow{F} & (Fa,Fc) \\
 \downarrow m & \searrow \alpha\Downarrow & \downarrow m' \\
 (a,c) & & (Ga,Gc) \\
 & \xrightarrow{G} &
 \end{array}
 \end{array}$$

• *Unit:*

$$\begin{array}{ccc}
 \mathbf{1} & \begin{array}{ccc}
 \nearrow I & & (a,a) \\
 \searrow \phi^F\Downarrow & \xrightarrow{F} & \Downarrow \alpha \\
 \searrow I' & & (Fa,Fa) = (Ga,Ga) \\
 & \xrightarrow{G} &
 \end{array} & = &
 \begin{array}{ccc}
 \mathbf{1} & \begin{array}{ccc}
 \nearrow I & & (a,a) \\
 \searrow \phi^G\Downarrow & \xrightarrow{G} & (a,a) \\
 \searrow I' & & (Fa,Fa) = (Ga,Ga) \\
 & \xrightarrow{G} &
 \end{array}
 \end{array}$$

3.6. DEFINITION. We now define composition of \mathcal{B} -icons. Vertical composition is inherited from \mathcal{B} , that is, components are composed vertically as in \mathcal{B} and axioms then follow from those in \mathcal{B} .

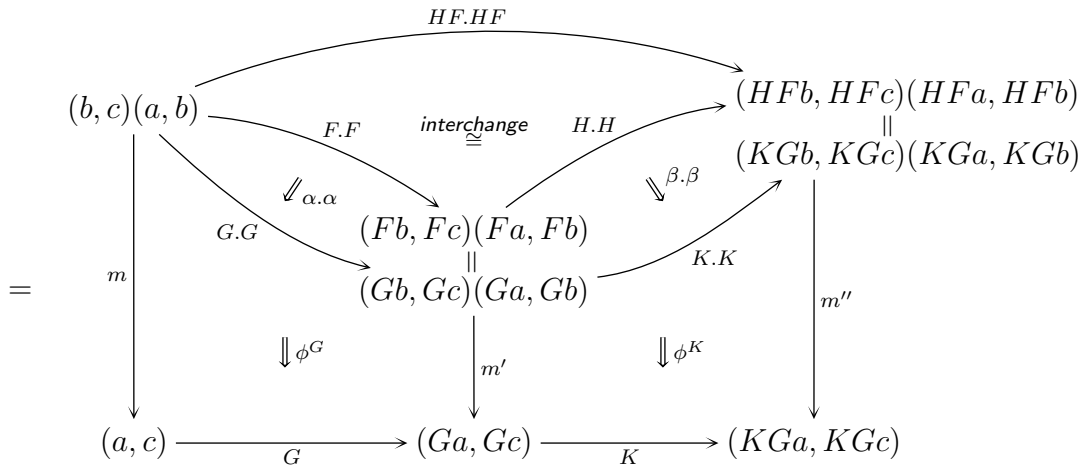
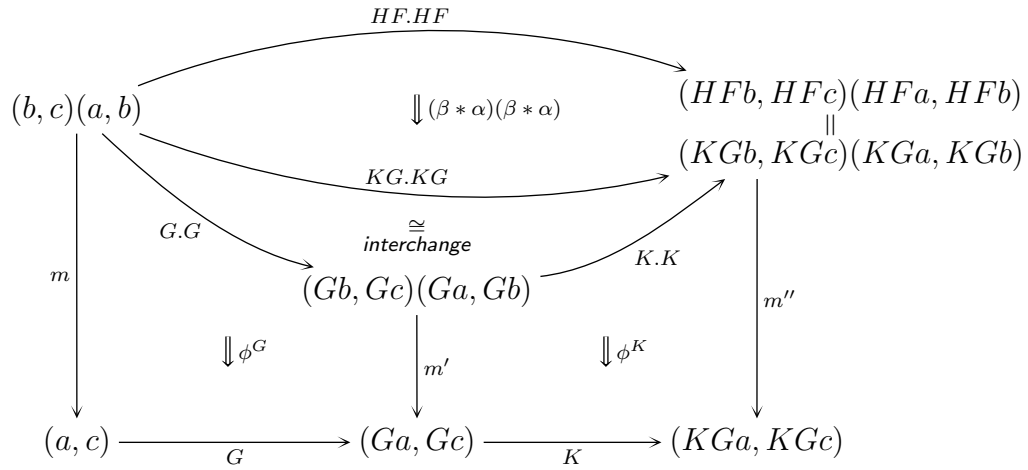
Horizontal composition is similar, but slightly more complicated. Given composable \mathcal{B} -icons

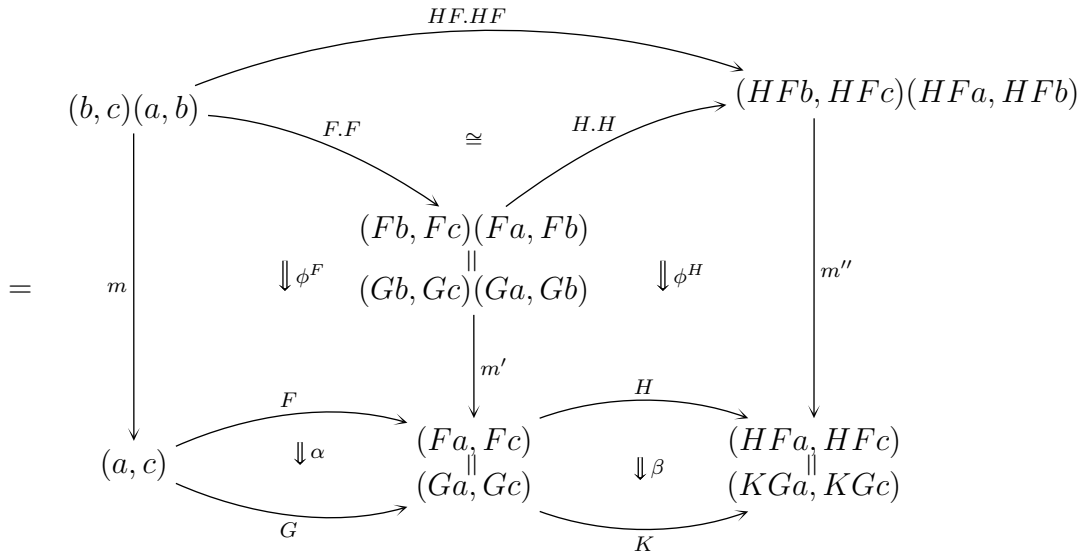
$$\begin{array}{ccccc}
 & \xrightarrow{F} & & \xrightarrow{H} & \\
 X & \xrightarrow{\Downarrow \alpha} & Y & \xrightarrow{\Downarrow \beta} & Z \\
 & \xrightarrow{G} & & \xrightarrow{K} &
 \end{array}$$

the component $(\beta * \alpha)_{a,b}$ is given by the horizontal composite

$$\begin{array}{ccccc}
 & \xrightarrow{F} & & \xrightarrow{H} & \\
 X(a,b) & \xrightarrow{\Downarrow \alpha_{a,b}} & Y(Fa,Fb) & \xrightarrow{\Downarrow \beta_{Fa,Fb}} & Z(HFa,HFb) \\
 & \xrightarrow{G} & & \xrightarrow{K} &
 \end{array}$$

noting that this makes sense as F and G agree on objects, and likewise H and K . We can then check that this satisfies the axioms for a \mathcal{B} -icon. The unit axiom follows immediately from the unit axioms for α and β . The composition axiom is as follows.





where the first equality follows from functoriality axioms and the second from the first \mathcal{B} -icon axiom applied on each side.

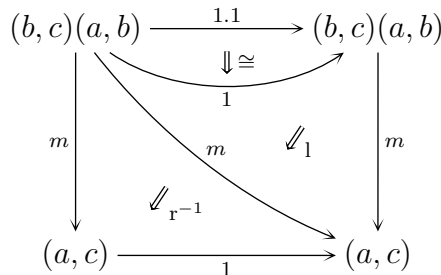
3.7. PROPOSITION. *Weak \mathcal{B} -categories, weak \mathcal{B} -functors and \mathcal{B} -icons organise themselves into a bicategory $\mathcal{B}\text{-Icon}$. Furthermore if the underlying bicategory of \mathcal{B} is a 2-category, then $\mathcal{B}\text{-Icon}$ is also a 2-category.*

PROOF. Given a pair of weak \mathcal{B} -categories X and Y , the weak \mathcal{B} -functors $X \rightarrow Y$ and \mathcal{B} -icons between those form a category—composition is vertical composition of \mathcal{B} -icons and the identity $1_F : F \Rightarrow F$ has all its components given by identity 2-cells in \mathcal{B} . The category axioms follow from those for the hom-categories in \mathcal{B} .

We have defined the action of the composition functor on cells in Definitions 3.4 and 3.6. For functoriality, interchange of \mathcal{B} -icons follows from interchange of 2-cells in \mathcal{B} . Likewise the fact that horizontal composition of units is a unit follows from the corresponding fact in \mathcal{B} .

We now define the identity weak \mathcal{B} -functor.

- On objects it is the identity function.
- On homs it is the identity 1-cell in \mathcal{B} .
- The constraint cells are given below.



$$\begin{array}{ccc}
 \mathbf{1} & \xrightarrow{I} & (a, a) \\
 & \searrow I & \nearrow 1 \\
 & & (a, a)
 \end{array}$$

$\Downarrow_{\Gamma^{-1}}$

The axioms for a weak \mathcal{B} -functor follow from coherence for the monoidal bicategory \mathcal{B} .

We now deal with the associativity and unit constraints. First note that in each case the constraint must be a \mathcal{B} -icon, so has components which are 2-cells of \mathcal{B} . We simply define these to be the associativity and unit constraints of \mathcal{B} , and it only remains to check that these components satisfy the \mathcal{B} -icon axioms, which is straightforward as we now indicate. For the first axiom we must check the following equality

$(HG)F.(HG)F$
 $\Downarrow_{a.a}$
 $H(GF).H(GF)$

$GF.GF$
 \cong
 interchange

$F.F$
 \Downarrow_{ϕ^F}
 m'

$G.G$
 \Downarrow_{ϕ^G}
 m''

$H.H$
 \Downarrow_{ϕ^H}
 m''

$F \rightarrow G \rightarrow H$

m (left vertical arrow), m'' (right vertical arrow)

$(HG)F.(HG)F$
 \cong
 interchange

$F.F$
 \Downarrow_{ϕ^F}
 m'

$G.G$
 \Downarrow_{ϕ^G}
 m''

$H.H$
 \Downarrow_{ϕ^H}
 m''

$F \rightarrow G \rightarrow H$

\Downarrow_a
 $H(GF)$

\cong (left of the second diagram)

which follows from coherence for the monoidal bicategory \mathcal{B} and naturality of a , the associator for the 1-cells in \mathcal{B} . The unit axiom follows similarly.

Finally, to verify the bicategory axioms for $\mathcal{B}\text{-Icon}$ we only need to check them componentwise, so they all follow from the bicategory axioms for \mathcal{B} . Since the coherence constraints are built from those in \mathcal{B} , it follows that if the constraints in \mathcal{B} are identities then the constraints in $\mathcal{B}\text{-Icon}$ are also identities. Thus if \mathcal{B} is a 2-category then $\mathcal{B}\text{-Icon}$ is also a 2-category. ■

3.8. THEOREM. *If \mathcal{B} is a symmetric monoidal bicategory then so is $\mathcal{B}\text{-Icon}$. Thus in particular the icon construction can be iterated.*

PROOF. Suppose \mathcal{B} is a symmetric monoidal bicategory, with symmetry

$$R = R_{ab} : a \otimes b \longrightarrow b \otimes a.$$

We already know that $\mathcal{B}\text{-Icon}$ is a bicategory, and we now show that it is symmetric monoidal.

We define a (weak) functor

$$\otimes : \mathcal{B}\text{-Icon} \times \mathcal{B}\text{-Icon} \longrightarrow \mathcal{B}\text{-Icon}.$$

DEFINITION OF \otimes ON WEAK \mathcal{B} -CATEGORIES. Let X, Y be weak \mathcal{B} -categories. Then $X \otimes Y$ has

- objects pairs $\langle x, y \rangle \in \text{ob} X \times \text{ob} Y$,
- $(X \otimes Y)(\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle) = X(x_0, x_1).Y(y_0, y_1)$ where on the right the dot denotes the tensor product in \mathcal{B} ,
- composition: given $x_0, x_1, x_2 \in X_0$ and $y_0, y_1, y_2 \in Y_0$, composition is defined as below

$$\begin{array}{c} X(x_1, x_2).Y(y_1, y_2).X(x_0, x_1).Y(y_0, y_1) \\ \downarrow 1.R.1 \\ X(x_1, x_2).X(x_0, x_1).Y(y_1, y_2).Y(y_0, y_1) \\ \downarrow m.m \\ X(x_0, x_2).Y(y_0, y_2), \end{array}$$

- identities: for all $x \in X, y \in Y$, a the unit 1-cell is the composite

$$\mathbf{1} \longrightarrow \mathbf{1.1} \xrightarrow{I_x.I_y} X(x, x).Y(y, y),$$

- associativity constraints:

$$\begin{array}{ccccc}
(x_2, x_3)(y_2, y_3)(x_1, x_2)(y_1, y_2)(x_0, x_1)(y_0, y_1) & \xrightarrow{1.R.1.1.1} & (x_2, x_3)(x_1, x_2)(y_2, y_3)(y_1, y_2)(x_0, x_1)(y_0, y_1) & \xrightarrow{m.m.1.1} & (x_1, x_3)(y_1, y_3)(x_0, x_1)(y_0, y_1) \\
\downarrow 1.1.1.R.1 & & \downarrow 1.1.R.1 & & \downarrow 1.R.1 \\
(x_2, x_3)(y_2, y_3)(x_1, x_2)(x_0, x_1)(y_1, y_2)(y_0, y_1) & \xrightarrow{1.R.1.1} & (x_2, x_3)(x_1, x_2)(x_0, x_1)(y_2, y_3)(y_1, y_2)(y_0, y_1) & \xrightarrow{m.1.m.1} & (x_1, x_3)(x_0, x_1)(y_1, y_3)(y_0, y_1) \\
\downarrow 1.1.m.m & & \downarrow 1.m.1.m & & \downarrow m.m \\
(x_2, x_3)(y_2, y_3)(x_0, x_2)(y_0, y_2) & \xrightarrow{1.R.1} & (x_2, x_3)(x_0, x_2)(y_2, y_3)(y_0, y_2) & \xrightarrow{m.m} & (x_0, x_3)(y_0, y_3)
\end{array}$$

\cong braid coherence \cong naturality of R \cong naturality of R \Downarrow a.a

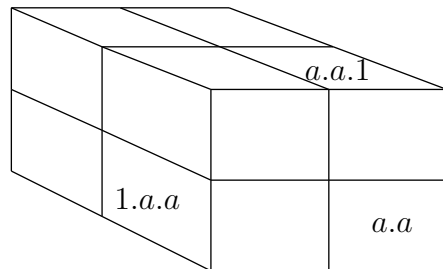
Note that in the above diagram, we have written R for various different instances of braidings. This is unambiguous given the sources and targets, but requires some care especially in the top left square.

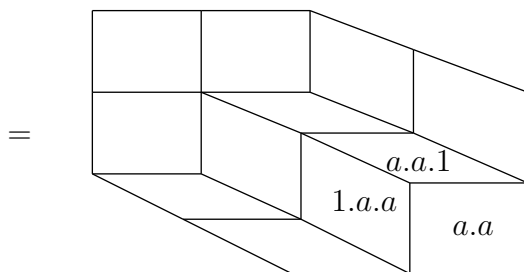
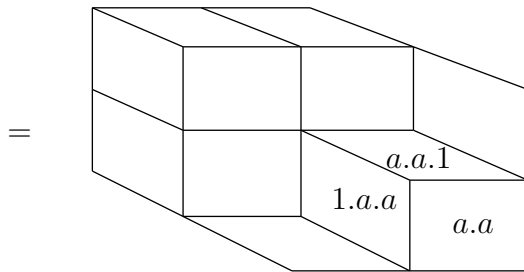
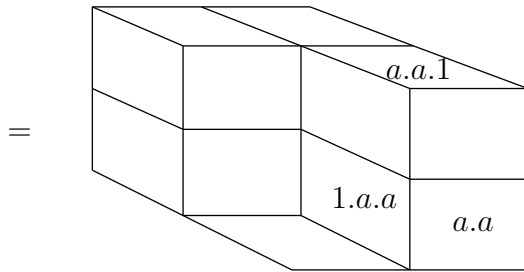
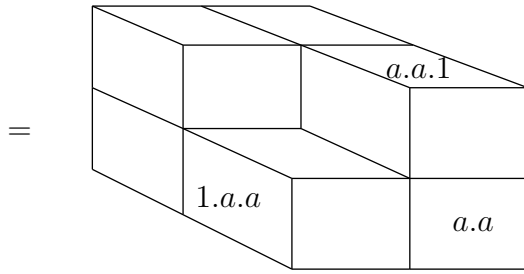
- unit constraints:

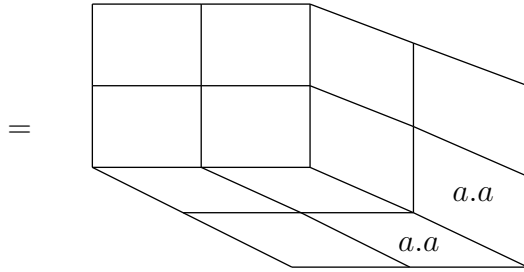
$$\begin{array}{ccccc}
 (x_0, x_1)(y_0, y_1) \cdot \mathbf{1} & \xrightarrow{\sim} & (x_0, x_1)(y_0, y_1) \cdot \mathbf{1} \cdot \mathbf{1} & \xrightarrow{1.I.I.I} & (x_0, x_1)(y_0, y_1)(x_0, x_0)(y_0, y_0) \\
 & \searrow^{\text{sym. } \cong \text{ coherence}} & \downarrow 1.R.1 & \text{naturality of } R \cong & \downarrow 1.R.1 \\
 & & (x_0, x_1) \cdot \mathbf{1} \cdot (y_0, y_1) \cdot \mathbf{1} & \xrightarrow{1.I.I.I} & (x_0, x_1)(x_0, x_0)(y_0, y_1)(y_0, y_0) \\
 & & & \searrow \sim & \downarrow m.m \\
 & & & & (x_0, x_1)(y_0, y_1) \\
 & \searrow \sim & & & \\
 & & & &
 \end{array}$$

$$\begin{array}{ccccc}
 \mathbf{1} \cdot (x_0, x_1)(y_0, y_1) & \xrightarrow{\sim} & \mathbf{1} \cdot \mathbf{1} \cdot (x_0, x_1)(y_0, y_1) & \xrightarrow{I.I.1.1} & (x_1, x_1)(y_1, y_1)(x_0, x_1)(y_0, y_1) \\
 & \searrow^{\text{sym. } \cong \text{ coherence}} & \downarrow 1.R.1 & \text{naturality of } R \cong & \downarrow 1.R.1 \\
 & & \mathbf{1} \cdot (x_0, x_1) \cdot \mathbf{1} \cdot (y_0, y_1) & \xrightarrow{I.I.1.1} & (x_1, x_1)(x_0, x_1)(y_1, y_1)(y_0, y_1) \\
 & & & \searrow \sim & \downarrow m.m \\
 & & & & (x_0, x_1)(y_0, y_1) \\
 & \searrow \sim & & & \\
 & & & &
 \end{array}$$

We check that this data for $X \otimes Y$ satisfies the axioms for a weak \mathcal{B} -category. Ignoring all coherence in \mathcal{B} , these are not difficult to check; they follow from the axioms for X and Y tensored together. The following diagram indicates how to prove the associativity axiom; we again use the cubical version of the axioms, as introduced in Definition 3.2, and omit almost all the labels as they can be inferred.







This completes the definition of $X \otimes Y$ as a weak \mathcal{B} -category.

DEFINITION OF \otimes ON WEAK \mathcal{B} -FUNCTORS. Given weak \mathcal{B} -functors

$$\begin{aligned} X &\xrightarrow{F} X' \\ Y &\xrightarrow{G} Y' \end{aligned}$$

we define

$$X \otimes Y \xrightarrow{F \otimes G} X' \otimes Y'$$

as follows:

- on objects: $(F \otimes G): \langle x, y \rangle \mapsto \langle Fx, Gy \rangle$
- on homs: $X(x_0, y_0).Y(y_0, y_1) \xrightarrow{F.G} X'(Fx_0, Fx_1).Y'(Gy_0, Gy_1)$
- constraints

$$\begin{array}{ccc} (x_1, x_2)(y_1, y_2)(x_0, x_1)(y_0, y_1) & \xrightarrow{FGFG} & (Fx_1, Fx_2)(Gy_1, Gy_2)(Fx_0, Fx_1)(Gy_0, Gy_1) \\ \downarrow 1R1 & \cong \text{ naturality of } R & \downarrow 1R1 \\ (x_1, x_2)(x_0, x_1)(y_1, y_2)(y_0, y_1) & \xrightarrow{FFGG} & (Fx_1, Fx_2)(Fx_0, Fx_1)(Gy_1, Gy_2)(Gy_0, Gy_1) \\ \downarrow mm & \Downarrow \phi\phi & \downarrow mm \\ (x_0, x_2)(y_0, y_2) & \xrightarrow{FG} & (Fx_0, Fx_2)(Gy_0, Gy_2) \end{array}$$

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{II} & (x, x)(y, y) \\ & \searrow \phi\phi & \downarrow FG \\ & I'I' & (Fx, Fx)(Gy, Gy) \end{array}$$

We now check that this data for $F \otimes G$ satisfies the axioms for a weak \mathcal{B} -functor. The calculations are similar to those for the weak \mathcal{B} -category axioms for $X \otimes Y$ above, essentially by tensoring together the axioms for F and G , and using coherence for \mathcal{B} braided monoidal category.

DEFINITION OF \otimes ON \mathcal{B} -ICONS. Suppose we have weak \mathcal{B} -functors

$$\begin{aligned} F, F' : X &\longrightarrow X' \\ G, G' : Y &\longrightarrow Y' \end{aligned}$$

agreeing on objects, and \mathcal{B} -icons

$$\begin{array}{ccc} & F & \\ X & \xrightarrow{\quad} & X' \\ & \Downarrow \alpha & \\ & F' & \end{array}$$

$$\begin{array}{ccc} & G & \\ Y & \xrightarrow{\quad} & Y' \\ & \Downarrow \beta & \\ & G' & \end{array}$$

we define the \mathcal{B} -icon

$$\begin{array}{ccc} & F \otimes G & \\ X \otimes Y & \xrightarrow{\quad} & X' \otimes Y' \\ & \Downarrow \alpha \otimes \beta & \\ & F' \otimes G' & \end{array}$$

to have components

$$\begin{array}{ccc} & F.G & \\ (x_0, x_1)(y_0, y_1) & \xrightarrow{\quad} & (Fx_0, Fx_1)(Gy_0, Gy_1) \\ & \Downarrow \alpha_{x_0x_1} \cdot \beta_{y_0y_1} & \\ & & (F'x_0, F'x_1)(G'y_0, G'y_1) \\ & F'.G' & \end{array}$$

and then the axioms for a \mathcal{B} -icon follow easily from naturality of R and the icon axioms for α and β tensored together.

This completes the definition of the weak functor

$$\otimes : \mathcal{B}\text{-Icon} \times \mathcal{B}\text{-Icon} \longrightarrow \mathcal{B}\text{-Icon}$$

on cells. Note that it follows immediately that this is locally functorial on hom-categories because this amounts to the functoriality of the tensor product of \mathcal{B} . We now define composition and unit constraints for the weak functoriality of \otimes . The principle is as before. We define putative \mathcal{B} -icons whose components are the corresponding constraints in \mathcal{B} ; as axioms are then inherited from \mathcal{B} , we just have to check that these components

really form \mathcal{B} -icons. These calculations are routine, and are generally similar to those in Definition 3.6 and Proposition 3.7, so we leave the details to the reader.

DEFINITION OF COMPOSITION CONSTRAINT FOR THE FUNCTOR \otimes . Given weak \mathcal{B} -functors

$$\begin{aligned} X &\xrightarrow{F} X' \xrightarrow{F'} X'' \\ Y &\xrightarrow{G} Y' \xrightarrow{G'} Y'' \end{aligned}$$

we need a constraint \mathcal{B} -icon

$$\begin{array}{ccc} & (F' \otimes G') \circ (F \otimes G) & \\ & \curvearrowright & \\ X \otimes Y & \Downarrow & X'' \otimes Y'' \\ & \curvearrowleft & \\ & (F' \circ F) \otimes (G' \circ G) & \end{array}$$

First observe that the source and target \mathcal{B} -functors agree on objects giving

$$(x, y) \mapsto (F'Fx, G'Gy)$$

so it makes sense to seek such a \mathcal{B} -icon. Now on homs the above \mathcal{B} -functors give us 1-cells in \mathcal{B}

$$\begin{aligned} X(x_0, x_1) &\xrightarrow{F_{x_0, x_1}} X'(Fx_0, Fx_1) \xrightarrow{F'_{Fx_0, Fx_1}} X''(F'Fx_0, F'Fx_1) \\ Y(y_0, y_1) &\xrightarrow{G_{y_0, y_1}} Y'(Gy_0, Gy_1) \xrightarrow{G'_{Gy_0, Gy_1}} Y''(G'Gy_0, G'Gy_1) \end{aligned}$$

so our constraint \mathcal{B} -icon needs components that are 2-cells in \mathcal{B} of the form

$$\begin{array}{ccc} & (F'_{Fx_0, Fx_1} \cdot G'_{Gy_0, Gy_1}) \circ (F_{x_0, x_1} \cdot G_{y_0, y_1}) & \\ & \curvearrowright & \\ X(x_0, x_1) \cdot Y(y_0, y_1) & \Downarrow & X''(F'Fx_0, F'Fx_1) \cdot Y''(G'Gy_0, G'Gy_1) \\ & \curvearrowleft & \\ & (F'_{Fx_0, Fx_1} \circ F_{x_0, x_1}) \cdot (G'_{Gy_0, Gy_1} \circ G_{y_0, y_1}) & \end{array}$$

so we can simply use the constraint for the functoriality of the tensor product in \mathcal{B} (recall that in all these diagrams the tensor product in \mathcal{B} is written as a dot). We must check this satisfies the axioms for a \mathcal{B} -icon; as before this follows from coherence in \mathcal{B} .

DEFINITION OF UNIT CONSTRAINT FOR THE FUNCTOR \otimes . Recall that for all objects X in \mathcal{B} we have an identity weak \mathcal{B} -functor 1_X . We need for all X, Y a coherence constraint

$$1_{X \otimes Y} \longrightarrow 1_X \otimes 1_Y.$$

As before, observe that the source and target functors agree on objects, both acting as the identity. So our constraint \mathcal{B} -icon needs components that are 2-cells in \mathcal{B} of the form

$$\begin{array}{ccc}
 & 1_{X(x_0, x_1).Y(y_0, y_1)} & \\
 & \curvearrowright & \\
 X(x_0, x_1).Y(y_0, y_1) & \Downarrow & X(x_0, x_1).Y(y_0, y_1) \\
 & \curvearrowleft & \\
 & 1_{X(x_0, x_1).1_{Y(y_0, y_1)}} &
 \end{array}$$

Again we use the constraint for the tensor product in \mathcal{B} . We must check this satisfies the axioms for a \mathcal{B} -icon, and again this follows from coherence in \mathcal{B} .

FUNCTORIALITY AXIOMS FOR \otimes . It suffices to check the axioms componentwise. Note that all the constraints have components given by constraint cells for the functoriality of the tensor product of \mathcal{B} as a functor

$$\mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}.$$

Hence by coherence for functors of bicategories, the axioms hold.

Thus we have defined a weak functor

$$\otimes: \mathcal{B}\text{-Icon} \times \mathcal{B}\text{-Icon} \longrightarrow \mathcal{B}\text{-Icon}.$$

UNIT OBJECT FOR THE MONOIDAL STRUCTURE ON $\mathcal{B}\text{-Icon}$. The unit object I in $\mathcal{B}\text{-Icon}$ has

- one object
- the hom object is the unit in \mathcal{B}
- composition is given by l .

All the rest of the structure is given by coherence. Note that we could instead use r for composition, which would give an equivalent object of $\mathcal{B}\text{-Icon}$.

CONSTRAINTS a, l, r FOR THE MONOIDAL STRUCTURE ON \mathcal{B} -Icon. We must define an adjoint equivalence (a, a^*, η, ϵ)

$$\begin{array}{ccc}
 & \xrightarrow{a} & \\
 (X \otimes Y) \otimes Z & & X \otimes (Y \otimes Z) \\
 & \xleftarrow{a^*} &
 \end{array}$$

where a and a^* are weak \mathcal{B} -functors, and η and ϵ must be \mathcal{B} -icons.

- On objects a and a^* are given by associativity in **Set**.
- On homs a and a^* are given by the adjoint equivalences for associativity of the tensor product in \mathcal{B} .

We show that this makes a into a weak \mathcal{B} -functor. First we must define constraints for a . The composition constraint is given by the composite below. Note that here X, Y, Z are shorthand for hom-objects of the weak \mathcal{B} -categories, with 0-cells suppressed. Also note that we have suppressed the associators necessary to make the given braidings applicable, as these are taken care of by coherence for the braiding.

$$\begin{array}{ccc}
 ((XY)Z)((XY)Z) & \xrightarrow{aa} & (X(YZ))(X(YZ)) \\
 \downarrow 1R1 & \text{braid coherence} & \downarrow 1R1 \\
 ((XY)(XY))(ZZ) & \cong & (XX)((YZ)(YZ)) \\
 \downarrow 1R111 & & \downarrow 111R1 \\
 ((XX)(YY))(ZZ) & \xrightarrow{a} & (XX)((YY)(ZZ)) \\
 \downarrow mmm & \text{naturality of } a & \downarrow mmm \\
 (XY)Z & \xrightarrow{a} & X(YZ)
 \end{array}$$

The unit constraint is defined similarly using coherence cells for the monoidal structure of \mathcal{B} and naturality of associator for the tensor product of \mathcal{B} . The axioms for a weak \mathcal{B} -functor follow from coherence in \mathcal{B} . The result for a^* follows similarly.

Next we define η and ϵ , the unit and counit for the adjoint equivalence. As usual, these are \mathcal{B} -icons with components given by the corresponding units and counits in \mathcal{B} ; all axioms are given by coherence in \mathcal{B} .

The constraints l and r follow in exactly the same way.

CONSTRAINTS π, μ, λ, ρ FOR THE MONOIDAL STRUCTURE ON \mathcal{B} -ICON. We now need the following constraints for a monoidal bicategory: π, μ, λ, ρ . In each case the components are the corresponding constraints from \mathcal{B} , and the \mathcal{B} -icon axioms follow from coherence in the usual manner.

Finally the monoidal bicategory axioms come from the monoidal bicategory axioms for \mathcal{B} together with the monoidal category axioms for **Set**. This completes the proof that **B-Icon** is a monoidal bicategory.

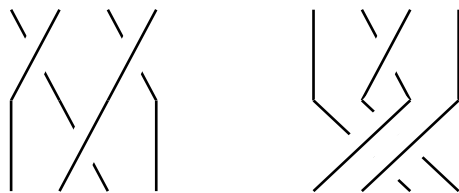
DEFINITION OF SYMMETRIC STRUCTURE. We now construct a braiding $R: X \otimes Y \rightarrow Y \otimes X$ for **B-Icon**.

- On objects R is the symmetry in **Set**.
- On homs R is the symmetry in \mathcal{B} .

It has the structure of a weak functor with constraints coming from the syllepsis in \mathcal{B} and some coherence cells as follows.

$$\begin{array}{ccc}
 XYXY & \xrightarrow{RR} & YXYX \\
 \downarrow 1R1 & \cong \text{symmetry} & \downarrow 1R1 \\
 XXYY & \xrightarrow{R} & YYXX \\
 \downarrow mm & \cong \text{naturality of } R & \downarrow mm \\
 XY & \xrightarrow{R} & YX
 \end{array}$$

Note that the top square is given as follows. The 1-cell source and target have the following underlying braids respectively.

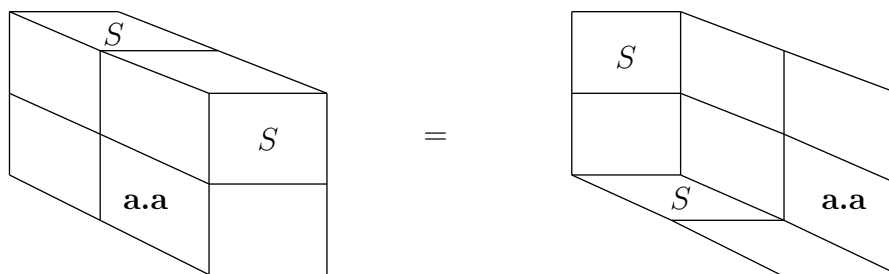


Comparing these braids, we see that there is an isomorphism between them using the syllepsis (so in particular to construct a braiding for **B-Icon** we need \mathcal{B} to have a sylleptic structure), and the isomorphism is unique by coherence for symmetric monoidal bicategories.

The unit constraint has the following form

$$\begin{array}{ccc}
 \mathbf{1} & \xrightarrow{I_X I_Y} & XY \\
 & \searrow I_Y I_X & \downarrow R \\
 & & YX
 \end{array}$$

and is given by a unique isomorphism from the braided structure. Using the cubical notation as in Definition 3.2 the first axiom for a weak \mathcal{B} -functor is



where S indicates a unique coherence 2-cell involving the syllepsis of \mathcal{B} , and as before $\mathbf{a.a}$ indicates the associativity constraint for a weak \mathcal{B} -category. The equality then follows from a calculation manipulating cubes in a similar fashion to earlier, using coherence for symmetric monoidal bicategories. The other axiom follows similarly.

Now we need a unit and counit. These come from the adjoint equivalence for R in the braided structure on \mathcal{B} . We just need to check that η and ϵ are \mathcal{B} -icons; this follows from coherence again, this time for symmetric monoidal bicategories. The modifications $R_{(-|-,-)}$ and $R_{(-,-|-)}$ are given on components by the corresponding cells in \mathcal{B} . We only need to check the icon axioms, and these follow from coherence for symmetric monoidal bicategories and the corresponding modification axioms for the braiding in \mathcal{B} . The braiding axioms follow from those in \mathcal{B} .

Finally we construct a syllepsis: the components come from the syllepsis in \mathcal{B} , so the axioms (and the symmetry axiom) are immediate from the same axioms in \mathcal{B} . We just need to check the icon axioms, and these follow from coherence for symmetric monoidal bicategories.

This completes the proof that $\mathcal{B}\text{-Icon}$ is a symmetric monoidal bicategory as required. ■

4. Higher dimensions

Now that we know $\mathcal{B}\text{-Icon}$ is a symmetric monoidal bicategory, we can iterate the icon construction to produce the higher-dimensional structures that motivated this work. Recall that putting $\mathcal{B} = \mathbf{Cat}$ gives \mathbf{Icon} , a convenient 2-category of bicategories. The idea is that iterating this gives us convenient 2-categories of 3-categories, 4-categories, and so on. In fact these are not strict n -categories, nor are they semi-strict in the usual sense—while they can be thought of as “ n -categories with extra strictness conditions”, in fact the coherence constraints have different shapes. For some purposes this might not be a critical difference, but it is crucial for our study of totalities of degenerate structures. We are finally able to give a satisfying account of doubly degenerate tricategories, as promised in [7]; we refer the reader to that paper for a discussion of the pitfalls of more naive approaches.

This suggests a general scheme for studying totalities of k -degenerate $(n+k)$ -categories, and comparing them with entries in the “Periodic Table” of n -categories.

4.1. ITERATION. The main example we have in mind starts the iteration with **Cat**, but we make the following general definition of iterated icons.

4.2. DEFINITION. We define **B-nIcon** for each $n \geq 0$ as follows.

- **B-0Icon** = \mathcal{B} .
- **B-nIcon** = **(B-(n-1)Icon)-Icon**.

Recall that if \mathcal{B} is a symmetric monoidal 2-category (that is, a symmetrical monoidal bicategory whose underlying bicategory is a 2-category) then **B-Icon** is a symmetric monoidal 2-category. Thus starting with $\mathcal{B} = \mathbf{Cat}$ we get symmetric monoidal 2-categories of n -categories. These are a subtly semi-strict version of n -category in which low-dimensional composition is strict. We will now exhibit this explicitly for $\mathcal{B} = \mathbf{Icon}$ with monoidal structure given by cartesian product. This gives the 2-category **Icon-Icon**.

A 0-cell X is a “weak **Icon**-category” (see Definition 3.2). We now unpack this definition to reveal what sort of 3-dimensional structure is produced.

- a set X_0 of objects,
- for all $a, b \in X_0$ a bicategory $X(a, b)$,
- composition: for all $a, b, c \in X_0$ a weak functor

$$\begin{array}{rcl}
 m_{abc} : X(b, c) \times X(a, b) & \longrightarrow & X(a, c) \\
 (g, f) & \mapsto & g \otimes f \\
 (\beta, \alpha) & \mapsto & \beta \otimes \alpha,
 \end{array}$$

- identities: for all $a \in X_0$ a weak functor

$$\begin{array}{rcl}
 I_a : \mathbf{1} & \longrightarrow & X(a, a) \\
 * & \mapsto & I_a
 \end{array}$$

where here $\mathbf{1}$ is the terminal bicategory,

- associativity constraints: for all $a, b, c, d \in X_0$ an invertible icon

$$\begin{array}{ccc}
 (X(c, d) \times X(b, c)) \times X(a, b) & \xrightarrow{m_{b,c,d} \times 1} & X(b, d) \times X(a, b) \\
 \cong \downarrow & & \downarrow m_{a,b,d} \\
 X(c, d) \times (X(b, c) \times X(a, b)) & & \\
 \downarrow 1 \times m_{a,b,c} & \swarrow a_{a,b,c,d} & \\
 X(c, d) \times X(a, c) & \xrightarrow{m_{a,c,d}} & X(a, d)
 \end{array}$$

and

- unit constraints: for all a, b invertible icons

$$\begin{array}{ccc}
 X(a, b) \times \mathbf{1} & \xrightarrow{1 \times I_a} & X(a, b) \times X(a, a) & & X(b, b) \times X(a, b) & \xleftarrow{I_b \times 1} & \mathbf{1} \times X(a, b) \\
 & \searrow \cong & \downarrow m_{a,a,b} & & \downarrow m_{a,b,b} & & \swarrow \cong \\
 & & X(a, b) & & X(a, b) & & \\
 & & \nearrow r_{a,b} & & \nwarrow \ell_{a,b} & & \\
 & & & & & &
 \end{array}$$

satisfying the axioms given in Definition 3.2.

The key now is that since a, ℓ and r are icons, some additional coherence is implicit, as follows. Each of these constraint icons has a source and target functor that must agree on objects. So we must have, for all $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

$$\begin{aligned}
 (h \otimes g) \otimes f &= h \otimes (g \otimes f) \\
 f \otimes I_a &= f \\
 I_b \otimes f &= f
 \end{aligned}$$

that is, composition of 1-cells in the a weak **Icon**-category is strict.

Then the 2-cell components of the constraint icons give the following data:

- for all composable triples of 2-cells as shown

$$\begin{array}{ccccc}
 & f & & g & & h & \\
 a & \curvearrowright & b & \curvearrowright & c & \curvearrowright & d \\
 & \Downarrow \alpha & & \Downarrow \beta & & \Downarrow \gamma & \\
 & f' & & g' & & h' &
 \end{array}$$

invertible 3-cells

$$\begin{array}{ccc}
 & (\gamma \otimes \beta) \otimes \alpha & \\
 & \curvearrowright & \\
 (h \otimes g) \otimes f & & (h' \otimes g') \otimes f' \\
 h \otimes (g \otimes f) & \xrightarrow{a_{\gamma\beta\alpha}} & h' \otimes (g' \otimes f') \\
 & \curvearrowright & \\
 & \gamma \otimes (\beta \otimes \alpha) & \\
 & \alpha \otimes 1 & \\
 & \curvearrowright & \\
 f \otimes I & & f' \otimes I \\
 f & \xrightarrow{r_\alpha} & f' \\
 & \curvearrowright & \\
 & \alpha &
 \end{array}$$

$$\begin{array}{ccc}
 & 1 \otimes \alpha & \\
 & \curvearrowright & \\
 I \otimes f & & I \otimes f' \\
 \parallel & \ell_\alpha \Downarrow & \parallel \\
 f & & f' \\
 & \curvearrowleft & \\
 & \alpha &
 \end{array}$$

natural in all arguments.

It is straightforward to characterise the 1- and 2-cells in **Icon-Icon** in a similar fashion.

4.3. REMARKS. We might regard a weak **Icon**-category as a “tricategory in which 1-cell composition is strict” as in [20], where the notion is called “iconic tricategory”. However, iconic tricategories and weak **Icon**-categories have differently shaped constraint cells for the following reason. Recall that we think of icons as “identity component oplax natural transformations” although this is not strictly true, as the identity components are omitted from the data, so the 2-cell data have different shapes, yielding different, stricter, composition. Thus in a weak **Icon**-category, this phenomenon applies to the constraint icons. For example, an associator 3-cell would look like

$$\begin{array}{ccc}
 (h \otimes g) \otimes f & \xrightarrow{(\gamma \otimes \beta) \otimes \alpha} & (h' \otimes g') \otimes f' \\
 \downarrow I & \swarrow a_{\gamma\beta\alpha} & \downarrow I \\
 h \otimes (g \otimes f) & \xrightarrow{\gamma \otimes (\beta \otimes \alpha)} & h' \otimes (g' \otimes f')
 \end{array}$$

instead of the globular shape above, and then all the axioms involving it would be much more complicated to take into account the unit 1-cells at the sides. The correct statement should be that there is a bicategory of iconic tricategories and appropriate higher cells between them, which is biequivalent to **Icon-Icon**. Shulman [20] alludes to this but does not attempt the technicalities of producing the correct higher cells and their composition.

A weak **Icon**-category is not only stricter than a general tricategory but also more streamlined. This is crucially what enables us to cut down the dimensions of the totality, cleanly describe the correct higher cells, and produce the correct degenerate structures corresponding to those in the periodic table.

Note, however, that if we consider icons between 2-categories and 2-functors (so everything is strict), then these issues go away: icons in this case really are identity component oplax natural transformations. Similarly **Gray**-categories really are a strict form of weak **Icon**-category—those for which

- the homs are 2-categories,
- the composition and unit functors are cubical, and
- the icons a, ℓ, r are identities.

Recall that in the usual tricategory of bicategories, 1-cells compose strictly. In fact, a naturally occurring example of a (large) weak **Icon**-category is given by the totality of bicategories, as follows.

4.4. **THEOREM.** *There is a large weak **Icon**-category with*

- 0-cells all (small) bicategories, and
- given bicategories X, Y , $\mathbf{Bicat}(X, Y)$ is the bicategory of weak functors, weak transformations and modifications from X to Y .

PROOF. We must give weak functors for units and composition, and icons for associativity and unit constraints. It is not difficult to modify the proofs given in [12, Section 5.1] by omitting the relevant identity components. ■

4.5. **EXAMPLE.** The category of bicategories and weak functors is monoidal under the cartesian product. A category enriched in this category can be viewed as a weak **Icon**-category in which a, ℓ and r are identity icons. Thus not only is 1-cell composition strict, but all composition along bounding 0-cells is strict. An example of such a structure is the tricategory of topological spaces as constructed in [11].

4.6. **DEGENERACY.** We now show that the framework of **Icon-Icon** is correct for studying doubly degenerate 3-dimensional structures. That is, restricting **Icon-Icon** to the doubly degenerate 0-cells gives us the 2-category of braided monoidal categories.

In fact, a careful study must proceed via the intermediate structure of “2-tuply monoidal categories”. Essentially, these are categories with two monoidal structures and interchange up to isomorphism; precisely, they are pseudomonoids in the monoidal 2-category of monoidal categories and weak functors.

Joyal and Street state that 2-tuply monoidal categories are the same as braided monoidal categories [14, Proposition 5.3 and Example 5.4] but do not give full details about the totalities of such structures.

As we saw in [7], passing between these two types of structure is quite delicate in a way that has a serious impact on doubly degenerate case. For this reason we separate our comparison into two distinct steps.

1. Compare doubly degenerate weak **Icon**-categories with 2-tuply monoidal categories.
2. Compare 2-tuply monoidal categories with braided monoidal categories.

The present work makes the first step immediate, resolving the difficulties that arise from more naive approaches. For the second we refer to [14].

4.7. **THEOREM.** *Let \mathcal{J} be the full sub-2-category of **Icon-Icon** whose 0-cells are the doubly degenerate weak **Icon**-categories. That is, the weak **Icon**-categories with only one object, and whose single hom-bicategory has only one 0-cell. Let $\mathbf{2MonCat}$ be the 2-category of pseudomonoids in **MonCat**.*

Then there is a 2-equivalence $\mathcal{J} \simeq \mathbf{2MonCat}$.

PROOF. We characterise \mathcal{J} explicitly and will see that the only difference between \mathcal{J} and $\mathbf{2MonCat}$ is in a choice of unique 0- and 1-cell.

An object of \mathcal{J} consists of

- a monoidal category M with unit I , say,
- a monoidal functor

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{U} & M \\ * & \mapsto & u, \end{array}$$

- a monoidal functor $\boxtimes: M \times M \rightarrow M$,
- monoidal natural isomorphisms

$$\begin{array}{ccc} (x \boxtimes y) \boxtimes z & \xrightarrow{a_{x,y,z}} & x \boxtimes (y \boxtimes z) \\ x \boxtimes u & \xrightarrow{r_x} & x \\ u \boxtimes x & \xrightarrow{\ell_x} & x \end{array}$$

and axioms making (M, u, r, ℓ, a) into a monoidal category. This is exactly the structure of a 2-tuply monoidal category.

A 1-cell $F: M \rightarrow M'$ of \mathcal{J} consists of

- a monoidal functor $F: M \rightarrow M'$ (monoidal with respect to \otimes),
- a monoidal isomorphism

$$\begin{array}{ccc} M \times M & \xrightarrow{F \times F} & M' \times M' \\ \boxtimes \downarrow & \phi_2 \swarrow & \downarrow \boxtimes' \\ M & \xrightarrow{F} & M' \end{array}$$

- a monoidal isomorphism

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{U} & M \\ & \searrow U' & \downarrow F \\ & & M' \end{array}$$

and axioms making (F, ϕ_2, ϕ_0) into a monoidal functor with respect to \boxtimes and \boxtimes' .

A 2-cell

$$\begin{array}{ccc}
 & F & \\
 M & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & M' \\
 & F' &
 \end{array}$$

of \mathcal{J} consists of a monoidal transformation α with respect to \otimes (because icons between degenerate bicategories are precisely monoidal transformations between the corresponding monoidal categories, together), and axioms making it monoidal with respect to \boxtimes .

Thus once we have forgotten the unique 0- and 1-cells from the doubly degenerate objects of \mathcal{J} , we have exactly the 0-, 1- and 2-cells of **2MonCat**. \blacksquare

Note that we could perform the icon construction again to produce the 2-category **Cat-3Icon = Icon-Icon-Icon**. Triply degenerate 0-cells of this are 3-tuply monoidal categories; Joyal and Street [14] observe that these correspond to symmetric monoidal categories.

4.8. **THEOREM.** *Let \mathcal{K} be the full sub-2-category of **Icon-Icon-Icon** whose 0-cells are the triply degenerate ones. Let **3MonCat** be the 2-category of pseudomonoids in **2MonCat**. Then there is a 2-equivalence $\mathcal{K} \simeq \mathbf{3MonCat}$.*

We now propose a general construction that should provide the correct totality of k -degenerate $(n+k)$ -categories for every n and k . The idea is that *a priori* the k -degenerate $(n+k)$ -categories form an $(n+k+1)$ -category, but when we regard them as n -categories with extra structure, they should form an $(n+1)$ -category; naive methods of making this dimension change do not produce the correct structure. However, iterating the icon construction does produce the correct structure.

The generalised icon construction described in this paper gives the case $n = 1$. The case for $n > 1$ involves the $(n+1)$ -category of weak n -categories, a structure that is not very well understood at present. Nevertheless, we conjecture the following generalisations of the above constructions.

4.9. **CONJECTURE.** *Given a symmetric monoidal $(n+1)$ -category \mathcal{E} , we can define a symmetric monoidal $(n+1)$ -category $\mathcal{E}\text{-Icon}$ of “categories weakly enriched in \mathcal{E} ”, weak functors, and icon-like higher morphisms.*

Note also that the weakness of the enrichment increases as n increases.

However, the higher morphisms must be iconic only for the lowest dimension of component, indexed by 0-cells. For example, a transformation between tricategories is in general given by

- for every 0-cell in the source, a 1-cell in the target,
- for every 1-cell in the source, a 2-cell in the target, and
- for every 2-cell in the source, a 3-cell in the target.

Thus the icon version omits the 1-cell data, and alters the shape of the 2- and 3-cell data. Likewise, a modification between tricategories is in general given by

- for every 0-cell in the source, a 2-cell in the target, and
- for every 1-cell in the source, a 3-cell in the target,

so an “icon-like” modification would omit the 2-cell data, and alter the shape of the 3-cell data accordingly. For higher cells with more dimensions of icon-like behaviour, we need to iterate the construction. We propose the following definition, although the formalism to make it precise does not yet exist.

4.10. CONJECTURE. *Fix $n \geq 1$. We define a symmetric monoidal $(n + 1)$ -category $\mathcal{E}_{n,k}$ for each $k \geq 0$ as follows.*

- $\mathcal{E}_{n,0} = n\text{-Cat}$, and
- $\mathcal{E}_{n,k} = \mathcal{E}_{n,k-1}\text{-Icon}$.

Thus objects of $\mathcal{E}_{n,k}$ are a bit like “slightly strict” $(n + k)$ -categories.

4.11. REMARK. This indexing is convenient for considering the k -degenerate version, but another point of view is to fix the total dimension and ask what lower dimensional totalities of m -category we have, lower than the canonical $(m + 1)$ -category thereof. The idea is that we should now get a k -category of m -categories for every $k \leq m$ as well, but for different values of k we have different starting points in the above iteration. Thus in each case we have a different notion of slightly strict m -category. This is different from the approach of [9] in which low-dimensional totalities of tricategories are studied but the tricategories and functors remain fully weak; constructing a tricategory of such is then very complex.

Finally we make the following conjectures for k -degenerate $(n + k)$ -categories ones.

4.12. CONJECTURE. *Given a symmetric monoidal $(n+1)$ -category \mathcal{E} there is a symmetric monoidal $(n + 1)$ -category $\mathbf{Mon}(\mathcal{E})$ of weak monoids in \mathcal{E} . Iterating this construction, we write*

- $\mathbf{0Mon}(\mathcal{E}) = \mathcal{E}$, and
- $\mathbf{(k+1)Mon}(\mathcal{E}) = \mathbf{Mon(kMon}(\mathcal{E}))$.

4.13. CONJECTURE. *Write $\mathcal{J}_{n,k}$ for the full sub $(n + 1)$ -category of $\mathcal{E}_{n,k}$ whose objects are the k -degenerate ones. Then there is an equivalence of $(n + 1)$ -categories*

$$\mathcal{J}_{n,k} \simeq \mathbf{kMon}(n\text{-Cat}).$$

It remains to compare the k -tuply monoidal n -categories with the appropriate entry in the Periodic Table.

References

- [1] John Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Jour. Math. Phys.*, 36:6073–6105, 1995.
- [2] Dror Bar-Natan. Non-associative tangles. *AMS/IP Stud. Adv. Math.*, 2.1:139–183, 1997.
- [3] Michael Batanin and Mark Weber. Algebras of higher operads as enriched categories. *Applied Categorical Structures*, 19(1):93–135, 2011.
- [4] Eugenia Cheng. Comparing operadic theories of n -category. *Homology, Homotopy and Applications*, 13(2):217–249, 2011. Also E-print 0809.2070.
- [5] Eugenia Cheng and Nick Gurski. The periodic table of n -categories for low dimensions I: degenerate categories and degenerate bicategories. In Batanin, Davydov, Johnson, Lack, and Neeman, editors, *Categories in Algebra, Geometry and Mathematical Physics, proceedings of Streetfest*, volume 431 of *Contemporary Math.*, pages 143–164. AMS, 2007. E-print 0708.1178.
- [6] Eugenia Cheng and Nick Gurski. Iterated icons, 2010. Presented at PSSL90, Braunschweig and CT2010, Genova.
- [7] Eugenia Cheng and Nick Gurski. The periodic table of n -categories for low dimensions II: degenerate tricategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 52:82–125, 2011. E-print 0706.2307.
- [8] Richard Garner. Homomorphisms of higher categories. *Advances in Mathematics*, 224(6):2269–2311, 2010.
- [9] Richard Garner and Nick Gurski. The low-dimensional structures formed by tricategories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146:551–589, 2009.
- [10] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion. E-print 1301.3191.
- [11] Nick Gurski. Loop spaces, and coherence for monoidal and braided monoidal bicategories. *Adv. Math.*, 226(5):4225–4265, 2011.
- [12] Nick Gurski. *Coherence in three-dimensional category theory*. Number 201 in Cambridge Tracts in Mathematics. Cambridge University Press, 2013.
- [13] Nick Gurski and Angélica Osorno. Infinite loop spaces, and coherence for symmetric monoidal bicategories. *Adv. Math.*, 246:1–32, 2013.

- [14] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102:20–78, 1983.
- [15] Stephen Lack. Icons. *Applied Categorical Structures*, 18(3):289–307, 2010. E-print 0711.4657.
- [16] Tom Leinster. A survey of definitions of n -category. *Theory and Applications of Categories*, 10:1–70, 2002.
- [17] Steve Lack and Simona Paoli. 2-nerves for bicategories. *K-Theory*, 38(2):153–175, 2008. E-print math.CT/0607271.
- [18] Christopher Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, Massachusetts Institute of Technology, 2009.
- [19] Carlos Simpson. *Homotopy Theory of Higher Categories*. No. 19 in New Mathematical Monographs, Cambridge University Press, 2011.
- [20] Michael Shulman. Not every pseudoalgebra is equivalent to a strict one. *Adv. Math.*, 229(3):2024–2041, 2012.
- [21] Todd Trimble. What are ‘fundamental n -groupoids’?, 1999. seminar at DPMMS, Cambridge, 24 August 1999.

*Department of Mathematics,
University of Sheffield*

Email: `e.cheng@sheffield.ac.uk`

`nick.gurski@sheffield.ac.uk`

This article may be accessed at <http://www.tac.mta.ca/tac/>

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at <http://www.tac.mta.ca/tac/> and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS The typesetting language of the journal is \TeX , and $\text{\LaTeX}2\epsilon$ strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at <http://www.tac.mta.ca/tac/>.

MANAGING EDITOR. Robert Rosebrugh, Mount Allison University: rrosebrugh@mta.ca

\TeX EDITOR. Michael Barr, McGill University: barr@math.mcgill.ca

ASSISTANT \TeX EDITOR. Gavin Seal, Ecole Polytechnique Fédérale de Lausanne: gavin_seal@fastmail.fm

TRANSMITTING EDITORS.

Clemens Berger, Université de Nice-Sophia Antipolis: cberger@math.unice.fr

Richard Blute, Université d' Ottawa: rblute@uottawa.ca

Lawrence Breen, Université de Paris 13: breen@math.univ-paris13.fr

Ronald Brown, University of North Wales: [ronnie.profbbrown\(at\)btinternet.com](mailto:ronnie.profbbrown(at)btinternet.com)

Valeria de Paiva: valeria.depaiva@gmail.com

Ezra Getzler, Northwestern University: [getzler\(at\)northwestern\(dot\)edu](mailto:getzler(at)northwestern(dot)edu)

Kathryn Hess, Ecole Polytechnique Fédérale de Lausanne: kathryn.hess@epfl.ch

Martin Hyland, University of Cambridge: M.Hyland@dpmms.cam.ac.uk

Anders Kock, University of Aarhus: kock@imf.au.dk

Stephen Lack, Macquarie University: steve.lack@mq.edu.au

F. William Lawvere, State University of New York at Buffalo: wlawvere@buffalo.edu

Tom Leinster, University of Edinburgh: Tom.Leinster@ed.ac.uk

Ieke Moerdijk, Radboud University Nijmegen: i.moerdijk@math.ru.nl

Susan Niefield, Union College: niefiels@union.edu

Robert Paré, Dalhousie University: pare@mathstat.dal.ca

Jiri Rosicky, Masaryk University: rosicky@math.muni.cz

Giuseppe Rosolini, Università di Genova: rosolini@disi.unige.it

Alex Simpson, University of Edinburgh: Alex.Simpson@ed.ac.uk

James Stasheff, University of North Carolina: jds@math.upenn.edu

Ross Street, Macquarie University: street@math.mq.edu.au

Walter Tholen, York University: tholen@mathstat.yorku.ca

Myles Tierney, Rutgers University: tierney@math.rutgers.edu

Robert F. C. Walters, University of Insubria: rfcwalters@gmail.com

R. J. Wood, Dalhousie University: rjwood@mathstat.dal.ca